THE POSITIVE DIMENSIONAL FIBRES OF THE PRYM MAP

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The fibres of positive dimension of the Prym map are characterized.

Let $C$ be an irreducible complex smooth curve of genus $g$. Let $\pi : \tilde{C} \rightarrow C$ be a connected unramified double covering of $C$.

The Prym variety associated to the covering is, by definition, the component of the origin of the Kernel of the norm map

$$P(\tilde{C}, C) = \text{Ker}(Nm_\pi)^0 \subset J\tilde{C}.$$  

It is a principally polarized abelian variety (p.p.a.v.) of dimension $g(\tilde{C}) - g = g - 1$.

One defines the Prym map

$$P_g : R_g \rightarrow A_{g-1}$$

$$(\tilde{C}, \pi) \mapsto P(\tilde{C}, C),$$

where $R_g$ is the coarse moduli space of the coverings $\pi$ as above and $A_{g-1}$ stands for the coarse moduli space of p.p.a.v.'s of dimension $g - 1$.

It is well-known that this map is generically injective for $g \geq 7$ (Friedman-Smith, Kanev). On the other hand this map is never injective; this is a consequence of the tetragonal construction due to Donagi (see [Do1] for a description of the construction). This fact is already implicit in the results of Mumford ([M]):

The coarse moduli space $R\mathcal{H}_g$ of unramified double coverings of smooth hyperelliptic curves of genus $g$ has $[\frac{g-1}{2}] + 1$ irreducible components $R\mathcal{H}_{g,t}$, $t = 0, ..., [\frac{g-1}{2}]$. For an element $(\tilde{C}, C) \in R\mathcal{H}_{g,t}$ there exist two hyperelliptic curves

$$p_1 : C_1 \rightarrow \mathbb{P}^1, \quad p_2 : C_2 \rightarrow \mathbb{P}^1$$

of genus $g(C_1) = t \leq g - t - 1 = g(C_2)$ such that

a) $\tilde{C} = C_1 \times_{p_1} C_2$ and

b) $C = \tilde{C}/(\sigma_1 \circ \sigma_2)$, where $\sigma_1$ (resp. $\sigma_2$) is the involution on $\tilde{C}$ attached to the branched covering $\tilde{C} \rightarrow C_1$ (resp. $\tilde{C} \rightarrow C_2$).
Mumford proves (loc. cit. p. 346) that one has an isomorphism of p.p.a.v.

\[ P(\tilde{C}, C) \cong JC_1 \times JC_2. \]

Consequently the fibres of the restriction of \( P_g \) to \( \mathcal{R}H_g \) have positive dimension. In fact \( P_g(\mathcal{R}H_{g,t}) \) is contained in the product \( JH_t \times JH_{g-t-1} \), where \( JH_s \) stands for the locus of Jacobians of hyperelliptic curves of genus \( s \). Thus

\[ \dim \mathcal{R}H_{g,t} = 2g - 1 > \dim JH_t \times JH_{g-t-1} = \begin{cases} 2g - 4 & \text{if } t \neq 0, \\ 2g - 3 & \text{if } t = 0. \end{cases} \]

On the other hand positive dimensional fibres also appear for some coverings of bi-elliptic curves (a curve is called bi-elliptic if it can be represented as a ramified double covering of an elliptic curve).

In this note we characterize the fibres of positive dimension of the Prym map. To state our theorem we need some notation: let \( \mathcal{R}B_g \) be the coarse moduli space of the unramified double coverings \( \pi : \tilde{C} \to C \) such that \( C \) is a smooth bi-elliptic curve of genus \( g \). This variety has \( \lfloor \frac{g-1}{2} \rfloor + 2 \) irreducible components

\[ \mathcal{R}B_g = \left( \bigcup_{t=0}^{\lfloor \frac{g-1}{2} \rfloor} \mathcal{R}B_{g,t} \right) \cup \mathcal{R}B'_g \]

(see [N] for more details).

We obtain:

**Theorem.** Assume \( g \geq 13 \). A fibre of \( P_g \) is positive dimensional at \((\tilde{C}, C)\) if and only if \( C \) is either hyperelliptic or \((\tilde{C}, C) \in \bigcup_{t \geq 1} \mathcal{R}B_{g,t} \).

**Proof.** If \( C \) is hyperelliptic we apply the results of Mumford. On the other hand, all the irreducible components of the fibres of \( P_g|_{\mathcal{R}B_{g,t}} \) are positive dimensional for \( t \geq 1 \) (see [N, §20]). This finishes one implication.

The first step to see the opposite implication is to prove that the curve \( C \) is tetragonal (i.e. there exists a \( g^1_4 \) on \( C \)).

Let \( \eta \in JC \) be the two-torsion point characterizing the covering and denote by \( L \) the line bundle \( \omega_C \otimes \eta \). It is easy to check that \( L \) is very ample if \( C \) is non-tetragonal. Let \( \Phi_L \) be the projective embedding of \( C \) defined by \( L \).

As in Beauville ([B, p. 379]), we replace \( \mathcal{R}_g \) and \( \mathcal{A}_{g-1} \) by the corresponding functors. Then, the Prym map defines a morphism of functors \( Pr_g \). Our
hypothesis on the fibre of $P_g$ implies that the cotangent map to $Pr_g$ at $(\tilde{C}, C)$ is not surjective. By loc. cit. Prop. (7.5), this map can be shown as the cup-product map 

\[ S^2 H^0(C, L) \rightarrow H^0(C, L^{\otimes 2}) \]

followed by the isomorphism induced in cohomology by $L^{\otimes 2} \cong \omega^{\otimes 2}$. Hence, the non-surjectivity implies that $\Phi_L(C)$ is not a projectively normal curve.

We recall Theorem 1 in [G-L]: If $L$ is very ample and

\[ \deg(L) \geq 2g + 1 - 2h^1(L) - \text{Cliff}(C), \]

then $\Phi_L(C)$ is projectively normal (where Cliff$(C)$ is the Clifford index of $C$).

Since $h^1(L) = 0$ and $\deg(L) = 2g - 2$ one obtains $\text{Cliff}(C) \leq 2$. By using Clifford’s Theorem and [Ma, Propositions 7 and 8], it follows that the curve either possess a $g_1^4$ or is plane curve of degree six. The second case contradicts $g \geq 13$.

Thus $C$ is tetragonal. Since $g \geq 13$ the results in [De] can be applied: either the fibre is finite (generically, three elements) or we are in one of the following three possibilities: $C$ is either hyperelliptic or bi-elliptic or trigonal.

Assume that $C$ is bi-elliptic. Theorems (9.4), (10.9) and (10.10) in [N] states that $P_{g-1}^{-1}(P(\tilde{C}, C))$ consists of two points for every $(\tilde{C}, C) \in \mathcal{R}B_{g,0} \cup \mathcal{R}B'_{g,1}$, hence

\[ (\tilde{C}, C) \in \bigcup_{t \geq 1} \mathcal{R}B_{g,t}. \]

To finish the proof we have to rule out the case: $C$ trigonal. In [R], Recillas (cf. also [Do2]) establishes an isomorphism

\[ \tau : \mathcal{R}T_g \cong \mathcal{M}_{g-1}^{\text{tet},0}, \]

where $\mathcal{R}T_g$ is the coarse moduli space of unramified double coverings of trigonal curves and $\mathcal{M}_{g-1}^{\text{tet},0}$ is the moduli space of pairs $(X, g_1^4)$ of tetragonal curves $X$ and a base-point-free tetragonal linear series on $X$ not containing divisors of the form $2x + 2y$. This map satisfies that

\[ \tau(\tilde{C}, C) = (X, g_1^4) \mapsto P(\tilde{C}, C) \cong JX \quad \text{(as p.p.a.v.)}. \]

Let us fix $(\tilde{C}, C)$ as above and let $(\tilde{D}, D) \in \mathcal{R}_g$ such that $P(\tilde{D}, D) \cong P(\tilde{C}, C) \cong JX$. Since $C$ is not hyperelliptic, then the singular locus of the theta divisor of $P(\tilde{D}, D)$ has codimension 3 by [M, p. 344]. In loc. cit. a list of the Prym varieties with such property appear. We obtain that $D$ is either trigonal or bi-elliptic. Since $P(\tilde{D}, D)$ is the Jacobian of a curve the bi-elliptic case contradicts [S].
Hence it suffices to prove that all the fibres of the restriction of $P_g$ to $\mathcal{RT}_g$ are zero dimensional. This follows from the bijection $\tau$. Indeed, a curve $X$ of genus $g \geq 12$ has at most one base-point-free $g^1_4$ without divisors of the form $2x + 2y$; otherwise there exists a map $f : X \to \mathbb{P}^1 \times \mathbb{P}^1$ and then either the genus is $\leq 9$ or $X$ is bi-elliptic. By [T, Lemma (4.3)] the linear series of degree 4 and dimension 1 on a bi-elliptic curve come from $g^1_2$ linear series on the elliptic curve, thus divisors of the forbidden form appear.

Now the classical Torelli Theorem says that

$$\mathcal{M}^{tet,0}_{g-1} \to \mathcal{A}_{g-1}$$

$$(X, g^1_4) \mapsto JX$$

is injective. Composing with $\tau$ we are done. □

**Remark.** Note that if one drops the hypothesis on the genus, at least one gets that the Clifford index of $C$ is $\leq 2$.

**References**


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