THE QUASI-LINEARITY PROBLEM FOR $C^*$-ALGEBRAS

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Let $A$ be a $C^*$-algebra with no quotient isomorphic to the algebra of all two-by-two matrices. Let $\mu$ be a quasi-linear functional on $A$. Then $\mu$ is linear if, and only if, the restriction of $\mu$ to the closed unit ball of $A$ is uniformly weakly continuous.

Introduction.

Throughout this paper, $A$ will be a $C^*$-algebra and $A$ will be the real Banach space of self-adjoint elements of $A$. The unit ball of $A$ is $A_1$ and the unit ball of $A$ is $A_1$. We do not assume the existence of a unit in $A$.

Definition. A quasi-linear functional on $A$ is a function $\mu : A \rightarrow \mathbb{R}$ such that, whenever $B$ is an abelian subalgebra of $A$, the restriction of $\mu$ to $B$ is linear. Furthermore $\mu$ is required to be bounded on the closed unit ball of $A$.

Given any quasi-linear functional $\mu$ on $A$ we may extend it to $A$ by defining

$$\tilde{\mu}(x + iy) = \mu(x) + i\mu(y)$$

whenever $x \in A$ and $y \in A$. Then $\tilde{\mu}$ will be linear on each maximal abelian $*$-subalgebra of $A$. We shall abuse our notation by writing $\mu$ instead of $\tilde{\mu}$.

When $A = M_2(\mathbb{C})$, the $C^*$-algebra of all two-by-two matrices over $\mathbb{C}$, there exist examples of quasi-linear functionals on $A$ which are not linear.

Definition. A local quasi-linear functional on $A$ is a function $\mu : A \rightarrow \mathbb{R}$ such that, for each $x$ in $A$, $\mu$ is linear on the smallest norm closed subalgebra of $A$ containing $x$. Furthermore $\mu$ is required to be bounded on the closed unit ball of $A$.

Clearly each quasi-linear functional on $A$ is a local quasi-linear functional. Surprisingly, the converse is false, even when $A$ is abelian (see Aarnes [2]). However when $A$ has a rich supply of projections (e.g. when $A$ is a von Neumann algebra) each local quasi-linear functional is quasi-linear [3].

The solution of the Mackey-Gleason Problem shows that every quasi-linear functional on a von Neumann algebra $\mathcal{M}$, where $\mathcal{M}$ has no direct summand of Type $I_2$, is linear [4, 5, 6]. This was first established for positive quasi-linear functionals by the conjunction of the work of Christensen [7] and...
Yeadon [11], and for $\sigma$-finite factors by the work of Paschciewicz [10]. All build on the fundamental theorem of Gleason [8].

Although quasi-linear functionals on general $C^*$-algebras seem much harder to tackle than the von Neumann algebra problem, we can apply the von Neumann results to make progress. In particular, we prove:

Let $\mathcal{A}$ be a $C^*$-algebra with no quotient isomorphic to $M_2(\mathbb{C})$. Let $\mu$ be a (local) quasi-linear functional on $\mathcal{A}$. Then $\mu$ is linear if, and only if, the restriction of $\mu$ to $A_1$, is uniformly weakly continuous.

1. Preliminaries: Uniform Continuity.

Let $X$ be a real or complex vector space. Let $\mathcal{F}$ be a locally convex topology for $X$. Let $V$ be a $\mathcal{F}$-open neighbourhood of 0. We call $V$ symmetric if $V$ is convex and, whenever $x \in V$ then $-x \in V$.

Let $B$ be a subset of $X$. A scalar valued function on $X$, $\mu$, is said to be uniformly continuous on $B$, with respect to the $\mathcal{F}$-topology, if, given any $\epsilon > 0$, there exists an open symmetric neighbourhood of 0, $V$, such that whenever $x \in B$, $y \in B$ and $x - y \in V$ then

$$|\mu(x) - \mu(y)| < \epsilon.$$

**Lemma 1.1.** Let $X$ be a Banach space and let $\mathcal{F}$ be any locally convex topology for $X$ which is stronger than the weak topology. Let $\mu$ be any bounded linear functional on $X$. Then $\mu$ is uniformly $\mathcal{F}$-continuous on $X$.

**Proof.** Choose $\epsilon > 0$. Let

$$V = \{x \in X : |\mu(x)| < \epsilon\} = \mu^{-1}\{\lambda : |\lambda| < \epsilon\}.$$

Then $V$ is open in the weak topology of $X$. Hence $V$ is a symmetric $\mathcal{F}$-open neighbourhood of 0 such that $x - y \in V$ implies

$$|\mu(x) - \mu(y)| = |\mu(x - y)| < \epsilon.$$


**Lemma 1.2.** Let $X$ be a subspace of a Banach space $Y$. Let $\mathcal{G}$ be a locally convex topology for $Y$ which is weaker than the norm topology. Let $\mathcal{F}$ be the relative topology induced on $X$ by $\mathcal{G}$. Let $B$ be a subset of $X$ and let $C$ be the closure of $B$ in $Y$, with respect to the $\mathcal{G}$-topology. Let $\mu : B \to C$ be uniformly continuous on $B$ with respect to the $\mathcal{F}$-topology. Then there exists
a function \( \tilde{\mu} : C \to C \) which extends \( \mu \) and which is uniformly \( G \)-continuous. Furthermore, if \( \mu \) is bounded on \( B \) then \( \tilde{\mu} \) is bounded on \( C \).

**Proof.** Since \( \mathcal{F} \) is the relative topology induced by \( G \), \( \mu \) is uniformly \( G \)-continuous on \( B \). Let \( K \) be the closure of \( \mu[B] \) in \( \mathbb{C} \). Then \( K \) is a complete metric space. So, see [9, page 125], \( \mu \) has a unique extension to \( \tilde{\mu} : C \to K \) where \( \tilde{\mu} \) is uniformly \( G \)-continuous.

If \( \mu \) is bounded on \( B \) then \( K \) is bounded and so \( \tilde{\mu} \) is bounded on \( C \). \( \square \)

**Lemma 1.3.** Let \( X \) be a Banach space. Let \( X_1 \) be the closed unit ball of \( X \) and let \( X_1^{**} \) be closed unit ball of \( X^{**} \). Let \( \mu : X_1 \to \mathbb{C} \) be a bounded function which is uniformly weakly continuous. Then \( \mu \) has a unique extension to \( \tilde{\mu} : X_1^{**} \to \mathbb{C} \) where \( \tilde{\mu} \) is bounded and uniformly weak*-continuous.

**Proof.** Let \( \mathcal{G} \) be the weak*-topology on \( X^{**} \). For each \( \phi \in X^* \)
\[ X \cap \{ x \in X^{**} : |\phi(x)| < 1 \} = \{ x \in X : |\phi(x)| < 1 \}. \]

So \( \mathcal{G} \) induces the weak topology on \( X \). So \( \mu \) is uniformly \( \mathcal{G} \)-continuous on \( X_1 \). Since \( X_1 \) is dense in \( X_1^{**} \), with respect to the \( \mathcal{G} \)-topology, it follows from Lemma 1.2 that \( \tilde{\mu} \) exists and has the required properties. \( \square \)

### 2. Algebraic Preliminaries.

**Lemma 2.1.** Let \( B \) be a non-abelian \( C^* \)-subalgebra of a von Neumann algebra \( \mathcal{M} \), where \( \mathcal{M} \) is of Type I\(_2\). Then \( B \) has a surjective homomorphism onto \( M_2(\mathbb{C}) \), the algebra of all two-by-two complex matrices.

**Proof.** We have \( \mathcal{M} = M_2(\mathbb{C}) \otimes C(S) \) where \( S \) is hyperstonian. For each \( s \in S \) there is a homomorphism \( \pi_S \) from \( \mathcal{M} \) onto \( M_2(\mathbb{C}) \) defined by
\[
\pi_S \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{11}(s) & x_{12}(s) \\ x_{21}(s) & x_{22}(s) \end{pmatrix}.
\]
Clearly, if \( \pi_S[B] \) is abelian for every \( s \) then \( B \) is abelian. So, for some \( s \), \( \pi_S[B] \) is a non-abelian*-subalgebra of \( M_2(\mathbb{C}) \) and so equals \( M_2(\mathbb{C}) \). \( \square \)

**Lemma 2.2.** Let \( \pi \) be a representation of a \( C^* \)-algebra \( \mathcal{A} \) on a Hilbert space \( H \). Let \( \mathcal{M} = \pi[\mathcal{A}]'' \) where the von Neumann algebra \( \mathcal{M} \) has a direct summand of Type I\(_2\). Then \( \mathcal{A} \) has a surjective homomorphism onto \( M_2(\mathbb{C}) \).

**Proof.** Let \( e \) be a central projection of \( \mathcal{M} \) such that \( e\mathcal{M} \) is of Type I\(_2\). Since \( \pi[\mathcal{A}] \) is dense in \( \mathcal{M} \) in the strong operator topology, \( e\pi[\mathcal{A}] \) is dense in \( e\mathcal{M} \). Since \( e\mathcal{M} \) is not abelian neither is \( e\pi[\mathcal{A}] \). So, by the preceding lemma, \( e\pi[\mathcal{A}] \), and hence \( \mathcal{A} \), has a surjective homomorphism onto \( M_2(\mathbb{C}) \). \( \square \)
3. Linearity.

We now come to our basic theorem.

**Theorem 3.1.** Let $\mathcal{A}$ be a $C^*$-algebra which has no quotient isomorphic to $M_2(\mathbb{C})$. Let $\pi$ be a representation of $\mathcal{A}$ on a Hilbert space $H$. Let $\mathcal{M}$ be the closure of $\mathcal{A}$ in the strong operator-topology of $L(H)$. Let $\mu$ be a local quasi-linear functional on $\pi[\mathcal{A}]$, which is uniformly continuous on the closed unit ball of $\pi[\mathcal{A}]$ with respect to the topology induced on $\pi[\mathcal{A}]$ by the strong operator topology of $L(H)$. Then $\mu$ is linear.

**Proof.** We may suppose, by restricting to a closed subspace of $H$ if necessary, that $\pi[\mathcal{A}]$ has an upward directed net converging, in the strong operator topology to the identity of $H$. Clearly $\pi[\mathcal{A}]$ has no quotient isomorphic to $M_2(\mathbb{C})$ for, otherwise, $M_2(\mathbb{C})$ would be a quotient of $\mathcal{A}$.

So, to simplify our notation we shall suppose that $\mathcal{A} = \pi[\mathcal{A}] \subset L(H)$.

Let $\mathcal{M}$ be the double commutant of $\mathcal{A}$ in $L(H)$. Let $M_1$ be the set of all self-adjoint elements in the unit ball of $M$. Then, by the Kaplansky Density Theorem, $A_1$ is dense in $M_1$ with respect to the strong operator-topology of $L(H)$.

Then, by Lemma 1.2, there exists $\overline{\mu} : M_1 \to \mathbb{C}$ such that $\overline{\mu}$ is an extension of $\mu \mid A_1$ and such that $\overline{\mu}$ is continuous with respect to the strong operator topology. Since $\mu[A_1]$ is bounded so, also, is $\overline{\mu}[M_1]$.

We know that for each $a \in A_1$ and each $t \in \mathbb{R}$,

$$\mu(ta) = t\mu(a).$$

We extend the definition of $\overline{\mu}$ to the whole of $M$ by defining

$$\overline{\mu}(x) = \|x\|\overline{\mu}\left(\frac{1}{\|x\|}x\right)$$

whenever $x \in M$ with $\|x\| > 1$. It is then easy to verify that if $(a_\lambda)$ is a bounded net in $A$ which converges to $x$ in the strong operator topology of $L(H)$ then

$$\mu(a_\lambda) \to \overline{\mu}(x).$$

Also, whenever $(x_n)(n = 1, 2, \ldots)$ is a bounded sequence in $M$, converging to $x$ in the strong operator topology, then

$$\overline{\mu}(x_n) \to \overline{\mu}(x).$$

Let $x$ be a fixed element of $M$ and let $(a_\lambda)$ be a bounded net in $A$ which converges to $x$ in the strong operator topology. Then, for each positive whole number $n$, $a_\lambda^n \to x^n$ in the strong operator topology. So $\mu(a_\lambda^n) \to \overline{\mu}(x^n)$. 

Let $\phi_1, \phi_2$ be polynomials with real coefficients and zero constant term. Then, since $\mu$ is a local quasi-linear functional,

$$\mu \{\phi_1(a_\lambda)\} + \mu \{\phi_2(a_\lambda)\} = \mu \{\phi_1 + \phi_2)(a_\lambda)\}.$$

Now

$$\phi_1(a_\lambda) \to \phi_1(x), \phi_2(a_\lambda) \to \phi_2(x).$$

and

$$(\phi_1 + \phi_2)(a_\lambda) \to (\phi_1 + \phi_2)(x)$$

in the strong operator topology. So

$$\bar{\mu} \{\phi_1(x)\} + \bar{\mu} \{\phi_2(x)\} = \bar{\mu} \{\phi_1(x) + \phi_2(x)\}.$$

Let $N(x)$ be the norm-closure of the set of all elements of the form $\phi(x)$, where $\phi$ is a polynomial with real coefficients and zero constant term. Then, since each norm convergent sequence is bounded and strongly convergent, $\bar{\mu}$ is linear on $N(x)$.

Let $p_1, p_2, \ldots, p_n$ be orthogonal projections in $M$. Let

$$x = p_1 + \frac{1}{2}p_2 + \ldots + \frac{1}{2^{n-1}}p_n + \frac{1}{2^n} \{1 - p_1 - p_2 - \ldots - p_n\}.$$

Then $(x^k)(k = 1, 2, \ldots)$ converges in norm to $p_1$. So $p_1$ is in $N(x)$. Then

$$\{(2x - 2p_1)^k\} (k = 1, 2, \ldots)$$

converges in norm to $p_2$. Similarly, $p_3, p_4, \ldots, p_n$ and $1 - p_1 - p_2 - \ldots - p_n$ are all in $N(x)$.

Let $\nu(p) = \bar{\mu}(p)$ for each projection $p$ in $M$. Then $\nu$ is a bounded finitely additive measure on the projections of $M$.

Since $\mathcal{A}$ has no quotient isomorphic to $M_2(\mathbb{C})$, it follows from Lemma 2.2 that $\mathcal{M}$ has no direct summand of Type I$_2$. Hence, by Theorem A of [4] or [6], $\nu$ extends to a bounded linear functional on $\mathcal{M}$, which we again denote by $\nu$. From the argument of the preceding paragraph, $\bar{\mu}$ and $\nu$ coincide on finite (real) linear combinations of orthogonal projections. Hence by norm-continuity and spectral theory, $\bar{\mu}(x) = \nu(x)$ for each $x \in M$. Thus $\mu$ is linear.

As an application of the above theorem, we shall see that when a quasi-linear functional $\mu$ has a "control functional", it is forced to be linear. We need a definition.
Definition. Let \( \phi \) be a positive linear functional in \( \mathcal{A} \) and let \( \mu \) be a quasi-linear functional on \( \mathcal{A} \). Then \( \mu \) is said to be uniformly absolutely continuous with respect to \( \phi \) if, given any \( \epsilon > 0 \) there can be found \( \delta > 0 \) such that, whenever \( b \in A_1 \) and \( c \in A_1 \) and \( \phi((b - c)^2) < \delta \), then \( |\mu(b) - \mu(c)| < \epsilon \).

Corollary 3.2. Let \( \mathcal{A} \) be a \( C^* \)-algebra which has no quotient isomorphic to \( M_2(\mathbb{C}) \). Let \( \mu \) be a local quasi-linear functional on \( \mathcal{A} \) which is uniformly absolutely continuous with respect to \( \phi \), where \( \phi \) is a positive linear functional in \( \mathcal{A}^* \). Then \( \mu \) is linear.

Proof. Let \((\pi, H)\) be the universal representation of \( \mathcal{A} \) on its universal representation space \( H \). We identify \( \mathcal{A} \) with its image under \( \pi \) and identify \( \pi[A]^\prime \) with \( \mathcal{A}^{**} \).

Let \( \xi \) be a vector in \( H \) which induces \( \phi \), that is, \( \phi(a) = \langle a\xi, \xi \rangle \) for each \( a \in \mathcal{A} \).

Choose \( \epsilon > 0 \). Then, by hypothesis, there exists \( \delta > 0 \) such that, whenever \( b \in A_1 \) and \( c \in A_1 \) with
\[
\|(b - c)\xi\|^2 < \delta
\]
then
\[
|\mu(b) - \mu(c)| < \epsilon.
\]
So \( \mu \) is uniformly continuous on \( A_1 \), with respect to the strong operator topology of \( L(H) \). Hence, by the preceding theorem \( \mu \) is linear.

Theorem 3.3. Let \( \mathcal{A} \) be a \( C^* \)-algebra with no quotient isomorphic to \( M_2(\mathbb{C}) \). Let \( \mu \) be a (local) quasi-linear functional on \( \mathcal{A} \). Then \( \mu \) is a bounded linear functional if, and only if, \( \mu \) is uniformly weakly continuous on the unit ball of \( \mathcal{A} \).

Proof. By Lemma 1.1 each bounded linear functional on \( \mathcal{A} \) is uniformly weakly continuous. We now assume that \( \mu \) is uniformly weakly continuous on \( A_1 \). Let \((\pi, H)\) be the universal representation of \( \mathcal{A} \). Let \( \mathcal{M} = \pi[A]^\prime \). Then \( A^{**} \) can be identified with \( \mathcal{M} \) and \( A^{**} \) with \( \mathcal{M} \).

By Lemma 1.3 there exists a function \( \overline{\mu} : M_1 \to \mathbb{C} \) which is uniformly continuous with respect to the weak*-topology on \( M_1 \) and such that \( \overline{\mu}|A_1 \) coincides with \( \mu|A_1 \).

The weak*-topology on \( M_1 \) coincides with the weak-operator topology of \( L(H) \), restricted to \( M_1 \). This is weaker than the strong operator-topology restricted to \( M_1 \). So \( \overline{\mu} \) is uniformly continuous on \( M_1 \) with respect to the strong operator topology of \( L(H) \). Thus \( \mu \) is uniformly continuous on \( A_1 \).
with respect to the strong operator topology of $L(H)$. Then, by Theorem 3.1, $\mu$ is linear.

\[ \square \]

References


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