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## ON $H^p$ -SOLUTIONS OF THE BEZOUT EQUATION

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**We obtain a sufficient condition on bounded holomorphic functions  $g_1, g_2$  in the unit disk for the existence of  $f_1, f_2$  in the Hardy space  $H^p$  such that  $1 = f_1g_1 + f_2g_2$ . The sharpness of this condition is also studied.**

1. Let  $\mathbb{D}$  be the unit disk in the complex plane,  $\mathbb{T}$  its boundary. For  $1 \leq p \leq \infty$ ,  $H^p$  denotes the Hardy-space of holomorphic functions in  $\mathbb{D}$  such that

$$\|f\|_p = \sup_r \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < +\infty \quad p < \infty$$

$$\|f\|_\infty = \sup_{|z| < 1} |f(z)|.$$

It is well-known ([7, p. 57]) that if  $f \in H^p$ , the non-tangential maximal function

$$Mf(e^{i\theta}) = \sup\{|f(z)|; z \in \Gamma(\theta)\}$$

$\Gamma(\theta)$  being the Stolz angle with vertex  $e^{i\theta}$ , belongs to  $L^p(\mathbb{T})$ .

In this paper, given  $g_1, g_2 \in H^\infty$ , we study the Bezout equation  $1 = f_1g_1 + f_2g_2$ . Concretely, we are interested in knowing the precise condition on  $g_1, g_2$  so that solutions  $f_1, f_2 \in H^p$  exist.

If  $|g|^2 = |g_1|^2 + |g_2|^2$ ,  $|f|^2 = |f_1|^2 + |f_2|^2$ , it follows from  $1 = f_1g_1 + f_2g_2$  that  $1 \leq |f||g|$  and hence

$$(C) \quad M(|g|^{-1}) \in L^p(\mathbb{T}).$$

It can be easily seen that this condition is sufficient if  $g_1$  or  $g_2$  is an interpolating Blaschke product. Nevertheless, we show in Section 2 that it is not sufficient in general. In fact for each  $\varepsilon > 0$  we exhibit  $g_1, g_2 \in H^\infty$  such that  $M(|g|^{-2+\varepsilon}) \in L^p(\mathbb{T})$  but no  $H^p$  solutions exist.

In Section 3 we obtain a general sufficient condition implying in particular the following:

**Theorem 1.** *Assume that for some  $\varepsilon > 0$*

$$M(|g|^{-2} |\log |g||^{2+\varepsilon}) \in L^p(\mathbb{T}).$$

Then there exist  $f_1, f_2 \in H^p$  such that  $1 = f_1g_1 + f_2g_2$ .

In Section 4, it is shown that the same method gives the following improvement on the problem considered by Wolff and also by Cegrell in [4].

**Theorem 2.** *Let  $f, g_1, g_2 \in H^\infty$  be such that*

$$|f| \leq \frac{|g|^2}{|\log |g||^{2+\varepsilon}} \quad \text{for some } \varepsilon > 0.$$

Then there are  $f_1, f_2 \in H^\infty$  such that  $f = f_1g_1 + f_2g_2$ .

The proofs rely essentially on: (a) An  $L^p$ -version of Wolff’s criteria for the existence of bounded solutions of the  $\bar{\partial}$ -equation, already used in [1]. (b) An improvement of Cegrell’s result in [4] on gradients of bounded holomorphic functions.

Both theorems hold of course for more than two generators, using the Koszul complex technique as in [7, p. 364]. Theorem 1 holds as well in the setting of the unit ball, but some modifications are needed (see [2]).

Finally, we mention that similar results to those stated here have been independently obtained by K.C. Lin in [8] and [9]. The authors thank the referee for pointing this out to them.

**2.** Before proceeding, we recall that a positive measure  $\mu$  on  $\mathbb{D}$  is called a *Carleson measure* if there exists  $K > 0$  such that

$$\mu(\{z : |z - e^{i\theta}| \leq r\}) \leq Kr \quad e^{i\theta} \in \mathbb{T}, \quad r > 0.$$

The smallest of such  $K$  is called the Carleson norm of  $\mu$ . Equivalently ([7, p. 32])  $\mu$  is a Carleson measure if and only if for all functions  $h$  in  $\mathbb{D}$

$$\iint_{\mathbb{D}} |h| d\mu \leq c \int_0^{2\pi} M(h) d\theta.$$

In some particular cases it is quite easy to see that the condition (C) is sufficient. For instance, if  $g_1$  is a Blaschke product with zeros  $z_n$ , the question is obviously equivalent to the interpolation problem

$$f_2(z_n) = \frac{1}{g_2(z_n)}, \quad \text{with } f_2 \in H^p.$$

Indeed,  $1 - f_2g_2$  belongs then to  $H^p$  and vanishes on  $\{z_n\}$ , so  $1 - f_2g_2 = f_1g_1$ ,  $f_1 \in H^p$ . In case  $g_1$  is an interpolating Blaschke product, this interpolation problem has a solution if and only if

$$\sum_n \frac{1}{|g_2(z_n)|^p} (1 - |z_n|) < +\infty,$$

(see [10] and also [5]). Let  $\delta_n$  denote the delta-mass at the point  $z_n$ . Since  $\sum(1 - |z_n|)\delta_n$  is a Carleson measure, (C) implies the above condition.

Next, we give examples showing that condition (C) is far from being sufficient.

**Theorem 3.** *Given  $1 \leq p < \infty$  and  $\varepsilon > 0$ , there exist bounded analytic functions  $g_1, g_2$  with  $M(|g|^{-2+\varepsilon}) \in L^p(\mathbb{T})$ , such that there exist no  $f_1, f_2 \in H^p$  satisfying  $f_1g_1 + f_2g_2 \equiv 1$ .*

*Proof.* We will denote by  $\rho(z, w)$  the pseudo-hyperbolic distance in the unit disc,  $\rho(z, w) = |z - w| |1 - \bar{w}z|^{-1}$ ,  $z, w \in \mathbb{D}$  and  $f^{(j)}$  the  $j$ -th derivative of a function  $f$ . Let  $N$  be a natural number such that  $(N + 1)\varepsilon > 1$ .

Let  $z_n = 1 - 2^{-n}$ ,  $n \geq 1$ , and take an  $H^\infty$ -interpolating sequence  $\{\alpha_n\}$ ,  $0 < \rho(\alpha_n, z_n)$  decreasing to 0, satisfying

$$(1) \quad \sum_n (1 - |z_n|)\rho(\alpha_n, z_n)^{-(N+1)p(2-\varepsilon)} < \infty,$$

$$(2) \quad \sum_n (1 - |z_n|)\rho(\alpha_n, z_n)^{-(2N+1)p} = \infty.$$

Let  $B_1$  and  $B_2$  be the Blaschke products with zeros  $\{z_n\}$  and  $\{\alpha_n\}$ . From now on, the letter  $c$  will denote different constants independent on  $n$ . Since  $B_2$  is an interpolating Blaschke product, one has

$$\inf_n \rho(z, \alpha_n) \geq |B_2(z)| \geq c \inf_n \rho(z, \alpha_n), \quad |z| < 1,$$

(see [7, p. 404]).

Now as in [3] take  $g_i = B_i^{N+1}$ ,  $i = 1, 2$ . Let  $I_n$  be the arc on the unit circle centered at 1 of length  $2(1 - |z_n|) = 2^{-n+1}$ . In estimating  $|g(z)|^{-1}$ , for  $z \in \Gamma(\theta)$ , the worst case occurs when  $z$  is one of the  $\{z_n\}$  or  $\{\alpha_n\}$ . Since  $\rho(\alpha_n, z_n)$  is decreasing, for  $e^{i\theta} \in I_n \setminus I_{n+1}$  one has

$$M(|g|^{-1}) \leq \frac{c}{\rho(\alpha_{n+1}, z_{n+1})^{N+1}}.$$

Hence, condition (1) implies  $M(|g|^{-2+\varepsilon}) \in L^p(\mathbb{T})$ . Now, assume there exist  $f_1, f_2 \in H^p$  satisfying  $f_1g_1 + f_2g_2 \equiv 1$ . Then,

$$f_2^{(N)}(z_n) = (B_2^{-N-1})^{(N)}(z_n), \quad n \geq 1.$$

Write

$$B_{2,n}(z) = \prod_{k \neq n} \frac{-\bar{\alpha}_k}{|\alpha_k|} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z}, \quad B_2^{-N-1}(z) = h(z)k(z)$$

where  $h(z) = (1 - \bar{\alpha}_n z)^{N+1} B_{2,n}(z)^{-N-1}$ ,  $k(z) = (z - \alpha_n)^{-N-1}$ . Then

$$(B_2^{-N-1})^{(N)}(z) = \sum_{j=0}^N \binom{N}{j} h^{(j)}(z) k^{(N-j)}(z).$$

Using Cauchy's formula on the disk of center  $z_n$  and radius  $4^{-1}(1 - |z_n|)$ , one gets

$$|h^{(j)}(z_n)| \leq c \frac{(1 - |\alpha_n|)^{N+1}}{(1 - |z_n|)^j} \leq c(1 - |z_n|)^{N+1-j}$$

and hence

$$\begin{aligned} |h^{(j)}(z_n)| |k^{(N-j)}(z_n)| &\leq c |z_n - \alpha_n|^{-2N+j-1} (1 - |z_n|)^{N+1-j} \\ &\leq c \rho(z_n, \alpha_n)^{-2N+j-1} (1 - |z_n|)^{-N}. \end{aligned}$$

For  $j = 0$ , one gets

$$\begin{aligned} |h(z_n)| |k^{(N)}(z_n)| &\geq c (1 - |z_n|)^{N+1} |z_n - \alpha_n|^{-2N-1} \\ &\geq c \rho(\alpha_n, z_n)^{-2N-1} (1 - |z_n|)^{-N}. \end{aligned}$$

Therefore, for large  $n$ ,

$$(3) \quad |f_2^{(N)}(z_n)| = |(B_2^{-N-1})^{(N)}(z_n)| \geq c (1 - |z_n|)^{-N} \rho(\alpha_n, z_n)^{-2N-1}.$$

Since  $f_2 \in H^p$ , the function

$$F(e^{i\theta}) = \left( \int_{\Gamma(\theta)} |f_2^{(N)}(z)|^2 (1 - |z|)^{2N-2} dm(z) \right)^{1/2}$$

belongs to  $L^p(\mathbb{T})$  ([11, p. 216]). For  $e^{i\theta} \in I_n \setminus I_{n+1}$ , since  $D_n = \{z \in \mathbb{D} : |z - z_n| < 4^{-1}(1 - |z_n|)\} \subset \Gamma(\theta)$ , one has

$$\begin{aligned} |F(e^{i\theta})|^2 &\geq \int_{D_n} |f_2^{(N)}(z)|^2 (1 - |z|)^{2N-2} dm(z) \\ &\geq c (1 - |z_n|)^{2N} |f_2^{(N)}(z_n)|^2, \quad e^{i\theta} \in I_n \setminus I_{n+1}. \end{aligned}$$

Using (3) and  $F \in L^p(\mathbb{T})$ , one gets

$$\infty > \sum (1 - |z_n|) \rho(\alpha_n, z_n)^{-(2N+1)p}$$

and this contradicts (2). □

**3.** In this section we will prove a generalization of Theorem 1 stated in the introduction.

**Lemma 1.** *If  $g$  is holomorphic on  $\overline{\mathbb{D}}$  and  $0 < p \leq 2$ ,*

$$\int_0^{2\pi} \{|g(e^{i\theta})|^p - |g(0)|^p\} \frac{d\theta}{2\pi} \leq 4 \int_0^{2\pi} |g(e^{i\theta}) - g(0)|^p \frac{d\theta}{2\pi}.$$

*Proof.* This is a general statement for a probability measure  $d\mu$  on  $X$  and measurable  $\varphi$

$$\int_X |\varphi|^p d\mu - \left| \int_X \varphi d\mu \right|^p \leq 4 \int_X \left| \varphi - \int_X \varphi d\mu \right|^p d\mu.$$

First notice that this is trivial for  $p \leq 1$  (with constant 1) and that for  $p = 2$  there is equality with constant 1, too. In general, and assuming without loss of generality that  $\int_X \varphi d\mu = 1$ , it follows for real  $\varphi$  integrating the inequality

$$|\varphi|^p - 1 \leq 3|\varphi - 1|^p + p(\varphi - 1).$$

For complex-valued  $\varphi = \varphi_1 + i\varphi_2$ , it follows from  $|\varphi|^p \leq |\varphi_1|^p + |\varphi_2|^p$  (for positive  $\varphi$  the inequality holds with constant 1). □

We start with a generalization of a result in [4]. Although we need it only for  $H^\infty$  functions we state it in full generality, for  $BMOA$  functions (see [7] for definitions). We denote by  $\|g\|_*$  the  $BMO$  norm of  $g(e^{i\theta})$ .

Let  $d\mu$  be a positive measure on  $[0, 1)$  such that  $\int_0^1 \frac{d\mu(\alpha)}{\alpha^2} < +\infty$ , and write

$$\tilde{\mu}(x) = \int_0^1 x^\alpha d\mu(\alpha), \quad x > 0.$$

**Lemma 2.** *If  $g \in BMOA$ ,  $\frac{|g'|^2}{|g|^2} \tilde{\mu}(|g|^2)(1 - |z|^2)$  is a Carleson measure with Carleson norm bounded by  $K\|g\|_*$ ,  $K$  depending on  $\mu$ .*

*Proof.* We consider the function

$$G(z) = \int_0^1 \frac{|g(z)|^{2\alpha}}{\alpha^2} d\mu(\alpha), \quad |z| < 1.$$

For  $\alpha > 0$ , a computation shows that  $\Delta|g|^{2\alpha} = 4\alpha^2|g'|^2|g|^{2\alpha-2}$  when  $g \neq 0$ , hence

$$\Delta G = 4|g'|^2|g|^{-2}\tilde{\mu}(|g|^2).$$

Without loss of generality we can assume that  $g$  is holomorphic on  $\overline{\mathbb{D}}$ . We argue like in [7, p. 327]. Let  $z_1, \dots, z_N$  be the non-zero zeros in  $\mathbb{D}$ , and let  $\Omega_\varepsilon$  be the domain  $\mathbb{D} \setminus \bigcup_{j=0}^N \Delta_j$  where  $\Delta_0 = \{|z| \leq \varepsilon\}$ ,  $\Delta_j = \{|z - z_j| \leq \varepsilon\}$ ,  $j = 1, \dots, N$ .

By Green's formula applied to the function  $G$  in  $\Omega_\varepsilon$

$$4 \iint_{\Omega_\varepsilon} |g'|^2 |g|^{-2} \tilde{\mu}(|g|^2) \log \frac{1}{|z|} dA(z) = \int_0^{2\pi} G(e^{i\theta}) d\theta - \sum_{j=0}^N \int_{\partial\Delta_j} \left( \frac{\partial}{\partial n} G \right) \log \frac{1}{|z|} - |G| \frac{\partial}{\partial n} \left( \log \frac{1}{|z|} \right) ds.$$

Let  $r = |z - z_j|$ ; then for  $z$  close to  $z_j$ ,

$$|g'(z)|^2 |g(z)|^{-2} \tilde{\mu}(|g(z)|^2) \leq Cr^{-2} \tilde{\mu}(r^2).$$

Now, the hypothesis on  $\mu$  translates to

$$\int_0^1 \frac{\tilde{\mu}(x)}{x} |\log x| dx < +\infty.$$

Hence  $|g'|^2 |g|^{-2} \tilde{\mu}(|g|^2) \log \frac{1}{|z|}$  is integrable on  $\mathbb{D}$ . Also, for  $|z - z_j| = \varepsilon$

$$|G(z)| \leq c \int_0^1 \frac{\varepsilon^{2\alpha}}{\alpha^2} d\mu(\alpha)$$

which tends to 0 when  $\varepsilon \rightarrow 0$ , and

$$|\nabla G(z)| \leq \frac{c}{\varepsilon} \int_0^1 \frac{\varepsilon^{2\alpha}}{\alpha} d\mu(\alpha)$$

which also tends to zero when multiplied by  $\varepsilon |\log \varepsilon|$ . At zero we obtain  $-2\pi |G(0)|$  as limit when  $\varepsilon \rightarrow 0$ . Therefore

$$\begin{aligned} & \frac{2}{\pi} \iint_{\mathbb{D}} |g'(z)|^2 |g(z)|^{-2} \tilde{\mu}(|g(z)|^2) \log \frac{1}{|z|} dA(z) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \frac{|g(e^{i\theta})|^{2\alpha}}{\alpha^2} d\mu(\alpha) d\theta - \int_0^1 \frac{|g(0)|^{2\alpha}}{\alpha^2} d\mu(\alpha) \\ &= \int_0^1 \frac{d\mu(\alpha)}{\alpha^2} \int_0^{2\pi} \{ |g(e^{i\theta})|^{2\alpha} - |g(0)|^{2\alpha} \} \frac{d\theta}{2\pi} \leq \quad (\text{by Lemma 1}) \\ &\leq 4 \int_0^1 \frac{d\mu(\alpha)}{\alpha^2} \int_0^{2\pi} |g(e^{i\theta}) - g(0)|^{2\alpha} \frac{d\theta}{2\pi} \leq \int_0^1 \frac{d\mu(\alpha)}{\alpha^2} (c \|g\|_*)^{2\alpha}. \end{aligned}$$

If  $\psi_w$  is an automorphism of the disc, applying this inequality to  $g \circ \psi_w$ , changing variables in the area integral and using the invariance of the *BMO* norm we get

$$\begin{aligned} & \sup_{w \in \mathbb{D}} \iint_{\mathbb{D}} |g'(z)|^2 |g(z)|^{-2} \tilde{\mu}(|g(z)|^2) \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - z\bar{w}|^2} dA(z) \\ & \leq c \int_0^1 \frac{\|g\|_*^{2\alpha}}{\alpha^2} d\mu(\alpha) < +\infty, \end{aligned}$$

and the result follows, by [7, p. 239]. □

Taking for  $\mu$  a delta-mass at  $\varepsilon$  we get Cegrell’s result [4] that  $|g'|^2 |g|^{\varepsilon-2} (1 - |z|^2)$  is a Carleson measure. Taking  $d\mu(\alpha) = \alpha^{1+\varepsilon} d\alpha$  one gets that

$$\frac{|g'|^2}{|g|^2 |\log |g||^{2+\varepsilon}} (1 - |z|^2)$$

is a Carleson measure for every  $\varepsilon > 0$ .

Next lemma is the  $L^p$ -version of Wolff’s criteria for bounded solutions of the  $\bar{\partial}$ -equation ([7, p. 322]).

**Lemma 3.** *Let  $1 \leq p \leq \infty$ , let  $G$  be a  $C^1$  function in  $\bar{\mathbb{D}}$  such that:*

- (a)  $G = \varphi_1 \psi_1$ , where  $M(\varphi_1) \in L^p$  and  $|\psi_1|^2 \log \frac{1}{|z|}$  is a Carleson measure.
- (b)  $\partial G = \varphi_2 \psi_2$ , where  $M(\varphi_2) \in L^p$  and  $|\psi_2| \log \frac{1}{|z|}$  is a Carleson measure.

*Then there exists a  $C^1$  function  $u$  in  $\mathbb{D}$ , continuous on  $\bar{\mathbb{D}}$  such that*

$$\frac{\partial u}{\partial \bar{z}} = G$$

and

$$\int_0^{2\pi} |u(e^{i\theta})|^p d\theta \leq C$$

where  $C$  depends only of the  $L^p$ -norms of  $M(\varphi_1), M(\varphi_2)$  and the Carleson norms of the measures in (a), (b).

*Proof.* We adapt Wolff’s proof for the case  $p = \infty$ . Let  $q$  be the conjugate exponent of  $p$ ,  $1 < q \leq \infty$ . By duality,

$$\inf \left\{ \|b\|_p : \frac{\partial b}{\partial \bar{z}} = G \right\} = \sup \left\{ \left| \frac{1}{2\pi} \int_0^{2\pi} F k d\theta \right| : k \in H_0^q, \|k\|_q \leq 1 \right\}$$

where  $F$  is a priori solution, say the one given by the Cauchy kernel, which is continuous on  $\overline{\mathbb{D}}$ . By Green's formula

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} Fk d\theta &= \frac{1}{2\pi} \iint_{\mathbb{D}} \Delta(Fk) \log \frac{1}{|z|} dA(z) \\ &= \frac{2}{\pi} \iint_{\mathbb{D}} k'(z)G(z) \log \frac{1}{|z|} dA(z) + \frac{2}{\pi} \iint_{\mathbb{D}} k(z) \frac{\partial G}{\partial z} \log \frac{1}{|z|} dA(z) = I_1 + I_2. \end{aligned}$$

We will prove now that if  $|\psi|^2 \log \frac{1}{|z|}$  is a Carleson measure with constant  $K$ , then

$$(4) \quad \iint_{\mathbb{D}} |k'(z)| |\varphi(z)| |\psi(z)| \log \frac{1}{|z|} dA(z) \leq C \|k\|_q \|M\varphi\|_p K$$

where  $C$  is an absolute constant. This will imply the required bound for  $I_1$ . For  $p = \infty, q = 1$  this holds true as shown by Wolff reducing the situation to  $k = g^2$  with  $g \in H^2$ . Alternatively a real-analysis proof can be obtained using the inequality, following from [6, Th. 1],

$$\iint_{\mathbb{D}} |k'(z)| |\psi(z)| \log \frac{1}{|z|} dA(z) \leq \int_0^{2\pi} A(k)(e^{i\theta})^{1/2} C(\psi)(e^{i\theta}) d\theta$$

where  $A(k)$  is the area function of  $k$

$$A(k)(e^{i\theta}) = \left( \iint_{\Gamma(e^{i\theta})} |k'(z)|^2 dA(z) \right)^{1/2}$$

and  $C(\psi)$  is given by

$$C(\psi)(e^{i\theta}) = \sup_{e^{i\theta} \in I} \left( \frac{1}{|I|} \iint_I |\psi|^2 \log \frac{1}{|z|} dA \right)^{1/2}$$

$\hat{I}$  being the tent over  $I$ . This method applies to situations where there is no factorization.

For  $p = 1, q = \infty$ , we use Schwarz inequality to bound the left member of (4) by

$$\left( \iint_{\mathbb{D}} |\varphi| |k'|^2 \log \frac{1}{|z|} dA(z) \right)^{1/2} \left( \iint_{\mathbb{D}} |\varphi| |\psi|^2 \log \frac{1}{|z|} dA(z) \right)^{1/2}$$

If  $k \in BMOA$ ,  $|k'|^2 \log \frac{1}{|z|} dA$  is a Carleson measure; since Carleson measures operate on functions with integrable non-tangential maximal function, (4) follows for  $p = 1, q = \infty$ . Next, consider the operator, for fixed  $\psi$

$$\varphi \mapsto L_\varphi$$

where  $L_\varphi(k) = \iint_{\mathbb{D}} k' \varphi \psi \log \frac{1}{|z|} dA(z)$ ; let  $T_\infty^p$  be the tent space ([6])

$$T_\infty^p = \{\varphi : M(\varphi) \in L^p\}.$$

We have shown that  $L$  is bounded from  $T_\infty^1$  to  $(BMOA)^*$  and from  $T_\infty^\infty$  to  $(H^1)^*$ . By interpolation, we conclude that  $L$  is bounded from  $T_\infty^p$  to  $(H^q)^*$  i.e.

$$|L_\varphi(k)| \leq C \|k\|_q \|M\varphi\|_p K$$

(alternatively,  $\varphi$  can be replaced by the harmonic extension of  $M\varphi$  and argue with the  $L^p$ -spaces rather than the tent spaces).

It remains to bound  $I_2$ . But

$$|I_2| \leq \iint_{\mathbb{D}} |k(z)| |\varphi_2(z)| |\psi_2(z)| \log \frac{1}{|z|} dA(z)$$

and this is easier: just note that  $M(k\varphi_2) \leq M(k)M(\varphi_2)$  is in  $L^1$  and use again that Carleson measures operate on such functions.  $\square$

Note that the lemma holds if  $G, \partial G$  are linear combinations  $\sum \varphi_i \psi_i$  with  $\varphi_i, \psi_i$  as above.

**Theorem 4.** *Let  $g_1, g_2 \in H^\infty$  such that  $|g|^2 = |g_1|^2 + |g_2|^2 > 0$ . Let  $\mu$  and  $\tilde{\mu}$  be as above. Assume that*

$$M\left(\frac{1}{|g|^2} \frac{1}{\tilde{\mu}(|g|^2)}\right) \in L^p(\mathbb{T}).$$

*Then there are  $f_1, f_2 \in H^p$  such that  $f_1 g_1 + f_2 g_2 = 1$ .*

*Proof.* By a standard regularization argument we may assume that  $g_1, g_2$  are holomorphic on  $\overline{\mathbb{D}}$ . The smooth solutions

$$\varphi_i = \frac{\overline{g}_i}{|g|^2}, \quad i = 1, 2$$

satisfy  $M(\varphi_i) \in L^p$  and the general holomorphic solutions are given by

$$f_1 = \varphi_1 + u g_2 \quad f_2 = \varphi_2 - u g_1$$

where  $u$  satisfies

$$\frac{\partial u}{\partial \bar{z}} = \frac{\overline{g}_1 \overline{g}_2 - \overline{g}_1 \overline{g}_2}{|g|^4} \stackrel{def}{=} G.$$

We need only check that  $G$  satisfies the hypothesis of Lemma 3. For (a) we can take, by Lemma 2,  $\psi_1$  to be

$$\psi_1 = \frac{\overline{g}_i}{|g|} \tilde{\mu}(|g|^2)^{1/2}, \quad |\psi_1| \leq \frac{|g'_i|}{|g_i|} \tilde{\mu}(|g_i|^2)^{1/2}$$

and

$$\varphi_1 = \frac{\overline{g_j}}{|g|^{3\tilde{\mu}}(|g|^2)^{1/2}}, \quad |\varphi_1| \leq \frac{1}{|g|^{2\tilde{\mu}}(|g|^2)^{1/2}}.$$

Similarly,  $\partial G$  is a linear combination of terms of type

$$\frac{\overline{g'_i} \overline{g'_j}}{|g|^6} g_k g_l$$

and we may take

$$\begin{aligned} \psi_2 &= \frac{\overline{g'_i} \overline{g'_j}}{|g|^2} \tilde{\mu}(|g|^2), \quad |\psi_2| \leq \frac{|g'_i|^2}{|g_i|^2} \tilde{\mu}(|g_i|^2) + \frac{|g'_j|^2}{|g_j|^2} \tilde{\mu}(|g_j|^2) \\ \varphi_2 &= \frac{g_k g_l}{|g|^4 \tilde{\mu}(|g|^2)}, \quad |\varphi_2| \leq \frac{1}{|g|^2 \tilde{\mu}(|g|^2)} \end{aligned}$$

using again Lemma 2. □

We note as a particular case of the theorem, corresponding to  $d\mu(\alpha) = \alpha^{1+\varepsilon} d\alpha$ , the sufficient condition

$$M\left(\frac{|\log |g||^{2+\varepsilon}}{|g|^2}\right) \in L^p$$

stated in the introduction.

4. Lemma 2 can be used as well to improve Cegrell's result on the equation  $f = f_1 g_1 + f_2 g_2$ :

**Theorem 5.** *If  $f, g_1, g_2 \in H^\infty$  satisfy*

$$|f| \leq |g|^2 \tilde{\mu}(|g|^2)$$

*there exist  $f_1, f_2 \in H^\infty$  such that  $f = f_1 g_1 + f_2 g_2$ .*

*Proof.* In this case it must be shown that the equation  $\bar{\partial}u = G$  where

$$G = f \frac{\overline{g'_1} \overline{g_2} - \overline{g_1} \overline{g'_2}}{(|g_1|^2 + |g_2|^2)^2}$$

has a bounded solution. In this case

$$|G| \leq \frac{|g'_1| + |g'_2|}{|g|} \tilde{\mu}(|g|^2)^{1/2} \leq \frac{|g'_1|}{|g_1|} \tilde{\mu}(|g_1|^2)^{1/2} + \frac{|g'_2|}{|g_2|} \tilde{\mu}(|g_2|^2),$$

and  $|G|^2(1 - |z|)$  is indeed a Carleson measure; similarly for  $\partial G$ .

### References

- [1] E. Amar, *On the corona problem*, The Journal of Geometric Analysis, Vol. 1. 4 (1991), 291-305.

- [2] E. Amar and J. Bruna, *On  $H^p$ -solutions of the Bezout equation in the ball*, to appear in *J. Fourier Analysis*.
- [3] J. Bourgain, *On finitely generated closed ideals in  $H^\infty(\mathbb{D})$* , *Ann. Inst. Fourier*, **35** 4 (1985), 163–174.
- [4] U. Cegrell, *A generalization of the corona theorem in the unit disc*, *Math. Z.*, **203** (1990).
- [5] J.A. Cima and G. Taylor, *On the equation  $f_1g_1 + f_2g_2 = 1$  in  $H^p$* , *Illinois J. Math.*, **11** (1967), 431–438.
- [6] R.R. Coiffmann, Y. Meyer and E.M. Stein, *Some new function spaces and their applications to Harmonic Analysis*, *J. Funct. Analysis*, **62** (1985), 304–333.
- [7] J.B. Garnett, *Bounded analytic functions*, Academic Press, 1981.
- [8] K.C. Lin, *On the constants of the corona theorem and the ideals of  $H^\infty$* , Preprint, University of Alabama.
- [9] K.C. Lin, *On the  $H^p$  solutions to the corona equation*, Preprint, University of Alabama.
- [10] H.S. Shapiro and A.L. Shields, *On some interpolation problems for analytic functions*, *Amer. J. Math.*, **83** (1961), 513–532.
- [11] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, 1970.

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