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## A FROBENIUS PROBLEM ON THE KNOT SPACE

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According to J.-L. Brylinski, there is a natural almost complex structure  $J$  on the space  $K$  of all knots in the Euclidean space  $R^3$ . The almost complex structure is formally integrable on  $K$ , i.e., the Nijenhuis tensor of  $J$  vanishes. The problem is whether  $J$  is integrable and hence  $K$  is a complex manifold. In this paper, we study the integrability of  $J$  explicitly in view point of a Frobenius problem.

### 1. Introduction

A knot is by definition a smooth imbedded circle in the Euclidean space  $R^3$ . The knot space is the space of all knots. In this paper, we study an integrability problem on the knot space which is as follows: According to Brylinski [3, 4], for any  $\gamma \in K$ , the tangent space  $T_\gamma K$  is the space of sections of the normal bundle of  $\gamma$  in  $R^3$ . A natural almost complex structure  $J$  is defined on  $K$  as a rotation of  $\frac{\pi}{2}$  in the normal plane bundle.  $J$  is formally integrable on  $K$ , i.e., the Nijenhuis tensor of  $J$  vanishes. Compared to the well-known theorem of Newlander-Nirenberg [17], the problem is whether  $J$  is integrable and hence  $K$  is a complex manifold.

A result of Drinfeld and LeBrun [3, 4] is that  $J$  is weakly integrable on the space  $K_0$  of real analytic knots, i.e., there are enough holomorphic functions on each local chart of  $K_0$ . In Lempert [15], the theory of twistor CR-manifolds is used to prove that  $J$  is weakly integrable on the space of real analytic knots in a real analytic 3-manifold with a real analytic metric. It is also proved that  $J$  is not integrable on the space  $K$  and  $K_0$ , i.e., there is no open set  $U \neq \emptyset$  on the knot space which is biholomorphic to an open set in  $T_\gamma K$  or  $T_\gamma K_0$ . LeBrun [14] has a similar result on the so-called space of world-sheets which are time-like 2-surfaces in 4-manifold with a Lorentzian metric.

In this paper, we define a natural local coordinate system on  $K$  and study the integrability of  $J$  explicitly in view point of a Frobenius problem. It will be shown that in the local coordinate system  $J$  can be written explicitly to see that it is real analytic and the  $\bar{\partial}$ -equation can be complexified to obtain a Frobenius problem and the Frobenius problem can be further reduced to a first order nonlinear partial differential equation in two dimensions. In the

case  $K_0$ , the equation is solvable and hence  $J$  is weakly integrable by the theorem of Cauchy-Kowalewska. In the case  $K$ , the equation is not solvable and thus the Frobenius problem is not integrable. (This does not implies that  $J$  is not integrable.) It is also explained that why the holomorphic functions on  $K_0$  fail to make a local chart by the implicit function theorem.

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## 2. The Knot Space $K$

In this section, some basic properties on the knot space  $K$  are collected and a natural coordinate system on  $K$  is defined on  $K$ . To formulate the almost complex structure  $J$  on  $K$ , the local basis on each of the local chart is also explicitly given. For a general knowledge on the knot space  $K$ , the reader may refer to Brylinski [5], which serves as the background of the paper.

**2.1. The knot space  $K$ .** The knot space  $K$  is roughly speaking the space of all knots in  $R^3$ . A precise identification of the space  $K$  is given as follows.

The knot space  $K$  has a close relation with the loop space  $L$ , i.e., the space of all smooth maps from the standard circle  $S^1$  to  $R^3$ , with the topology of uniform convergence of the map and all its derivatives. It is well-known that  $L$  is a Fréchet space, and the orientation preserving diffeomorphism group of  $S^1$  acts on  $L$  as a reparametrization. Restricted on the space  $L^*$  of imbedded loops, the action is free and the quotient space is a smooth Fréchet manifold. The knot space  $K$  is thus defined to be the quotient space.

An element in  $K$  is a closed oriented imbedded curve in  $R^3$ . For any  $\gamma \in K$ , denote  $l$  the arc length of  $\gamma$  and  $s$  an arc-length parametrization of  $\gamma$ . For convenience, a parametrization  $\theta$  of  $\gamma$  is called standard, if

$$\frac{ds}{d\theta} = l, (0 \leq \theta \leq 1).$$

An elementary fact is that different arc-length or standard parametrizations of  $\gamma$  differ only by a constant.

For any  $\gamma \in K$ , let  $N_\gamma$  denote the normal bundle of  $\gamma$  in  $R^3$ . A basic fact is that the tangent space  $T_\gamma K$  is the space  $\Gamma(N_\gamma)$  of sections of  $N_\gamma$ . This can be understood as follows: Since  $L^*$  is an open submanifold in  $L$ , for any  $\gamma \in L^*$ ,

$$T_\gamma L^* \simeq C^\infty(S^1, R^3).$$

Modulo the tangent factor to the knot,  $T_\gamma K = \Gamma(N_\gamma)$ .

For any  $\gamma \in K$ , denote by  $N_\delta(\gamma)$  the tubular neighborhood of  $\gamma$  with radius  $\delta$  in  $R^3$ . Note that, when  $\delta > 0$  is small,  $N_\delta(\gamma)$  is imbedded in  $R^3$ , the space  $\mathcal{N}_\delta(\gamma)$  of knots in  $R^3$  with image in  $N_\delta(\gamma)$  is an open neighborhood of  $\gamma$  in  $K$ . Note also that  $\mathcal{N}_\delta(\gamma)$  can be identified as the space of sections  $h$  of  $N_\gamma$  with  $C^0$ -norm  $\|h\|_{C^0} < \delta$ .

$$(2.1) \quad \mathcal{N}_\delta(\gamma) \simeq \{h \in \Gamma(N_\gamma) : \|h\|_{C^0} < \delta\}.$$

Similarly, the space  $\mathcal{N}_\delta^1(\gamma)$  of knots in  $R^3$ , which can be identified as

$$(2.2) \quad \mathcal{N}_\delta^1(\gamma) \simeq \{h \in \Gamma(N_\gamma) : \|h\|_{C^1} < \delta\},$$

is also an open neighborhood of  $\gamma$  in  $K$ .

**2.2. A local coordinate system on  $K$ .** To define a local coordinate system on the knot space  $K$ , recall the basic theory of frénet of curves in  $R^3$  as follows. Note that an element  $\gamma \in K$  is a closed imbedded curve in  $R^3$ , the curvature  $\kappa$  of  $\gamma$  is a well-defined continuous function along  $\gamma$ .  $\kappa$  has nonnegative values and may be zero somewhere on  $\gamma$ . Denote by  $K^*$  the space of knots in  $R^3$  with curvature  $\kappa > 0$  everywhere, i.e,

$$K^* = \{\gamma \in K : \kappa > 0\}.$$

There is first the following:

**Lemma 2.1.** *The space  $K^*$  is open and dense in  $K$ .*

*Proof.* Clearly  $K^*$  is an open set in  $K$ . To show that  $K^*$  is dense in  $K$ , the idea is that, for any  $\gamma \in K$ , even  $\kappa$  vanishes somewhere on  $\gamma$ , a generic small twist of the curve has positive curvature everywhere. In another word, a certain generic perturbation of  $\gamma$  is in  $K^*$ .

To describe the perturbation, note first that  $N_\gamma$  is a trivial plane bundle, there are two sections  $\tilde{e}_2, \tilde{e}_3$  of  $N_\gamma$  which form a basis of  $\Gamma(N_\gamma)$ . Let  $\theta$  be a standard parametrization of  $\gamma$ ; then the perturbation  $\tilde{\gamma}$  is a twist by the normal frame field as follows:

$$\tilde{\gamma}(\theta) = \gamma(\theta) + f_2(\theta)\tilde{e}_2(\theta) + f_3(\theta)\tilde{e}_3(\theta),$$

where  $f_2, f_3$  are smooth periodic functions in  $\theta$ . Note that  $\frac{d\tilde{\gamma}}{d\theta}$  involves  $f_2, f_3$  and their first derivatives, when  $\delta > 0$  is small, and

$$\|f_2\|_{C^1} < \delta, \|f_3\|_{C^1} < \delta,$$

$\frac{d\tilde{\gamma}}{d\theta} \neq 0$  everywhere. Denote by  $\tilde{s}$  an arc-length parametrization of  $\tilde{\gamma}$ . Then for a generic perturbation  $(f_2, f_3)$ ,  $\frac{d^2\tilde{\gamma}}{d\tilde{s}^2} \neq 0$  everywhere. Thus  $\kappa(\tilde{\gamma}) > 0$ ,  $\tilde{\gamma} \in K^*$ . This shows that  $K^*$  is dense in  $K$ . Lemma 2.1 is proved.

To define a local coordinate system on  $K$ , for any  $\gamma \in K^*$ , fix an arc-length parametrization  $s$  and a standard parametrization  $\theta$  of  $\gamma$ . Note that the Frenét frame  $\{e_1, e_2, e_3\}$  is well-defined along  $\gamma$ , where

$$(2.3) \quad \begin{cases} e_1 = \frac{d\gamma}{ds} \\ \frac{de_1}{ds} = \kappa e_2 \\ e_3 = e_1 \times e_2. \end{cases}$$

Recall the following Frenét formula:

$$(2.4) \quad \begin{cases} \frac{de_1}{ds} = \kappa e_2 \\ \frac{de_2}{ds} = -\kappa e_1 + \tau e_3 \\ \frac{de_3}{ds} = -\tau e_2. \end{cases}$$

Recall that the open neighborhood  $\mathcal{N}_\delta(\gamma)$  of  $\gamma$  is identified as (2.1). For any  $\tilde{\gamma} \in \mathcal{N}_\delta(\gamma)$ ,  $\tilde{\gamma}$  corresponds to a section  $z(\theta) \in \Gamma(N_\gamma)$ . Note that  $z(\theta)$  can be written as

$$z(\theta) = x(\theta)e_2 + y(\theta)e_3;$$

where  $x(\theta)$ ,  $y(\theta)$  are smooth periodic functions in  $\theta$ . Expand  $x(\theta)$  and  $y(\theta)$  as Fourier series

$$(2.5) \quad \begin{aligned} x(\theta) &= x_0 + \sum_{k=1}^{\infty} x_{2k-1} \sin(2k\pi\theta) + x_{2k} \cos(2k\pi\theta), \\ y(\theta) &= y_0 + \sum_{k=1}^{\infty} y_{2k-1} \sin(2k\pi\theta) + y_{2k} \cos(2k\pi\theta), \end{aligned}$$

then a local coordinate of  $\tilde{\gamma} = \gamma + z(\theta) \in \mathcal{N}_\delta(\gamma)$  can be given as the Fourier coefficients  $\{x_k, y_k : k \in N\}$ .

To define the local coordinate system on  $K$ , it is left to show that the collection

$$(2.6) \quad \{\mathcal{N}_\delta(\gamma) : \gamma \in K^*, \delta > 0\}$$

is an open cover on  $K$ . Needless to say, in (2.6),  $\delta > 0$  is chosen small so that the tubular neighborhood  $N_\delta(\gamma)$  is imbedded in  $R^3$ .

**Lemma 2.2.**  $K = \cup_{\gamma \in K^*, \delta > 0} \mathcal{N}_\delta(\gamma)$ .

*Proof.* For any  $\gamma \in K$ , choose  $\delta > 0$  and a sequence  $\{\gamma_n\}$  in  $K^*$  so that  $N_\delta(\gamma)$  is imbedded and  $\gamma_n \rightarrow \gamma$  in  $C^0$ -norm. Choose  $n$  large such that  $N_{\frac{\delta}{2}}(\gamma_n)$  is also imbedded; then  $\gamma \in \mathcal{N}_{\frac{\delta}{2}}(\gamma_n)$ .

Similarly, the collection

$$(2.7) \quad \{\mathcal{N}_\delta^1(\gamma) : \gamma \in K^*, \delta > 0\}$$

is an open cover on  $K$ . Thus (2.7) also defines a local coordinate system on  $K$ . This is the local coordinate system we will use.

**2.3. A local basis on the local patch.** To formulate the almost complex structure  $J$  in local coordinates, a local basis  $\{X_k, Y_k : k \in N\}$  on  $K$  will be defined in this section. It will be also shown that  $\{X_k, Y_k : k \in N\}$  is the local basis, i.e.,  $X_k = \partial_{x_k}, Y_k = \partial_{y_k}$  for all  $k \in N$ .

To define  $X_0$ , consider the normal vector field  $e_2 = e_2(\theta)$  along  $\gamma$ . Note that  $e_2$  can be regarded as a tangent vector on  $K$  at  $\gamma$ . It is defined that  $X_0(\gamma) = e_2$ . For any  $\tilde{\gamma} \in \mathcal{N}_\delta^1(\gamma)$ , to define  $X_0(\tilde{\gamma})$ , translate the vector field  $e_2 = e_2(\theta)$  along  $\gamma$  onto  $\tilde{\gamma}$ . Note that  $e_2$  may not remain in  $T_{\tilde{\gamma}}K$ , i.e.,  $e_2(\theta)$  may have both normal component  $\bar{e}_2$  and tangential component  $e_2^T$  along  $\tilde{\gamma}$ . It is defined that  $X_0 = \bar{e}_2$ .  $\bar{e}_2$  will be explicitly computed later.

To define  $Y_0$  on  $\mathcal{N}_\delta^1(\gamma)$ , consider the normal vector field  $e_3 = e_3(\theta)$  along  $\gamma$ . The translated vector field  $e_3(\theta)$  along  $\tilde{\gamma}$  may have both normal component  $\bar{e}_3$  and tangential component  $e_3^T$ . It is defined that  $Y_0 = \bar{e}_3$ .  $\bar{e}_3$  will be also explicitly computed later.

Similarly, for any  $k \in N$ , consider the translated vector field  $\sin(2k\pi\theta)e_2(\theta)$  along  $\tilde{\gamma}$ . Note that the normal component is  $\sin(2k\pi\theta)\bar{e}_2$  and the tangential component is  $\cos(2k\pi\theta)e_2^T$ . It is defined that

$$(2.8) \quad X_{2k-1} = \sin(2k\pi\theta)\bar{e}_2.$$

There are also the following definitions:

$$(2.9) \quad \begin{aligned} X_{2k} &= \cos(2k\pi\theta)\bar{e}_2, Y_{2k-1} = \sin(2k\pi\theta)\bar{e}_3, \\ Y_{2k} &= \cos(2k\pi\theta)\bar{e}_3 (k \in N). \end{aligned}$$

**Proposition 2.3.**  $\{X_k, Y_k : k \in N\}$  defined above is the local basis on the local patch  $\mathcal{N}_\delta^1(\gamma)$  when  $\delta > 0$  is small, i.e.,

$$(2.10) \quad X_k = \partial_{x_k}, Y_k = \partial_{y_k},$$

where  $\{x_k, y_k\}$  is the local coordinates defined as (2.5).

*Proof.* Notice that  $X_k, Y_k$  are in fact inherited from the base vectors on the loop space  $L$ . To be precise, let

$$L' = \{\gamma \in L^* : \kappa(\gamma) > 0\}.$$

Then  $L'$  is an open subset in  $L$ . For any  $\gamma \in L'$ , let  $\{e_1, e_2, e_3\}$  be the Frenét frame along  $\gamma$ . Note that, for any  $\tilde{\gamma}$  in a neighborhood of  $\gamma$  in  $L$ ,  $\tilde{\gamma}$  can be written as

$$\tilde{\gamma} = \gamma + \sum_{i=1}^3 h_i e_i$$

for some smooth periodic functions  $h_1, h_2, h_3$ . Thus, a local coordinate of  $\tilde{\gamma}$  can be given as the coefficients of the Fourier expansion of  $h = (h_1, h_2, h_3)$ ;  $e_1, e_2, e_3$  are all local base vectors on  $L$ . Modulo the factor with values in the Virasoro algebra,  $\bar{e}_2, \bar{e}_3$  are both local base vectors on  $\mathcal{N}_\delta^1(\gamma)$ ,

$$\bar{e}_2 = \partial_{x_0}, \bar{e}_3 = \partial_{y_0}.$$

Similarly, the other  $X_k, Y_k$ 's are also base vectors on  $\mathcal{N}_\delta^1(\gamma)$ ,

$$X_k = \partial_{x_k}, Y_k = \partial_{y_k} (k \in N).$$

**Remark.** Notice that  $\|e_2^T\|_{C^0}$  and  $\|e_3^T\|_{C^0}$  involve the first derivatives of  $x(\theta)$  and  $y(\theta)$ . To ensure  $\bar{e}_2, \bar{e}_3 \neq 0$  and linear independent along  $\tilde{\gamma}$ ,  $\bar{e}_2$  and  $\bar{e}_3$  are defined only on the small local patch  $\mathcal{N}_\delta^1(\gamma)$ . On the other hand, it is a remark that these local patches do give an open cover on  $K$  and thus defines a local coordinate system on  $K$ . The proof is similar to that of Lemma 2.2.

$\bar{e}_2$  and  $\bar{e}_3$  are now explicitly computed as follows. For  $\gamma \in K^*$ , denote by  $l, \kappa, \tau$  the arc length, curvature and torsion of  $\gamma$ , and  $s, \theta$  an arc-length and standard parameter of  $\gamma$ , also  $\{e_1, e_2, e_3\}$  the Frenét frame along  $\gamma$ . For any  $\tilde{\gamma} \in \mathcal{N}_\delta^1(\gamma)$ ,

$$(2.11) \quad \tilde{\gamma} = \gamma + x(\theta)e_2 + y(\theta)e_3,$$

let  $\tilde{s}$  denote the arc-length parametrization of  $\tilde{\gamma}$ ,  $\bar{e}_1 = \frac{d\tilde{\gamma}}{d\tilde{s}}$  the unit tangent field along  $\tilde{\gamma}$ .

To compute  $\bar{e}_2$  and  $\bar{e}_3$ , differentiate (2.11). By the Frenét formula,

$$\bar{e}_1 \frac{d\tilde{s}}{d\theta} = l(1 - \kappa x)e_1 + (x' - l\tau y)e_2 + (y' + l\tau x)e_3.$$

For convenience, introduce

$$(2.12) \quad \lambda_1 = l(1 - \kappa x), \lambda_2 = x' - l\tau y, \lambda_3 = y' + l\tau x;$$

then there are the following identities:

$$\frac{d\tilde{s}}{d\theta} = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{\frac{1}{2}},$$

$$(2.13) \quad \tilde{e}_1 = (\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3) / (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{\frac{1}{2}}.$$

Notice that

$$\bar{e}_2 = e_2 - \langle e_2, \tilde{e}_1 \rangle \tilde{e}_1,$$

$$\bar{e}_3 = e_3 - \langle e_3, \tilde{e}_1 \rangle \tilde{e}_1,$$

$\bar{e}_2$  and  $\bar{e}_3$  are given as:

$$\bar{e}_2 = [-\lambda_1 \lambda_2 e_1 + (\lambda_1^2 + \lambda_3^2) e_2 - \lambda_2 \lambda_3 e_3] / (\lambda_1^2 + \lambda_2^2 + \lambda_3^2),$$

$$(2.14) \quad \bar{e}_3 = [-\lambda_1 \lambda_3 e_1 - \lambda_2 \lambda_3 e_2 + (\lambda_1^2 + \lambda_2^2) e_3] / (\lambda_1^2 + \lambda_2^2 + \lambda_3^2).$$

Notice that both  $\bar{e}_2$  and  $\bar{e}_3$  are linear combinations of  $e_1, e_2, e_3$  with coefficients which are real analytic functions in  $\kappa, \tau, x(\theta), y(\theta)$  and the first derivatives  $x'(\theta), y'(\theta)$ .

### 3. The almost complex structure J

On the knot space  $K$ , there is a genuine almost complex structure  $J$ . Recall that, for any  $\gamma \in K$ ,  $T_\gamma K = \Gamma(N_\gamma)$ .  $J_\gamma$  is defined as the rotation of  $\frac{\pi}{2}$  in the plane bundle. In [3-5], it is proved by Brylinski that  $J$  is formally integrable, i.e., the Nijenhuis tensor of  $J$  vanishes on  $K$ . In this section,  $J$  is formulated explicitly in local coordinates. This means to compute the action of  $J$  on the local basis  $\{X_k, Y_k\}$  defined as (2.8) and (2.9). In this way  $J$  is shown real analytic on  $K$ .

To compute  $J(X_k)$  and  $J(Y_k)$ , for any  $\gamma \in K^*$ , fix a standard parametrization  $\theta$  for  $\gamma$  and the Frenét frame  $\{e_1, e_2, e_3\}$  along  $\gamma$ . For any  $\tilde{\gamma} \in \mathcal{N}_\delta^1(\gamma)$ ,

$$\tilde{\gamma} = \gamma + x(\theta)e_2 + y(\theta)e_3,$$

let  $\tilde{s}$  be the arc-length parametrization for  $\tilde{\gamma}$  and  $\tilde{e}_1 = \frac{d\tilde{\gamma}}{d\tilde{s}}$ .

Recall that  $\tilde{e}_1, \bar{e}_2$  and  $\bar{e}_3$  are computed as (2.13) and (2.14). Since  $J$  is the rotation of  $\frac{\pi}{2}$ ,

$$J(\bar{e}_2) = \tilde{e}_1 \times \bar{e}_2,$$

$$(3.1) \quad J(\bar{e}_3) = \tilde{e}_1 \times \bar{e}_3.$$

Substituting (2.13) and (2.14) to (3.1),  $J(\bar{e}_2)$  and  $J(\bar{e}_3)$  are computed as follows:

$$J(\bar{e}_2) = (\lambda_2 e_1 - \lambda_1 e_2) / (\lambda_1^2 + \lambda_2^2 + \lambda_3^2),$$

$$(3.2) \quad J(\bar{e}_3) = (\lambda_1 e_3 - \lambda_3 e_1) / (\lambda_1^2 + \lambda_2^2 + \lambda_3^2),$$

where  $\lambda_1, \lambda_2$  and  $\lambda_3$  are given as (2.12). Written as linear combinations

$$(3.3) \quad J(\bar{e}_2) = a e_2 + b e_3, J(\bar{e}_3) = c \bar{e}_2 + d \bar{e}_3,$$

the coefficients are then given as follows:

$$(3.4) \quad \begin{aligned} a &= \frac{\lambda_2 \lambda_3}{\lambda_1 (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{\frac{1}{2}}}, \\ b &= \frac{\lambda_1^2 + \lambda_3^2}{\lambda_1 (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{\frac{1}{2}}}, \\ c &= -\frac{\lambda_1^2 + \lambda_2^2}{\lambda_1 (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{\frac{1}{2}}}, \\ d &= -\frac{\lambda_2 \lambda_3}{\lambda_1 (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^{\frac{1}{2}}}. \end{aligned}$$

Let  $A$  denote the  $2 \times 2$  matrix defined as

$$(3.5) \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note that, for any  $X = g_2 \bar{e}_2 + g_3 \bar{e}_3 \in T_\gamma K$ ,

$$(3.6) \quad JX = \tilde{e}_1 \times X = g_2 \bar{e}_2 + g_3 \bar{e}_3.$$

Denote by  $X = (g_2, g_3)$ ,  $J$  is then given as

$$(3.7) \quad JX = XA.$$

$J$  is represented by the matrix  $A$  and can be compared to the almost complex structure in two dimensions.

At the origin of  $\mathcal{N}_\delta^1(\gamma)$ ,  $A$  is the standard matrix

$$(3.8) \quad A_\gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that  $A$  is a  $2 \times 2$  matrix with entries which are real analytic functions in  $\kappa, \tau, x(\theta), y(\theta)$  and the first derivatives  $x'(\theta), y'(\theta)$ .  $J$  is a well-defined and smooth almost complex structure on  $K$ . There is further the following:

**Proposition 3.**  $J$  is a real analytic almost complex structure on  $K$ .

*Proof.* With  $J$  given explicitly as above, the proof is omitted.

**Remark.** Since  $A$  involves the first derivatives  $x'(\theta)$  and  $y'(\theta)$ , for any  $\gamma \in K$ ,

$$J_\gamma : T_\gamma K \rightarrow T_\gamma K$$

make sense as an endomorphism only when  $K$  is equipped with the smooth Fréchet topology.

To end the section, the formula (3.7) is explained as follows. Note that for fixed  $\tilde{\gamma} \in \mathcal{N}_\delta^1(\gamma)$ , the entries of  $A$  are smooth periodic functions. Expand the entries as Fourier series

$$a = a_0 + \sum_{k=1}^\infty a_{2k-1} \sin(2k\pi\theta) + a_{2k} \cos(2k\pi\theta),$$

$$b = b_0 + \sum_{k=1}^\infty b_{2k-1} \sin(2k\pi\theta) + b_{2k} \cos(2k\pi\theta),$$

then  $J(\bar{e}_2)$  is actually given as

$$(3.9) \quad J(\bar{e}_2) = a\bar{e}_2 + b\bar{e}_3 = \sum_{k=0}^\infty a_k X_k + b_k Y_k.$$

$J(\bar{e}_3)$  can be formulated similarly as (3.9).

#### 4. The $\bar{\partial}$ -Equation and the Frobenius Problem

In this section, we formulate the  $\partial$ -equation corresponding to the almost complex structure  $J$  which is conjugate to the  $\bar{\partial}$ -equation. Recall that  $J$  is real analytic on  $K$ , the  $\partial$ -equation can be complexified into a Frobenius equation. Since  $J$  is represented by the  $2 \times 2$  matrix  $A$ , and the entries of  $A$  involves the first derivatives of the coordinate functions, the Frobenius equation can be reduced to a first order nonlinear partial differential equation. In the next section we solve the nonlinear equation and prove that  $J$  is weakly integrable on the space  $K_0$  of real analytic knots and in Section 6 we prove that the Frobenius equation is not solvable on  $K$ . Note that the Frobenius equation is stronger than the  $\partial$ -equation: When the former is solvable, so is the latter. Conversely, if the  $\partial$ -equation is solvable and the solutions are real analytic, the complexified solutions satisfy the Frobenius equation.

**4.1. The  $\partial$ -equation.** A few notations are fixed first to formulate the  $\partial$ -equation. First, since  $J$  is an almost complex structure on  $K$ , for any  $\gamma \in K$ ,  $J_\gamma^2 = -I$  as an endomorphism on  $T_\gamma K$ , where  $I$  is the identity map. Let  $T_\gamma^C K$  be the complexified tangent space

$$T_\gamma^C K = T'_\gamma K \oplus T''_\gamma K,$$

where

$$(4.1) \quad T'_\gamma K = \{X - iJ_\gamma X : X \in T_\gamma K\}$$

is the  $i$ -eigenspace of  $J_\gamma : T_\gamma K \rightarrow T_\gamma K$  and

$$(4.2) \quad T''_\gamma K = \{X + iJ_\gamma X : X \in T_\gamma K\}$$

is the  $(-i)$ -eigenspace of  $J_\gamma$ .

Let  $T'K = \cup_{\gamma \in K} T'_\gamma K$ . Then  $T'K$  is a subbundle of  $T^C K$ . It is well-known that  $T'K$  is closed under the Lie bracket if and only if  $J$  is formally integrable, i.e., the Nijenhuis tensor of  $J$  vanishes. Similarly  $T''K = \cup_{\gamma \in K} T''_\gamma K$  is also a subbundle of  $T^C K$ .

For fixed  $\gamma \in K^*$  and  $\tilde{\gamma} \in \mathcal{N}_\delta^1(\gamma)$ ,  $T'_\gamma K$  is given as

$$(4.3) \quad T'_\gamma K = \{X - iXA : X \in C^\infty(S^1, R^2)\},$$

in local coordinates, where  $A$  is the  $2 \times 2$  matrix (3.5). Note that  $T'_\gamma K$  is spanned by

$$\{X_k - iX_k A : k \in N\}$$

since  $J^2 = -I$ .

Introduce complex coordinates

$$z_k = x_k + iy_k, \bar{z}_k = x_k - iy_k,$$

$$\partial_{z_k} = \frac{1}{2}(X_k - iY_k),$$

$$(4.4) \quad \partial_{\bar{z}_k} = \frac{1}{2}(X_k + iY_k) (k \in N).$$

$X_k - iX_k A$  is computed as

$$(4.5) \quad (1 + b - ia)\partial_{z_k} + (1 - b - ia)\partial_{\bar{z}_k}.$$

Since  $J$  is formally integrable, the collection

$$(4.6) \quad \left\{ \partial_{z_k} + \frac{1 - b - ia}{1 + b - ia} \partial_{\bar{z}_k} : k \in N \right\}$$

is close under the Lie bracket. Thus elements in (4.6) are commutative. It will be proved in the next section that for any  $k \in N$ , the  $\partial$ -equation

$$(4.7) \quad \partial_{z_k} + \frac{1 - b - ia}{1 + b - ia} \partial_{\bar{z}_k} = \partial_{\zeta_k}$$

is solvable on the space  $K_0$ .

**4.2. The Frobenius equation.** Recall that  $a, b$  are real analytic functions in  $\kappa, \tau$ , and the coordinate functions  $x(\theta), y(\theta)$  and their first derivatives. Without confusion, denote by

$$(4.8) \quad a = a(z(\theta), \bar{z}(\theta)), b = b(z(\theta), \bar{z}(\theta)).$$

Note that both  $a$  and  $b$  can be complexified as  $a(z(\theta), w(\theta))$  and  $b(z(\theta), w(\theta))$ , (4.6) can be complexified as

$$(4.9) \quad \left\{ \partial_{z_k} + \frac{1 - b(z, w) - ia(z, w)}{1 + b(z, w) - ia(z, w)} \partial_{w_k} \right\}$$

on the complexified local patch

$$(4.10) \quad \mathcal{N}_{\delta, C}^1(\gamma) = \left\{ f \in \Gamma(N_\gamma^C) : \|f\|_{C^1} < \delta \right\}.$$

Note that a smooth map on  $\mathcal{N}_{\delta, C}^1(\gamma)$  can be written as

$$(4.11) \quad \phi(z(\theta), w(\theta)) = \phi_0 + \sum_{k=1}^{\infty} \phi_{2k-1} \sin(2k\pi\theta) + \phi_{2k} \cos(2k\pi\theta)$$

with  $\phi_k$ 's are functions in  $\{z_k, w_k : k \in N\}$ . Let  $D_1\phi = \left(\frac{\partial\phi_i}{\partial z_k}\right)$  denote the Jacobian matrix. The Frobenius equation is then

$$(4.12) \quad \begin{cases} D_1\phi(z, w) = \frac{1 - b(z, \phi) - ia(z, \phi)}{1 + b(z, \phi) - ia(z, \phi)} \\ \phi(0, w) = w. \end{cases}$$

The reader may compare our formulation with Lang [12].

**4.3. The reduction of the Frobenius equation.** To solve the Frobenius equation (4.12), as Lang [12], for any  $(z, w) \in \mathcal{N}_{\delta, C}^1(\gamma)$ , let  $\psi$  be the map defined as

$$(4.13) \quad \psi(t, z, w) = \phi(tz, w).$$

By the equation (4.12),  $\psi(t, z, w)$  satisfies the following ordinary differential equation in the ét space  $C^\infty(S^1, R^2)$ :

$$(4.14) \quad \begin{cases} \frac{d\psi}{dt} = \frac{(1 - b(tz, \psi) - ia(tz, \psi))z}{1 + b(tz, \psi) - ia(tz, \psi)} \\ \psi(0, z, w) = w. \end{cases}$$

Recall that  $a(z(\theta)$  and  $w(\theta)), b(z(\theta), w(\theta))$  are  $C^\omega$ -functions in  $\kappa, \tau, z(\theta), w(\theta)$  and  $z'(\theta), w'(\theta)$ . The Frobenius equation (4.14) involves  $\psi, \frac{\partial\psi}{\partial t}$  and  $\frac{\partial\psi}{\partial\theta}$ , it is a nonlinear partial differential equation of the first order.

**Proposition 4.** *For  $z, w \in C^\infty(S^1, R^2)$ , the Frobenius equation (4.12) has a unique solution  $\phi(z, w)$  iff (4.14) has a unique solution  $\psi(t, z, w)$  for  $0 \leq t \leq 1$ . The relation between the solutions is*

$$(4.15) \quad \phi(z, w) = \psi(1, z, w).$$

*Proof.* Similar to that of [12] or [11]. □

### 5. The Weak Integrability on $K_0$

In this section, we solve the Frobenius equation (4.12) and prove that  $J$  is weakly integrable on the space  $K_0$  of real analytic knots. By the construction in Section 4, the proof is quite easy by the theorem of Cauchy-Kowalewska. Since  $K_0$  is equipped with the  $C^\omega$ -topology, we need to pay attention to analytical details. An explanation is also given in the section that the holomorphic functions on  $K_0$  fail to make a local chart on  $K_0$  by the inverse theorem of Nash and Moser.

**5.1. The  $C^\omega$ -topology on  $K_0$ .** The precise definition of  $K_0$  is given as follows. Let  $L_0$  be the space of  $C^\omega$ -loops in  $R^3$  and  $L_0^*$  be the space of imbedded  $C^\omega$ -loops in  $R^3$ . Then the orientation preserving  $C^\omega$ -diffeomorphism group of  $S^1$  act freely on the space  $L_0^*$  and  $K_0$  is defined as the quotient space.

The  $C^\omega$ -topology on  $L_0$  is given as follows. Note that for any  $\gamma(t) \in L_0$ ,  $\gamma(t)$  can be extended analytically over a certain annulus

$$A_{\epsilon_0} = \{z \in C : 1 - \epsilon_0 < |z| < 1 + \epsilon_0\}.$$

Let

$$(5.1) \quad \mathcal{N}_{\epsilon, \delta}(\gamma) = \{\tilde{\gamma} \in L_0 : \|\tilde{\gamma} - \gamma\|_{C^0(A_\epsilon)} < \delta\}$$

with  $\epsilon < \epsilon_0$ . As Brylinski [5], the collection

$$\{\mathcal{N}_{\epsilon, \delta}(\gamma) : \epsilon > 0, \delta > 0\}$$

define a local basis of the  $C^\omega$ -topology at  $\gamma$ . Note that  $\gamma \in L$  is in  $L_0$  iff the arc-length parametrization  $\gamma(s)$  or the standard parametrization  $\gamma(\theta)$  is  $C^\omega$ .

As Lemma 2.1, let  $L'_0$  be the space of  $C^\omega$ -loops with curvature  $\kappa > 0$  everywhere. Then  $L'_0$  is an open and dense set in  $L_0$ . Thus the collection

$$(5.2) \quad \{\mathcal{N}_{\epsilon,\delta}(\gamma) : \gamma \in L'_0, \epsilon > 0, \delta > 0\}$$

gives an open cover on  $L_0$  and hence a local coordinate system on  $L_0$ . Note that the  $C^\omega$ -topology is a finer one than the smooth Fréchet topology, because by the Cauchy formula, for any  $\epsilon, \epsilon'$  with  $0 < \epsilon < \epsilon'$ , all the  $C^n$ -norm of  $\gamma \in L_0$  on  $A_\epsilon$  can be bounded by  $\|\gamma\|_{C^0(A_{\epsilon'})}$ , as Theorem 14.6 of [20].

Descending to the quotient topology, for any  $\gamma \in K_0$ , define again  $\mathcal{N}_{\epsilon,\delta}(\gamma)$  by (5.1). Then the collection

$$(5.3) \quad \{\mathcal{N}_{\epsilon,\delta}(\gamma) : \gamma \in K'_0, \epsilon > 0, \delta > 0\}$$

is an open cover on  $K_0$  and gives a local coordinate system on  $K_0$ . As (2.5), (2.8) and (2.9), let  $x(\theta)$  and  $y(\theta)$  denote the local coordinate functions, and

$$(5.4) \quad \{X_k, Y_k : k \in N\}$$

be the local basis on  $\mathcal{N}_{\epsilon,\delta}(\gamma)$ .

**5.2. The weak integrability on  $K_0$ .** For any  $\gamma \in K_0$ , the tangent space  $T_\gamma K_0$  is the space  $\Gamma_0(N_\gamma)$  of  $C^\omega$ -sections of the normal bundle  $N_\gamma$ . Let  $J$  be defined as the rotation in  $\frac{\pi}{2}$  in  $\Gamma_0(N_\gamma)$ . The computations in Section 4 can be translated on  $K_0$ ; As (3.7),  $J$  is represented by the  $2 \times 2$  matrix  $A$  with entries (3.4).

**Proposition 5.1.**  *$J$  is a well-defined, formally integrable almost complex structure on  $K_0$  and is real analytic.*

*Proof.* For any  $\epsilon, \epsilon'$  with  $0 < \epsilon < \epsilon'$ , since

$$(5.5) \quad \|z^{(k)}\|_{C^0(A_\epsilon)} \leq C \|z\|_{C^0(A_{\epsilon'})},$$

for any  $k \in N$ , the matrix  $A$  defines a smooth map in the  $C^\omega$ -topology. Thus  $J$  is well-defined on  $K_0$ .  $J$  is formally integrable by Brylinski [3, 4]. With  $J$  given explicitly as (3.4), the proof of the analyticity is omitted.

**Theorem 5.2** (Drinfeld, LeBrun). *The almost complex structure  $J$  is weakly integrable on the space  $K_0$ , i.e., for any  $k \in N$ , the  $\partial$ -equation (4.7) is solvable and the holomorphic differentials  $\{\partial_{\zeta_k} : k \in N\}$  is weakly dense in  $T^*K_0$ .*

*Proof.* By Proposition 5.1,  $J$  is real analytic on  $K_0$ . As Section 4, the  $\partial$ -equation can be complexified into a Frobenius equation and the Frobenius

equation can be further reduced to a first order nonlinear partial differential equation

$$(5.6) \quad \begin{cases} \frac{\partial \psi}{\partial t} = \frac{1 - b(tz, \psi) - ia(tz, \psi)}{1 + b(tz, \psi) - ia(tz, \psi)} z \\ \psi(0, z, w) = w. \end{cases}$$

Note that, for any  $\gamma \in K_0^*$ ,  $\kappa, \tau$  are both real analytic functions in  $\theta$ , and for any  $(z, w) \in C^\omega(S^1, R^2)$ , the nonlinear PDE (5.6) is a real analytic system. By the theorem of Cauchy-Kowalewska, (5.6) has a unique solution  $\psi(t, z, w)$  for small  $t \geq 0$ . By rescaling,  $\psi(t, z, w)$  is defined on  $0 \leq t \leq 1$ . By Proposition 4, the Frobenius equation

$$(5.7) \quad \begin{cases} D_1 \phi = \frac{1 - b(z, \phi) - ia(z, \phi)}{1 + b(z, \phi) - ia(z, \phi)} \\ \phi(0, w) = w \end{cases}$$

has a unique solution for any  $(z, w) \in \mathcal{N}_{\epsilon, \delta}(\gamma) \otimes C$  when  $\delta > 0$  is small. The  $\partial$ -equation is thus solvable, the holomorphic functions are given by  $\zeta_k = z_k + \phi_k(z, \bar{z})$ .

Let  $\Phi$  be the map on  $\mathcal{N}_{\epsilon, \delta}(\gamma)$  defined as

$$(5.8) \quad \Phi(z_k) = z_k + \phi_k(z, \bar{z}).$$

Notice that

$$(5.9) \quad D\Phi(z, \bar{z}) = \frac{2(1 - ia)}{1 + b - ia},$$

involves the first derivatives of the coordinate functions.  $D\Phi$  is invertible on  $\mathcal{N}_{\epsilon, \delta}(\gamma)$ . Thus the germ of holomorphic differentials is weakly dense in  $T^*K_0$ . Theorem 5.2 is thus proved.

**5.3. On the inverse function theorem.** In this section, it is shown that the inverse function theorem of Nash and Moser fails to implies that  $\Phi$  defined as (5.8) is a local diffeomorphism and thus  $J$  is integrable on  $K$ . The reader may refer to Hamilton [8] for the exact statement of the inverse function theorem. Roughly speaking, the inverse function theorem works in the tame category. As in Hamilton [8], the space  $C^\omega(S^1, R^2)$  with the  $C^\omega$ -topology is a tame Fréchet space. It will be shown that the map  $\Phi$  fails to satisfy the tameness conditions.

It is a remark that however  $\Phi$  and the inverse of  $D\Phi$  both satisfy the tameness estimates in the  $C^\infty$ -topology. By (5.9), the inverse of  $D\Phi$  is an ordinary differential operator. As Corollary 2.2.7 of Part II of [8], the inverse

is a tame map on  $\mathcal{N}_{\epsilon,\delta}(\gamma)$ . To prove that  $\Phi$  is tame, we solve the equation (5.6). (We will return to this practise of solving (5.6) more specifically in Section 6, and here we are brief.) As in Garabedian [6] and John [9], (5.6) can be solved by integrating a system of ordinary differential equations which describes the characteristic curves with initial conditions given by  $w$ . As Theorem 3.2.1 of Part II of [8], the tameness estimates of  $\Phi$  can be easily established.

**Proposition 5.3.** *The inverse of  $D\Phi$  defined as (5.9) is not a tame map in the  $C^\omega$ -topology.*

*Proof.* Consider the Frobenius equation around the circle

$$\gamma(\theta) = \frac{1}{2\pi}(\cos(2\pi\theta), \sin(2\pi\theta), 0)$$

and let  $x(\theta) = 0$ . Note that  $l = 1, \kappa = 2\pi, \tau = 0$ ,

$$(5.10) \quad \lambda_1 = 1, \lambda_2 = 0, \lambda_3 = y'(\theta),$$

the matrix  $A$  has the the following explicit entries:

$$(5.11) \quad \begin{aligned} a = 0, b = \sqrt{1 + y'(\theta)^2}, \\ c = -\frac{1}{\sqrt{1 + y'(\theta)^2}}, d = 0. \end{aligned}$$

Thus the inverse of  $D\Phi$  is computed as

$$(5.12) \quad (D\Phi)_{|(0,y)}^{-1} = \frac{1}{2} \left( 1 + \sqrt{1 + y'^2} \right).$$

Note that  $(D\Phi)^{-1}$  has a nonlinear term  $y'^2$ . As Example 2.1.3 of Part II of [8], it is not a tame map in the  $C^\omega$ -topology.

### 6. The Frobenius Problem on $\mathbf{K}$

In this section, we give an explicit form of the Frobenius equation

$$(6.1) \quad \begin{cases} \frac{\partial \psi}{\partial t} = \frac{1 - b(tz, \psi) - ia(tz, \psi)}{1 + b(tz, \psi) - ia(tz, \psi)} z \\ \psi(0, z, w) = w \end{cases}$$

and prove an insolvability of the equation. By Proposition 4, the Frobenius problem on  $K$  is thus not integrable.

**6.1. An explicit form of the Frobenius equation.** Consider the Frobenius equation (6.1) around the standard circle

$$\gamma(\theta) = (\cos(2\pi\theta), \sin(2\pi\theta), 0).$$

Then  $l = 2\pi$ ,  $\kappa = 1$ ,  $\tau = 0$ ,

$$(6.2) \quad \lambda_1 = 2\pi(1 - x(\theta)), \lambda_2 = x'(\theta), \lambda_3 = y'(\theta).$$

Introducing

$$(6.3) \quad \mu_2 = \frac{\lambda_2}{\lambda_1}, \mu_3 = \frac{\lambda_3}{\lambda_1},$$

the matrix  $A$  has the following entries:

$$(6.4) \quad \begin{aligned} a(z, \bar{z}) &= \frac{\mu_2\mu_3}{\sqrt{1 + \mu_2^2 + \mu_3^2}}, \\ b(z, \bar{z}) &= \frac{1 + \mu_3^2}{\sqrt{1 + \mu_2^2 + \mu_3^2}}, \\ c(z, \bar{z}) &= -\frac{1 + \mu_2^2}{\sqrt{1 + \mu_2^2 + \mu_3^2}}, \\ d(z, \bar{z}) &= -\frac{\mu_2\mu_3}{\sqrt{1 + \mu_2^2 + \mu_3^2}}. \end{aligned}$$

All the entries can be complexified as  $a = a(z, w)$  etc..

To find a simple form of the equation, let  $z = \delta > 0$ . Then

$$(6.5) \quad \begin{aligned} 1 + \mu_2^2 + \mu_3^2 &= 1 + \frac{z'w'}{\lambda^2} = 1, \\ 1 + b - ia &= 2 + \mu_3^2 - i\mu_2\mu_3 = 2, \\ 1 - b - ia &= -\mu_3^2 - i\mu_2\mu_3 = \frac{w'^2}{2\lambda_1^2}. \end{aligned}$$

The equation (6.1) has the explicit form

$$(6.6) \quad \begin{cases} \frac{\partial\psi}{\partial t} = \frac{\delta\left(\frac{\partial\psi}{\partial\theta}\right)^2}{16\pi^2(1 - \delta t - \psi)^2} \\ \psi|_{t=0} = w(\theta). \end{cases}$$

Introduce  $\varphi = \frac{1}{1 - \delta t - \psi}$  and  $v = \frac{1}{1 - w}$ . Denote again by  $t$  for  $\frac{t}{16\pi^2}$ . Then (6.6) is converted as

$$(6.7A) \quad \begin{cases} \frac{\partial\varphi}{\partial t} = \varphi^2 + \left(\frac{\partial\varphi}{\partial\theta}\right)^2 \\ \psi|_{t=0} = v. \end{cases}$$

(6.7B)

**6.2. A simpler example.** The equation (6.7) is a first order, nonlinear equation. As in Garabedian [6] and John [9], when  $w$  is a real function, (6.6) can be explicitly solved by integrating a system of ordinary differential equations which describes the characteristic curves. This is also the case when  $w$  is real analytic. To show that the Frobenius equation is not solvable, (6.7) will be shown unsolvable for certain  $v$ . To illustrate the idea of proof, consider first the equation

$$(6.8A) \quad \begin{cases} \frac{\partial \psi}{\partial t} = \left(\frac{\partial \psi}{\partial \theta}\right)^2 \\ \psi|_{t=0} = v. \end{cases}$$

**Proposition 6.1.** *Let  $v(\theta)$  be a smooth function on  $S^1$  with  $\text{Im}v'(\theta) \neq 0$  on  $S^1$ . Then the equation (6.8) is solvable iff  $v \in C^\omega$ .*

*Proof.* The “if” part is by the theorem of Cauchy-Kowalevaska. To prove the “only if” part, note that (6.8) can be explicitly solved. Let  $\eta = \frac{\partial \psi}{\partial \theta}$ . Then (6.8) is converted into the quasi-linear Burger equation

$$(6.9A) \quad \begin{cases} \frac{\partial \eta}{\partial t} = 2\eta \frac{\partial \eta}{\partial \theta} \\ \eta|_{t=0} = v', \end{cases}$$

and the solution is given as

$$(6.10) \quad \eta(t, \theta) = v'(\theta + 2t\eta(t, \theta)).$$

Assume that  $v'$  is never real and (6.8) has a solution  $\eta(\theta, t)$ . Then for small  $t$ ,  $\eta(\theta, t)$  is also never real. By (6.10),  $v'$  is extended over a certain annulus  $A_\epsilon$ . To show that the extension is holomorphic, let

$$(6.11) \quad \eta = \eta_1 + i\eta_2, v' = v'_1 + iv'_2$$

and rewrite (6.10) as

$$(6.12) \quad \eta(t, \theta) = v'(\theta + 2t\eta_1, 2t\eta_2).$$

Differentiating the explicit function (6.12),

$$(6.13) \quad \begin{cases} \frac{\partial \eta}{\partial t}|_{t=0} = 2\eta_1 \frac{\partial v}{\partial \zeta_1} + 2\eta_2 \frac{\partial v}{\partial \zeta_2} \\ \frac{\partial \eta}{\partial \theta}|_{t=0} = \frac{\partial v}{\partial \zeta_1}. \end{cases}$$

Substituting (6.13) to (6.9),

$$(6.14) \quad \eta_2 \left( \frac{\partial v'}{\partial \zeta_1} + i \frac{\partial v'}{\partial \zeta_2} \right) = 0.$$

Since  $\eta_2 \neq 0$  for small  $t$ ,  $v'$  satisfies the Cauchy-Riemann equation,  $v' \in C^\omega$ .

**6.3. An unsolvability of the Frobenius equation.** Denote by  $S$  the set of complex valued, smooth functions  $v(\theta)$  on  $S^1$  such that

$$v' \neq 0, \operatorname{Im} \frac{v^2 + v'^2}{v'} \neq 0$$

on  $S^1$ . The following proposition shows that the Frobenius equation is not solvable on the knot space  $K$ .

**Proposition 6.2.** (6.7) is unsolvable for generic  $v \in S$ .

*Proof.* To find the general solution for (6.7), let  $p = \frac{\partial \varphi}{\partial \theta}$ ,  $q = \frac{\partial \varphi}{\partial t}$  and rewrite the equation as

$$(6.15) \quad F(\theta, t, \varphi, p, q) = q - \varphi^2 - p^2 = 0.$$

As [6] and [9], (6.7) is solved by integrating

$$(6.16) \quad \begin{cases} \frac{d\theta}{ds} = F_p = -2p \\ \frac{dt}{ds} = F_q = 1 \\ \frac{d\varphi}{ds} = pF_p + qF_q = q - 2p^2 \\ \frac{dp}{ds} = -F_\theta - pF_\varphi = 2p\varphi \\ \frac{dq}{ds} = -F_t - qF_\varphi = 2q\varphi \end{cases}$$

with the initial condition

$$(6.17) \quad \begin{cases} \theta|_{s=0} = \tau \\ t|_{s=0} = 0 \\ \varphi|_{s=0} = v(\tau) \\ p|_{s=0} = v'(\tau) \\ q|_{s=0} = v^2(\tau) + v'^2(\tau). \end{cases}$$

Where  $s$  and  $\tau$  are two parameters.

To integrate (6.16), note first that (6.16) implies  $t = s$ . The last two equations of (6.16) imply that  $\frac{p}{q}$  is independent of  $s$ ,

$$(6.18) \quad q = \frac{v^2 + v'^2}{v'} p.$$

Substituting (6.18) to (6.16), (6.16) is reduced as

$$(6.19) \quad \begin{cases} \frac{d\theta}{dt} = -2p \\ \frac{d\varphi}{dt} = \frac{v^2 + v'^2}{v'}p - 2p^2 \\ \frac{dp}{dt} = 2p\varphi. \end{cases}$$

Let  $\lambda$  and  $\mu$  be functions defined as

$$(6.20) \quad \lambda(\tau) = \frac{v^2 + v'^2}{v'}, \mu(\tau) = \frac{v}{v'}.$$

Then the equation (6.19) implies

$$(6.21) \quad \frac{d\varphi}{dp} = \frac{\lambda - 2p}{2\psi}, \varphi^2 = \lambda p - p^2.$$

Substituting (6.21) to (6.19),  $p$  is integrated as

$$(6.22) \quad \begin{aligned} \frac{dp}{dt} &= 2p\sqrt{\lambda p - p^2}, \\ t &= \int_{v'}^p \frac{du}{2u\sqrt{\lambda u - u^2}}, \\ p(\tau, t) &= \frac{\lambda(\tau)}{1 + (\lambda t - \mu)^2}. \end{aligned}$$

Substituting (6.21) to (6.19), (6.7) is solved;

$$(6.23A) \quad \begin{cases} \theta(\tau, t) = \tau - 2 \tan^{-1}(\lambda u - \mu) \Big|_0^t \\ (6.23B) \quad \varphi(\tau, t) = -\frac{\lambda(\lambda t - \mu)}{1 + (\lambda t - \mu)^2}. \end{cases}$$

Note that  $\tau = \tau(\theta, t)$  is determined by (6.23A) implicitly.

When  $Im\lambda \neq 0$  on  $S^1$ , similar to the case of Burger equation, if the equation (6.7) has a solution, then (6.23A) implies that  $\tau$  is a complex variable and thus  $\mu(\tau), \lambda(\tau)$  are both forced to extend over a certain annulus  $A_\epsilon$ . Assume that  $\lambda$  and  $\mu$  are extended as

$$(6.24) \quad \lambda = \lambda(\tau, \bar{\tau}), \mu = \mu(\tau, \bar{\tau}).$$

Let  $z(\tau, t)$  and  $\gamma(\tau, t)$  be functions defined as

$$(6.25) \quad z(\tau, t) = \lambda t - \mu, \gamma(\tau, t) = \frac{2\lambda}{1+z^2}.$$

Introduce

$$(6.26) \quad \alpha = 1 - \frac{2}{1+z^2} \frac{\partial z}{\partial \tau} - \frac{2}{1+\mu^2} \frac{\partial \mu}{\partial \tau},$$

$$\beta = -\frac{2}{1+z^2} \frac{\partial z}{\partial \bar{\tau}} - \frac{2}{1+\mu^2} \frac{\partial \mu}{\partial \bar{\tau}}.$$

Differentiating (6.23A),

$$\frac{\partial \tau}{\partial t} = \frac{\bar{\alpha}\gamma - \beta\bar{\gamma}}{|\alpha|^2 - |\beta|^2},$$

$$(6.27) \quad \frac{\partial \tau}{\partial \theta} = \frac{\bar{\alpha} - \beta}{|\alpha|^2 - |\beta|^2}.$$

Note that  $\alpha \sim 1$  and  $\beta \sim 0$  for small  $t$ .

Differentiating (6.23B) follows that

$$(6.28) \quad \frac{\partial \varphi}{\partial t} = \frac{-z}{1+z^2} \frac{\partial \lambda}{\partial t} - \frac{\lambda(1-z^2)}{(1+z^2)^2} \frac{\partial z}{\partial t}.$$

Substitute

$$(6.29) \quad \frac{\partial z}{\partial t} = t \frac{\partial \lambda}{\partial t} - \frac{\partial \mu}{\partial t} + \lambda,$$

$$\frac{\partial \lambda}{\partial t} = \frac{\partial \lambda}{\partial \tau} \frac{\partial \tau}{\partial t} + \frac{\partial \lambda}{\partial \bar{\tau}} \frac{\partial \bar{\tau}}{\partial t},$$

$$(6.30) \quad \frac{\partial \mu}{\partial t} = \frac{\partial \mu}{\partial \tau} \frac{\partial \tau}{\partial t} + \frac{\partial \mu}{\partial \bar{\tau}} \frac{\partial \bar{\tau}}{\partial t}$$

and (6.27) to (6.28).  $\frac{\partial \varphi}{\partial t} - \varphi^2$  is then computed as

$$(6.31) \quad \frac{-\lambda^2}{(1+z^2)^2} - \frac{\bar{\alpha}\gamma - \beta\bar{\gamma}}{|\alpha|^2 - |\beta|^2} \left\{ \frac{-\mu z^2 + 2z + \mu}{(1+z^2)^2} \frac{\partial \lambda}{\partial \tau} - \frac{\lambda(1-z^2)}{(1+z^2)^2} \frac{\partial \mu}{\partial \tau} \right\}$$

$$- \frac{\alpha\bar{\gamma} - \bar{\beta}\gamma}{|\alpha|^2 - |\beta|^2} \left\{ \frac{-\mu z^2 + 2z + \mu}{(1+z^2)^2} \frac{\partial \lambda}{\partial \bar{\tau}} - \frac{\lambda(1-z^2)}{(1+z^2)^2} \frac{\partial \mu}{\partial \bar{\tau}} \right\}.$$

Similarly  $\frac{\partial \varphi}{\partial \theta}$  is computed as

$$(6.32) \quad -\frac{\bar{\alpha} - \beta}{|\alpha|^2 - |\beta|^2} \left\{ \frac{-\mu z^2 + 2z + \mu}{(1 + z^2)^2} \frac{\partial \lambda}{\partial \tau} - \frac{\lambda(1 - z^2)}{(1 + z^2)^2} \frac{\partial \mu}{\partial \tau} \right\} \\ -\frac{\alpha - \bar{\beta}}{|\alpha|^2 - |\beta|^2} \left\{ \frac{-\mu z^2 + 2z + \mu}{(1 + z^2)^2} \frac{\partial \lambda}{\partial \bar{\tau}} - \frac{\lambda(1 - z^2)}{(1 + z^2)^2} \frac{\partial \mu}{\partial \bar{\tau}} \right\}.$$

Note that  $\beta$  is a linear function in  $\frac{\partial \lambda}{\partial \bar{\tau}}$  and  $\frac{\partial \mu}{\partial \bar{\tau}}$ ,  $\alpha$  is a similar function in  $\frac{\partial \lambda}{\partial \tau}$  and  $\frac{\partial \mu}{\partial \tau}$ . Multiplying the equation (6.7) by

$$(|\alpha|^2 - |\beta|^2)^2 (1 + z^2)^4 (1 + \bar{z}^2)^2$$

it follows that  $\frac{\partial \lambda}{\partial \bar{\tau}}$ ,  $\frac{\partial \mu}{\partial \bar{\tau}}$  and  $t$  satisfy a polynomial equation

$$(6.33) \quad P \left( t, \lambda, \mu, \frac{\partial \lambda}{\partial \tau}, \frac{\partial \mu}{\partial \tau}, \frac{\partial \lambda}{\partial \bar{\tau}}, \frac{\partial \mu}{\partial \bar{\tau}} \right) = 0.$$

Consider  $P$  as a polynomial in  $\frac{\partial \lambda}{\partial \bar{\tau}}$  and  $\frac{\partial \mu}{\partial \bar{\tau}}$ . Note that, when

$$(6.34) \quad \frac{\partial \lambda}{\partial \bar{\tau}} = 0, \frac{\partial \mu}{\partial \bar{\tau}} = 0,$$

$\lambda$ ,  $\mu$  and thus  $v$  and  $v'$  are holomorphically extended, (6.23) gives a solution to (6.7). Hence (6.34) is a solution to (6.33), the constant part  $P_0$  of  $P$  vanishes identically.

$$(6.35) \quad P_0 \left( t, \lambda, \mu, \frac{\partial \lambda}{\partial \tau}, \frac{\partial \mu}{\partial \tau} \right) = 0.$$

Let  $H = P - P_0$ .

By (6.31) and (6.32),  $P_0$  can be easily computed as a complete square. Valued at  $t = 0$ , (6.35) implies the equation

$$(6.36) \quad \mu \frac{\partial \lambda}{\partial \tau} + \frac{\lambda(1 - \mu^2)}{1 + \mu^2} \frac{\partial \mu}{\partial \tau} = \lambda.$$

(In fact, (6.35) and (6.36) are equivalent.) Consider the equation  $H = 0$ . Valued at  $t = 0$ , the equation implies either the linear equation

$$(6.37A) \quad \mu \frac{\partial \lambda}{\partial \bar{\tau}} + \frac{\lambda(1 - \mu^2)}{1 + \mu^2} \frac{\partial \mu}{\partial \bar{\tau}} = 0.$$

or the linear equation

$$(6.37B) \quad \mu \frac{\partial \lambda}{\partial \bar{\tau}} + \frac{\lambda(1 - \mu^2)}{1 + \mu^2} \frac{\partial \mu}{\partial \bar{\tau}} = \frac{2\bar{\lambda}}{1 + \bar{\mu}^2}.$$

The vanishing of the coefficient of the highest  $t$ -power in  $H$  implies the equation

$$(6.38) \quad \left(1 - \frac{2}{1 + \mu^2} \frac{\partial \mu}{\partial \tau} + \frac{2}{1 + \bar{\mu}^2} \frac{\overline{\partial \mu}}{\partial \bar{\tau}}\right) \left(-\mu \frac{\partial \lambda}{\partial \tau} + \lambda \frac{\partial \mu}{\partial \tau}\right) + \frac{2}{1 + \mu^2} \frac{\partial \mu}{\partial \bar{\tau}} \left(-\mu \frac{\partial \lambda}{\partial \tau} + \lambda \frac{\partial \mu}{\partial \tau}\right) = 0.$$

Where the first and second term of (6.38) are derived from those terms of (6.32), since the highest  $t$ -power appears in the square of (6.32). By (6.36) and (6.37A), substituting  $\frac{\partial \lambda}{\partial \tau}$  and  $\frac{\partial \lambda}{\partial \bar{\tau}}$  to (6.38),

$$(6.39) \quad \frac{2}{1 + \bar{\mu}^2} \frac{2\lambda}{1 + \mu^2} \left| \frac{\partial \mu}{\partial \bar{\tau}} \right|^2 = 0.$$

Thus  $\frac{\partial \mu}{\partial \bar{\tau}} = 0, \frac{\partial \lambda}{\partial \bar{\tau}} = 0. v \in C^\omega$ .

The complicated case is the equation (6.37B). In this case, substituting again (6.36) and (6.37B) to (6.38) follows that  $\frac{\partial \mu}{\partial \tau}$  is a quadratic equation in  $\frac{\partial \mu}{\partial \bar{\tau}}$ .

$$(6.40) \quad \frac{\partial \mu}{\partial \tau} = Q \left( \lambda, \mu, \frac{\partial \mu}{\partial \bar{\tau}} \right).$$

To complete the proof of Proposition 6.2, it will be shown that, the equation  $H = 0$  implies two more nontrivial equations which can be reduced to different polynomial equations

$$(6.41) \quad \begin{aligned} \Gamma_1 \left( \lambda, \mu, \frac{\partial \mu}{\partial \bar{\tau}} \right) &= 0, \\ \Gamma_2 \left( \lambda, \mu, \frac{\partial \mu}{\partial \bar{\tau}} \right) &= 0 \end{aligned}$$

which are in the single variable  $\frac{\partial \mu}{\partial \bar{\tau}}$  by (6.36), (6.37B) and (6.40). Thus by restricting  $\lambda$  and  $\mu$  on the real line  $\tau = \bar{\tau}$ , for generic  $v(\theta)$ , (6.41) has no solution. (The resultant can be used to check that  $\Gamma_1$  and  $\Gamma_2$  have no common solutions.)

The two polynomials can be indeed chosen in the following way. Regard  $H$  as a polynomial in  $z$  instead of  $t$ . Then the two polynomials are deduced from the constant term and the coefficient of  $z$ . An explicit calculation can be given to show that the polynomials are in fact different; the calculation is lengthy however not difficult, may we omit the details here. Proposition 6.2 is thus proved.

**Corollary 6.3.** *The Frobenius problem on  $K$  is not integrable.*

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