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**THE QUASI-LINEARITY PROBLEM FOR  $C^*$ -ALGEBRAS**

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## THE QUASI-LINEARITY PROBLEM FOR $C^*$ -ALGEBRAS

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Let  $\mathcal{A}$  be a  $C^*$ -algebra with no quotient isomorphic to the algebra of all two-by-two matrices. Let  $\mu$  be a quasi-linear functional on  $\mathcal{A}$ . Then  $\mu$  is linear if, and only if, the restriction of  $\mu$  to the closed unit ball of  $\mathcal{A}$  is uniformly weakly continuous.

### Introduction.

Throughout this paper,  $\mathcal{A}$  will be a  $C^*$ -algebra and  $A$  will be the real Banach space of self-adjoint elements of  $\mathcal{A}$ . The unit ball of  $A$  is  $A_1$  and the unit ball of  $\mathcal{A}$  is  $\mathcal{A}_1$ . We do not assume the existence of a unit in  $\mathcal{A}$ .

**Definition.** A *quasi-linear functional* on  $A$  is a function  $\mu : A \rightarrow \mathbb{R}$  such that, whenever  $B$  is an abelian subalgebra of  $A$ , the restriction of  $\mu$  to  $B$  is linear. Furthermore  $\mu$  is required to be bounded on the closed unit ball of  $A$ .

Given any quasi-linear functional  $\mu$  on  $A$  we may extend it to  $\mathcal{A}$  by defining

$$\tilde{\mu}(x + iy) = \mu(x) + i\mu(y)$$

whenever  $x \in A$  and  $y \in A$ . Then  $\tilde{\mu}$  will be linear on each maximal abelian \*-subalgebra of  $\mathcal{A}$ . We shall abuse our notation by writing ' $\mu$ ' instead of ' $\tilde{\mu}$ '.

When  $\mathcal{A} = M_2(\mathbb{C})$ , the  $C^*$ -algebra of all two-by-two matrices over  $\mathbb{C}$ , there exist examples of quasi-linear functionals on  $\mathcal{A}$  which are not linear.

**Definition.** A *local quasi-linear functional* on  $A$  is a function  $\mu : A \rightarrow \mathbb{R}$  such that, for each  $x$  in  $A$ ,  $\mu$  is linear on the smallest norm closed subalgebra of  $A$  containing  $x$ . Furthermore  $\mu$  is required to be bounded on the closed unit ball of  $A$ .

Clearly each quasi-linear functional on  $A$  is a local quasi-linear functional. Surprisingly, the converse is false, even when  $A$  is abelian (see Aarnes [2]). However when  $A$  has a rich supply of projections (e.g. when  $\mathcal{A}$  is a von Neumann algebra) each local quasi-linear functional is quasi-linear [3].

The solution of the Mackey-Gleason Problem shows that every quasi-linear functional on a von Neumann algebra  $\mathcal{M}$ , where  $\mathcal{M}$  has no direct summand of Type  $I_2$ , is linear [4, 5, 6]. This was first established for positive quasi-linear functionals by the conjunction of the work of Christensen [7] and

Yeadon [11], and for  $\sigma$ -finite factors by the work of Paschciewicz [10]. All build on the fundamental theorem of Gleason [8].

Although quasi-linear functionals on general  $C^*$ -algebras seem much harder to tackle than the von Neumann algebra problem, we can apply the von Neumann results to make progress. In particular, we prove:

Let  $\mathcal{A}$  be a  $C^*$ -algebra with no quotient isomorphic to  $M_2(\mathbb{C})$ . Let  $\mu$  be a (local) quasi-linear functional on  $A$ . Then  $\mu$  is linear if, and only if, the restriction of  $\mu$  to  $A_1$ , is uniformly weakly continuous.

### 1. Preliminaries: Uniform Continuity.

Let  $X$  be a real or complex vector space. Let  $\mathcal{F}$  be a locally convex topology for  $X$ . Let  $V$  be a  $\mathcal{F}$ -open neighbourhood of 0. We call  $V$  *symmetric* if  $V$  is convex and, whenever  $x \in V$  then  $-x \in V$ .

Let  $B$  be a subset of  $X$ . A scalar valued function on  $X$ ,  $\mu$ , is said to be *uniformly continuous* on  $B$ , with respect to the  $\mathcal{F}$ -topology, if, given any  $\epsilon > 0$ , there exists an open symmetric neighbourhood of 0,  $V$ , such that whenever  $x \in B$ ,  $y \in B$  and  $x - y \in V$  then

$$|\mu(x) - \mu(y)| < \epsilon.$$

**Lemma 1.1.** *Let  $X$  be a Banach space and let  $\mathcal{F}$  be any locally convex topology for  $X$  which is stronger than the weak topology. Let  $\mu$  be any bounded linear functional on  $X$ . Then  $\mu$  is uniformly  $\mathcal{F}$ -continuous on  $X$ .*

*Proof.* Choose  $\epsilon > 0$ . Let

$$\begin{aligned} V &= \{x \in X : |\mu(x)| < \epsilon\} \\ &= \mu^{-1}\{\lambda : |\lambda| < \epsilon\}. \end{aligned}$$

Then  $V$  is open in the weak topology of  $X$ . Hence  $V$  is a symmetric  $\mathcal{F}$ -open neighbourhood of 0 such that  $x - y \in V$  implies

$$|\mu(x) - \mu(y)| = |\mu(x - y)| < \epsilon.$$

□

**Lemma 1.2.** *Let  $X$  be a subspace of a Banach space  $Y$ . Let  $\mathcal{G}$  be a locally convex topology for  $Y$  which is weaker than the norm topology. Let  $\mathcal{F}$  be the relative topology induced on  $X$  by  $\mathcal{G}$ . Let  $B$  be a subset of  $X$  and let  $C$  be the closure of  $B$  in  $Y$ , with respect to the  $\mathcal{G}$ -topology. Let  $\mu : B \rightarrow \mathbb{C}$  be uniformly continuous on  $B$  with respect to the  $\mathcal{F}$ -topology. Then there exists*

a function  $\bar{\mu} : C \rightarrow \mathbb{C}$  which extends  $\mu$  and which is uniformly  $\mathcal{G}$ -continuous. Furthermore, if  $\mu$  is bounded on  $B$  then  $\bar{\mu}$  is bounded on  $C$ .

*Proof.* Since  $\mathcal{F}$  is the relative topology induced by  $\mathcal{G}$ ,  $\mu$  is uniformly  $\mathcal{G}$ -continuous on  $B$ . Let  $K$  be the closure of  $\mu[B]$  in  $\mathbb{C}$ . Then  $K$  is a complete metric space. So, see [9, page 125],  $\mu$  has a unique extension to  $\bar{\mu} : C \rightarrow K$  where  $\bar{\mu}$  is uniformly  $\mathcal{G}$ -continuous.

If  $\mu$  is bounded on  $B$  then  $K$  is bounded and so  $\bar{\mu}$  is bounded on  $C$ .  $\square$

**Lemma 1.3.** *Let  $X$  be a Banach space. Let  $X_1$  be the closed unit ball of  $X$  and let  $X_1^{**}$  be closed unit ball of  $X^{**}$ . Let  $\mu : X_1 \rightarrow \mathbb{C}$  be a bounded function which is uniformly weakly continuous. Then  $\mu$  has a unique extension to  $\bar{\mu} : X_1^{**} \rightarrow \mathbb{C}$  where  $\bar{\mu}$  is bounded and uniformly weak\*-continuous.*

*Proof.* Let  $\mathcal{G}$  be the weak\*-topology on  $X^{**}$ . For each  $\phi \in X^*$

$$X \cap \{x \in X^{**} : |\phi(x)| < 1\} = \{x \in X : |\phi(x)| < 1\}.$$

So  $\mathcal{G}$  induces the weak topology on  $X$ . So  $\mu$  is uniformly  $\mathcal{G}$ -continuous on  $X_1$ . Since  $X_1$  is dense in  $X_1^{**}$ , with respect to the  $\mathcal{G}$ -topology, it follows from Lemma 1.2 that  $\bar{\mu}$  exists and has the required properties.  $\square$

## 2. Algebraic Preliminaries.

**Lemma 2.1.** *Let  $\mathcal{B}$  be a non-abelian  $C^*$ -subalgebra of a von Neumann algebra  $\mathcal{M}$ , where  $\mathcal{M}$  is of Type  $I_2$ . Then  $\mathcal{B}$  has a surjective homomorphism onto  $M_2(\mathbb{C})$ , the algebra of all two-by-two complex matrices.*

*Proof.* We have  $\mathcal{M} = M_2(\mathbb{C}) \overline{\otimes} C(S)$  where  $S$  is hyperstonian. For each  $s \in S$  there is a homomorphism  $\pi_s$  from  $\mathcal{M}$  onto  $M_2(\mathbb{C})$  defined by

$$\pi_s \begin{Bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{Bmatrix} = \begin{Bmatrix} x_{11}(s) & x_{12}(s) \\ x_{21}(s) & x_{22}(s) \end{Bmatrix}.$$

Clearly, if  $\pi_s[\mathcal{B}]$  is abelian for every  $s$  then  $\mathcal{B}$  is abelian. So, for some  $s$ ,  $\pi_s[\mathcal{B}]$  is a non-abelian\*-subalgebra of  $M_2(\mathbb{C})$  and so equals  $M_2(\mathbb{C})$ .  $\square$

**Lemma 2.2.** *Let  $\pi$  be a representation of a  $C^*$ -algebra  $\mathcal{A}$  on a Hilbert space  $H$ . Let  $\mathcal{M} = \pi[\mathcal{A}]''$  where the von Neumann algebra  $\mathcal{M}$  has a direct summand of Type  $I_2$ . Then  $\mathcal{A}$  has a surjective homomorphism onto  $M_2(\mathbb{C})$ .*

*Proof.* Let  $e$  be a central projection of  $\mathcal{M}$  such that  $e\mathcal{M}$  is of Type  $I_2$ . Since  $\pi[\mathcal{A}]$  is dense in  $\mathcal{M}$  in the strong operator topology,  $e\pi[\mathcal{A}]$  is dense in  $e\mathcal{M}$ . Since  $e\mathcal{M}$  is not abelian neither is  $e\pi[\mathcal{A}]$ . So, by the preceding lemma,  $e\pi[\mathcal{A}]$ , and hence  $\mathcal{A}$ , has a surjective homomorphism onto  $M_2(\mathbb{C})$ .  $\square$

### 3. Linearity.

We now come to our basic theorem.

**Theorem 3.1.** *Let  $\mathcal{A}$  be a  $C^*$ -algebra which has no quotient isomorphic to  $M_2(\mathbb{C})$ . Let  $\pi$  be a representation of  $\mathcal{A}$  on a Hilbert space  $H$ . Let  $\mathcal{M}$  be the closure of  $\mathcal{A}$  in the strong operator-topology of  $L(H)$ . Let  $\mu$  be a local quasi-linear functional on  $\pi[\mathcal{A}]$ , which is uniformly continuous on the closed unit ball of  $\pi[\mathcal{A}]$  with respect to the topology induced on  $\pi[\mathcal{A}]$  by the strong operator topology of  $L(H)$ . Then  $\mu$  is linear.*

*Proof.* We may suppose, by restricting to a closed subspace of  $H$  if necessary, that  $\pi[\mathcal{A}]$  has an upward directed net converging, in the strong operator topology to the identity of  $H$ . Clearly  $\pi[\mathcal{A}]$  has no quotient isomorphic to  $M_2(\mathbb{C})$  for, otherwise,  $M_2(\mathbb{C})$  would be a quotient of  $\mathcal{A}$ .

So, to simplify our notation we shall suppose that  $\mathcal{A} = \pi[\mathcal{A}] \subset L(H)$ .

Let  $\mathcal{M}$  be the double commutant of  $\mathcal{A}$  in  $L(H)$ . Let  $M_1$  be the set of all self-adjoint elements in the unit ball of  $M$ . Then, by the Kaplansky Density Theorem,  $A_1$  is dense in  $M_1$  with respect to the strong operator-topology of  $L(H)$ .

Then, by Lemma 1.2, there exists  $\bar{\mu} : M_1 \rightarrow \mathbb{C}$  such that  $\bar{\mu}$  is an extension of  $\mu \upharpoonright A_1$  and such that  $\bar{\mu}$  is continuous with respect to the strong operator topology. Since  $\mu[A_1]$  is bounded so, also, is  $\bar{\mu}[M_1]$ .

We know that for each  $a \in A_1$  and each  $t \in \mathbb{R}$ ,

$$\mu(ta) = t\mu(a).$$

We extend the definition of  $\bar{\mu}$  to the whole of  $M$  by defining

$$\bar{\mu}(x) = \|x\| \bar{\mu} \left( \frac{1}{\|x\|} x \right)$$

whenever  $x \in M$  with  $\|x\| > 1$ . It is then easy to verify that if  $(a_\lambda)$  is a bounded net in  $A$  which converges to  $x$  in the strong operator topology of  $L(H)$  then

$$\mu(a_\lambda) \rightarrow \bar{\mu}(x).$$

Also, whenever  $(x_n)(n = 1, 2, \dots)$  is a bounded sequence in  $M$ , converging to  $x$  in the strong operator topology, then

$$\bar{\mu}(x_n) \rightarrow \bar{\mu}(x).$$

Let  $x$  be a fixed element of  $M$  and let  $(a_\lambda)$  be a bounded net in  $A$  which converges to  $x$  in the strong operator topology. Then, for each positive whole number  $n$ ,  $a_\lambda^n \rightarrow x^n$  in the strong operator topology. So  $\mu(a_\lambda^n) \rightarrow \bar{\mu}(x^n)$ .

Let  $\phi_1, \phi_2$  be polynomials with real coefficients and zero constant term. Then, since  $\mu$  is a local quasi-linear functional,

$$\mu \{ \phi_1(a_\lambda) \} + \mu \{ \phi_2(a_\lambda) \} = \mu \{ (\phi_1 + \phi_2)(a_\lambda) \}.$$

Now

$$\phi_1(a_\lambda) \rightarrow \phi_1(x), \phi_2(a_\lambda) \rightarrow \phi_2(x).$$

and

$$(\phi_1 + \phi_2)(a_\lambda) \rightarrow (\phi_1 + \phi_2)(x)$$

in the strong operator topology. So

$$\bar{\mu} \{ \phi_1(x) \} + \bar{\mu} \{ \phi_2(x) \} = \bar{\mu} \{ \phi_1(x) + \phi_2(x) \}.$$

Let  $N(x)$  be the norm-closure of the set of all elements of the form  $\phi(x)$ , where  $\phi$  is a polynomial with real coefficients and zero constant term. Then, since each norm convergent sequence is bounded and strongly convergent,  $\bar{\mu}$  is linear on  $N(x)$ .

Let  $p_1, p_2, \dots, p_n$  be orthogonal projections in  $M$ .

Let

$$x = p_1 + \frac{1}{2}p_2 + \dots + \frac{1}{2^{n-1}}p_n + \frac{1}{2^n} \{ 1 - p_1 - p_2 - \dots - p_n \}.$$

Then  $(x^k) (k = 1, 2, \dots)$  converges in norm to  $p_1$ . So  $p_1$  is in  $N(x)$ . Then

$$\{ (2x - 2p_1)^k \} (k = 1, 2, \dots)$$

converges in norm to  $p_2$ . Similarly,  $p_3, p_4, \dots, p_n$  and  $1 - p_1 - p_2 - \dots - p_n$  are all in  $N(x)$ .

Let  $\nu(p) = \bar{\mu}(p)$  for each projection  $p$  in  $M$ . Then  $\nu$  is a bounded finitely additive measure on the projections of  $M$ .

Since  $\mathcal{A}$  has no quotient isomorphic to  $M_2(\mathbb{C})$ , it follows from Lemma 2.2 that  $\mathcal{M}$  has no direct summand of Type  $I_2$ . Hence, by Theorem A of [4] or [6],  $\nu$  extends to a bounded linear functional on  $\mathcal{M}$ , which we again denote by  $\nu$ . From the argument of the preceding paragraph,  $\bar{\mu}$  and  $\nu$  coincide on finite (real) linear combinations of orthogonal projections. Hence by norm-continuity and spectral theory,  $\bar{\mu}(x) = \nu(x)$  for each  $x \in M$ . Thus  $\mu$  is linear.  $\square$

As an application of the above theorem, we shall see that when a quasi-linear functional  $\mu$  has a "control functional", it is forced to be linear. We need a definition.

**Definition.** Let  $\phi$  be a positive linear functional in  $\mathcal{A}^*$  and let  $\mu$  be a quasi-linear functional on  $\mathcal{A}$ . Then  $\mu$  is said to be *uniformly absolutely continuous with respect to  $\phi$*  if, given any  $\epsilon > 0$  there can be found  $\delta > 0$  such that, whenever  $b \in A_1$  and  $c \in A_1$  and  $\phi((b - c)^2) < \delta$ , then  $|\mu(b) - \mu(c)| < \epsilon$ .

**Corollary 3.2.** *Let  $\mathcal{A}$  be a  $C^*$ -algebra which has no quotient isomorphic to  $M_2(\mathbb{C})$ . Let  $\mu$  be a local quasi-linear functional on  $\mathcal{A}$  which is uniformly absolutely continuous with respect to  $\phi$ , where  $\phi$  is a positive linear functional in  $\mathcal{A}^*$ . Then  $\mu$  is linear.*

*Proof.* Let  $(\pi, H)$  be the universal representation of  $\mathcal{A}$  on its universal representation space  $H$ . We identify  $\mathcal{A}$  with its image under  $\pi$  and identify  $\pi[\mathcal{A}]''$  with  $\mathcal{A}^{**}$ .

Let  $\xi$  be a vector in  $H$  which induces  $\phi$ , that is,

$$\phi(a) = \langle a\xi, \xi \rangle \text{ for each } a \in \mathcal{A}.$$

Choose  $\epsilon > 0$ . Then, by hypothesis, there exists  $\delta > 0$  such that, whenever  $b \in A_1$  and  $c \in A_1$  with

$$\|(b - c)\xi\|^2 < \delta$$

then

$$|\mu(b) - \mu(c)| < \epsilon.$$

So  $\mu$  is uniformly continuous on  $A_1$ , with respect to the strong operator topology of  $L(H)$ . Hence, by the preceding theorem  $\mu$  is linear.  $\square$

**Theorem 3.3.** *Let  $\mathcal{A}$  be a  $C^*$ -algebra with no quotient isomorphic to  $M_2(\mathbb{C})$ . Let  $\mu$  be a (local) quasi-linear functional on  $\mathcal{A}$ . Then  $\mu$  is a bounded linear functional if, and only if,  $\mu$  is uniformly weakly continuous on the unit ball of  $\mathcal{A}$ .*

*Proof.* By Lemma 1.1 each bounded linear functional on  $\mathcal{A}$  is uniformly weakly continuous. We now assume that  $\mu$  is uniformly weakly continuous on  $A_1$ . Let  $(\pi, H)$  be the universal representation of  $\mathcal{A}$ . Let  $\mathcal{M} = \pi[\mathcal{A}]''$ . Then  $\mathcal{A}^{**}$  can be identified with  $\mathcal{M}$  and  $\mathcal{A}^{**}$  with  $M$ .

By Lemma 1.3 there exists a function  $\bar{\mu} : M_1 \rightarrow \mathbb{C}$  which is uniformly continuous with respect to the weak\*-topology on  $M_1$  and such that  $\bar{\mu}|_{A_1}$  coincides with  $\mu|_{A_1}$ .

The weak\*-topology on  $M_1$  coincides with the weak-operator topology of  $L(H)$ , restricted to  $M_1$ . This is weaker than the strong operator-topology restricted to  $M_1$ . So  $\bar{\mu}$  is uniformly continuous on  $M_1$  with respect to the strong operator topology of  $L(H)$ . Thus  $\mu$  is uniformly continuous on  $A_1$

with respect to the strong operator topology of  $L(H)$ . Then, by Theorem 3.1,  $\mu$  is linear.  $\square$

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# PACIFIC JOURNAL OF MATHEMATICS

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---

A class of incomplete non-positively curved manifolds	1
BRIAN BOWDITCH	
The quasi-linearity problem for $C^*$ -algebras	41
L. J. BUNCE and JOHN DAVID MAITLAND WRIGHT	
Distortion of boundary sets under inner functions. II	49
JOSE LUIS FERNANDEZ PEREZ, DOMINGO PESTANA and JOSÉ RODRÍGUEZ	
Irreducible non-dense $A_1^{(1)}$ -modules	83
VJACHESLAV M. FUTORNY	
$M$ -hyperbolic real subsets of complex spaces	101
GIULIANA GIGANTE, GIUSEPPE TOMASSINI and SERGIO VENTURINI	
Values of Bernoulli polynomials	117
ANDREW GRANVILLE and ZHI-WEI SUN	
The uniqueness of compact cores for 3-manifolds	139
LUKE HARRIS and PETER SCOTT	
Estimation of the number of periodic orbits	151
BOJU JIANG	
Factorization of $p$ -completely bounded multilinear maps	187
CHRISTIAN LE MERDY	
Finitely generated cohomology Hopf algebras and torsion	215
JAMES PEICHENG LIN	
The positive-dimensional fibres of the Prym map	223
JUAN-CARLOS NARANJO	
Entropy of a skew product with a $Z^2$ -action	227
KYEWON KOH PARK	
Commuting co-commuting squares and finite-dimensional Kac algebras	243
TAKASHI SANO	
Second order ordinary differential equations with fully nonlinear two-point boundary conditions. I	255
H. BEVAN THOMPSON	
Second order ordinary differential equations with fully nonlinear two-point boundary conditions. II	279
H. BEVAN THOMPSON	
The flat part of non-flat orbifolds	299
FENG XU	



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