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**CORRECTION TO: “FREE BANACH-LIE ALGEBRAS,  
COUNIVERSAL BANACH-LIE GROUPS, AND MORE”**

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**We correct a proof of the fact that the free Banach-Lie algebra on a normed space of dimension  $\geq 2$  is centreless, and observe that, as a corollary, every Banach-Lie algebra is a factor algebra of a Banach-Lie algebra faithfully representable in a Banach space.**

1. All the major results of our paper [2] are based on the following statement, which appears as a part of Theorem 2.1.

**Theorem A.** *The free Banach-Lie algebra on a normed space  $E$  is either trivial (if  $\dim E = 0$ ), or one-dimensional (if  $\dim E = 1$ ), or centreless.*

Unfortunately, the proof of the above result presented in [2] is unsatisfactory, and it was Professor W.T. van Est who has kindly drawn the author's attention to this fact. Below we present a correct proof of Theorem A.

A 1973 investigation [4] of van Est and Świerczkowski was partly motivated by the question: is every Banach-Lie algebra a factor algebra of a Banach-Lie algebra faithfully representable in a Banach space? We can answer this in the positive.

Indeed, every Banach-Lie algebra  $\mathfrak{g}$  is a factor Banach-Lie algebra of a free Banach-Lie algebra [2]. Since centreless Banach-Lie algebras are exactly those whose adjoint representation is faithful, the following direct corollary of Theorem A holds.

**Theorem B.** *Every Banach-Lie algebra is a factor algebra of a Banach-Lie algebra admitting a faithful representation in a Banach space.*

2. Denote by  $\mathbb{K}$  the basic field (either  $\mathbb{R}$  or  $\mathbb{C}$ ), and let  $E$  be a normed space. For an  $n > 0$ , let  $\mathcal{A}_n(E) = E^{\otimes n} \equiv E \otimes_{\pi} E \otimes_{\pi} \cdots \otimes_{\pi} E$  be an  $n$ -fold (non-completed) projective tensor product. ([3, III.6.3.]) Endow the space  $\mathcal{B}^n(E)$  of all  $n$ -linear continuous functionals on  $E^n$  with a norm:

$$\|f\| \stackrel{def}{=} \sup\{|f(x_1, \dots, x_n)| : \|x_i\| \leq 1, i = 1, 2, \dots, n\}.$$

The spaces  $\mathcal{A}_n(E)$  and  $\mathcal{B}^n(E)$  admit a canonical pairing, which determines an isometric embedding of  $\mathcal{A}_n(E)$  into the strong dual  $\mathcal{B}^n(E)'$  ([3, exer. III.21, (a)]).

Let  $\mathcal{A}_+(E)$  stand for the free associative (non-unital) algebra on  $E$ ,  $\mathcal{A}_+(E) = \bigoplus_{n=1}^\infty \mathcal{A}_n(E)$ , endowed with an  $l_1$ -type norm,  $\left\| \sum_{n=1}^k x_n \right\| = \sum_{n=1}^k \|x_n\|$ . Denote by  $\hat{\mathcal{A}}_+(E)$  the Banach associative algebra completion of  $\mathcal{A}_+(E)$ . It is easy to verify that  $\hat{\mathcal{A}}_+(E)$  contains an isometric copy of  $E$  in such a way that an arbitrary linear contraction  $f$  from  $E$  to an associative algebra  $A$  endowed with a complete submultiplicative norm extends to a unique algebra homomorphism  $\bar{f}: \hat{\mathcal{A}}_+(E) \rightarrow A$  of norm  $\leq 1$ . We call  $\hat{\mathcal{A}}_+(E)$  the *free Banach algebra* on  $E$ . Denote the Banach space completion of  $\mathcal{A}_n(E)$  by  $\hat{\mathcal{A}}_n(E)$ ; then  $\hat{\mathcal{A}}_+(E)$  is the  $l_1$ -type sum of  $\hat{\mathcal{A}}_n(E)$ ,  $n = 1, 2, \dots$

It is clear that the free Banach-Lie algebra  $\mathcal{FL}(E)$  is naturally isometric to the  $l_1$ -type direct sum of a family of complete normed spaces  $\mathcal{FL}_n(E)$ ,  $n \in \mathbb{N}$ , where  $\mathcal{FL}_n(E)$  is the completion of  $\mathcal{L}_n(E)$ . (Here  $\mathcal{L}_1(E) = E$ , and the linear subspaces  $\mathcal{L}_n(E)$  of the free Lie algebra,  $\mathcal{L}(E) = \bigoplus_{k=1}^\infty \mathcal{L}_k(E)$ , on the vector space  $E$  [1], are defined in a usual recursive fashion.) The symbol  $E_{1/2}$  will stand for the normed space  $(E, (1/2)\|\cdot\|)$ . We will denote by  $\hat{\mathcal{A}}_+^{\downarrow}(E)$  an algebra obtained from the free Banach algebra  $\hat{\mathcal{A}}_+(E_{1/2})$  by doubling its norm. The doubled norm  $\|\cdot\|^{\downarrow}$  is Lie-submultiplicative, and the identity map  $\text{Id}_E$  extends to a contracting Lie algebra morphism  $i: \mathcal{FL}(E) \rightarrow \hat{\mathcal{A}}_+^{\downarrow}(E)$ . The restriction of  $i$  to  $\mathcal{L}(E)$  is well known to be mono [2]. Since the identity map  $\mathcal{A}_+^{\downarrow}(E) \rightarrow \hat{\mathcal{A}}_+(E_{1/2})$  has norm  $1/2$ , its composition with  $i$  is a contracting Lie algebra homomorphism  $\mathcal{FL}(E) \rightarrow \hat{\mathcal{A}}_+(E_{1/2})$ , which we denote by  $i$  as well.

**Assertion 1.** *Let  $n = 1, 2, \dots$ . The restriction  $i_n$  of  $i: \mathcal{FL}(E) \rightarrow \hat{\mathcal{A}}_+(E_{1/2})$  to  $\mathcal{FL}_n(E)$  is an isomorphic embedding of normed spaces; namely, for each  $x \in \mathcal{FL}_n(E)$  one has*

$$(1) \quad \|i(x)\|_{\hat{\mathcal{A}}_n(E_{1/2})} \leq \|x\| \leq \frac{2^n}{n} \|i(x)\|_{\hat{\mathcal{A}}_n(E_{1/2})}.$$

*Proof.* Define recursively the  $n$ -fold commutator,  $[x_1, \dots, x_n]$ , by  $[x_1, \dots, x_n] = [[x_1, \dots, x_{n-1}], x_n]$ . The map  $(x_1, \dots, x_n) \mapsto [x_1, \dots, x_n]$  from  $E \times E \times \dots \times E \subset \mathcal{A}_n(E)$  to  $\mathcal{L}_n(E)$  is  $n$ -linear and

$$\|[x_1, \dots, x_n]\| \leq \|x_1\|_E \dots \|x_n\|_E = 2^n \frac{1}{2} \|x_1\|_E \dots \frac{1}{2} \|x_n\|_E.$$

Therefore, it extends to a unique bounded linear operator (of norm  $\leq 2^n$ ),  $\nu: \hat{\mathcal{A}}_n(E_{1/2}) \rightarrow \mathcal{FL}_n(E)$ , having the property that if  $x_1, x_2, \dots, x_n \in E$ , then

$\nu(x_1x_2 \dots x_n) = [x_1, \dots, x_n]$ . The restriction of  $\nu$  to  $\mathcal{A}_n(E)$  is the familiar *Specht-Wever map* [1].

While the left hand side inequality in (1) follows from a definition of  $i$ , suppose that  $\|i(x)\| \leq 1$ . One can assume without loss in generality that  $x \in \mathcal{L}_n(E)$ . Let an  $\varepsilon > 0$  be arbitrary. For a finite collection of elements  $x_{i,j} \in E$  one has ([3, III.6.3])  $i(x) = \sum_i x_{i,1}x_{i,2} \dots x_{i,n}$  and

$$\|i(x)\| \geq \sum_i \frac{1}{2} \|x_{i,1}\|_E \frac{1}{2} \|x_{i,2}\|_E \dots \frac{1}{2} \|x_{i,n}\|_E - \varepsilon$$

that is,

$$\sum_i \|x_{i,1}\|_E \|x_{i,2}\|_E \dots \|x_{i,n}\|_E \leq 2^n(1 + \varepsilon),$$

and therefore

$$\left\| \sum_i [x_{i,1}, x_{i,2}, \dots, x_{i,n}] \right\| \leq 2^n(1 + \varepsilon).$$

According to the Specht-Wever theorem [1],  $\nu(x) \equiv \sum_i [x_{i,1}, x_{i,2}, \dots, x_{i,n}] = nx$ , whence  $x = (1/n) \sum_i [x_{i,1}, x_{i,2}, \dots, x_{i,n}]$  and  $\|x\| \leq (2^n/n)(1 + \varepsilon)$ . Since  $\varepsilon > 0$  is arbitrary, one has  $\|x\| \leq 2^n/n$ , as desired.  $\square$

**3. Proof of Theorem A.** We can assume that  $\dim E \geq 2$ . Let an  $x \in \mathcal{FL}(E)$ ,  $x \neq 0$  be arbitrary,  $x = \sum_{k=1}^\infty x_k$ , where  $x_k \in \mathcal{FL}_n(E)$ . For at least one  $n = 1, 2, \dots$ , one has  $x_n \neq 0$ . It remains to find a  $z \in E$  such that  $[z, x_k] \neq 0$ , for clearly then  $[z, x] \neq 0$  as well. If  $E$  is of finite dimension, then such is  $\mathcal{L}_n(E)$ ; if an element  $x \in \mathcal{FL}_n(E) \equiv \mathcal{L}_n(E)$  commutes with every element of  $E$ , it must belong to the centre of  $\mathcal{L}(E)$ , which is trivial if  $\dim E > 1$ . In infinite dimensions, however, this argument fails (which was essentially author’s blunder in [2]).

Denote by  $\text{ad}$  the adjoint representation of  $\mathcal{FL}(E)$  in the underlying Banach space,  $\mathcal{FL}(E)_+$ .

**Assertion 2.** Assume that  $\dim E = \infty$ . Let  $n \in \mathbb{N}$  and let  $x \in \mathcal{FL}_n(E)$ ,  $\|x\| = 1$ . Then  $\|\text{ad } x\| \geq n2^{-n}$ .

*Proof.* Since  $\|\text{ad}\| \leq 1$ , it is enough to check the desired property for  $x \in \mathcal{L}_n(E)$ : indeed, the unit sphere of  $\mathcal{L}_n(E)$  is dense in the unit sphere of  $\mathcal{FL}_n(E)$ , and if  $x_n \rightarrow x$  as  $n \rightarrow \infty$ , then  $\text{ad } x_n \rightarrow \text{ad } x$  in  $\text{End}(\mathcal{FL}(E)_+)$  and  $\|\text{ad } x_n\| \rightarrow \|\text{ad } x\|$ .

The norm of  $i(x)$  in  $\hat{\mathcal{A}}_+(E_{1/2})$  is  $\geq n2^{-n}$ , according to Assertion 1. Assume that  $i(x) = \sum_i x_{1,i} \otimes x_{2,i} \otimes \dots \otimes x_{n,i}$ , where  $x_{j,i} \in E$ . Let an  $\varepsilon > 0$  be arbitrary.

Choose an  $f \in \mathcal{B}^n(E_{1/2})$  with  $\|f\| \leq 1$  and  $|f(i(x))| \geq n2^{-n} - \varepsilon$ . Due to infinite-dimensionality of  $E$ , there exists a linear functional  $g: E \rightarrow \mathbb{K}$  of norm 1 with  $g(x_{n,i}) = 0$  for all (finitely many) values of  $i$ . Let  $y \in E$  be such that  $\|y\| = 1$  and  $g(y) = 1$ . (The kernel of  $g$ , being one-dimensional, admits a projection from  $E$  of norm 1.) The mapping  $f \otimes g$  of the form  $a \otimes b \mapsto f(a) \cdot g(b)$ ,  $a \in E_{1/2}^n$ ,  $b \in E_{1/2}$ , is an  $(n+1)$ -linear functional of norm  $\leq 1$  on  $E_{1/2}^{n+1}$ . Since

$$(f \otimes g)(y \otimes i(x)) = \sum_i f(y \otimes x_{1,i} \otimes \cdots \otimes x_{n-1,i}) \cdot g(x_{n,i}) = 0,$$

one has

$$\begin{aligned} |(f \otimes g)(i(x) \otimes y - y \otimes i(x))| &= |(f \otimes g)(i(x) \otimes y)| \\ &= |f(i(x))| \cdot 1 \geq \frac{n}{2^n} - \varepsilon, \end{aligned}$$

and in view of arbitrariness of  $\varepsilon > 0$ , the norm of the element  $i([y, x]) = i(x)y - yi(x)$  in  $\hat{\mathcal{A}}_+(E_{1/2})$  is  $\geq n2^{-n}$ . In view of Assertion 1, the norm of  $[y, x]$  in  $\mathcal{FL}_n(E)$  is  $\geq n2^{-n}$  as well.  $\square$

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