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REBECCA A. HERB

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REBECCA A. HERB

Let F be a ρ-adic field and E a cyclic extension of F of degree d corresponding to the character κ of $F^\times$. For any positive integer m, we consider $H = GL(m, E)$ as a subgroup of $G = GL(md, F)$. In this paper we discuss matching of orbital integrals between H and G. Specifically, ordinary orbital integrals corresponding to regular semisimple elements of H are matched with orbital integrals on G which are twisted by the character κ. For the general situation we only match functions which are smooth and compactly supported on the regular set. If the extension E/F is unramified, we are able to match arbitrary smooth, compactly supported functions.

§1. Introduction.

Let F be a locally compact, non-discrete, nonarchimedean local field of characteristic zero. Let κ be a unitary character of $F^\times$ of order d, and let E be the cyclic extension of F corresponding to κ. Let m and n be positive integers with $n = md$ and write $G = GL(n, F)$, $H = GL(m, E)$. H can be identified with a subgroup of G. In this paper we discuss matching of orbital integrals between H and G. Specifically, ordinary orbital integrals corresponding to regular semisimple elements of H are matched with orbital integrals on G which are twisted by the character κ. For the general situation we only match functions which are smooth and compactly supported on the regular set. If the extension E/F is unramified, we are able to match arbitrary smooth, compactly supported functions.

Extend κ to a character of G by $\kappa(g) = \kappa(\det g)$ and let

$$G_0 = \{g \in G : \kappa(g) = 1\}.$$

G_0 is an open normal subgroup of G of finite index and $H \subset G_0$. Let $C_c^\infty(G)$ denote the set of locally constant, compactly supported, complex-valued functions on G. For any $\gamma \in G$ we let G_γ denote the centralizer of $\gamma \in G$. If $G_\gamma \subset G_0$, let

$$\Lambda_\kappa^G(f, \gamma) = \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)\kappa(x)dx, f \in C_c^\infty(G),$$

be the twisted orbital integral of f over the orbit of γ. If $G_\gamma \not\subset G_0$, set $\Lambda^G_\kappa(f,\gamma) = 0$. Clearly for all $x, \gamma \in G, f \in C_c^\infty(G)$,

$$\Lambda^G_\kappa(f, x\gamma x^{-1}) = \kappa(x)\Lambda^G_\kappa(f,\gamma).$$

Similarly we define

$$\Lambda^H(f,\gamma) = \int_{H_\gamma\backslash H} f(x^{-1}\gamma x)dx, f \in C_c^\infty(H), \gamma \in H,$$

the ordinary orbital integral of f over the H-orbit of γ.

The main results of this paper are the following theorems. Let G' denote the set of regular semisimple elements of G and $C_c^\infty(G')$ the subset of all $f \in C_c^\infty(G)$ with support in G'.

Theorem 1.1.

(i) *Let $f_G \in C_c^\infty(G')$. Then there is $f_H \in C_c^\infty(H \cap G')$ such that for all $\gamma \in H \cap G'$,*

$$\Lambda^G_\kappa(f_G,\gamma) = \Lambda^H(f_H,\gamma).$$

(ii) *Conversely, suppose $f_H \in C_c^\infty(H \cap G')$ such that*

$$\Lambda^H(f_H, x\gamma x^{-1}) = \kappa(x)\Lambda^H(f_H,\gamma)$$

for all $x \in G, \gamma \in H \cap G'$ such that $x\gamma x^{-1} \in H$. Then there is $f_G \in C_c^\infty(G')$ such that for all $\gamma \in H \cap G'$,

$$\Lambda^G_\kappa(f_G,\gamma) = \Lambda^H(f_H,\gamma).$$

In the case that κ is unramified, a stronger version of Theorem 1.1 can be proven using results of [**W2, Hn**]. Let Δ^H_G be the transfer factor defined as in [**W2**].

Theorem 1.2. *Assume that κ is unramified.*

(i) *Let $f_G \in C_c^\infty(G)$. Then there is $f_H \in C_c^\infty(H)$ such that for all $\gamma \in H \cap G'$,*

$$\Delta^H_G(\gamma)\Lambda^G_\kappa(f_G,\gamma) = \Lambda^H(f_H,\gamma).$$

(ii) *Conversely, suppose $f_H \in C_c^\infty(H)$ such that*

$$\Lambda^H(f_H, x\gamma x^{-1}) = \Delta^H_G(x\gamma x^{-1})\Delta^H_G(\gamma)^{-1}\kappa(x)\Lambda^H(f_H,\gamma)$$

for all $x \in G, \gamma \in H \cap G'$ such that $x\gamma x^{-1} \in H$. Then there is $f_G \in C_c^\infty(G)$ such that for all $\gamma \in H \cap G'$,

$$\Delta^H_G(\gamma)\Lambda^G_\kappa(f_G,\gamma) = \Lambda^H(f_H,\gamma).$$

The matching theorems for twisted orbital integrals will be used in another paper to prove character formulas relating twisted characters on G to ordinary characters on H. These will generalize the lifting theorem proven by Kazhdan [**K**] in the case that $m = 1$. It will be shown in that paper that

$$\Delta_G^H\left(x\gamma x^{-1}\right) = \Delta_G^H(\gamma)\kappa(x)^{-1}$$

for all $x \in N_G(H), \gamma \in H \cap G'$. Thus when $x \in N_G(H)$, the condition on f_H in Theorem 1.2, (ii), is just

$$\Lambda^H\left(f_H, x\gamma x^{-1}\right) = \Lambda^H\left(f_H, \gamma\right)$$

for all $\gamma \in H \cap G'$. Since Λ^H is an ordinary orbital integral, this is automatic when $x \in H$.

The proof of Theorem 1.1 is routine using an easy extension of results in [**V**] to the twisted case and techniques as in [**A-C**, 1.3]. The proof of Theorem 1.2 uses the fundamental lemma proven by [**W2**, **Hn**]. Assume that κ is unramified. Let $K = GL(n, R)$ where R is the ring of integers of F and let $\mathcal{H}(G)$ denote the Hecke algebra of functions in $C_c^\infty(G)$ which are K bi-invariant. Similarly, we define $\mathcal{H}(H)$, the Hecke algebra of H. Let $b : \mathcal{H}(G) \to \mathcal{H}(H)$ be the homomorphism defined in [**W2**]. The following theorem was proven by Waldspurger [**W2**] when the algebra $F(\gamma)$ is a product of tamely ramified extensions of F and was extended to the general case (as well as to the case of characteristic F not zero) by Henniart [**Hn**].

Theorem 1.3 (Waldspurger, Henniart). *Let $\phi \in \mathcal{H}(G), \gamma \in H \cap G'$. Then*

$$\Delta_G^H(\gamma)\Lambda_\kappa^G(\phi, \gamma) = \Lambda^H(b\phi, \gamma).$$

Theorem 1.2 follows from Theorem 1.3 as follows. First, using standard techniques, it is enough to prove a matching of orbital integrals in a neighborhood of each semisimple element s of H. Further, by passing to centralizers, it is easy to reduce to the case that $s = 1$. The matching in a neighborhood of $s = 1$ is a result of the following theorems which show that all germs in a neighborhood of the identity come from Hecke functions.

Theorem 1.4 [**W1**, **Hr**]. *Let $u_1, ..., u_p$ be a complete set of representatives for the unipotent conjugacy classes of H. Then there are $\phi_1, ..., \phi_p \in \mathcal{H}(H)$ such that*

$$\Lambda^H\left(\phi_i, u_j\right) = \begin{cases} 1, & \text{if } 1 \le i = j \le p; \\ 0, & \text{if } 1 \le i \ne j \le p. \end{cases}$$

Using the results of [**V**] we obtain the following corollary.

Corollary 1.5. *Let $u_1, ..., u_p, \phi_1, ..., \phi_p$ be as above. Let $f \in C_c^\infty(H)$. Then there is a neighborhood U of 1 in H so that for all $\gamma \in U$,*

$$\Lambda^H(f,\gamma) = \sum_{i=1}^{p} \Lambda^H(f,u_i)\,\Lambda^H(\phi_i,\gamma).$$

Let u be a unipotent element of G. If $G_u \not\subset G_0$, then $\Lambda_\kappa^G(f,u) = 0$ for all $f \in C_c^\infty(G)$. It is easy to show that the unipotent conjugacy classes $\mathcal{O}(u)$ of G for which $G_u \subset G_0$ are in bijective correspondance with the unipotent conjugacy classes of H.

Theorem 1.6 [Hr]. *Let $v_1, ..., v_p$ be a complete set of representatives for the unipotent conjugacy classes in G such that $G_{v_i} \subset G_0$. Then there are $\psi_1, ..., \psi_p \in \mathcal{H}(G)$ such that*

$$\Lambda_\kappa^G(\psi_i, v_j) = \begin{cases} 1, & \text{if } 1 \le i = j \le p; \\ 0, & \text{if } 1 \le i \ne j \le p. \end{cases}$$

An easy extension of germ expansions to the twisted case yields the following corollary.

Corollary 1.7. *Let $v_1, ..., v_p, \psi_1, ..., \psi_p$ be as above. Let $f \in C_c^\infty(G)$. Then there is a neighborhood U of 1 in G so that for all $\gamma \in U$,*

$$\Lambda_\kappa^G(f,\gamma) = \sum_{i=1}^{p} \Lambda_\kappa^G(f,v_i)\,\Lambda_\kappa^G(\psi_i,\gamma).$$

The organization of the paper is as follows.

In §2 we extend many of the results of Vignéras [**V**] to the case of twisted orbital integrals.

In §3 we use the results of §2 to prove Theorems 1.1 and 1.2.

I would like to thank the Mathematics Departments of the University of Toronto and the University of Wisconsin for their hospitality during the preparation of this paper.

§2. Twisted Orbital Integrals.

Let $G = GL(n,F)$ and let κ be a unitary character of $F^\times$ of order d, d a divisor of n. In this section we do not assume that κ is unramified. We extend κ to a character of G by setting $\kappa(g) = \kappa(\det g), g \in G$. Let $G_0 = \{g \in G : \kappa(g) = 1\}$. Then G_0 is an open normal subgroup of finite index in

G. For any $x \in G$ we let G_x denote the centralizer of $x \in G$. If $G_x \subseteq G_0$, we let

$$\Lambda_\kappa(f,x) = \int_{G_x \backslash G} f\left(g^{-1}xg\right)\kappa(g)dg, f \in C_c^\infty(G), x \in G$$

be the twisted orbital integral of f over the orbit of x. If $G_x \not\subseteq G_0$, we let $\Lambda_\kappa(f,x) = 0$ for all $f \in C_c^\infty(G)$. (We assume measures are normalized as in [**V**, 1.h].)

In this section we will extend results of Vignéras on orbital integrals to the twisted case. For $x \in G$, define the normalizing factor $d(x)$ as in [**V**, 1.g]. We will also write

$$F_\kappa(f,x) = d(x)\Lambda_\kappa(f,x).$$

Let s be a semisimple element in G. Then as in [**V**, 1.j] we write A_s for the set of all elements x of G with semisimple part (of the Jordan decomposition of x) conjugate to s. Let $A_s = \cup \mathcal{O}\left(su_i\right), 1 \leq i \leq m$, be the standard decomposition as in [**V**, 1.j] where $\mathcal{O}(x)$ denotes the G orbit of $x \in G$. For $x \in G_0$ we will write $\mathcal{O}_0(x)$ for the G_0 orbit of x.

Lemma 2.1. *Fix $1 \leq i \leq m$ and suppose that $G_{su_i} \subseteq G_0$. Then there is $f_i \in C_c^\infty(G)$ such that*

$$F_\kappa\left(f_i, su_j\right) = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

Proof. As in [**V**, 1.k], for each $1 \leq i \leq m$ there is a compact open subset K_i in G so that $su_i \in K_i$, and $K_i \cap \mathcal{O}\left(su_j\right) = \emptyset, 1 \leq j \leq i-1$. Now suppose that $G_{su_i} \subseteq G_0$. Then

$$\mathcal{O}\left(su_i\right) \approx G_{su_i}\backslash G \approx G_{su_i}\backslash G_0 \times G_0\backslash G \approx \mathcal{O}_0\left(su_i\right) \times G_0\backslash G$$

so that $\mathcal{O}_0\left(su_i\right)$ is open and closed in $\mathcal{O}\left(su_i\right)$. Thus there is $K_i' \subseteq K_i$ compact open in G so that $su_i \in K_i', K_i' \cap \mathcal{O}\left(su_i\right) \subseteq \mathcal{O}_0\left(su_i\right)$. Now if f_i' is the characteristic function of K_i', then $F_\kappa\left(f_i', su_i\right) \neq 0$ because there can be no cancellation in the integral, and $F_\kappa\left(f_i', su_j\right) = 0, 1 \leq j \leq i-1$. Now using a standard Graham-Schmidt type procedure we can obtain f_i's as in the lemma. □

Lemma 2.2. *Let $s \in G$ be semisimple and suppose that $f \in C_c^\infty(G)$ satisfies $F_\kappa(f,x) = 0$ for all $x \in A_s$. Then there is a neighborhood V_f of s in G such that $F_\kappa(f,x) = 0$ for all $x \in V_f$.*

Proof. We follow the proof of [**K**, 3.8]. Let $S = C_c^\infty\left(A_s\right)$. Since A_s is G-invariant, G acts on S by $g\ \tilde{f}(x) = \tilde{f}\left(g^{-1}xg\right), g \in G, x \in A_s, \tilde{f} \in S$. Since

A_s is closed in G, restriction gives a mapping $\pi : C_c^\infty(G) \to S$. Let S' be the dual of S and let $\Lambda = \left\{\lambda \in S' : \lambda\left(g \cdot \tilde{f}\right) = \kappa(g)\lambda\left(\tilde{f}\right), \forall g \in G, \tilde{f} \in S\right\}$. Then since G has only a finite number of orbits in A_s we see that Λ is generated by the $\lambda_i, 1 \le i \le m$, where $\lambda_i(\pi(f)) = F_\kappa(f, su_i)$. Let $S_\kappa = \left\{\tilde{f} \in S : \lambda\left(\tilde{f}\right) = 0, \forall \lambda \in \Lambda\right\}$. Then S_κ is the set of all finite sums of functions of the form $g \cdot \tilde{f} - \kappa(g)\tilde{f}$.

Now let $f \in C_c^\infty(G)$ such that $F_\kappa(f, su_i) = 0, 1 \le i \le m$. Then $\tilde{f} = \pi(f) \in S_\kappa$ so there are $g_1, ..., g_k \in G, \tilde{f}_1, ..., \tilde{f}_k \in S$, such that $\tilde{f} = \sum_{i=1}^k g_i \cdot \tilde{f}_i - \kappa(g_i)\tilde{f}_i$. Let $f_i \in C_c^\infty(G)$ such that $\pi(f_i) = \tilde{f}_i$, and let $\phi = f - \sum_{i=1}^k g_i \cdot f_i + \kappa(g_i) f_i$. Then $\pi(\phi) = 0$ so by [**V**, 2.4] there is an open, G-invariant neighborhood V_f of s such that ϕ is zero on V_f. Thus $F_\kappa(\phi, x) = 0$ for all $x \in V_f$. But for all $x \in G$, $F_\kappa(f, x) = F_\kappa(\phi, x)$. □

Renumber $u_1, ..., u_m$ so that $su_i, 1 \le i \le k$, are the orbits of A_s such that $G_{su_i} \subseteq G_0, 1 \le i \le k$. Suppose $f_1, ..., f_k \in C_c^\infty(G)$ satisfy $F_\kappa(f_i, su_j) = \delta_{ij}, 1 \le i, j \le k$, and $f_1', ..., f_k' \in C_c^\infty(G)$ satisfy $\Lambda_\kappa(f_i', su_j) = \delta_{ij}, 1 \le i, j \le k$.

Lemma 2.3. *Let $f \in C_c^\infty(G)$. Then there is a neighborhood V_f of s in G so that*

$$F_\kappa(f, x) = \sum_{i=1}^k F_\kappa(f, su_i) F_\kappa(f_i, x)$$

and

$$\Lambda_\kappa(f, x) = \sum_{i=1}^k \Lambda_\kappa(f, su_i) \Lambda_\kappa(f_i', x)$$

for all $x \in V_f$.

Proof. Let $f' = f - \sum_{i=1}^k F_\kappa(f, su_i) f_i$. Then $F_\kappa(f', su_j) = 0, 1 \le j \le k$. Thus by Lemma 2.2 there is a neighborhood V_f of s such that $F_\kappa(f', x) = 0$ for all $x \in V_f$. □

As in [**V**, 1.m], for any $s \in G$ semisimple, we let T be the center of $M = G_s$. Let $u \in Z_G(T)$ be unipotent. Then (T, u) is called a standard couple. For any subset X of G, let X^{reg} denote the subset of elements $x \in X$ such that the dimension of the conjugacy class of x is greater than or equal to the dimension of the conjugacy class of any $y \in X$.

We can now extend Theorems A and B of [**V**, 1.n] to the twisted case.

Theorem 2.4. *(A) Let $f \in C_c^\infty(G)$ and let $F(x) = F_\kappa(f, x), x \in G$. Let (T, u) be any standard couple. Then F has the following properties.*

(i) $F(gxg^{-1}) = \kappa(g)F(x), \forall x, g \in G;$

(ii) *the restriction of F to Tu^{reg} is locally constant;*

(iii) *the restriction of F to Tu has compact support;*

(iv) *for every $s \in T$ there is a neighborhood V_F of s in T such that for $t \in V_F \cap T^{reg}$,*

$$F(tu) = \sum_{i=1}^{k} F(su_i) F_\kappa(f_i, tu)$$

where $su_i, f_i, 1 \leq i \leq k$ are defined as in Lemma 2.3.

(B) Conversely, if F is a function on G satisfying (i)-(iv) *above, then there is $f \in C_c^\infty(G)$ such that $F(x) = F_\kappa(f, x)$ for all $x \in G$.*

Proof. Part (A) follows from Lemma 2.3 and [**V**, 2.7]. It also follows easily from [**V**, 2.7] that if $f' \in C_c^\infty(T^{reg})$ transforms according to κ under the action of $W(Tu) = N_G(Tu)/Z_G(Tu)$, then there is $f \in C_c^\infty(\mathcal{O}(T^{reg}))$ such that $f'(t) = F_\kappa(f, t)$ for all $t \in T^{reg}$. Now the proof of (B) follows by an induction argument as in [**V**, 2.8]. □

We can use Theorem 2.4 to obtain the following localization result. Let $T_1, ..., T_r$ be a complete set of Cartan subgroups of G, up to G-conjugacy. Let $X = \cup_{i=1}^r T_i \subseteq G$.

Lemma 2.5. *Let V be a closed and open subset of X such that $\mathcal{O}(V) \cap X = V$. Then given $f \in C_c^\infty(G)$ there is $f_V \in C_c^\infty(G)$ such that*

$$F_\kappa(f, \gamma) = F_\kappa(f_V, \gamma), \gamma \in V$$

and

$$F_\kappa(f_V, \gamma) = 0, \gamma \in X \backslash V.$$

Proof. Let $F(x) = F_\kappa(f, x), x \in G$. For any $x \in G$, write $x = s(x)u(x)$ for the Jordan decomposition of x. Define

$$F_V(x) = \begin{cases} F(x), & \text{if } s(x) \in \mathcal{O}(V); \\ 0, & \text{otherwise.} \end{cases}$$

Then for any $x, g \in G, s(gxg^{-1}) = gs(x)g^{-1} \in \mathcal{O}(V)$ if and only if $s(x) \in \mathcal{O}(V)$. Thus if $s(x) \notin \mathcal{O}(V)$ we have $F_V(x) = F_V(gxg^{-1}) = 0$. If $s(x) \in \mathcal{O}(V)$ we have $F_V(gxg^{-1}) = F(gxg^{-1}) = \kappa(g)F(x) = \kappa(g)F_V(x)$. Thus F_V satisfies (i) of Theorem 2.4.

Let (T, u) be any standard couple. We can assume that $T \subseteq T_i \subseteq X$ for some T_i. Let $V_T = V \cap T$. It is open and closed in T. Let χ_V be the characteristic function of $V_T u$. It is a locally constant function. Further $F_V|_{Tu} = F|_{Tu} \cdot \chi_V$ since, using our assumption that $\mathcal{O}(V) \cap X = V$, for every $tu \in Tu, t \in \mathcal{O}(V)$ if and only if $t \in V_T$. Thus F_V satisfies (ii) and (iii) of Theorem 2.4.

Finally, fix $s \in T$. If $s \notin V_T$, there is a neighborhood U of s in T such that $U \cap V_T = \emptyset$. Now $F_V(su_i) = 0$ for all i and $F_V(tu) = 0$ for all $t \in U$. Thus F_V satisfies the germ expansion in U. If $s \in V_T$, then let V_F be a neighborhood of s in T such that for all $t \in V_F \cap T^{reg}$,

$$F(tu) = \sum_i F(su_i) F_\kappa(f_i, tu).$$

Let $V_{F_V} = V_F \cap V_T$. Then for all $t \in V_{F_V}, F_V(tu) = F(tu)$. Also $F_V(su_i) = F(su_i)$ for all i. Thus F_V also satifies (iv). □

Let $s \in G$ be an arbitrary semisimple element. Let $\{T_1, ..., T_r\}$ be representatives for the Cartan subgroups of G, up to G-conjugacy, such that $s \in T_i, 1 \le i \le r$. Let M be the centralizer of s in G. Then $T_i \subseteq M, 1 \le i \le r$, and for any $\psi \in C_c^\infty(M), \gamma \in T_i \cap G'$, we can define

$$\Lambda_\kappa^M(\psi, \gamma) = \int_{T_i \backslash M} \psi(m^{-1}\gamma m) \kappa(m) dm$$

if $T_i \subset G_0$ and $\Lambda_\kappa^M(\psi, \gamma) = 0$ if $T_i \not\subset G_0$.

Lemma 2.6.

(i) *Let $f \in C_c^\infty(G)$. Then there are neighborhoods V_i of s in T_i and $\psi \in C_c^\infty(M)$ so that for all $1 \le i \le r, \gamma \in V_i \cap G'$,*

$$\Lambda_\kappa^G(f, \gamma) = \Lambda_\kappa^M(\psi, \gamma).$$

(ii) *Let $\psi \in C_c^\infty(M)$. Then there are neighborhoods V_i of s in T_i and $f \in C_c^\infty(G)$ so that for all $1 \le i \le r, \gamma \in V_i \cap G'$,*

$$\Lambda_\kappa^G(f, \gamma) = \Lambda_\kappa^M(\psi, \gamma).$$

Proof. The proof is an easy generalization of the argument used in [**V**, 2.5]. Define $su_j, f_j, 1 \le j \le k$ as in Theorem 2.4. Let T be the center of M.

Fix $f \in C_c^\infty(G)$ and let $\Omega = supp\ f$. Then using [**HC**], there are neighborhoods V_i of s in T_i and an open, compact subset $\omega \subseteq M\backslash G$ so that $g^{-1}V_i g \cap \Omega = \emptyset, 1 \le i \le r$, unless $Mg \in \omega$. Further, as in [**V**, 2.5], there is a neighborhood V of s in T and an open, compact subset $C \subseteq M\backslash G$ so that $g^{-1}Vu_j g \cap \Omega = \emptyset, 1 \le j \le k$, unless $Mg \in C$. Choose $\alpha \in C_c^\infty(G)$ so that

$$\tilde{\alpha}(g) = \int_M \alpha(mg) dm = \begin{cases} 1, & \text{if } Mg \in C \cup \omega; \\ 0, & \text{if } Mg \notin C \cup \omega. \end{cases}$$

Define

$$\psi(m)=\int_G \alpha(x)\kappa(x)f\left(x^{-1}mx\right)dx, m\in M.$$

Then $\psi\in C_c^\infty(M)$, and it is easy to check that for all $1\leq i\leq r, \gamma\in V_i\cap G'$,

$$\Lambda_\kappa^G(f,\gamma)=\Lambda_\kappa^M(\psi,\gamma).$$

Further, for all $1\leq j\leq k, \gamma\in V$,

$$\Lambda_\kappa^G\left(f,\gamma u_j\right)=\Lambda_\kappa^M\left(\psi,\gamma u_j\right).$$

This proves part (i) of the Lemma.

Define $su_j, f_j, 1\leq j\leq k$ as above and let $f'_j=d\left(su_j\right)f_j$. Then the f'_j satisfy $\Lambda_\kappa^G\left(f'_j,su_l\right)=\delta_{jl}, 1\leq j,l\leq k$. To prove part (ii), we use (i) to choose neighborhoods V_i of s in $T_i, 1\leq i\leq r$ and V of s in T, and functions $\psi_j\in C_c^\infty(M), 1\leq j\leq k$, so that for all $1\leq j\leq k, 1\leq i\leq r, \gamma\in V_i\cap T'_i$,

$$\Lambda_\kappa^G\left(f'_j,\gamma\right)=\Lambda_\kappa^M\left(\psi_j,\gamma\right).$$

Further, for all $1\leq l\leq k, \gamma\in V$,

$$\Lambda_\kappa^G\left(f'_j,\gamma u_l\right)=\Lambda_\kappa^M\left(\psi_j,\gamma u_l\right).$$

Thus the functions ψ_j satisfy

$$\Lambda_\kappa^M\left(\psi_j,su_l\right)=\begin{cases}1, & \text{if } j=l;\\ 0, & \text{if } j\neq l.\end{cases}$$

Now fix $\psi\in C_c^\infty(M)$. As in [**V**, 2.5], the orbital decomposition of $A_{s,M}$ and A_s can be represented by the same elements $su_1,\ldots,su_m$. Also $M_{su_i}=G_{su_i}$ and $M_0=M\cap G_0$, so that $M_{su_i}\subseteq M_0$ if and only if $G_{su_i}\subseteq G_0$. Thus we can also take $su_1,\ldots,su_k$ the same for M and G. Thus using Lemma 2.3 applied to M there is a neighborhood U of s in M so that for all $m\in U$,

$$\Lambda_\kappa^M(\psi,m)=\sum_{j=1}^k\Lambda_\kappa^M\left(\psi_j,m\right)\Lambda_\kappa^M(\psi,su_j).$$

Define $f\in C_c^\infty(G)$ by

$$f(g)=\sum_{j=1}^k\Lambda_\kappa^M\left(\psi,su_j\right)f'_j(g), g\in G.$$

Then for all $\gamma \in G$,

$$\Lambda_\kappa^G(f,\gamma) = \sum_{j=1}^{k} \Lambda_\kappa^M(\psi, su_j)\, \Lambda_\kappa^G\left(f_j',\gamma\right).$$

But now we have

$$\Lambda_\kappa^G\left(f_j',\gamma\right) = \Lambda_\kappa^M(\psi_j,\gamma), \gamma \in V_i \cap T_i', 1 \le i \le r, 1 \le j \le k,$$

so that

$$\Lambda_\kappa^G(f,\gamma) = \sum_{j=1}^{k} \Lambda_\kappa^M(\psi, su_j)\, \Lambda_\kappa^M(\psi_j,\gamma).$$

Thus for $\gamma \in V_i \cap U \cap T_i', 1 \le i \le r$, we have

$$\Lambda_\kappa^G(f,\gamma) = \Lambda_\kappa^M(\psi,\gamma).$$

□

§3. Matching Theorems.

Let $G = GL(n,F), K = GL(n,R)$, and let κ be a unitary character of $F^\times$ of order d, d a divisor of n. Unless otherwise noted we will assume that κ is unramified.

As in Theorem 1.6 we let $u_1, ..., u_k$ represent the unipotent conjugacy classes with $G_{u_i} \subset G_0$, and $\phi_1, ..., \phi_k \in \mathcal{H}(G)$ satisfy $\Lambda_\kappa(\phi_i, u_j) = \delta_{ij}$. The following lemma is a special case of Lemma 2.3.

Lemma 3.1. *Let $f \in C_c^\infty(G)$. Then there is a neighborhood U of 1 in G so that*

$$\Lambda_\kappa(f,\gamma) = \sum_{i=1}^{k} \Lambda_\kappa(f, u_i)\, \Lambda_\kappa(\phi_i,\gamma)$$

for all $\gamma \in U \cap G'$.

Now let E be the cyclic extension of order d of F corresponding to κ and let $H = GL(m,E), md = n$. Fix an embedding of H in G as in [**W2**]. Then for $\gamma \in H$ we can define both the ordinary orbital integral $\Lambda^H(f,\gamma), f \in C_c^\infty(H)$, and the twisted orbital integral $\Lambda_\kappa^G(f,\gamma), f \in C_c^\infty(G)$.

Write $\mathcal{H}(G), \mathcal{H}(H)$ for the Hecke algebras of G and H respectively. Let $b : \mathcal{H}(G) \to \mathcal{H}(H)$ be the homomorphism of $\mathcal{H}(G)$ onto $\mathcal{H}(H)$ defined as in [**W2**], and define the transfer factor Δ_G^H as in [**W2, Hl**]. The following theorem was proven by Waldspurger [**W2**] for F of characteristic zero and $F(\gamma)$ tamely ramified over F, and was extended by Henniart [**Hn**].

Theorem 3.2 (Waldspurger, Henniart). *Let* $f \in \mathcal{H}(G), \gamma \in H \cap G'$. *Then*

$$\Delta_G^H(\gamma)\Lambda_\kappa^G(f,\gamma) = \Lambda^H(bf,\gamma).$$

Write Z_G for the center of G.

Theorem 3.3. *Let* $z \in Z_G$.

(i) *Let* $f_G \in C_c^\infty(G)$. *Then there are a neighborhood* U *of* z *in* H *and* $f_H \in C_c^\infty(H)$ *so that*

$$\Delta_G^H(\gamma)\Lambda_\kappa^G(f_G,\gamma) = \Lambda^H(f_H,\gamma)$$

for all $\gamma \in U \cap G'$.

(ii) *Let* $f_H \in C_c^\infty(H)$. *Then there are a neighborhood* U *of* z *in* H *and* $f_G \in C_c^\infty(G)$ *so that*

$$\Delta_G^H(\gamma)\Lambda_\kappa^G(f_G,\gamma) = \Lambda^H(f_H,\gamma)$$

for all $\gamma \in U \cap G'$.

Proof. Suppose first that $z = 1$ is the identity. Define $u_1, ..., u_k \in G$, $\phi_1, ..., \phi_k \in \mathcal{H}(G)$, as in Lemma 3.1. Let $f_G \in C_c^\infty(G)$ and let V be a neighborhood of 1 in G so that

$$\Lambda_\kappa^G(f_G,\gamma) = \sum_{i=1}^k \Lambda_\kappa^G(f_G,u_i)\,\Lambda_\kappa^G(\phi_i,\gamma)$$

for all $\gamma \in V \cap G'$. Define $f_H \in C_c^\infty(H)$ by

$$f_H = \sum_{i=1}^k \Lambda_\kappa^G(f_G,u_i)\,(b\phi_i)\,.$$

Let $U = V \cap H$. Then using Theorem 3.2, for all $\gamma \in U \cap G'$,

$$\begin{aligned}\Delta_G^H(\gamma)\Lambda_\kappa^G(f_G,\gamma) &= \sum_{i=1}^k \Lambda_\kappa^G(f_G,u_i)\,\Delta_G^H(\gamma)\Lambda_\kappa^G(\phi_i,\gamma)\\ &= \sum_{i=1}^k \Lambda_\kappa^G(f_G,u_i)\,\Lambda^H(b\phi_i,\gamma) = \Lambda^H(f_H,\gamma)\,.\end{aligned}$$

Now let $u'_1, ..., u'_k \in H, \phi'_1, ..., \phi'_k \in \mathcal{H}(H)$ be defined as in Theorem 1.4 so that $u'_1, ..., u'_k$ represent the unipotent conjugacy classes in H and

$\Lambda^H\left(\phi_i', u_j'\right) = \delta_{ij}$. Let $f_H \in C_c^\infty(H)$ and let U be a neighborhood of 1 in H so that

$$\Lambda^H(f_H, \gamma) = \sum_{i=1}^{k} \Lambda^H(f_H, u_i')\,\Lambda^H(\phi_i', \gamma)$$

for all $\gamma \in U \cap H'$. Choose $\phi_1, ..., \phi_k \in \mathcal{H}(G)$ so that $b\phi_i = \phi_i', 1 \leq i \leq k$, and define

$$f_G = \sum_{i=1}^{k} \Lambda^H(f_H, u_i')\,\phi_i.$$

Then as above

$$\begin{aligned}\Delta_G^H(\gamma)\Lambda_\kappa^G(f_G, \gamma) &= \sum_{i=1}^{k} \Lambda^H(f_H, u_i')\,\Delta_G^H(\gamma)\Lambda_\kappa^G(\phi_i, \gamma) \\ &= \sum_{i=1}^{k} \Lambda^H(f_H, u_i')\,\Lambda^H(\phi_i', \gamma) = \Lambda^H(f_H, \gamma)\end{aligned}$$

for all $\gamma \in U \cap G'$.

To extend the result to arbitrary $z \in Z_G$, we use right translation by z as in [**V**, 2.5]. □

We want to extend the matching of Theorem 3.3 to a matching which is valid for every $\gamma \in H \cap G'$. In order to do this, we need to be able to match orbital integrals in the neighborhood of any semisimple element of H.

Let $s \in H$ be an arbitrary semisimple element. Let M_G be the centralizer of s in G and let M_H be the centralizer of s in H.

Lemma 3.4.

(i) *Let $\psi_G \in C_c^\infty(M_G)$. Then there are a neighborhood U of s in M_H and $\psi_H \in C_c^\infty(M_H)$ so that for all $\gamma \in U \cap G'$,*

$$\Delta_H^G(\gamma)\Lambda_\kappa^{M_G}(\psi_G, \gamma) = \Lambda^{M_H}(\psi_H, \gamma).$$

(ii) *Let $\psi_H \in C_c^\infty(M_H)$. Then there are a neighborhood U of s in M_H and $\psi_G \in C_c^\infty(M_G)$ so that for all $\gamma \in U \cap G'$,*

$$\Delta_H^G(\gamma)\Lambda_\kappa^{M_G}(\psi_G, \gamma) = \Lambda^{M_H}(\psi_H, \gamma).$$

Proof. Write $M_G = \prod_{i=1}^k GL(n_i, F_i)$ where the F_i are extensions of degree r_i of F and $\sum_{i=1}^k n_i r_i = n$. For each $1 \leq i \leq k$, let κ_i be the character of $F_i^\times$ given by $\kappa_i(\lambda) = \kappa\left(N_{F_i/F}(\lambda)\right)$. Now the center T_G of M_G is isomorphic to

$\prod_{i=1}^{k} F_i^{\times}$. For $\lambda_i \in F_i^{\times}, 1 \leq i \leq k$, write $a(\lambda_1, ..., \lambda_k)$ for the corresponding element of T_G. Then

$$\kappa(a(\lambda_1, ..., \lambda_k)) = \prod_{i=1}^{k} \kappa_i(\lambda_i^{n_i}).$$

Let d_i be the order of κ_i. Then if there is $1 \leq i \leq k$ such that d_i does not divide n_i, there is $a \in T_G$ so that $\kappa(a) \neq 1$. But since $s \in H$ is semisimple, it is contained in some Cartan subgroup T of H. But every Cartan subgroup of H is a Cartan subgroup of G so that $T_G \subseteq T$. Thus $T_G \subseteq H$ so that $\kappa(a) = 1$ for all $a \in T_G$. Thus d_i divides n_i for all i. Write $n_i = m_i d_i, 1 \leq i \leq k$ and let E_i be the extension of F_i corresponding to κ_i. It is the minimal extension of F_i containing E. Now $M_H = \prod_{i=1}^{k} GL(m_i, E_i)$.

Thus $M_G = \prod_{i=1}^{k} GL(n_i, F_i)$ and $M_H = \prod_{i=1}^{k} GL(m_i, E_i)$ are products of groups $G_i = GL(n_i, F_i)$, $H_i = GL(m_i, E_i)$ of the same type as our original groups G and H. Further, if $g = (g_1, g_2, ..., g_k) \in M_G = \prod G_i$, then $\det_G g = \prod N_{F_i/F}(\det_{G_i} g_i)$ so that $\kappa(g) = \prod \kappa_i(g_i)$. Thus κ-twisted orbital integrals on M_G are the products of κ_i-twisted orbital integrals on the factors G_i. Now since $s \in M_H$ is central in M_G, we can apply Theorem 3.3 to match functions $\psi_G \in C_c^{\infty}(M_G)$ in a neighborhood of s with functions $\psi_H' \in C_c^{\infty}(M_H)$ using the transfer factor $\Delta_{M_G}^{M_H}$. Thus to complete the proof of the lemma it suffices to show that there is a neighborhood U of s in M_H so that $\Delta_G^H \left(\Delta_{M_G}^{M_H}\right)^{-1}$ is constant and non-zero on $U \cap G'$, so we can also match using the transfer factor Δ_G^H. This is proven in Lemmas 3.5 and 3.6 below. $\square$

In order to complete the proof of Lemma 3.4, we must define the transfer factors. For $\gamma, \delta \in H$, let $c_1, ..., c_m$, respectively $d_1, ..., d_m$ denote the eigenvalues of γ, resp. δ, in some extension of E. As in [**W2, Hl**] we set

$$r(\gamma, \delta) = \prod_{i,j=1}^{m} (c_i - d_j).$$

Then for all $\gamma \in H \cap G'$, we define

$$\Delta_G^{H,1}(\gamma) = \left| \prod_{\sigma,\tau \in \mathcal{G}(E/F), \sigma \neq \tau} r(\sigma\gamma, \tau\gamma) \right|_F^{\frac{1}{2}} \left| \det_G(\gamma) \right|_F^{\frac{(m-n)}{2}}$$

where $\mathcal{G}(E/F)$ denotes the Galois group of E/F. Further, we set

$$\Delta_G^{H,2}(\gamma) = 1$$

for all $\gamma \in H$ if d is odd. If d is even, let σ_+ be the unique element of order 2 in $\mathcal{G}(E/F)$ and let ν_E denote the valuation in E. Then we define

$$\Delta_G^{H,2}(\gamma) = (-1)^{\nu_E(r(\gamma,\sigma_+\gamma))}$$

for all $\gamma \in H$. Finally, for all $\gamma \in H \cap G'$, we define

$$\Delta_G^H(\gamma) = \Delta_G^{H,1}(\gamma)\Delta_G^{H,2}(\gamma).$$

We now return to the notation of Lemma 3.4 so that $s \in H$ is an arbitrary semisimple element with centralizers M_G and M_H in G and H respectively.

Lemma 3.5. *There is a neighborhood U of s in M_H so that $\Delta_G^{H,1}\left(\Delta_{M_G}^{M_H,1}\right)^{-1}$ is constant and non-zero on $U \cap G'$.*

Proof. For $\gamma \in H \cap G'$, let $c_1, ..., c_m$ denote the eigenvalues of γ considered as an element of $H = GL(m, E)$ and let $d_1, ..., d_n$ denote its eigenvalues considered as an element of $G = GL(n, F)$. Define

$$\Delta_H(\gamma) = \prod_{1 \le i \ne j \le m} (c_i - c_j), \quad \Delta_G(\gamma) = \prod_{1 \le i \ne j \le n} (d_i - d_j).$$

Fix $\gamma \in H \cap G'$. For each $\sigma \in \mathcal{G}(E/F)$, let $c(i, \sigma), 1 \le i \le m$, denote the eigenvalues of $\sigma\gamma$ as an element of $H = GL(m, E)$. Then as an element of $G = GL(n, F), \gamma$ has eigenvalues $c(i, \sigma), 1 \le i \le m, \sigma \in \mathcal{G}(E/F)$. Thus we can rewrite

$$\prod_{\sigma,\tau \in \mathcal{G}(E/F), \sigma \ne \tau} r(\sigma\gamma, \tau\gamma) = \Delta_G(\gamma) N_{E/F} \Delta_H(\gamma)^{-1}$$

and

$$\Delta_G^{H,1}(\gamma) = |\Delta_G(\gamma)|_F^{\frac{1}{2}} |\Delta_H(\gamma)|_E^{\frac{-1}{2}} \left|\det_G(\gamma)\right|_F^{\frac{(m-n)}{2}}.$$

Now use the notation in the proof of Lemma 3.4 so that we have $M_H = \prod H_i, M_G = \prod G_i$, where for $1 \le i \le k$, $H_i = GL(m_i, E_i), G_i = GL(n_i, F_i)$. Then for any $\gamma = \prod \gamma_i \in M_H \cap M_G'$, we have

$$\begin{aligned} \Delta_{M_G}^{M_H,1}(\gamma) &= \prod \Delta_{G_i}^{H_i,1}(\gamma_i) \\ &= \prod_i |\Delta_{G_i}(\gamma_i)|_{F_i}^{\frac{1}{2}} |\Delta_{H_i}(\gamma_i)|_{E_i}^{\frac{-1}{2}} \left|\det_{G_i}(\gamma_i)\right|_{F_i}^{\frac{(m_i-n_i)}{2}}. \end{aligned}$$

Thus

$$\Delta_G^{H,1}(\gamma)\ \Delta_{M_G}^{M_H,1}(\gamma)^{-1} = |\Delta_G(\gamma)|_F^{\frac{1}{2}} \prod_i |N_{F_i/F}\Delta_{G_i}(\gamma_i)|_F^{\frac{-1}{2}}$$
$$\times |\Delta_H(\gamma)|_E^{\frac{-1}{2}} \prod_i |N_{E_i/E}\Delta_{H_i}(\gamma_i)|_E^{\frac{1}{2}}$$
$$\times \left|\det_G(\gamma)\right|_F^{\frac{(m-n)}{2}} \prod_i \left|\det_{G_i}(\gamma_i)\right|_{F_i}^{\frac{(n_i-m_i)}{2}}.$$

We can index the eigenvalues of γ in $GL(n,F)$ as $c(i,j,t), 1 \leq i \leq k, 1 \leq j \leq n_i, 1 \leq t \leq r_i$, so that

$$\prod_i N_{F_i/F}\Delta_{G_i}(\gamma_i) = \prod_i \prod_t \prod_{j \neq j'} [c(i,j,t) - c(i,j',t)].$$

Then we have

$$\Delta_G(\gamma) \prod_i N_{F_i/F}\Delta_{G_i}(\gamma_i)^{-1} = \prod_{(i,t)\neq(i',t')} \prod_{j,j'} [c(i,j,t) - c(i',j',t')].$$

Thus $\gamma \mapsto \Delta_G(\gamma)\prod_i N_{F_i/F}\Delta_{G_i}(\gamma_i)^{-1}$ extends to a continuous function on M_G. Further, when $\gamma = s$, γ_i is central in G_i for all i so that $c(i,j,t) = c(i,j',t)$ for all i,j,j',t. But since $M_G = \prod G_i$ is the full centralizer of s in G we have $c(i,j,t) \neq c(i',j',t')$ if $(i,t) \neq (i',t')$. Thus $\Delta_G(\gamma)\prod_i N_{F_i/F}\Delta_{G_i}(\gamma_i)^{-1}$ is non-zero at $\gamma = s$. Similarly, we see that $\gamma \mapsto \Delta_H(\gamma)\prod_i N_{E_i/E}\Delta_{H_i}(\gamma_i)^{-1}$ extends to a continuous function on M_H which is non-zero at $\gamma = s$. Finally, the determinant factors are certainly continuous and non-zero on all of M_H. Thus

$$\gamma \mapsto \Delta_G^{H,1}(\gamma)\Delta_{M_G}^{M_H,1}(\gamma)^{-1}$$

extends to a function which is constant in a neighborhood of s in M_H. □

Lemma 3.6. *There is a neighborhood U of s in M_H so that $\Delta_G^{H,2}\left(\Delta_{M_G}^{M_H,2}\right)^{-1}$ is constant and non-zero on $U \cap G'$.*

Proof. We first need to derive an alternate formula for $\Delta_G^{H,2}$. Let σ_0 be a generator of $\mathcal{G}(E/F)$. For all $\gamma \in H$ we define

$$\tilde{\Delta}(\gamma) = \prod_{0 \leq i < j \leq d-1} r\left(\sigma_0^i\gamma, \sigma_0^j\gamma\right).$$

Then for each $\gamma \in H \cap G', \tilde{\Delta}(\gamma)$ is an element of $E^\times$. Clearly $r(\delta,\gamma) = (-1)^m r(\gamma,\delta)$ for all $\gamma,\delta \in H$. Thus it is easy to see that

$$\sigma_0\tilde{\Delta}(\gamma) = (-1)^{m(d-1)}\tilde{\Delta}(\gamma),\ \forall \gamma \in H.$$

If $m(d-1)$ is even we let $e_0 = 1$. Suppose that $m(d-1)$ is odd. Then d is even. Define $E_2 = \{e \in E : \sigma_0^2 e = e\}$. Then E_2/F is a cyclic extension of degree 2 and we can choose a unit $e_0 \in E_2$ such that $E_2 = F[e_0], \sigma_0 e_0 = -e_0$. With these choices of e_0 we have $e_0\tilde{\Delta}(\gamma) \in F$ for all $\gamma \in H$. We now claim that

$$\Delta_G^{H,2}(\gamma) = \kappa\left(e_0\tilde{\Delta}(\gamma)\right), \quad \gamma \in H \cap G'.$$

Let η be an unramified character of $E^\times$ which extends κ. Thus for all $e \in E^\times, \eta(e) = \zeta^{\nu_E(e)}$ where ζ is a primitive d^{th} root of unity. Now since e_0 is a unit we have

$$\Delta_G^{H,2}(\gamma) = \eta\left(e_0\tilde{\Delta}(\gamma)\right) = \eta\left(\tilde{\Delta}(\gamma)\right).$$

Now

$$\prod_{\sigma \neq \tau} r(\sigma\gamma, \tau\gamma) = \pm\tilde{\Delta}(\gamma)^2$$

so that

$$\nu_E\left(\tilde{\Delta}(\gamma)\right) = \frac{1}{2}\nu_E\left(\prod_{\sigma \neq \tau} r(\sigma\gamma, \tau\gamma)\right).$$

But

$$\prod_{\sigma \neq \tau} r(\sigma\gamma, \tau\gamma) = N_{E/F}\left(\prod_{\tau \neq 1} r(\gamma, \tau\gamma)\right).$$

Thus

$$\nu_E\left(\tilde{\Delta}(\gamma)\right) = \frac{1}{2}\prod_{1 \leq i \leq d-1} \nu_E\left(N_{E/F} r\left(\gamma, \sigma_0^i\gamma\right)\right).$$

But for any $1 \leq i \leq d-1$,

$$\begin{aligned}\nu_E\left(N_{E/F} r\left(\gamma, \sigma_0^{d-i}\gamma\right)\right) &= \nu_E\left(N_{E/F}\sigma_0^{d-i} r\left(\sigma_0^i\gamma, \gamma\right)\right)\\ &= \nu_E\left(N_{E/F} r\left(\gamma, \sigma_0^i\gamma\right)\right).\end{aligned}$$

Further, $\nu_E\left(N_{E/F}(e)\right) = d\nu_E(e)$ for all $e \in E^\times$. Thus, calculating modulo d, we have

$$\nu_E\left(\tilde{\Delta}(\gamma)\right) \equiv \begin{cases} \frac{d}{2}\nu_E\left(r\left(\gamma, \sigma_0^{\frac{d}{2}}\gamma\right)\right), & \text{if } d \text{ is even;}\\ 0, & \text{if } d \text{ is odd.}\end{cases}$$

Now when d is even $\sigma_0^{\frac{d}{2}} = \sigma_+$ so we can conclude that

$$\eta\left(\tilde{\Delta}(\gamma)\right) = \begin{cases} (-1)^{\nu_E(r(\gamma, \sigma_+\gamma))}, & \text{if } d \text{ is even;}\\ 1, & \text{if } d \text{ is odd.}\end{cases}$$

This completes the proof that $\Delta_G^{H,2}(\gamma) = \kappa\left(e_0\tilde{\Delta}(\gamma)\right), \quad \gamma \in H \cap G'$.

Similarly, for all $\gamma = \prod_i \gamma_i \in \prod H_i$, we have

$$\Delta_{M_G}^{M_H,2}(\gamma) = \prod \kappa_i\left(e_{0,i}\tilde{\Delta}_i(\gamma_i)\right)$$

where $e_{0,i}, \tilde{\Delta}_i$ are defined for the pair H_i, G_i. Since $\kappa_i = \kappa \circ N_{F_i/F}$, for $\gamma = \prod \gamma_i \in M_H \cap G'$ we have

$$\Delta_G^{H,2}(\gamma)\Delta_{M_G}^{M_H,2}(\gamma)^{-1} = \kappa\left(e_0\tilde{\Delta}(\gamma)\prod N_{F_i/F}\left(e_{0,i}\tilde{\Delta}_i(\gamma_i)\right)^{-1}\right).$$

Thus $\Delta_G^{H,2}\left(\Delta_{M_G}^{M_H,2}\right)^{-1}$ will extend to a function which is constant and nonzero in a neighborhood of s if we can show that

$$\gamma \mapsto \tilde{\Delta}(\gamma)\prod N_{F_i/F}\left(e_{0,i}\tilde{\Delta}_i(\gamma_i)\right)^{-1}$$

extends to a continuous function on M_H which is not zero at $\gamma = s$. Note that, using the notation in the proof of Lemma 3.6, we have

$$\tilde{\Delta}(\gamma)^2 = \pm\prod_{\sigma\neq\tau} r(\sigma\gamma, \tau\gamma) = \pm\Delta_G(\gamma)N_{E/F}\Delta_H(\gamma)^{-1}.$$

Thus the analysis proceeds exactly as in Lemma 3.6. That is, $\prod N_{F_i/F}\left(e_{0,i}\tilde{\Delta}_i(\gamma_i)\right)^{-1}$ cancels out exactly the terms in $\tilde{\Delta}(\gamma)$ which are zero when $\gamma = s$. □

Let $T_1, ..., T_k$ denote the Cartan subgroups of H containing s, up to G-conjugacy.

Lemma 3.7.

(i) *Let $f_G \in C_c^\infty(G)$. Then there are neighborhoods V_i of s in T_i and $f_H \in C_c^\infty(H)$ so that for all $1 \le i \le k, \gamma \in V_i \cap G'$,*

$$\Delta_H^G(\gamma)\Lambda_\kappa^G(f_G, \gamma) = \Lambda^H(f_H, \gamma).$$

(ii) *Let $f_H \in C_c^\infty(H)$. Then there are neighborhoods V_i of s in T_i and $f_G \in C_c^\infty(G)$ so that for all $1 \le i \le k, \gamma \in V_i \cap G'$,*

$$\Delta_H^G(\gamma)\Lambda_\kappa^G(f_G, \gamma) = \Lambda^H(f_H, \gamma).$$

Proof. This follows easily from combining Lemmas 3.4 and 2.6. □

Locally there is no obstruction to matching twisted orbital integrals on G with ordinary orbital integrals on H. However, if $f_H \in C_c^\infty(H)$ is to match orbital integrals with f_G for all $h \in H \cap G'$, we must have

$$(*) \qquad \Lambda^H\left(f_H, xhx^{-1}\right) = \kappa(x)\Delta_H^G\left(xhx^{-1}\right)\Delta_H^G(h)^{-1}\Lambda^H\left(f_H, h\right)$$

for all $h \in H \cap G'$ and $x \in G$ such that $xhx^{-1} \in H$.

Theorem 3.8.

(i) *Let $f_G \in C_c^\infty(G)$. Then there is $f_H \in C_c^\infty(H)$ so that for all $\gamma \in H \cap G'$,*

$$\Delta_H^G(\gamma)\Lambda_\kappa^G\left(f_G, \gamma\right) = \Lambda^H\left(f_H, \gamma\right).$$

(ii) *Let $f_H \in C_c^\infty(H)$ satisfying (*). Then there is $f_G \in C_c^\infty(G)$ so that for all $\gamma \in H \cap G'$,*

$$\Delta_H^G(\gamma)\Lambda_\kappa^G\left(f_G, \gamma\right) = \Lambda^H\left(f_H, \gamma\right).$$

Proof. (i) Let $T_1, ..., T_k$ be a complete set of Cartan subgroups of H up to H-conjugacy. For each i, let Ω_i be the support of $\Lambda_\kappa^G\left(f_G, \cdot\right)$ restricted to T_i. Let $X = \cup T_i$ and $\Omega = \cup \Omega_i$. Then Ω is a compact subset of X. For each $s \in X$, use Lemma 3.7 to find $U(s)$, a compact open neighborhood of s in X, and $f_s \in C_c^\infty(H)$ such that

$$\Delta_H^G(\gamma)\Lambda_\kappa^G\left(f_G, \gamma\right) = \Lambda^H\left(f_s, \gamma\right), \gamma \in U(s) \cap G'.$$

Note that since both sides are invariant under H-conjugacy, the equality is in fact valid for all $\gamma \in \mathcal{O}_H(U(s)) \cap G'$. Write $U'(s) = \mathcal{O}_H(U(s)) \cap X$.

Since Ω is compact, there are $s_1, ..., s_p$ so that $\Omega \subseteq \cup_{i=1}^p U'\left(s_i\right)$. By shrinking if necessary we can assume that the $U'\left(s_i\right)$ are disjoint. Now by Lemma 2.5 applied to ordinary orbital integrals on H, there are $f_i \in C_c^\infty(H), 1 \leq i \leq p$, so that

$$\Lambda^H\left(f_i, \gamma\right) = \begin{cases} \Lambda^H\left(f_{s_i}, \gamma\right), & \text{if } \gamma \in U'\left(s_i\right); \\ 0, & \text{if } \gamma \in X \backslash U'\left(s_i\right). \end{cases}$$

Let $f_H = \sum_{i=1}^p f_i$. Then for $\gamma \in X \cap G'$, if $\gamma \in U'\left(s_i\right)$, then

$$\Lambda^H\left(f_H, \gamma\right) = \Lambda_H\left(f_{s_i}, \gamma\right) = \Delta_H^G(\gamma)\Lambda_\kappa^G\left(f_G, \gamma\right).$$

If $\gamma \notin \cup_{i=1}^p U'\left(s_i\right)$, then $\gamma \notin \Omega$ so that

$$\Lambda_\kappa^G\left(f_G, \gamma\right) = 0 = \Lambda^H\left(f_H, \gamma\right).$$

(ii) Let $T_1, ..., T_k$ be a complete set of Cartan subgroups of H up to G-conjugacy. For each i, let Ω_i be the support of $\Lambda^H(f_H, \cdot)$ restricted to T_i. Let $X = \cup T_i$ and $\Omega = \cup \Omega_i$. Then Ω is a compact subset of X. For each $s \in X$, use Lemma 3.7 to find $U(s)$, a compact open neighborhood of s in X, and $f_s \in C_c^\infty(G)$ such that

$$\Delta_H^G(\gamma)\Lambda_\kappa^G(f_s, \gamma) = \Lambda^H(f_H, \gamma), \gamma \in U(s) \cap G'.$$

Note that since both sides transform in the same way with respect to G-conjugacy, the equality is in fact valid for all $\gamma \in \mathcal{O}_G(U(s)) \cap H \cap G'$. Write $U'(s) = \mathcal{O}_G(U(s)) \cap X$. Now the proof is finished in the same way as that of (i) using Lemma 2.5. □

If we drop the assumption that E/F is unramified, we can obtain a weaker version of Theorem 3.8 as follows. Let s be a semisimple element of H and as before let $T_1, ..., T_r$ be the Cartan subgroups of G which contain s, up to G-conjugacy. Suppose that $M_G = M_H$. Then $T_i \subseteq M_G = M_H \subseteq H$ for all $1 \le i \le r$. We can use the results of §2 to prove the following lemma.

Lemma 3.9. *Suppose $s \in H$ is a semisimple element such that $M_G = M_H$.*

(i) *Let $f_G \in C_c^\infty(G)$. Then there are neighborhoods V_i of s in T_i and $f_H \in C_c^\infty(H)$ so that for all $1 \le i \le r, \gamma \in V_i \cap G'$,*

$$\Lambda_\kappa^G(f_G, \gamma) = \Lambda^H(f_H, \gamma).$$

(ii) *Let $f_H \in C_c^\infty(H)$. Then there are neighborhoods V_i of γ_0 in T_i and $f_G \in C_c^\infty(G)$ so that for all $1 \le i \le r, \gamma \in V_i \cap G'$,*

$$\Lambda_\kappa^G(f_G, \gamma) = \Lambda^H(f_H, \gamma).$$

Proof. For part (i), use Lemma 2.6 to match $f_G \in C_c^\infty(G)$ with $\psi_G \in C_c^\infty(M_G)$. Now use Vignéras's version of Lemma 2.6 [**V**] applied to H and ordinary orbital integrals to match $\psi_H = \psi_G \in C_c^\infty(M_H)$ with $f_H \in C_c^\infty(H)$. For part (ii) go backwards. □

Suppose that $s \in H \cap G'$. Then $M_G = M_H$ is a Cartan subgroup of H and G, so that we can apply Lemma 3.9 in a neighborhood of s. Thus if we restrict our attention to functions supported on such points, we can use Lemmas 3.9 and 2.5 to prove the following theorem.

Theorem 3.10.

(i) *Let $f_G \in C_c^\infty(G')$. Then there is $f_H \in C_c^\infty(H \cap G')$ so that for all $\gamma \in H \cap G'$,*

$$\Lambda_\kappa^G(f_G, \gamma) = \Lambda^H(f_H, \gamma).$$

(ii) *Let* $f_H \in C_c^\infty(H \cap G')$ *such that*

$$\Lambda^H\left(f_H, x\gamma x^{-1}\right) = \kappa(x)\Lambda^H\left(f_H, \gamma\right)$$

for all $\gamma \in H \cap G', x \in G$ *such that* $x\gamma x^{-1} \in H$. *Then there is* $f_G \in C_c^\infty(G')$ *so that for all* $\gamma \in H \cap G'$,

$$\Lambda_\kappa^G\left(f_G, \gamma\right) = \Lambda^H\left(f_H, \gamma\right).$$

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UNIVERSITY OF MARYLAND
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