ON MODULI OF INSTANTON BUNDLES ON $\mathbb{P}^{2n+1}$

Vincenzo Ancona and G. Ottaviani

Let $M_{\mathbb{P}^{2n+1}(k)}$ be the moduli space of stable instanton bundles on $\mathbb{P}^{2n+1}$ with $c_2 = k$. We prove that $M_{\mathbb{P}^{2n+1}(2)}$ is smooth, irreducible, unirational and has zero Euler-Poincaré characteristic, as it happens for $\mathbb{P}^3$. We find instead that $M_{\mathbb{P}^{5}(3)}$ and $M_{\mathbb{P}^{5}(4)}$ are singular.

1. Definition and preliminaries.

Instanton bundles on a projective space $\mathbb{P}^{2n+1}(\mathbb{C})$ were introduced in [OS] and [ST]. In [AO] we studied their stability, proving in particular that special symplectic instanton bundles on $\mathbb{P}^{2n+1}$ are stable, and that on $\mathbb{P}^5$ every instanton bundle is stable.

In this paper we study some moduli spaces $M_{\mathbb{P}^{2n+1}(k)}$ of stable instanton bundles on $\mathbb{P}^{2n+1}$ with $c_2 = k$. For $k = 2$ we prove that $M_{\mathbb{P}^{2n+1}(2)}$ is smooth, irreducible, unirational and has zero Euler-Poincaré characteristic (Theor. 3.2), just as in the case of $\mathbb{P}^3$ [Har].

We find instead that $M_{\mathbb{P}^{5}(k)}$ is singular for $k = 3, 4$ (theor. 3.3), which is not analogous with the case of $\mathbb{P}^3$ [ES], [P]. To be more precise, all points corresponding to symplectic instanton bundles are singular. Theor. 3.3 gives, to the best of our knowledge, the first example of a singular moduli space of stable bundles on a projective space. The proof of Theorem 3.3 needs help from a personal computer in order to calculate the dimensions of some cohomology group [BaS].

We recall from [OS], [ST] and [AO] the definition of instanton bundle on $\mathbb{P}^{2n+1}(\mathbb{C})$.

**Definition 1.1.** A vector bundle $E$ of rank $2n$ on $\mathbb{P}^{2n+1}$ is called an instanton bundle of quantum number $k$ if

(i) The Chern polynomial is $c_t(E) = (1 - t^2)^{-k} = 1 + kt^2 + (k+1)t^2 + \ldots$

(ii) $E(q)$ has natural cohomology in the range $-2n - 1 \leq q \leq 0$ (that is $h^i(E(q)) \neq 0$ for at most one $i = i(q)$)

(iii) $E|_r \simeq O_r^{2n}$ for a general line $r$.

Every instanton bundle is simple [AO]. There is the following characterization:
Theorem 1.2 ([ST], [AO]). A vector bundle $E$ of rank $2n$ on $\mathbb{P}^{2n+1}$ satisfies the properties (i) and (ii) if and only if $E$ is the cohomology of a monad

\[(1.1) \quad \mathcal{O}(-1)^k \xrightarrow{A} \mathcal{O}^{2n+2k} \xrightarrow{B} \mathcal{O}(1)^k.\]

With respect to a fixed system of homogeneous coordinates the morphism $A$ (resp. $B$) of the monad can be identified with a $k \times (2n + 2k)$ (resp. $(2n + 2k) \times k$) matrix whose entries are homogeneous polynomials of degree 1. Then the conditions that (1.1) is a monad are equivalent to:

$A, B$ have rank $k$ at every point $x \in \mathbb{P}^{2n+1}$, $A \cdot B = 0$.

Definition 1.3. A bundle $S$ appearing in an exact sequence:

\[(1.2) \quad 0 \to S^* \to \mathcal{O}^d \xrightarrow{B} \mathcal{O}(1)^c \to 0\]

is called a Schwarzenberger type bundle (STB).

The kernel bundle $\text{Ker } B$ in the monad (1.1) is the dual of a STB.

Definition 1.4. An instanton bundle is called special if it arises from a monad (1.1) where the morphism $B$ is defined in some system of homogeneous coordinates $(x_0, \ldots, x_n, y_0, \ldots, y_n)$ on $\mathbb{P}^{2n+1}$ by the matrix

\[B = \begin{bmatrix} x_0 & \cdots & \cdots & x_n \\ \vdots & \ddots & \ddots & \vdots \\ x_n & x_0 & \cdots & x_n \\ y_0 & \vdots & \ddots & \vdots \\ y_n & y_0 & \cdots & y_n \end{bmatrix}\]
Example 1.5. Take

\[
A = \begin{bmatrix}
  y_n & \cdots & y_0 & -x_n & \cdots & -x_0 \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  y_0 & \cdots & -x_n & \cdots & -x_0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
x_0 \\
\vdots \\
x_n \\
y_0 \\
\vdots \\
y_n
\end{bmatrix}
\]

\[
E = \text{Ker } B / \text{Im } A
\]

is a special instanton bundle.

Property (iii) of the definition 1.1 can be checked by the following:

Theorem 1.6 [OS]. Let \( E = \text{Ker } B / \text{Im } A \) as in (1.1). Let \( r \) be the line joining two distinct points \( P, Q \in \mathbb{P}^{2n+1} \). Then

\[
E|_r \simeq \mathcal{O}_r^{2n} \iff A(P) \cdot B(Q)
\]

is an invertible matrix.

Example 1.7. Consider the special instanton bundle \( E \) of the example 1.5. Let \( P = (1, 0, \ldots ; 0, \ldots, 0) \), \( Q = (0, \ldots ; 0, \ldots, 1) \). Then

\[
A(P) = \begin{bmatrix}
  & & & -1 \\
  & \ddots & & \\
  & & 1 & \\
-1 & & & 
\end{bmatrix}
\]

\[
B(Q) = \begin{bmatrix}
1 \\
\ddots \\
& 1
\end{bmatrix}
\]

and \( A(P) \cdot B(Q) = \begin{bmatrix}
  & & & -1 \\
  & \ddots & & \\
  & & 1 & \\
-1 & & & 
\end{bmatrix} \) is invertible. Hence \( E \) is trivial on the line \( \{ x_1 = \ldots = x_n = y_0 = \ldots = y_{n-1} = 0 \} \).

Proposition 1.8. Let \( E \) be an instanton bundle as in (1.1). Then

\[
H^2(\mathcal{O} \otimes \mathcal{O}^*) = H^2[\text{Ker } B \otimes (\text{Ker } A^t)]
\]
Proof. See [AO] Theorem 3.13 and Remark 2.22. □

Remark 1.9. If $E \cong E^*$, then

$$H^2(E \otimes E^*) = H^2[(\text{Ker } A^t) \otimes (\text{Ker } A^t)] = H^2[(\text{Ker } B) \otimes (\text{Ker } B)].$$

Remark 1.10. The single complex associated with the double complex obtained by tensoring the two sequences

$$0 \to \text{Ker } A^t \to \mathcal{O}^{2n+2k} \xrightarrow{A^t} \mathcal{O}(1)^k \to 0$$
$$0 \to \text{Ker } B^t \to \mathcal{O}^{2n+2k} \xrightarrow{B^t} \mathcal{O}(1)^k \to 0$$

gives the resolution

$$0 \to (\text{Ker } A^t) \otimes (\text{Ker } B) \to \mathcal{O}^{2n+2k} \otimes \mathcal{O}^{2n+2k}$$
$$\to \mathcal{O}^{2n+2k} \otimes \mathcal{O}(1)^k \otimes \mathcal{O}(1)^k \otimes \mathcal{O}^{2n+2k} \to \mathcal{O}(1)^k \otimes \mathcal{O}(1)^k \to 0$$

where $\alpha = (A^t \otimes \text{id}, \text{id} \otimes B)$.

Hence

$$H^2(E \otimes E^*) = \text{Coker } H^0(\alpha)$$

and its dimension can be computed using [BaS]. For the convenience of the reader we sketch the steps needed in the computations.

$A, B^t$ are given by $k \times (2n + 2k)$ matrices whose entries are linear homogeneous polynomials.

$$A \otimes \text{Id}_k = (a_1, \ldots, a_{k(2n+2k)})$$

and

$$\text{Id}_k \otimes B^t = (b_1, \ldots, b_{k(2n+2k)})$$

are both $k^2 \times (2n + 2k)k$ matrices. Let

$$C = (a_1, \ldots, a_{k(2n+2k)}, b_1, \ldots, b_{k(2n+2k)}).$$

We will denote by $\text{syz}_m C$ the dimension of the space of the syzygies of $C$ of degree $m$. Then

$$h^2(E \otimes E^*) = h^0(\mathcal{O}(2)^{k^2}) - (4n + 4k)h^0(\mathcal{O}(1)^k) + \text{syz}_1 C$$

$$= k(n+1)[k(2n-5) - 8n] + \text{syz}_1 C$$

$$h^1(E \otimes E^*) = h^2(E \otimes E^*) + 1 - k^2 + 8n^2k - 4n^2 + 3nk^2 - 2nk^2$$

$$= 1 - 6k^2 - 8kn - 4n^2 + \text{syz}_1 C.$$
Note also that \( h^0(E(1)) = \text{syz}_1 B^t - k \) and \( h^0(E^*(1)) = \text{syz}_1 A - k \).

**Remark 1.11.** In the same way we obtain
\[
\begin{align*}
\text{h}^1(E \otimes E^*(-1)) &= \text{syz}_0 C \\
\text{h}^2(E \otimes E^*(-1)) &= 2k(nk - 2n - k) + \text{syz}_0 C.
\end{align*}
\]

**2. Example on \( \mathbb{P}^5 \).**

Let \((a, b, c, d, e, f)\) be homogeneous coordinates in \( \mathbb{P}^5 \).

**Example 2.1.** \((k = 3)\) Let

\[
B^t = \begin{bmatrix}
  a & b & c & d & e & f \\
  a & b & c & d & e & f \\
  a & b & c & d & e & f
\end{bmatrix},
\]

\[
A = \begin{bmatrix}
  f & e & d & -c & -b & -a \\
  f & e & d & -c & -b & -a \\
  f & e & d & -c & -b & -a
\end{bmatrix}.
\]

The corresponding monad gives a special symplectic instanton bundle on \( \mathbb{P}^5 \) with \( k = 3 \). With the notation of remark 1.10, using \([\text{BaS}]\) we can compute syz\(_0\) \( C = 14 \), syz\(_1\) \( C = 174 \). Hence \( h^2(E \otimes E^*) = 3 \) from the formulas of Remark 1.10. Moreover \( h^0(E(1)) = 4 \).

**Example 2.2.** \((k = 3)\) Let \( B^t \) as in the Example 2.1 and

\[
A = \begin{bmatrix}
  f & e & d & -c & -b & -a \\
  e & d & 2f & -b & -a & -2c \\
  d & f & -a & -c & -b
\end{bmatrix}.
\]

We have \( \text{syz}_0 C = 10, \text{syz}_1 C = 171 \). Hence \( h^2(E \otimes E^*) = 0 \). We can compute also the syzygies of \( B^t \) and \( A \) and we get \( h^0(E(1)) = 4, h^0(E^*(1)) = 3 \), hence \( E \) is not self-dual.

**Example 2.3.** \((k = 4)\) Let

\[
B^t = \begin{bmatrix}
  a & b & c & d & e & f \\
  a & b & c & d & e & f \\
  a & b & c & d & e & f \\
  a & b & c & d & e & f
\end{bmatrix},
\]

\[
A = \begin{bmatrix}
  f & e & d & -c & -b & -a \\
  f & e & d & -c & -b & -a \\
  f & e & d & -c & -b & -a \\
  f & e & d & -c & -b & -a
\end{bmatrix}.
\]
$E$ is a special symplectic instanton bundle with $k = 4$. We compute

$$h^2(E \otimes E^*) = 12.$$ 

**Example 2.4.** $(k = 4)$ Let $B^4$ as in the Example 2.3. Let

$$A = \begin{bmatrix}
  f & e & d & -c & -b & -a \\
  e & d & 2f & -b & -a & -2c \\
  3d & f & e & -3a & -c & -b \\
  f & e & d & -c & -b & -a
\end{bmatrix}.$$ 

In this case $h^2(E \otimes E^*) = 6$, $h^0(E(1)) = 4$, $h^0(E^*(1)) = 3$.

**Example 2.5.** $(k = 4)$ Let $B^4$ as in the Example 2.3. Let

$$A = \begin{bmatrix}
  f & e & d & -c & -b & -a \\
  e & d & 2f & -b & -a & -2c \\
  3d & f & e & -3a & -c & -b \\
  5d & f & e & d + f & e & -5a & -c & -b & -a & -c & -b
\end{bmatrix}.$$ 

Now $H^2(E \otimes E^*) = 0$, $h^0(E(1)) = 4$, $h^0(E^*(1)) = 2$.

3. **On the singularities of moduli spaces.**

The stable Schwarzenberger type bundles on $\mathbb{P}^m$ (see (1.2)) form a Zariski open subset of the moduli space of stable bundles. Let $N_{p_m}(k, q)$ be the moduli space of stable STB whose first Chern class is $k$ and whose rank is $q$. The following proposition is easy and well known:

**Proposition 3.1.** The space $N_{p_m}(k, q)$ is smooth, irreducible of dimension $1 - k^2 - (q + k)^2 + k(q + k)(m + 1)$.

We denote by $\text{MI}_{p_{2n+1}}(k)$ the moduli space of stable instanton bundles with quantum number $k$. It is an open subset of the moduli space of stable $2n$-bundles on $\mathbb{P}^{2n+1}$ with Chern polynomial $(1 - t^2)^{-k}$.

On $\mathbb{P}^5$ (as on $\mathbb{P}^3$) all instanton bundles are stable by [AO], Theorem 3.6. $\text{MI}_{p_{2n+1}}(2)$ is smooth ([AO] Theorem 3.14), unirational of dimension $4n^2 + 12n - 3$ and has zero Euler-Poincaré characteristic ([BE], [K]).

**Theorem 3.2.** The space $\text{MI}_{p_{2n+1}}(2)$ is irreducible.

*Proof.* The moduli space $N = N_{p_{2n+1}}(2, n + 2)$ of stable STB of rank $2n + 2$ and $c_1 = 2$ is irreducible of dimension $4n^2 + 8n - 3$ by Prop. 3.1
For a given instanton bundle $E$ there is a STB $S$ associated with $E$, which is stable ([AO], Theorem 2.8) and unique (ibid., Prop. 2.17). It is easy to prove that the map $\pi : M \rightarrow N$ defined by $\pi([E]) = [S]$ is algebraic, moreover $\pi$ is dominant by [ST]. If $m = [E] \in M$, the fiber $\pi^{-1}(\pi(m))$ is a Zariski open subset of the grassmannian of planes in the vector space $H^0(\mathbb{P}^{2n+1}, S^*(1))$, where $\pi(m) = [S]$; by the Theorem 3.14 of [AO], $h^0(\mathbb{P}^{2n+1}, S^*(1)) = 2n + 2$, hence $\dim \pi^{-1}(\pi(m)) = 4n$.

In order to prove that $M$ is irreducible, we suppose by contradiction that there are at least two irreducible components $M_0$ and $M_1$ of $M$. Then $M_0 \cap M_1 = \emptyset$ ($M$ is smooth), $\pi(M_0)$ and $\pi(M_1)$ are constructible subset of $N$ by Chevalley’s theorem. Looking at the dimensions of $M_0, M_1, N$ and the fibers of $\pi$ we conclude that both $\pi(M_0)$ and $\pi(M_1)$ must contain an open subset of $N$, which implies $\pi(M_0) \cap \pi(M_1) \neq \emptyset$ by the irreducibility of $N$. This is a contradiction because the fibers of $\pi$ are connected.

For $n \geq 2$ and $k \geq 3$, it is no longer true that $M_{\mathbb{P}^{2n+1}}(k)$ is smooth. In fact on $\mathbb{P}^5$ we have:

**Theorem 3.3.** The space $M_{\mathbb{P}^5}(k)$ is singular for $k = 3, 4$. To be more precise, the irreducible component $M_0(k)$ of $M_{\mathbb{P}^5}(k)$ containing the special instanton bundles is generically reduced of dimension $54(k = 3)$ or $65(k = 4)$, and $M_{\mathbb{P}^5}(k)$ is singular at the points corresponding to special symplectic instanton bundles.

**Proof.** Let $E_0$ be the special instanton bundle on $\mathbb{P}^5$ of the Example 2.2($k = 3$) or of the Example 2.5($k = 4$). Then $h^2(E_0 \otimes E_0^*) = 0$ and $M_0(k)$ is smooth at the point corresponding to $E_0$, of dimension $h^1(E_0 \otimes E_0^*) = 54(k = 3)$ or $65(k = 4)$. In particular, $M_0(k)$ is generically reduced. If $E_1$ is a special symplectic instanton bundle on $\mathbb{P}^5$, the computations in 2.1 and 2.3 show that $h^2(E_1 \otimes E_1^*) = 3(k = 3)$ or $12(k = 4)$, and $h^1(E_1 \otimes E_1^*) = 57$ or $77$ respectively. Hence $M_{\mathbb{P}^5}(k)$ is singular at $E_1$ for $k = 3$ and $4$.

**Remark 3.4.** It is natural to conjecture that $M_{\mathbb{P}^{2n+1}}(k)$ is singular for all $n \geq 2$ and $k \geq 3$.

**Theorem 3.5.** Let $E$ be an instanton bundle on $\mathbb{P}^{2n+1}$ with $c_2(E) = k$. Then $h^1(E(t)) = 0$ for $t \leq -2$ and $k - 1 \leq t$.

**Proof.** The result is obvious for $t \leq -2$. It is sufficient to prove $h^1(S^*(t)) = 0$ for $t \geq k - 1$. We have

$$S^*(t) = \bigwedge^{2n+k-1} S(t - k).$$
Taking wedge products of (1.2) we have the exact sequence
\[
0 \to O(t + 1 - 2n - 2k)^{\alpha_0} \to \ldots \to O(t - k - 1)^{\alpha_{2n+k-2}} \to O(t - k)^{\alpha_{2n+k-1}} \to \bigwedge^{2n+k-1} S(t - k) \to 0
\]
for suitable \( \alpha_i \in \mathbb{N} \) and from this sequence we can conclude.

Ellia proves Theorem 3.5 in the case of \( \mathbb{P}^3 \) ([E], Prop. IV.1). He also remarks that the given bound is sharp. This holds on \( \mathbb{P}^{2n+1} \) as it is shown by the following theorem, which points out that the special symplectic instanton bundles are the "furthest" from having natural cohomology.

**Theorem 3.6.** Let \( E \) be a special symplectic instanton bundle on \( \mathbb{P}^{2n+1} \) with \( c_2 = k \). Then
\[
h^1(E(t)) \neq 0 \text{ for } -1 \leq t \leq k - 2.
\]

**Proof.** For \( n = 1 \) the thesis is immediate from the exact sequence
\[
0 \to O(t - 1) \to E(t) \to J_C(t + 1) \to 0
\]
where \( C \) is the union of \( k + 1 \) disjoint lines in a smooth quadric surface. Then the result follows by induction on \( n \) by considering the sequence
\[
0 \to E(t - 2) \to E(t - 1)^2 \to E(t) \to E(t)|_{\mathbb{P}^{2n-1}} \to 0
\]
and the fact that, for a particular choice of the subspace \( \mathbb{P}^{2n-1} \), the restriction \( E|_{\mathbb{P}^{2n-1}} \) splits as the direct sum of a rank-2 trivial bundle and a special symplectic instanton bundle on \( \mathbb{P}^{2n-1}([ST] 5.9) \).

**Remark 3.7.** In [OT] it is proved that if \( E_k \) is a special symplectic instanton bundle on \( \mathbb{P}^5 \) with \( c_2 = k \) then \( h^1(\text{End } E_k) = 20k - 3 \).

In the following table we summarize what we know about the component \( M_0(k) \subset \text{MIP}_5(k) \) containing \( E_k \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>( h^1( E_k \otimes E_k^* ) )</th>
<th>( h^2( E_k \otimes E_k^* ) )</th>
<th>( \text{dim } M_0(k) )</th>
<th>( \text{MIP}_5(k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14</td>
<td>0</td>
<td>14</td>
<td>open subset of ( \mathbb{P}^{14} )</td>
</tr>
<tr>
<td>2</td>
<td>37</td>
<td>0</td>
<td>37</td>
<td>smooth, irreduc., unirat.</td>
</tr>
<tr>
<td>3</td>
<td>57</td>
<td>3</td>
<td>54</td>
<td>singular</td>
</tr>
<tr>
<td>4</td>
<td>77</td>
<td>12</td>
<td>65</td>
<td>singular</td>
</tr>
<tr>
<td>( k \geq 2 )</td>
<td>( 20k - 3 )</td>
<td>( 3(k - 2)^2 )</td>
<td>( ? )</td>
<td>( ? )</td>
</tr>
</tbody>
</table>
References


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Added in proof. After this paper has been written we received a preprint of R. Miró-Roig and J. Orus-Lacort where they prove that the conjecture stated in the Remark 3.4 is true.
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Correction to: “Asymptotic radial symmetry for solutions of $\Delta u + e^u = 0$ in a punctured disc”

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