IRREDUCIBLE NON-DENSE $A_1^{(1)}$-MODULES

Vjacheslav M. Futorny
IRREDUCIBLE NON-DENSE $A_1^{(1)}$-MODULES

V.M. FUTORNY

We study the irreducible weight non-dense modules for Affine Lie Algebra $A_1^{(1)}$ and classify all such modules having at least one finite-dimensional weight subspace. We prove that any irreducible non-zero level module with all finite-dimensional weight subspaces is non-dense.

1. Introduction.

Let $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ and $\mathcal{G} = \mathcal{G}(A)$ is the associated Kac-Moody algebra over the complex numbers $\mathbb{C}$ with Cartan subalgebra $H \subset \mathcal{G}$, 1-dimensional center $\mathbb{C}c \subset H$ and root system $\Delta$.

A $\mathcal{G}$-module $V$ is called a weight if $V = \bigoplus_{\lambda \in H^*} V_\lambda$, $V_\lambda = \{v \in V \mid hv = \lambda(h)v\}$ for all $h \in H$. If $V$ is an irreducible weight $\mathcal{G}$-module then $c$ acts on $V$ as a scalar. We will call this scalar the level of $V$. For a weight $\mathcal{G}$-module $V$, set $P(V) = \{\lambda \in H^* \mid V_\lambda \neq 0\}$.

Let $Q = \sum_{\varphi \in \Delta} \mathbb{Z}\varphi$. It is clear that if a weight $\mathcal{G}$-module $V$ is irreducible then $P(V) \subset \lambda + Q$ for some $\lambda \in H^*$. An irreducible weight $\mathcal{G}$-module $V$ is called dense if $P(V) = \lambda + Q$ for some $\lambda \in H^*$, and non-dense otherwise.

Irreducible dense modules whose weight spaces are all one-dimensional were classified by S. Spirin [1] for the algebra $A_1^{(1)}$ and by D. Britten, F. Lemire, F. Zorzitto [2] in the general case. It follows from [2] that such modules exist only for algebras $A_n^{(1)}$, $C_n^{(1)}$. V. Chari and A. Pressley constructed a family of irreducible integrable dense modules with all infinite-dimensional weight spaces. These modules can be realized as tensor product of standard highest weight modules with so-called loop modules [3].

In the present paper we study irreducible non-dense weight $\mathcal{G}$-modules. We use Kac [4] as a basic reference for notation, terminology and preliminary results. Our main result is the classification of all irreducible non-dense $\mathcal{G}$-modules having at least one finite-dimensional weight subspace. This includes, in particular, all irreducible highest weight modules. Moreover, we show that this classification includes all irreducible modules of non-zero level whose weight spaces are all finite-dimensional.
The paper is organized as follows. In Section 3 we study generalized Verma modules $M_\alpha^\varepsilon(\lambda, \gamma)$, $\alpha$ is a real root, $\lambda \in H^*$, $\gamma \in \mathbb{C}$, $\varepsilon \in \{+, -\}$ which do not necessarily have a highest weight (cf. [5]). By making use of the generalized Casimir operator and generalized Shapovalov form we obtain the criteria of irreducibility for the modules $M_\alpha^\varepsilon(\lambda, \gamma)$ without highest weight (Theorem 3.11).

In Section 4 we classify all irreducible $\mathbb{Z}$-graded modules for the Heisenberg subalgebra $G \subset \mathcal{G}$ with at least one finite-dimensional graded component. Irreducible $G$-modules with trivial action of $c$ were described earlier in [6].

Let $\delta \in \Delta$ such that $\mathbb{Z}\delta - \{0\}$ is the set of all imaginary roots in $\Delta$. Following [6] we introduce in Section 5 the category $\tilde{O}(\alpha)$ of weight $\mathcal{G}$-modules $\tilde{V}$ such that $P(\tilde{V}) \subset \bigcup_{i=1}^{\ell} \{\lambda_i - k\alpha + n\delta \mid k, n \in \mathbb{Z}, k \geq 0\}$ where $\lambda_i \in H^*$, but without any restriction on the action of the center (unlike in [6] where the trivial action of the center is required). The irreducible objects in $\tilde{O}(\alpha)$ are the unique quotients of $\mathcal{G}$-modules $M_\alpha(\lambda, V)$, where $\lambda \in H^*$, $V$ is irreducible $\mathbb{Z}$-graded $G$-module. Modules $M_\alpha(\lambda, C)$, with $\lambda(c) = 0$ were studied in [7-9]. If $\lambda(c) \neq 0$ and at least one graded component of $V$ is finite-dimensional then the module $M_\alpha(\lambda, V)$ is irreducible [8, 9]. In Section 6 we classify all irreducible non-dense $\mathcal{G}$-modules with at least one finite-dimensional weight subspace (Theorem 6.2). It turns out that these modules are the quotients of the modules of type $M_\alpha^\varepsilon(\lambda, \gamma)$ or $M_\alpha(\lambda, V)$. Moreover, any irreducible $\mathcal{G}$-module of non-zero level whose weight spaces are all finite-dimensional is the quotient of $M_\alpha^\varepsilon(\lambda, \gamma)$ for some real root $\alpha$, $\lambda \in H^*$, $\gamma \in \mathbb{C}$, $\varepsilon \in \{+, -\}$ (Theorem 6.3).

2. Preliminaries.

We have the root space decomposition for $\mathcal{G} : \mathcal{G} = H \oplus \sum_{\varphi \in \Delta} \mathcal{G}_\varphi$, where dim $\mathcal{G}_\varphi = 1$ for all $\varphi \in \Delta$. Denote by $\mathcal{U}(\mathcal{G})$ the universal enveloping algebra of $\mathcal{G}$, by $W$ the Weyl group and by $\langle , \rangle$ the standard non-degenerate symmetric bilinear form on $\mathcal{G}$ [4, Theorem 3.2]. Let $\Delta^{re}$ be the set of real roots in $\Delta$ and $\Delta^{im}$ be the set of imaginary roots in $\Delta$. Fix $\alpha \in \Delta^{re}$ and consider a subalgebra $\mathcal{G}(\alpha) \subset \mathcal{G}$ generated by $\mathcal{G}_\alpha$ and $\mathcal{G}_{-\alpha}$. Then $\mathcal{G}(\alpha) \simeq sl(2)$ and we fix in $\mathcal{G}(\alpha)$ a standard basis $e_\alpha, e_{-\alpha}, h_\alpha = [e_\alpha, e_{-\alpha}]$ where $[h_\alpha, e_{\pm\alpha}] = \pm 2e_{\pm\alpha}$. We will use the following realization of $\mathcal{G}$:

$$\mathcal{G} = \mathcal{G}(\alpha) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

with $[x \otimes t^n + ac + bd, y \otimes t^m + a_1c + b_1d] = [x, y] \otimes t^{n+m} + bmy \otimes t^m - b_1nx \otimes t^n + n\delta_{n,-m}(x, y)c$, for all $x, y \in \mathcal{G}(\alpha), a, b, a_1, b_1 \in \mathbb{C}$. Then $H = Ch_\alpha \oplus \mathbb{C}c \oplus \mathbb{C}d$. 

Denote by $\delta$ the element of $H^*$ defined by: $\delta(ha) = \delta(c) = 0$ and $\delta(d) = 1$. Then $\Delta^m = Z\delta - \{0\}$ and $\pi = \{\alpha, \delta - \alpha\}$ is a basis of $\Delta$. Let $\Delta_+ = \Delta_+(\pi)$ be the set of all positive roots with respect to $\pi$. The root system $\Delta$ can be described in the following way: $\Delta = \{\pm \alpha + n\delta \mid n \in \mathbb{Z}\} \cup \{n\delta \mid n \in \mathbb{Z} - \{0\}\}$.

We have $\mathcal{G}_{\pm \alpha + n\delta} = \mathcal{G}_{\pm \alpha} \otimes t^n$, $n \in \mathbb{Z}$, $\mathcal{G}_{n\delta} = Ch_\alpha \otimes t^n$, $n \in \mathbb{Z} - \{0\}$. Set $e_{\alpha + n\delta} = e_\alpha \otimes t^n$, $e_{-\alpha + n\delta} = e_{-\alpha} \otimes t^n$, $n \in \mathbb{Z}$, $e_{m\delta} = h_\alpha \otimes t^m$, $m \in \mathbb{Z} - \{0\}$. Then $[e_{k\delta}, e_{m\delta}] = 2k\delta_{k,-m}c$, $[e_{k\delta}, e_{m\delta} + n\delta] = \pm 2e_{\pm\alpha + (n+k)\delta}$, $[e_{\alpha + k\delta}, e_{-\alpha + m\delta}] = \delta_{k,-m}(h_\alpha + kc) + (1 - \delta_{k,-m})e_{k+m}\delta$ for any $k, m \in \mathbb{Z}$.

For a Lie algebra $\mathfrak{A}$, $S(\mathfrak{A})$ will denote the corresponding symmetric algebra. We will identify the algebra $\mathcal{U}(H) = S(H)$ with the ring of polynomials $\mathbb{C}[H^*]$ and denote by $\sigma$ the involutive antiautomorphism on $\mathcal{U}(\mathcal{G})$ such that $\sigma(e_\alpha) = e_{-\alpha}$, $\sigma(e_{-\alpha}) = e_\alpha - \delta$. Set $\mathcal{N}_\alpha^+ = \sum_{\varphi \in \Delta_+} \mathcal{G}_\varphi$, $\mathcal{N}_\alpha^- = \sum_{\varphi \in \Delta_+} \mathcal{G}_{-\varphi}$.


The center of $\mathcal{U}(\mathcal{G}(\alpha))$ is generated by the Casimir element $z_\alpha = (h_\alpha + 1)^2 + 4e_{-\alpha}e_\alpha$. Denote

$$
\mathcal{N}_\alpha^+ = \sum_{\varphi \in \Delta_+ - \{\alpha\}} \mathcal{G}_\varphi, \quad \mathcal{N}_\alpha^- = \sum_{\varphi \in \Delta_+ - \{\alpha\}} \mathcal{G}_{-\varphi},
$$

$$
T_\alpha = S(H) \otimes \mathbb{C}[z_\alpha], \quad E_\alpha^\varepsilon = (H + \mathcal{G}(\alpha)) \otimes \mathcal{N}_\alpha^\varepsilon, \quad \varepsilon \in \{+, -\}.
$$

Let $\lambda \in H^*$, $\varepsilon \in \mathbb{C}$. Consider the 1-dimensional $T_\alpha$-module $\mathbb{C}v_\lambda$ with the action $(h \otimes z_\alpha^n)v_\lambda = h(\lambda)\gamma^n v_\lambda$ for any $h \in S(H)$, and construct an $H + \mathcal{G}(\alpha)$-module

$$
V(\lambda, \gamma) = \mathcal{U}(\mathcal{G}(\alpha) + H) \mathop{\otimes}_{T_\alpha} \mathbb{C}v_\lambda.
$$

It is clear that the module $V(\lambda, \gamma)$ has a unique irreducible quotient $V_{\lambda, \gamma}$.

**Proposition 3.1.**

(i) If $V$ is an irreducible weight $H + \mathcal{G}(\alpha)$-module then $V \simeq V_{\lambda, \gamma}$ for some $\lambda \in H^*$, $\gamma \in \mathbb{C}$.

(ii) $V_{\lambda, \gamma} \simeq V_{\lambda', \gamma'}$ if and only if $\gamma = \gamma'$, $\lambda' = \lambda + n\alpha$, $n \in \mathbb{Z}$, $\gamma \neq (\lambda(h_\alpha) + 2\ell + 1)^2$ for all integers $\ell$, $0 \leq \ell < n$ if $n \geq 0$ or for all integers $\ell$, $n \leq \ell < 0$ if $n < 0$.

**Proof.** This is essentially the classification of irreducible weight $sl(2)$-modules.  \(\square\)

Let $\lambda \in H^*$, $\gamma \in \mathbb{C}$, $\varepsilon \in \{+, -\}$. Consider $V_{\lambda, \gamma}$ as $E_\alpha^\varepsilon$-module with trivial action of $\mathcal{N}_\alpha^\varepsilon$ and construct the $\mathcal{G}$-module

$$
M_\alpha^\varepsilon(\lambda, \gamma) = \mathcal{U}(\mathcal{G}) \mathop{\otimes}_{\mathcal{U}(E_\alpha^\varepsilon)} V_{\lambda, \gamma}.
$$
associated with $\alpha, \lambda, \gamma, \epsilon$.

The module $M^\epsilon_\alpha(\lambda, \gamma)$ is called a generalized Verma module. Notice that $V_{\lambda, \gamma}$ does not have to be finite-dimensional.

**Proposition 3.2.**

(i) $M^\epsilon_\alpha(\lambda, \gamma)$ is a free $\sigma(U(\mathcal{N}^\epsilon_\alpha))$-module with all finite-dimensional weight subspaces.

(ii) $M^\epsilon_\alpha(\lambda, \gamma)$ has a unique irreducible quotient, $L^\epsilon_\alpha(\lambda, \gamma)$.

(iii) $M^\epsilon_\alpha(\lambda, \gamma) \simeq M^\epsilon'_\alpha(\lambda', \gamma')$ if and only if $\epsilon = \epsilon'$, $\gamma = \gamma'$, $\lambda' = \lambda + n\alpha, n \in \mathbb{Z}$ and $\gamma \neq (\lambda(h_\alpha) + 2\ell + 1)^2$ for all $\ell \in \mathbb{Z}$, $0 \leq \ell < n$ or for all $\ell \in \mathbb{Z}$, $n \leq \ell < 0$ if $n < 0$.

**Proof.** Follows from the construction of $G$-module $M^\epsilon_\alpha(\lambda, \gamma)$ and Proposition 3.1. \qed

Let $R_\lambda = \{(\lambda(h_\alpha) + 2\ell + 1)^2 \mid \ell \in \mathbb{Z}\}$. Recall that $V$ is called a highest weight module with respect to $\mathcal{N}_+$ and with highest weight $\lambda \in H^*$ if $V = U(\mathcal{G})v, v \in V_\lambda$ and $V_{\lambda + \varphi} = 0$ for all $\varphi \in \Delta_+(\pi)$. Proposition 3.2, (iii) implies that $M^\epsilon_\alpha(\lambda, \gamma)$ and $L^\epsilon_\alpha(\lambda, \gamma)$ are highest weight modules with respect to some choice of basis of $\Delta$ and, therefore, are the quotients of Verma modules [4], if and only if $\gamma \in R_\lambda$. The theory of highest weight modules was developed in [4, 10].

**Corollary 3.3.**

(i) Let $V$ be an irreducible weight $G$-module, $0 \neq v \in V_\lambda$ and $\mathcal{N}^\epsilon_\alpha v = 0$. Then $V \simeq L^\epsilon_\alpha(\lambda, \gamma)$ for some $\gamma \in \mathbb{C}$.

(ii) Let $\lambda \not\in R_\lambda$. $L^\epsilon_\alpha(\lambda, \gamma) \simeq L^\epsilon_\alpha(\lambda', \gamma')$ if and only if $\epsilon = \epsilon'$, $\alpha' = \alpha$ or $\alpha' = -\alpha, \gamma = \gamma'$, $\lambda' = \lambda + n\alpha, n \in \mathbb{Z}$ and $\gamma \neq (\lambda(h_\alpha) + 2\ell + 1)^2$ for all $\ell \in \mathbb{Z}$, $0 \leq \ell < n$ or for all $\ell \in \mathbb{Z}$, $n \leq \ell < 0$ if $n < 0$.

**Proof.** Since $V$ is irreducible $G$-module, $V' = U(\mathcal{G}(\alpha))v$ is an irreducible $G(\alpha)$-module and $V \simeq \sigma(U(\mathcal{N}^\epsilon_\alpha))V'$. Then $V$ is a homomorphic image of $M^\epsilon_\alpha(\lambda, \gamma)$ for some $\gamma \in \mathbb{C}$ and, thus, $V \simeq L^\epsilon_\alpha(\lambda, \gamma)$ which proves (i). (ii) follows from Proposition 3.2, (iii). \qed

From now on we will consider the modules $M^+_\alpha(\lambda, \gamma)(= M(\lambda, \gamma))$. All the results for the modules $M^-_\alpha(\lambda, \gamma)$ can be proved analogously. Set $z = z_\alpha$.

For $\lambda \in H^*, \gamma \in \mathbb{C}$ and integer $n \geq 0$ we denote by $z(n)$ the restriction of $z$ to the subspace $M(\lambda, \gamma)_{\lambda - n(\delta - \alpha)}$.

**Proposition 3.4.** If $\gamma \neq (\lambda(h_\alpha) + 2\ell + 1)^2$ for all $0 \leq \ell < 2n$ then $\text{Spec } z(n) = \{(2k \pm \sqrt{\gamma})^2 \mid k \in \mathbb{Z}, 0 \leq k \leq n\}$.

**Proof.** Denote $V_n = M(\lambda, \gamma)_{\lambda - n(\delta - \alpha)}, n > 0$. One can easily show that $V_n = e_{\alpha - \delta}V_{n-1} + e_{-\delta}e_{\alpha}V_{n-1} + e_{-\alpha - \delta}e_{\alpha}^2V_{n-1}$. Let $V_{n-1} = \oplus V_{n-1}(\tau), \tau \in \mathbb{C}$,
where \( V_{n-1}(\tau) = \{ v \in V_{n-1} | \exists N : (z(n-1) - \tau)^N v = 0 \} \). Then the subspace \( e_{\alpha-\delta}V_{n-1}(\tau) + e_{-\delta}e_{\alpha}V_{n-1}(\tau) + e_{-\alpha-\delta}e_{\alpha}^2V_{n-1}(\tau) \subset V_n \) is \( z(n) \)-invariant and \( z(n) \) has on it the eigenvalues \( \tau \) and \( (2 \pm \sqrt{\tau})^2 \), thanks to the condition \( \gamma \neq (\lambda(h_{\alpha}) + 2\ell + 1)^2 \), \( 0 \leq \ell < 2n \), which implies that \( z(n) \) has eigenvalues \( (2k \pm \sqrt{\gamma})^2 \), \( 0 \leq k \leq n \).

\( \square \)

**Corollary 3.5.** If \( \gamma \notin R_\lambda \) then \( e_\alpha \) and \( e_{-\alpha} \) act injectively on \( M(\lambda, \gamma) \).

**Proof.** If \( \gamma \notin R_\lambda \) then Spec \( z(n) \cap R_{\lambda-n}\beta} = \emptyset \) for all integer \( n \geq 0 \) by Proposition 3.4 and, therefore, \( e_\alpha \) and \( e_{-\alpha} \) act injectively on \( M(\lambda, \gamma) \). \( \square \)

Fix \( \rho \in H^* \) such that \( (\rho, \alpha) = 1, (\rho, \delta) = 2 \). Since \( M(\lambda, \gamma) \) is a restricted module, i.e. for every \( v \in M(\lambda, \gamma) \), \( G\varphi v = 0 \) for all but a finite number of positive roots \( \varphi \), we have well-defined action of a generalized Casimir operator \( \Omega \) on \( M(\lambda, \gamma) \) [4]:

\[
\Omega v = (\mu + 2\rho, \mu)v + 2 \sum_{\varphi \in \Delta_+} e_{-\varphi}e_\varphi v, \ v \in M(\lambda, \gamma)_\mu,
\]

where \( e_{-\varphi} \in G_{-\varphi} \), \( (e_{-\varphi}, e_\varphi) = 1, \varphi \in \Delta_+ \). Set \( \tilde{\Omega} = 2\Omega \) id.

Let \( s_\alpha \in W, s_\alpha(\mu) = \mu - (\mu, \alpha)\alpha, \mu \in H^* \).

**Lemma 3.6.** For a \( G \)-module \( M(\lambda, \gamma) \)

\[
\tilde{\Omega} = [(\lambda + 2\rho + s_\alpha(\lambda + 2\rho), \lambda) + \gamma] \text{id}.
\]

**Proof.** Follows from [4, Th.2.6] and definition of \( \tilde{\Omega} \). \( \square \)

**Lemma 3.7.** Let \( n > 0, \beta = \delta - \alpha, 0 \neq v \in M(\lambda, \gamma)_{\lambda-n\beta}, \gamma \neq (\lambda(h_{\alpha}) + 2\ell + 1)^2 \) for all \( 0 \leq \ell < 2n \) and \( N_\alpha^+v = 0 \) Then \( k^2\gamma = (n(\lambda(c) + 2) - k^2)^2 \) for some \( k \in \mathbb{Z}, 0 \leq k \leq n \).

**Proof.** It follows from Lemma 3.6 that \( z(n)v = \gamma'v \) and

\[
(\lambda - n\beta + 2\rho + s_\alpha(\lambda - n\beta + 2\rho), \lambda - n\beta) + \gamma' = (\lambda + 2\rho + s_\alpha(\lambda + 2\rho), \lambda) + \gamma
\]

which implies

\[
\gamma' = \gamma + 4n(\lambda(c) + 2).
\]

But, \( \gamma' = (2k \pm \sqrt{\gamma})^2 \) for some \( k \in \mathbb{Z}, 0 \leq k \leq n \) by Proposition 3.4. Therefore, \( k^2\gamma = (n(\lambda(c) + 2) - k^2)^2 \) which completes the proof. \( \square \)

**Corollary 3.8.** Let \( \lambda \in H^*, \gamma \in C - R_\lambda \). If \( k^2\gamma \neq (n(\lambda(c) + 2) - k^2)^2 \) for all \( n, k \in \mathbb{Z}, n > 0, 0 \leq k \leq n \) then \( G \)-module \( M(\lambda, \gamma) \) irreducible.

**Proof.** If the \( G \)-module \( M(\lambda, \gamma) \) has a non-trivial submodule \( M \), then \( M \) contains a non-zero vector \( v \) of weight \( \lambda - n(\delta - \alpha) \), \( n > 0 \), such that \( N_\alpha^+v = 0 \). Now, the statement follows from Lemma 3.7. \( \square \)
Consider the following decomposition of $\mathcal{U}(G)$:

$$\mathcal{U}(G) = (\mathcal{N}_-\mathcal{U}(G) + \mathcal{U}(G)\mathcal{N}_+^+) \oplus T_\alpha \mathbb{C}[e_\alpha]e_\alpha \oplus T_\alpha \mathbb{C}[e_{-\alpha}]e_{-\alpha} \oplus T_\alpha.$$ 

Let $j$ be the projection of $\mathcal{U}(G)$ to $T_\alpha$. Introduce the generalized Shapiro form $P$, a symmetric bilinear form on $\mathcal{U}(G)$ with values in $T_\alpha$, as follows (cf. [11]):

$$F(x,y) = j(\sigma(x)y), \quad x,y \in \mathcal{U}(G).$$

The algebra $\mathcal{U}(G)$ is $Q$-graded: $\mathcal{U}(G) = \mathcal{N}_-^Q \oplus \mathcal{U}(G)\mathcal{N}_+^Q$. It is clear that $F(\mathcal{U}(G)_{\eta_1}, \mathcal{U}(G)_{\eta_2}) = 0$ if $\eta_1 \neq \eta_2$. Denote $\mathcal{U}(N_-) = \mathcal{U}(G)\mathcal{N}_-^Q$ and let $F_\eta$ be a restriction of $F$ to $\mathcal{U}(N_-)_{-\eta}$.

For $\lambda \in H^*$, $\gamma \in \mathbb{C}$, consider the linear map $\theta_{\lambda,\gamma} : T_\alpha \to \mathbb{C}$ defined by $\theta_{\lambda,\gamma}(h \otimes z^n) = h(\lambda)\gamma^n$ for any $h \in S(H)$, $n \in \mathbb{Z}_+$. 

Set $\lambda_k = \lambda + k\alpha$, $k \in \mathbb{Z}$. Let $\mu = \lambda - n(\delta - \alpha) \in P(M(\lambda, \gamma))$, $n \in \mathbb{Z}_+$ and $\gamma \neq (h(\alpha) + 2s + 1)^2$ for all integer $s$, $0 \leq s < 2n$. Then $\lambda_{2n} \in P(M(\lambda, \gamma))$, $M(\lambda, \gamma)_{\lambda_{2n}} = \mathcal{C}v_n$ and $M(\lambda, \gamma)_{\mu} = \mathcal{U}(N_-)_{-n(\alpha + \delta)}v_n$. Set $F^{(n)} = F_{n(\alpha + \delta)}$. We define a a bilinear $\mathbb{C}$-valued form $F^0_{\mu}$ on $M(\lambda, \gamma)_{\mu}$ as follows:

$$F^0_{\mu}(u_1v_n, u_2v_n) = \theta_{\lambda_{2n},\gamma}(F^{(n)}(u_1, u_2)), \quad u_1, u_2 \in \mathcal{U}(N_-)_{-n(\alpha + \delta)}.$$ 

One can see that $\dim L(\lambda, \gamma)_{\mu} = \text{rank } F^0_{\mu}$.

**Lemma 3.9.** Let $\lambda \in H^*$, $\gamma \in \mathbb{C} - R_{\lambda}$. The following conditions are equivalent:

(i) $M(\lambda, \gamma)$ is irreducible.

(ii) $F^0_{\lambda - n(\delta - \alpha)}$ is non-degenerate for all integers $n > 0$.

(iii) $\theta_{\lambda_{2n},\gamma}(\det F^{(n)}) \neq 0$ for all integers $n > 0$.

**Proof.** Follows from the Corollary 3.5. □

Consider in $T_\alpha$ the following polynomials: $f_{m,k} = k^2z - (m(c + 2) - k^2)^2$, $g_s = z - (h(\alpha) + 2s + 1)^2$, $s, m, k \in \mathbb{Z}$, $0 \leq k \leq m$. Lemma 3.7 implies that if $\theta_{\lambda,\gamma}(g_s) \neq 0$ for all $s \in \mathbb{Z}$, $0 \leq s < 2n$ and $\theta_{\lambda_{2m},\gamma}(f_{m,k}) \neq 0$ for all $m, k \in \mathbb{Z}$, $0 < m \leq n$, $0 \leq k \leq m$, then $M(\lambda, \gamma)_{\lambda - n(\delta - \alpha)} = L(\lambda, \gamma)_{\lambda - n(\delta - \alpha)}$ and $\theta_{\lambda_{2m},\gamma}(\det F^{(n)}) \neq 0$. We conclude that the polynomial $\det F^{(n)}$ is not identically equal to zero and has its zeros in the union of zeros of polynomials $f_{m,k}, 0 < m \leq n, 0 \leq k \leq m, g_s, 0 \leq s \leq 2n$. Therefore, $\det F^{(n)}$ is a product of factors of type $f_{m,k}$ and $g_s$.

**Lemma 3.10.** Let $n, m \in \mathbb{Z}$, $n \geq 0$, $0 < m \leq n$. Then $f_{m,k}$ is a factor of $\det F^{(n)}$ if and only if $k$ is a divisor of $m$ or $k = 0$.

**Proof.** Assume that $k$ is a divisor of $m$ or $k = 0$. Set $r = 2n + 2m + k$. Consider $\lambda \in H^*$ and $\gamma \in \mathbb{C} - \mathbb{Z}$ such that $\theta_{\lambda,\gamma}(f_{m,k}) = \theta_{\lambda,\gamma}(g_r) = 0$. For integer $s \geq 0$
set \( v_s = \lambda - s\alpha \). Then \( \theta_{\nu_s, \gamma}(f_{m,k}) = \theta_{\nu_s, \gamma}(g_{r+s}) = 0 \) and \( \nu_s(h_{\alpha}) \notin \mathbb{Z} \), which implies that \( \theta_{\nu_s, \gamma}(g_{\ell}) \neq 0 \) for all \( \ell \in \mathbb{Z} \), \( \ell < r+s \). Thus, the form \( F_{\nu_s-i\beta}^0 \), \( \beta = \delta - \alpha \) is defined for all \( s \geq 0 \), \( 0 < \ell \leq n \) and \( M(\nu_s, \gamma) \simeq M(\lambda_r) \), \( s \geq 0 \) by Proposition 3.2, (iii), where \( M(\lambda_r) \) is the Verma module with highest weight \( \lambda_r = \lambda + r\alpha \). Therefore, \( M(\nu_s, \gamma) \simeq M(\lambda_r) \) as \( T_\alpha \)-modules. The operator \( z(m) \) has eigenvectors \( w^+, w^- \in M(\lambda_r) \) with eigenvalues \( \gamma^+ = (\lambda(h_{\alpha}) + 4(n+m+k)+1)^2 \) and \( \gamma^- = (\lambda(h_{\alpha}) + 4(n+m)+1)^2 \) respectively. Since \( \theta_{\nu_s, \gamma}(f_{m,k}) = 0 \), then

\[
\gamma^* = \gamma + 4m(\lambda(c) + 2) \in \{\gamma^+, \gamma^-\}
\]

and

\[
(v_s+2\rho+s_\alpha(v_s+2\rho), v_s) + \gamma = (v_s-m\beta+2\rho+s_\alpha(v_s-m\beta+2\rho), v_s-m\beta) + \gamma^*.
\]

Let \( w^*_s \in \{w^+_s, w^-_s\} \) and \( z(m)w^*_s = \gamma^*w^*_s \). Then

\[
\tilde{\Omega}w^*_s = [(v_s-m\beta+2\rho+s_\alpha(v_s-m\beta+2\rho), v_s-m\beta) + \gamma^*]w^*_s
\]

by Lemma 3.6. But, \( w^*_s \in M(\lambda_r) \) and

\[
\tilde{\Omega}w^*_s = (2(\lambda_r+2\rho, \lambda_r)+1)w^*_s
\]

by Corollary 2.6 in [4]. Hence

\[
2(\lambda_r+2\rho, \lambda_r) + 1 = (v_s-m\beta+2\rho+s_\alpha(v_s-m\beta+2\rho), v_s-m\beta) + \gamma^*
\]

and

\[
(\lambda_r+2\rho, \lambda_r) = (\lambda_r+2\rho-\tau^*, \lambda_r-\tau^*)
\]

where \( \tau^* = m\delta - k\alpha \) if \( \gamma^* = \gamma^+ \) and \( \tau^* = m\delta + k\alpha \) if \( \gamma^* = \gamma^- \). If \( k \) divides \( m \) or \( k = 0 \) then \( \tau^* \) is a quasiroot and \( D = Hom_\mathfrak{g}(M(\lambda_r-\tau^*), M(\lambda_r)) \neq 0 \) [10, Prop. 4.1].

Let \( 0 \neq \chi \in D \). Then \( \chi(M(\lambda_r-\tau^*)) \cap M(\lambda_r)_{v_s-n\beta} \neq 0 \) and therefore, \( \theta_{\lambda_{2n-s}, \gamma}(\text{det } F^{(n)}) = 0 \) for any integer \( s \geq 0 \). It implies that if \( \lambda \in H^* \), \( \gamma \in \mathbb{C} - \mathbb{Z} \) and \( \theta_{\lambda, \gamma}(f_{m,k}) = 0 \) then \( \theta_{\lambda, \gamma}(\text{det } F^{(n)}) = 0 \). Thus, \( f_{m,k} \) is a factor of \( \text{det } F^{(n)} \). Conversely, suppose that \( f_{n,k} \) is a factor of \( \text{det } F^{(n)} \), \( k \neq 0 \) and \( k \) is not a divisor of \( n \). Let \( r = 4n + k \). Consider a pair \( (\lambda, \gamma) \in H^* \times (\mathbb{C} - \mathbb{Z}) \) such that \( \theta_{\lambda, \gamma}(f_{n,k}) = \theta_{\lambda, \gamma}(g_r) = 0 \) but \( \theta_{\lambda, \gamma}(f_{p,q}) \neq 0 \) for all \( 0 < p < n \), \( 0 \leq q \leq p \) (such \( \lambda \) and \( \gamma \) always exist). Then \( \theta_{\lambda, \gamma}(\text{det } F^{(n)}) = 0 \) and the Verma module \( M(\lambda_r) \) has an irreducible subquotient with highest weight \( \lambda_r - \tau^* \), where \( \tau^* \) is one of \( n\delta + k\alpha, n\delta - k\alpha \). But, this contradicts the Theorem 2 in [10]. Therefore, \( f_{n,k} \) can not be a factor of \( \text{det } F^{(n)} \) if \( k \neq 0 \) and \( k \) is not a divisor of \( n \).
Let now $0 < m < n$, $0 < k < m$, $k$ is not a divisor of $m$ and $f_{m,k}$ is a factor of $\det F^{(n)}$. Consider a pair $(\lambda, \gamma) \in H^* \times \mathbb{C}$ such that $\theta_{\lambda,\gamma}(f_{m,k}) = 0$, $\theta_{\lambda,\gamma}(f_{p,q}) \neq 0$ for all $p, q \in \mathbb{Z}$, $0 < p \leq n$, $0 \leq q \leq p$, $(p,q) \neq (m,k)$ and $\theta_{\lambda,\gamma}(g_s) \neq 0$ for all $s \in \mathbb{Z}$. As it was shown above $f_{m,k}$ is not a factor of $\det F^{(m)}$ which implies that $\theta_{\lambda,\gamma}(\det F^{(m)}) \neq 0$. Now it follows from Lemma 3.7 that $M(\lambda, \gamma)_{\lambda - n\beta} = L(\lambda, \gamma)_{\lambda - n\beta}$ and $\theta_{\lambda,\gamma}(\det F^{(n)}) \neq 0$. But, this contradicts the assumption that $f_{m,k}$ is a factor of $\det F^{(n)}$. The Lemma is proved. \(\square\)

For $n \in \mathbb{Z}$, $n > 0$ denote $X_n = \{0\} \cup \{k \in \mathbb{Z}_+ \mid \frac{n}{k} \in \mathbb{Z}\}$.

**Theorem 3.11.** Let $\lambda \in H^*$, $\gamma \in \mathbb{C} - R_\lambda$. $G$-module $M(\lambda, \gamma)$ is irreducible if and only if $k^2\gamma \neq (n(\lambda(c) + 2) - k^2)^2$ for all $n \in \mathbb{Z}$, $n > 0$, $k \in X_n$.

**Proof.** Follows from Lemmas 3.9 and 3.10. \(\square\)

### 4. Irreducible representations of the Heisenberg subalgebra.

Consider the Heisenberg subalgebra $G = \mathbb{C}c \oplus \sum_{k \in \mathbb{Z}} G_{k\delta} \subset G$. It is a \(\mathbb{Z}\)-graded algebra with $\deg c = 0$, $\deg e_{k\delta} = k$. This gradation induces a \(\mathbb{Z}\)-gradation on the universal enveloping algebra $\mathcal{U}(G) : \mathcal{U}(G) = \bigoplus_{i \in \mathbb{Z}} \mathcal{U}_i$.

In this section we study the irreducible \(\mathbb{Z}\)-graded $G$-modules. The central element $c$ acts as a scalar on each such module. In general, we say that a $G$-module $V$ is a module of level $a \in \mathbb{C}$ if $c$ acts on $V$ as a multiplication by $a$.

#### 4.1. $G$-Modules of non-zero level.

Let $G_+ = \sum_{k > 0} G_{k\delta}$, $G_- = \sum_{k < 0} G_{k\delta}$. For $a \in \mathbb{C}^* = \mathbb{C} - \{0\}$, let $\mathbb{C}v_a$ be the 1-dimensional $G_\varepsilon \oplus \mathbb{C}c$-module for which $G_\varepsilon v_a = 0$, $cv_a = av_a$, $\varepsilon \in \{+,-\}$. Consider the $G$-module

$$M^\varepsilon(a) = \mathcal{U}(G) \bigotimes_{\mathcal{U}(G_\varepsilon \oplus \mathbb{C}c)} \mathbb{C}v_a$$

associated with $a$ and $\varepsilon$.

The module $M^\varepsilon(a)$ is a \(\mathbb{Z}\)-graded: $M^\varepsilon(a) = \sum_{i \in \mathbb{Z}} M^\varepsilon(a)_i$ where

$$M^\varepsilon(a)_i = (\sigma(\mathcal{U}(G_\varepsilon))) \cap \mathcal{U}_i \otimes v_a.$$

**Proposition 4.1.**

(i) The $G$-module $M^\varepsilon(a)$ is irreducible.

(ii) $M^\varepsilon(a)$ is a $\sigma(\mathcal{U}(G_\varepsilon))$-free module.
(iii) \( \dim M^\varepsilon(a)_i = P(|i|) \) where \( P(n) \) is a partition function.

**Proof.** (ii) and (iii) follow directly from the definition of \( M^\varepsilon(a) \). Since \( a \neq 0 \) one can easily show that for any non-zero \( u \in \sigma(U(G_\varepsilon)) \) there exists \( u' \in U(G_\varepsilon) \) such that \( 0 \neq u'u_\varepsilon \in M^\varepsilon(a)_0 \) which implies (i) and completes the proof. \( \square \)

**Lemma 4.2.** If \( V \) is a \( \mathbb{Z} \)-graded \( G \)-module of level \( a \in \mathbb{C}^* \) and \( \dim V_i < \infty \) for at least one \( i \in \mathbb{Z} \) then

\[
\text{Spec } e_\delta e_{-\delta} |_V \subset \{2ma \mid m \in \mathbb{Z} \}.
\]

**Proof.** Let \( v \in V_j \) be a non-zero eigenvector of \( e_\delta e_{-\delta} \) with eigenvalue \( b \) and \( b \neq 2ma \) for all \( m \in \mathbb{Z} \). Since \( a \neq 0 \), if \( e_n \delta v = 0 \) then \( e_{-n} \delta v \neq 0 \), \( n \in \mathbb{Z} - \{0\} \). Denote \( Y = \{n \in \mathbb{Z} - \{0, 1\} \mid e_n \delta v \neq 0 \} \). We may assume without lost of generality that \( j = i \) and \( |Y \cap \mathbb{Z}_+| = \infty \). Elements \( e_\delta \) and \( e_{-\delta} \) act injectively on the subspace spanned by \( e_\delta^k v, e_{-\delta}^k v, k \in \mathbb{Z} \). Then, for each \( k \in Y \cap \mathbb{Z}_+ \), \( e_\delta e_{-\delta}(e_{k_\delta} v) = be_{k_\delta} v \) and \( 0 \neq e_{k_\delta} e_{k_\delta} v \in V_i \). Set \( w_k = e_{k_\delta} e_{k_\delta} v \). Then \( e_\delta e_{-\delta} w_k = (b + 2ka)w_k \), \( k \in Y \cap \mathbb{Z}_+ \). This contradicts the assumption that \( \dim V_i < \infty \). Therefore, \( b = 2ma \) for some \( m \in \mathbb{Z} \). \( \square \)

For a \( \mathbb{Z} \)-graded \( G \)-module \( V \) and \( j \geq 0 \) denote by \( V^{[j]} \) the \( \mathbb{Z} \)-graded \( G \)-module with \( (V^{[j]})_i = V_{i-j}, i \in \mathbb{Z} \).

We describe now all irreducible \( \mathbb{Z} \)-graded \( G \)-modules of non-zero level with finite-dimensional components.

**Proposition 4.3.**

(i) Let \( V \) be an irreducible \( \mathbb{Z} \)-graded \( G \)-module of level \( a \in \mathbb{C}^* \) such that \( \dim V_i < \infty \) for at least one \( i \in \mathbb{Z} \). Then \( V^{[j]} \simeq M^\varepsilon(a) \) for some \( \varepsilon \in \{+, -\}, j \in \mathbb{Z} \).

(ii) \( \text{Ext}^1((M^\varepsilon(a))^{[j]}, M^\varepsilon'(a)) = 0 \) for any \( j \in \mathbb{Z}, \varepsilon, \varepsilon' \in \{+, -\} \).

**Proof.** (i) By Lemma 4.2 \( \text{Spec } X |_V \subset \{2ma \mid m \in \mathbb{Z} \} \) where \( X \) stands for \( e_\delta e_{-\delta} \). Let \( V_i \neq 0 \), \( n \) be an integer with maximal absolute value such that \( 2na \in \text{Spec } X |_{V_i} \) and let \( 0 \neq v \in V_i \), \( Xv = 2nav \). Assume that \( n > 0 \). Then \( e_{k_\delta} v = 0 \) for all \( k > 1 \). Indeed, if \( e_{k_\delta} v \neq 0 \) for some \( k > 1 \) then \( X(e_{k_\delta} v) = e_{k_\delta} Xv = 2na e_{k_\delta} v \) and \( 2(n + k)a \) is an eigenvalue of \( X \) on \( V_i \) which contradicts the assumption. Therefore, \( e_{k_\delta} v = 0 \) for all \( k > 1 \). Consider the element \( \tilde{v} = e_{\delta}^{n-1} v \neq 0 \). Then \( e_{-\delta} e_\delta \tilde{v} = e_{k_\delta} \tilde{v} = 0, k > 1 \). If \( e_\delta \tilde{v} \neq 0 \) then \( v_p = e_\delta^p \tilde{v} \neq 0, e_{k_\delta} v_p = 0 \) and, hence \( e_{-\delta} e_\delta v_p \neq 0 \) for all \( p > 0, k > 1 \). This would imply that \( \dim V_i = \infty \). Therefore, \( e_\delta \tilde{v} = 0 \) and \( V = U(G) \tilde{v} \simeq M^+(a) \) up to a shifting of gradation. If \( n \leq 0 \) then, clearly,
$V \simeq M^-(a)$ up to a shifting of gradation. Suppose that $V_i = 0$ but, for example, $V_{i-1} \neq 0$. Then $e_{k\delta}v = 0$ for any non-zero $v \in V_{i-1}$ for all $k > 0$ and thus $V = \mathcal{U}(G)v \simeq M^+(a)$ up to a shifting of gradation. This completes the proof of (i).

(ii) Follows from the proof of (i) and Proposition 4.1, (ii). \hfill \Box 

**Lemma 4.4.** Every finitely-generated $\mathbb{Z}$-graded $G$-module $V$ of level $a \in \mathbb{C}^*$ such that $\dim V_i < \infty$ for at least one $i \in \mathbb{Z}$ has a finite length.

**Proof.** If $V_i = 0$ then statement follows from Proposition 4.3. Let $V_i \neq 0$, $n$ be an integer with maximal absolute value such that $2na \in \text{Spec } e\delta e_{-\delta} |_{V_i}$ and $v$ be a corresponding eigenvector. It follows from the proof of Proposition 4.3, (i) that $V' = \mathcal{U}(G)v \simeq M^\varepsilon(a)$ up to a shifting of gradation. Consider a $G$-module $\bar{V} = V/V'$. Then $\dim \bar{V}_i < \dim V_i$ and we can complete the proof by induction on $\dim \bar{V}_i$. \hfill \Box 

Now we are in the position to establish the completely reducibility for for finitely-generated $G$-modules of non-zero level with finite-dimensional components.

**Proposition 4.5.** Every finitely-generated $\mathbb{Z}$-graded $G$-module $V$ of a non-zero level such that $\dim V_i < \infty$ for at least one $i \in \mathbb{Z}$ is completely reducible.

**Proof.** Follows from Lemma 4.4 and Proposition 4.3. \hfill \Box 

4.2. $G$-modules of level zero. The irreducible $G$-modules of level zero are classified by V. Chari [6]. We recall this classification.

Let $\tilde{G} = \mathcal{U}(G)/\mathcal{U}(G)c$ and let $g : \mathcal{U}(G) \rightarrow \tilde{G}$ be the canonical homomorphism. For $r > 0$ consider a $\mathbb{Z}$-graded ring $L_r = \mathbb{C}[t^r, t^{-r}]$, $\deg t = 1$ and denote by $P_r$ the set of graded ring epimorphisms $\Lambda : \tilde{G} \rightarrow L_r$ with $\Lambda(1) = 1$. Let $L_0 = \mathbb{C}$ and $\Lambda_0 : \tilde{G} \rightarrow \mathbb{C}$ is a trivial homomorphism such that $\Lambda_0(1) = 1$, $\Lambda_0(g(e_{k\delta})) = 0$ for all $k \in \mathbb{Z} - \{0\}$. Set $P_0 = \{\Lambda_0\}$.

Given $\Lambda \in P_r$, $r \geq 0$ define a $G$-module structure on $L_r$ by:

$$e_{k\delta}t^{rs} = \Lambda(g(e_{k\delta}))t^{rs}, \quad k \in \mathbb{Z} - \{0\}, \quad ct^{rs} = 0, \ s \in \mathbb{Z}.$$

Denote this $G$-module by $L_{r,\Lambda}$.

**Proposition 4.6.**

(i) Let $V$ be an irreducible $\mathbb{Z}$-graded $G$-module of level zero. Then $V \simeq L_{r,\Lambda}$ for some $r \geq 0$, $\Lambda \in P_r$ up to a shifting of gradation.

(ii) $L_{r,\Lambda} \simeq L_{r',\Lambda'}$ if and only if $r = r'$ and there exists $b \in \mathbb{C}^*$ such that $\Lambda(g(e_{k\delta})) = b^k \Lambda'(g(e_{k\delta}))$, $k \in \mathbb{Z} - \{0\}$. 

Proof. (i) is essentially Lemma 3.6 in [6]; (ii) follows from [6, Prop. 3.8].

Remark 4.7. All the results of Section 4, except Proposition 4.1 (iii), are hold for the Heisenberg subalgebra of an arbitrary Affine Lie Algebra.

5. The category $\tilde{O}(\alpha)$.

Let $\alpha \in \pi$. Following [6] we define category $\tilde{O}(\alpha)$ to be the category of weight $\mathcal{G}$-modules $M$ satisfying the condition that there exist finitely many elements $\lambda_1, \ldots, \lambda_r \in H^*$ such that $P(M) \subseteq \bigcup_{i=1}^r D(\lambda_i)$ where

$$D(\lambda_i) = \{ \lambda_i + k\alpha + n\delta \mid k, n \in \mathbb{Z}, k \leq 0 \}.$$

Notice that the trivial action of $c$, as in [6], is no longer required. It is clear that $\tilde{O}(\alpha)$ is closed under the operations of taking submodules, quotients and finite direct sums.

Denote $B_{\alpha} = \sum_{n \in \mathbb{Z}} \mathcal{G}_{\alpha+n\delta}$. Then $\mathcal{G} = B_{-\alpha} \oplus (H + G) \oplus B_{\alpha}$.

Let $V$ be an irreducible $\mathbb{Z}$-graded $G$-module of level $a \in \mathbb{C}$ and let $\lambda \in H^*$, $\lambda(c) = a$. Then we can define a $B = (H + G) \oplus B_{\alpha}$-module structure on $V$ by setting: $hv_i = (\lambda + i\delta)(h)v_i$, $B_{\alpha}v_i = 0$ for all $h \in H$, $v_i \in V_i$, $i \in \mathbb{Z}$.

Consider the $\mathcal{G}$-module

$$M_\alpha(\lambda, V) = \mathcal{U}(\mathcal{G}) \bigotimes_{\mathcal{U}(B)} V$$

associated with $\alpha, \lambda, V$.

Proposition 5.1.

(i) The $\mathcal{G}$-module $M_\alpha(\lambda, V)$ is $S(B_{-\alpha})$-free.

(ii) $M_\alpha(\lambda, V)$ has a unique irreducible quotient $L_\alpha(\lambda, V)$.

(iii) $P(M_\alpha(\lambda, V)) = (D(\lambda) - \{ \lambda + n\delta \mid n \in \mathbb{Z} \}) \cup P(V) \subset D(\lambda)$.

(iv) $M_\alpha(\lambda, V) \simeq M_{\alpha'}(\lambda', V')$ if and only if $\alpha' \in \{ \alpha + n\delta \mid n \in \mathbb{Z} \}$ and there exists $i \in \mathbb{Z}$ such that $\lambda = \lambda' + i\delta$ and $V^{[i]} \simeq V'$ as graded $G$-modules.

Proof. Follows from the construction of $\mathcal{G}$-module $M_\alpha(\lambda, V)$. $\square$

Now we describe the classes of isomorphisms of irreducible modules in $\tilde{O}(\alpha)$.

Proposition 5.2.

(i) Let $\tilde{V}$ be an irreducible object in $\tilde{O}(\alpha)$. Then there exist $\lambda \in H^*$ and an irreducible $G$-module $V$ such that $\tilde{V} \simeq L_\alpha(\lambda, V)$. 

Proof. (i) is essentially Lemma 3.6 in [6]; (ii) follows from [6, Prop. 3.8].

Remark 4.7. All the results of Section 4, except Proposition 4.1 (iii), are hold for the Heisenberg subalgebra of an arbitrary Affine Lie Algebra.
(ii) \( L_\alpha(\lambda, V) \simeq L_\alpha(\lambda', V') \) if and only if there exists \( i \in \mathbb{Z} \) such that \( \lambda = \lambda' + i\delta \) and \( V[i] \simeq V' \) as graded \( G \)-modules.

**Proof.** One can see that \( \tilde{V} \) contains a non-zero element \( v \in \tilde{V}_\lambda \) such that \( B_\lambda v = 0 \). Then \( V = \mathcal{U}(G)v \) is an irreducible \( \mathbb{Z} \)-graded \( G \)-module and \( \tilde{V} \simeq \mathcal{U}(B_{-\alpha})V \). This implies that \( \tilde{V} \) is a homomorphic image of \( M_\alpha(\lambda, V) \) and, therefore, is isomorphic to \( L_\alpha(\lambda, V) \), which proves (i). Part (ii) follows from Proposition 5.1, (iv). \( \square \)

**Lemma 5.3.** If \( 0 < \dim L_\alpha(\lambda, V)_\mu < \infty \) for some \( \mu \in H^* \) then \( \dim V_i < \infty \) for all \( i \in \mathbb{Z} \).

**Proof.** If \( \lambda(c) = 0 \) then \( V[j] \simeq L_{r,\Lambda} \) for some \( r \geq 0, \Lambda \in P_r, j \in \mathbb{Z} \) by Proposition 4.6 and, hence \( \dim V_i \leq 1 \) for all \( i \in \mathbb{Z} \). Let \( \lambda(c) = a \in \mathbb{C}^* \) and \( V[j] \simeq M^\varepsilon(a) \), for any \( j \in \mathbb{Z}, \varepsilon \in \{+,-\} \). By Proposition 4.3, (i), \( \dim V_i = \infty \) for all \( i \). If \( a \in Q_+ \) (\( a \notin Q_+ \) respectively) then \( \lambda(h_\alpha) - na \notin \mathbb{Z}_+ \) for all integer \( n \geq n_0 \) (\( n \leq n_0 \) respectively) and for some \( n_0 \in \mathbb{Z} \). Thus, \( e_{\alpha-n\delta}\varepsilon_{\alpha+n\delta} \) acts injectively on \( L_\alpha(\lambda, V) \) for all \( n \geq n_0 \) (\( n \leq n_0 \) respectively) which implies that \( \dim L_\alpha(\lambda, V)_\mu = \infty \). But, this contradicts the assumption. We conclude that \( V[j] \simeq M^\varepsilon(a) \) for some \( j \in \mathbb{Z}, \varepsilon \in \{+,-\} \) and \( \dim V_i < \infty \) for all \( i \in \mathbb{Z} \). \( \square \)

**Theorem 5.4.** Let \( \tilde{V} \in \tilde{O}(\alpha) \) be an irreducible.

(i) \([6]\) If \( \tilde{V} \) is of level zero then \( \tilde{V} \simeq L_\alpha(\lambda, L_{r,\Lambda}) \) for some \( \lambda \in H^* \), \( \lambda(c) = 0, r \geq 0, \Lambda \in P_r \).

(ii) If \( \tilde{V} \) is of level \( a \in \mathbb{C}^* \) and \( \dim \tilde{V}_\mu < \infty \) for at least one \( \mu \in P(\tilde{V}) \) then \( \tilde{V} \simeq L_\alpha(\lambda, M^\varepsilon(a)) \) for some \( \lambda \in H^* \), \( \lambda(c) = a, \varepsilon \in \{+,-\} \).

**Proof.** (i) follows from Propositions 5.2 and 4.6, while (ii) follows from Lemma 5.3, Propositions 5.2 and 4.3. \( \square \)

In some cases we can describe the structure of modules \( L_\alpha(\lambda, V) \).

Let \( \lambda(c) = 0, r = 0, \Lambda = \Lambda_0, L_{0,\Lambda_0} \simeq \mathbb{C} \). Set \( \tilde{M}(\lambda) = M_\alpha(\lambda, \mathbb{C}) \). Notice that \( \tilde{M}(\lambda) \simeq S(B_{-\alpha}) \) as vector spaces and, therefore, \( P(\tilde{M}(\lambda)) = \{\lambda - n\alpha + k\delta \mid k, n \in \mathbb{Z}, n > 0\} \cup \{\lambda\} \) and

\[
\dim \tilde{M}(\lambda)_{\lambda-n\alpha+k\delta} = \infty, n > 1, \dim \tilde{M}(\lambda)_\lambda = \dim \tilde{M}(\lambda)_{\lambda-\alpha+k\delta} = 1, k \in \mathbb{Z}.
\]

**Proposition 5.5.**

(i) \( L_\alpha(\lambda, C) \simeq \tilde{M}(\lambda) \) if and only if \( \lambda(h_\alpha) \neq 0 \).

(ii) If \( \lambda(h_\alpha) = 0 \) then \( L_\alpha(\lambda, C) \) is a trivial one-dimensional module.

**Proof.** Proposition follows from \([7, \text{Proposition 6.2}]\) and is also proved in \([8]\). \( \square \)
Let $\lambda(c) = a \in \mathbb{C}^*$. Set $M^\varepsilon(\lambda, a) = M_\alpha(\lambda, M^\varepsilon(a))$. We have

$$P(M^\varepsilon(\lambda, a)) = \{\lambda - k\alpha + n\delta \mid k, n \in \mathbb{Z}, k > 0\} \cup \{\lambda - \varepsilon n\delta \mid n \in \mathbb{Z}_+\}$$

and

$$\dim M^\varepsilon(\lambda, a)_{\lambda - k\alpha + n\delta} = \infty, k > 0, n \in \mathbb{Z}, \dim M^\varepsilon(\lambda, a)_{\lambda - \varepsilon n\delta} = P(n), n \in \mathbb{Z}_+.$$

**Proposition 5.6.** [8, 9] $L_\alpha(\lambda, M^\varepsilon(a)) \simeq M^\varepsilon(\lambda, a)$.

Recall, that $G$-module $\tilde{V}$ is called *integrable* if $e_{\pm\alpha}$ and $e_{\pm(\delta - \alpha)}$ act locally nilpotently on $\tilde{V}$. All irreducible integrable $G$-modules in $\tilde{O}(\alpha)$ of level zero were classified in [6]. In fact, they are the only integrable modules in $\tilde{O}(\alpha)$.

**Corollary 5.7.** If $\tilde{V}$ is irreducible integrable $G$-module in $\tilde{O}(\alpha)$ then $\tilde{V}$ is of level zero.

**Proof.** Suppose $\tilde{V}$ is of level $a \neq 0$. Since $\tilde{V}$ is integrable, it follows from Proposition 5.6 that $\tilde{V} \neq L_\alpha(\lambda, M^\varepsilon(a)), \varepsilon \in \{+, -\}$. Then $\tilde{V} \simeq L_\alpha(\lambda, V)$ and for any $k \in \mathbb{Z}_+$ there exist $i > k, j < -k$ such that $V_i \neq 0, V_j \neq 0$. Now the same arguments as in the proof of Lemma 5.3 show that $e_{-\alpha}$ and $e_{\delta - \alpha}$ are not locally nilpotent on such module and, therefore, $\tilde{V}$ has a zero level. □

**Remark.** (i) The structure of modules $L_\alpha(\lambda, L_{r,\Lambda})$, $r > 0$ is unclear in general. Some examples were considered in [1, 12].

(ii) Most of the results of Section 5 can be generalized for an arbitrary Affine Lie Algebra [6, 7, 12].


**Definition.** An irreducible weight $G$-module $V$ is called dense if $P(V) = \lambda + Q$ for some $\lambda \in H^*$ and non-dense otherwise.

In this section we classify all irreducible non-dense $G$-modules with at least one finite-dimensional weight subspace. Our main result is the following Theorem.

**Theorem 6.2.** If $\tilde{V}$ is an irreducible non-dense $G$-module with at least one finite-dimensional weight subspace then $\tilde{V}$ belongs to one of the following disjoint classes:

(i) highest weight modules with respect to some choice of $\pi$;

(ii) $L_\alpha^e(\lambda, \gamma), \alpha \in \Delta^{re}, \lambda \in H^*, \gamma \in \mathbb{C} - R_\lambda, \varepsilon \in \{+, -\}$;

(iii) $L_\alpha(\lambda, L_{r,\Lambda}), \alpha \in \Delta^{re}, \lambda \in H^*, \lambda(c) = 0, r \geq 0, \Lambda \in P_r$. 
Moreover, we can describe the irreducible $G$-modules of non-zero level with finite-dimensional weight subspaces.

**Theorem 6.3.** Let $\tilde{V}$ be an irreducible $G$-module of level $a \neq 0$ with all finite-dimensional weight subspaces. Then $\tilde{V} \simeq L_\alpha^c(\lambda, \gamma)$ for some $\alpha \in \Delta^{re}, \lambda \in H^*, \lambda(c) = a, \gamma \in \mathbb{C}, \varepsilon \in \{+, -\}.$

**Remark 6.4.** Theorems 6.2, 6.3 imply that in order to complete the classification of all weight irreducible $G$-modules one has to study the following classes:

(i) Modules of type $L_\alpha(\lambda, V)$ where $V$ is a graded irreducible $G$-module of non-zero level with all infinite-dimensional components.

(ii) Dense $G$-modules of zero level.

(iii) Dense $G$-modules of non-zero level with an infinite-dimensional weight subspace.

These classification problems are still open.

The proof of Theorem 6.2 is based on some preliminary results. We start with the following Definition.

**Definition 6.5.** A subset $P \subset \Delta$ is called closed if $\beta_1, \beta_2 \in P, \beta_1 + \beta_2 \in \Delta$ imply $\beta_1 + \beta_2 \in P$. A closed subset $P \subset \Delta$ is called a partition if $P \cap -P = \emptyset, P \cup -P = \Delta$.

**Lemma 6.6.** Let $P$ be a partition, $P \ni \delta$, $P_* = P \cap \Delta^{re}, \beta \in \Delta^{re}$.

(i) If $|P_* \cap \{\beta + k\delta \mid k \in \mathbb{Z}_+\}| < \infty$ or $|P_* \cap \{-\beta + k\delta \mid k \in \mathbb{Z}\}| < \infty$, then $P_* = \{\varphi + n\delta \mid n \in \mathbb{Z}\}$ for some $\varphi \in \Delta^{re}$.

(ii) If $|P_* \cap \{\beta + k\delta \mid k \in \mathbb{Z}\}| = |P_* \cap \{-\beta + k\delta \mid k \in \mathbb{Z}_+\}| = \infty$, then $P = \Delta_+ (\tilde{\pi})$ for some basis $\tilde{\pi}$ of $\Delta$.

**Proof.** Recall that $\Delta = \{\pm \beta + k\delta \mid k \in \mathbb{Z}\} \cup \{n\delta \mid n \in \mathbb{Z} - \{0\}\}$. It follows from [7] that there exist $w \in W$ and $\beta' \in \Delta^{re}$ such that

\[ wP = \{\beta' + k\delta \mid k \in \mathbb{Z}\} \cup \{k\delta \mid k > 0\} \]

or

\[ wP = \{\beta' + n\delta, -\beta' + k\delta \mid n \geq 0, k > 0\} \cup \{k\delta \mid k > 0\} = \Delta_+(\pi') \]

where $\pi' = \{\beta', \delta - \beta'\}$. Then

\[ P = \{w^{-1}\beta' + k\delta \mid k \in \mathbb{Z}\} \cup \{k\delta \mid k > 0\} \]
Definition 6.7. A non-zero element $v$ of a $G$-module $V$ is called admissible if $N^\varphi v = 0$ or $B^\varphi v = 0$, for some $\varphi \in \Delta^r$, $\varepsilon \in \{+, -\}$.

Lemma 6.8. If the $G$-module $V$ contains a non-zero vector $v \in V_{\lambda}$ such that $e^\varphi v = 0$ and $\lambda + k\delta \notin P(V)$ for some $\varphi \in \Delta^r$, $k \in \mathbb{Z} - \{0\}$ then $V$ contains an admissible vector.

Proof. We will assume that $k > 0$. The case $k < 0$ can be considered analogously. We prove the Lemma by the induction on $k$. Let $k = 1$. Then we have $e^{\varphi + m\delta}v = e^\delta v = 0$ for all $m \geq 0$. If $e^{\varphi - i\delta}v = 0$ for all $i > 0$ then $B^\varphi v = 0$ and $v$ is admissible. Let $e^{\varphi - n\delta}v \neq 0$ for some $n > 0$ and $e^{\varphi - i\delta}v = 0$, $0 \leq i < n$. Set $\tilde{v} = e^{\varphi - n\delta}v \neq 0$. Then $e^{\varphi - i\delta}\tilde{v} = e^\delta\tilde{v} = e^{-\varphi + (n+1)\delta}\tilde{v} = 0$, $i < n$ and, thus, $e^{\varphi}\tilde{v} = 0$ for any $\psi \in \tilde{P} = \{\varphi - i\delta, -\varphi + (n + j + 1)\delta, (j + 1)\delta | i < n, j \geq 0\}$. One can see that $\tilde{P} \cup \{-\varphi + n\delta\}$ is a partition and $\tilde{P} = \Delta_+ (\tilde{\pi}) - \{\varphi'\}$ for some $\varphi' \in \Delta^r$, $\tilde{\pi} = \{\varphi', \delta - \varphi'\}$, by Lemma 6.6. Hence, $N^\varphi\tilde{v} = 0$ which proves the Lemma for $k = 1$.

Assume now that the Lemma is proved for all $0 < k' < k$ and consider two cases:

(i) There exists $n \in \mathbb{Z}$, $0 < n < k$ such that $e^{\varphi + i\delta}v = 0$ for all $0 \leq i < n$ but $e^{\varphi + m\delta}v \neq 0$. Then $e^{\varphi + i\delta}\tilde{v} = e^{-\varphi + (k-n)\delta}\tilde{v} = 0$, $0 \leq i < n$ where $\tilde{v} = e^{\varphi + n\delta}v$ and $e^{-\varphi + (k-n)\delta}\tilde{v} \in V_{\lambda - k\delta} = 0$. If $k - n = 1$ or $k - n > 1$ and $e^{-\varphi + \delta}\tilde{v} = 0$ then $N^+_\lambda v = 0$ and $\tilde{v}$ is admissible. Let $k - n > 1$ and $v' = e^{-\varphi + \delta}\tilde{v} \neq 0$. Then $v' \in V_{\lambda'}$, $e^{\varphi'}v' = 0$, $\lambda' + (k - n - 1)\delta \notin P(V)$ where $\lambda' = \lambda + (n + 1)\delta$, $\varphi' = -\varphi + (k - n)\delta$ and $V$ has an admissible element by the induction hypotheses.

(ii) Let $e^{\varphi + i\delta}v = 0$ for all $0 \leq i \leq k$. Since $e_k\delta v = 0$ we have $e^{\varphi + i\delta}v = 0$ for all $i \geq 0$. If $\tilde{v}_m = e^{m\delta}v \neq 0$ for some $0 < m < k$ then $\tilde{v}_m \in V_{\lambda'}$, $\lambda' = \lambda + m\delta$, $e^\varphi\tilde{v}_m = 0$, $\lambda' + (k - m)\delta \notin P(V)$ and we can apply induction. Assume that $\tilde{v}_m = 0$ for all $0 < m < k$. Then we have $e^{\varphi + i\delta}v = e^{m\delta}v = 0$, $i \geq 0$, $0 < m < k$. If $e^{\varphi - j\delta}v = 0$ for all $j > 0$ then $B^\varphi v = 0$ and $v$ is admissible. Otherwise, let $n$ be a minimal positive integer such that $\tilde{v} = e^{\varphi - n\delta}v \neq 0$. Then $e^{\varphi - j\delta}\tilde{v} = e^{-\varphi + (n+k)\delta}\tilde{v} = e_{i\delta}\tilde{v} = 0$, $i \geq 0$, $j < n$. Assume that $e^{-\varphi + (n+1)\delta}\tilde{v} = 0$. We have $e^\varphi\tilde{v} = 0$ for any $\psi \in \tilde{P} = \{\varphi - j\delta, -\varphi + (n + m)\delta, m\delta | j < n, m > 0\}$. The set $\tilde{P} \cup \{-\varphi + n\delta\}$ is a partition, $|\tilde{P}^r \cap \{\varphi + i\delta | i \geq 0\}| = 1 = |\tilde{P}^r \cap \{-\varphi + i\delta | i > 0\}| = \infty$ and, therefore, $\tilde{P} = \Delta_+ (\tilde{\pi}) - \{\varphi'\}$ for some $\varphi' \in \Delta^r$, $\tilde{\pi} = \{\varphi', \delta - \varphi'\}$ by Lemma 6.6. We conclude that $N^\varphi\tilde{v} = 0$ and $\tilde{v}$ is admissible. Finally, suppose that $v' = e^{-\varphi + (n+1)\delta}\tilde{v} \neq 0$. Then $v' \in V_{\lambda'}$, $e^\varphi v' = 0$, $\lambda' + (k - 1)\delta \notin P(V)$ where $\lambda'$ stands for $\lambda + \delta$ and, thus $V$ has an admissible element by the assumption of induction. This completes the proof of Lemma. \qed
Proposition 6.9. Let $V$ be an irreducible non-dense $\mathbb{G}$-module. Then $V$ contains an admissible element.

Proof. Let $\lambda \in P(V)$ and $\lambda + \varphi \not\in P(V)$ for some $\varphi \in \Delta$. We can assume that $\varphi \in \Delta^{re}$. Indeed, let $\varphi = \delta$. If $e_{\alpha}v = e_{\delta - \alpha}v = 0$ for some $0 \neq v \in V$, $\alpha \in \Delta^{re}$ then $V$ is a highest weight module with respect to $\{\alpha, \delta - \alpha\}$ and $v$ is admissible. If, for example, $e_{\alpha}v \neq 0$ then $\lambda' = \lambda + \alpha \in P(V)$ and $\lambda' + (\delta - \alpha) \not\in P(V)$. Hence, we can assume that $\lambda + \varphi \not\in P(V)$, $\varphi \in \Delta^{re}$. Let $0 \neq v \in V$. If $v' = e_{\varphi - n\delta}v \neq 0$ for some $n \in \mathbb{Z} - \{0\}$ then $e_{\varphi}v' = 0, v' \in V$, $\lambda = \lambda + \varphi - n\delta, \lambda + n\delta \not\in P(V)$ and Proposition follows from Lemma 6.8. If $e_{\varphi - n\delta}v = 0$ for all $n \in \mathbb{Z}$ then $B_{\varphi}v = 0$ and $v$ is admissible. 

Corollary 6.10. If $\tilde{V}$ is an irreducible non-dense $\mathbb{G}$-module then either $\tilde{V} \simeq L_{\alpha}(\lambda, \gamma)$ or $\tilde{V} \simeq L_{\alpha}(\lambda, V)$ for some $\alpha \in \Delta^{re}$, $\lambda \in H^*, \gamma \in \mathbb{C}, \epsilon \in \{+, -\}$ and irreducible $\mathbb{G}$-module $V$.

Proof. Follows from Proposition 6.9, Corollary 3.3 (i) and Proposition 5.2.

Now Theorem 6.2 follows from Corollary 6.6 and Theorem 5.4.

Proof of Theorem 6.3. Let $\mu \in P(\tilde{V})$. Consider the $\mathbb{G}$-submodule $V = \mathcal{U}(G)\tilde{V}_{\mu} \subset \tilde{V}$. Then it follows from Proposition 4.5 that $V$ is completely reducible and moreover each irreducible component is isomorphic to $M^{\epsilon}(a)$, $\epsilon \in \{+, -\}$ up to a shifting of gradation by Proposition 4.3, (i). Denote by $V^+$ the sum of all irreducible components of $V$ isomorphic to $M^{\epsilon}(a)$ and assume that $V^+ \neq 0$. Let $0 \neq v \in V^+ \cap \tilde{V}_{\chi}, \chi \in P(\tilde{V})$ and $V^+ \cap \tilde{V}_{\chi + \delta} = 0$. We will show that for any $\alpha \in \Delta^{re}$ there exists $m_{\alpha} \in \mathbb{Z}_+$ such that $e_{\alpha + m_{\delta}}v = 0$ for all $m \geq m_{\alpha}$. Indeed, let $v_0 = e_{\alpha}v \neq 0$. Consider the $\mathbb{G}$-module $\mathcal{U}(G)v_0$ which is again completely reducible by Proposition 4.5. If $e_{k\delta}v \neq 0$ for all $k > 0$ then $v_k = e_{\delta}^k v_0 \neq 0$ for all $k > 0$. But, for big enough $k$, $v_k$ will belong to the direct sum of irreducible components of $\mathcal{U}(G)v_0$ each of which is isomorphic to $M^{-}(a)$ up to a shifting of gradation. This contradicts Proposition 4.1, (ii), since $e_{\delta}^2 v_k = 2^{k+2} e_{\alpha + (k+2)\delta}v = 2e_{2\delta}v_k$. Thus, there exists $m_{\alpha} \geq 0$ such that $e_{\alpha + m_{\delta}}v = 0$ and, therefore, $e_{\alpha + m_{\delta}}v = 0$ for any $m \geq m_{\alpha}$.

Suppose that $\chi + \delta \in P(\tilde{V})$. Since $\tilde{V}$ is irreducible there exists $0 \neq u \in \mathcal{U}(G)$ such that $0 \neq uv \in \tilde{V}_{\chi + \delta}$. It follows from the discussion above that $e_{n\delta}uv = 0$ for big enough $n \in \mathbb{Z}_+$. The $\mathbb{G}$-submodule $V' = \mathcal{U}(G)uv$ is completely reducible by Proposition 4.5 and since $V^+ \cap \tilde{V}_{\chi + \delta} = 0$, any irreducible component $L \subset V'$ such that $L \cap \tilde{V}_{\chi + \delta} = 0$ is isomorphic to $M^{-}(a)$ up to a shifting of gradation. Hence, $e_{n\delta} \tilde{v} \neq 0$ for any non-zero $\tilde{v} \in V' \cap \tilde{V}_{\chi + \delta}$ by Proposition 4.1, (ii) and $e_{n\delta}uv \neq 0$ in particular. This contradiction implies that $\chi + \delta \not\in P(\tilde{V})$ and therefore $\tilde{V}$ is a non-dense
IRREDUCIBLE NON-DENSE MODULES

\(G\)-module. Applying Theorem 6.2 we conclude that \(\hat{V} \simeq L_\alpha^\varepsilon(\lambda, \gamma)\) for some \(\alpha \in \Delta^{re}, \lambda \in H^*, \lambda(c) = a, \gamma \in C, \varepsilon \in \{+,-\}\) which completes the proof. \(\square\)

Acknowledgement.

The author gratefully acknowledges the support of the Natural Sciences and Engineering Research Council of Canada.

References


Received July 12, 1993 and revised December 2, 1993.

KIEV UNIVERSITY
KIEV, 252617, UKRAINE
E-mail address: FUTORNY@UNI-ALG.KIEV.UA
A class of incomplete non-positively curved manifolds
  BRIAN BOWDITCH 1

The quasi-linearity problem for \( C^* \)-algebras
  L. J. BUNCE and JOHN DAVID MAITLAND WRIGHT 41

Distortion of boundary sets under inner functions. II
  JOSE LUIS FERNANDEZ PEREZ, DOMINGO PESTANA and JOSÉ RODRÍGUEZ 49

Irreducible non-dense \( A^{(1)}_1 \)-modules
  VJACHESLAV M. FUTorny 83

\( M \)-hyperbolic real subsets of complex spaces
  GIULIANA GIGANTE, GIUSEPPE TOMASSINI and SERGIO VENTURINI 101

Values of Bernoulli polynomials
  ANDREW GRANVILLE and ZHI-WEI SUN 117

The uniqueness of compact cores for 3-manifolds
  LUKE HARRIS and PETER SCOTT 139

Estimation of the number of periodic orbits
  BOJU JIANG 151

Factorization of \( p \)-completely bounded multilinear maps
  CHRISTIAN LE MERDY 187

Finitely generated cohomology Hopf algebras and torsion
  JAMES PEICHENG LIN 215

The positive-dimensional fibres of the Prym map
  JUAN-CARLOS NARANJO 223

Entropy of a skew product with a \( Z^2 \)-action
  KYEWON KOH PARK 227

Commuting co-commuting squares and finite-dimensional Kac algebras
  TAKASHI SANO 243

Second order ordinary differential equations with fully nonlinear two-point boundary conditions. I
  H. BEVAN THOMPSON 255

Second order ordinary differential equations with fully nonlinear two-point boundary conditions. II
  H. BEVAN THOMPSON 279

The flat part of non-flat orbifolds
  FENG XU 299