

*Pacific
Journal of
Mathematics*

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Volume 172 No. 1

January 1996

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We consider the entropy of a dynamical system of a skew product \widehat{T} on $X_1 \times X_2$ where there is a Z^2 -action on the fiber X_2 . If the Z^2 -action comes from a Cellular Automaton map, then the contribution of the fiber to the entropy of the skew product is the directional entropy in the direction of the integral of a skewing function φ from X_1 to Z^2 .

1. Introduction.

J. Milnor has defined the notion of directional entropy in the study of dynamics of Cellular Automata [Mi1], [Mi2]. When the notion is applied to a Z^n action it is considered to be a generalization of the entropy of non co-compact subgroups of Z^n .

In the case of a Z^2 -action, we denote the generators of the groups by $\{U, V\}$. Let P be a generating partition under the Z^2 -action. We write $P_{i,j} = U^i V^j P$. If a subgroup is generated by $U^p V^q$, then there is a natural way to compute the entropy of $U^p V^q$ as a Z -action on the space. Milnor extended this idea to define the entropy of a vector by embedding Z^2 to the ambient vector space R^2 as follows.

$$h(\vec{v}) = \sup_{B: \text{bounded set}} \overline{\lim}_{t \rightarrow \infty} \frac{1}{t} H \left(\bigvee_{(i,j) \in B + [0,t]\vec{v}} P_{i,j} \right).$$

Given a vector \vec{v} , we let θ_o be the angle between two vectors \vec{v} and $(1, 0)$. Let $w = \frac{1}{\tan \theta_o}$ so that $(w, 1)$ is a scalar multiple of the vector \vec{v} . It is easy to see that

$$h(\vec{v}) = \lim_{m \rightarrow \infty} \lim_{t \rightarrow \infty} \frac{1}{t} H \left(\bigvee_{j=0}^{[ty]} \bigvee_{m+jw < i < m+jw} P_{i,j} \right),$$

where $[a]$ denote the greatest integer $\leq a$.

We note that if $\vec{v} = (p, q)$, then $h(\vec{v}) = h(U^p V^q)$. And it is easy to see that directional entropy is a homogeneous function, that is $h(c\vec{v}) = ch(\vec{v})$ for any $c \in R$.

Directional entropy in the case of a Z^2 -action generated by a Cellular Automaton map has been investigated in [Pa1, Pa3] and [Si]. D. Lind

defined a cone entropy, denoted by $h^c(\vec{v})$, of a vector \vec{v} . Given a vector $\vec{v} = (x, y)$ and a small angle θ , we consider the vectors $\vec{v}_\theta = (x_\theta, y)$ and $\vec{v}_{-\theta} = (x_{-\theta}, y)$ where x_θ and $x_{-\theta}$ satisfy $\frac{y}{x_\theta} = \tan(\theta_o + \theta)$ and $\frac{y}{x_{-\theta}} = \tan(\theta_o - \theta)$ respectively. Cone entropy is defined as follows.

$$h^c(\vec{v}) = \lim_{\theta \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} H \left(\bigvee_{j=0}^{[ny]} \bigvee_{x_{-\theta} \leq i \leq j x_\theta} P_{i,j} \right).$$

From the definition, it is clear that we have $h^c(\vec{v}) \geq h(\vec{v})$.

We say that a Z^2 -action is generated by a Cellular Automaton if one of the generators of the Z^2 -action, say V , is a block map (a finite code) of U . That is, $(V(x))_i$ depends only on the coordinates $x_{-r}, x_{-r+1}, \dots, x_r$ [He]. We call r the size of the block map V . We will show that in the case of a Z^2 -action generated by a Cellular Automaton map, the directional entropy and the cone entropy are the same (Theorem 1).

Let (X_1, ζ_1, μ_1, G) and (X_2, ζ_2, μ_2, H) be two ergodic measure preserving dynamical systems with finite entropy, where G and H denote the respective group. Given an integrable skewing function $\varphi : X_1 \rightarrow H$, we define a skew product G -action \hat{T} on $(X_1 \times X_2, \zeta_1 \times \zeta_2, \mu_1 \times \mu_2)$ such that $\hat{T}^g(x, y) = (T^g x, F^{\varphi(x)} y)$ where T denotes the G -action of X_1 and F denotes the H -action on X_2 . When we have $G = H = Z$, then the entropy of \hat{T} has been extensively studied by many people (e.g. [Ab], [Ad], [Ma, Ne]). It is well known in this case that $h(\hat{T}) = h(T) + |\int \varphi d\mu| h(F)$. The above formula says that, as we expect, the fiber contribution to the entropy is $|\int \varphi d\mu| h(F)$.

We investigate the entropy of \hat{T} when $G = Z$ and $H = Z^2$. Note that the above formula cannot hold when the acting group on the fiber is a more general group, say Z^2 . First of all, $\int \varphi d\mu$ is in general a vector. Secondly, if the skewing function takes a constant value, say $(1, 1)$, then the fiber contribution should come from the entropy of UV , not necessarily from the whole Z^2 -action. We prove that if the fiber Z^2 -action is generated by a Cellular Automaton map, then we have the analogous theorem (Theorem 2) to the case when $H = Z$.

We may mention that directional entropy can be also defined in a topological setting. D. Lind constructed an example whose topological entropy does not satisfy the analogue of our Theorem 3 [Li]. His example involves a Z^2 -action which is not generated by a Cellular Automaton map. It is not clear that Theorem 3 holds for topological entropy when we have a Z^2 -action on the fiber generated by a Cellular Automaton map. Lind's example is not interesting in the measure theoretic sense because it has the trivial invariant measure.

We have constructed a counterexample which does not satisfy Theorem 3

[Pa2]. For the example we explicitly construct the base transformation and use the Z^2 -action due to Thouvenot [Th] on the fiber. Both of them are constructed by cutting and staking method. It would be interesting to find out how generally Theorem 3 holds. For example, it is unknown if Theorem 3 is true when we have a topological Markov shift which does not satisfy the condition of Corollary 4. We are more interested in the case when the topological Markov shift has 0-entropy as a Z^2 -action.

Although Theorem 2 and 4 are more general than Theorem 1 and 3, we will prove Theorem 1 and 3 because their proofs are easier and more geometric. It is also easy to see the proofs of Theorem 2 and 4 from those of Theorem 1 and 3.

We would like to thank Professor D. Ornstein for helpful discussions and the Referee for many valuable comments.

2. Cone entropy.

Throughout the section we assume that our Z^2 -action is generated by a Cellular Automaton map. We denote by $H^m(\vec{v})$

$$\lim_{n \rightarrow \infty} \frac{1}{n} H \left(\bigvee_{j=0}^{n-1} \bigvee_{i=-m+jw}^{m+jw} P_{i,j} \right).$$

Note that $H^m(\vec{v})$ is independent of the size of the vector \vec{v} . Let τ denote $H(P_{0,0})$.

Lemma 1. $H^m(\vec{v}) = H^{m'}(\vec{v})$ if $m, m' > 2r + w$.

Proof. Case 1. \vec{v} is not a scalar multiple of $(1, 0)$.

Suppose $m' \geq m$. Clearly from the definition we have $H^{m'}(\vec{v}) \geq H^m(\vec{v})$. Hence it is enough to show $H^{m'}(\vec{v}) \leq H^m(\vec{v})$. Note that

$$\begin{aligned} H^m(\vec{v}) &= \lim_{n \rightarrow \infty} \frac{1}{n} H \left(\bigvee_{j=0}^{n-1} \bigvee_{-m+jw \leq i \leq m+jw} P_{i,j} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} H \left(\bigvee_{-m+jw \leq i \leq m+jw} P_{i,j} \mid \bigvee_{0 \leq k < j} \bigvee_{-m+kw \leq i \leq m+kw} P_{i,k} \right) \end{aligned}$$

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ H \left(\bigvee_{-m \leq i \leq m} P_{i,0} \right) \right. \\
 &\quad + \sum_{j=1}^{n-1} H \left(\bigvee_{jw \leq i \leq (j-1)w+r} P_{i,j} \left| \bigvee_{0 \leq k < j} \bigvee_{kw \leq i \leq 2m+kw} P_{i,k} \right. \right) \\
 &\quad + \sum_{j=1}^{n-1} H \left(\bigvee_{(j-1)w-r \leq i \leq jw} P_{i,j} \left| \bigvee_{1 \leq k < j} \bigvee_{-2m+kw \leq i \leq kw} P_{i,k} \right. \right. \\
 &\quad \left. \left. \bigvee_{-2m+jw \leq i \leq -2m+(j-1)w+r} P_{i,j} \right) \right\}.
 \end{aligned}$$

We make the following observations:

(1)

$$\lim_{n \rightarrow \infty} \frac{1}{n} H \left(\bigvee_{-m \leq i \leq m} P_{i,0} \right) = 0 = \lim_{n \rightarrow \infty} \frac{1}{n} H \left(\bigvee_{-m' \leq i \leq m'} P_{i,0} \right).$$

(2)

$$\begin{aligned}
 &H \left(\bigvee_{jw \leq i < (j-1)w+r} P_{i,j} \left| \bigvee_{0 \leq k < j} \bigvee_{kw \leq i \leq 2m+kw} P_{i,k} \right. \right) \\
 &\geq H \left(\bigvee_{jw \leq i < (j-1)w+r} P_{i,j} \left| \bigvee_{0 \leq k < j} \bigvee_{kw < i \leq 2m'+kw} P_{i,k} \right. \right),
 \end{aligned}$$

because we condition on more information.

(3) By the same reason, we have

$$\begin{aligned}
 &H \left(\bigvee_{(j-1)w-r \leq i \leq jw} P_{i,j} \left| \bigvee_{0 \leq k < j} \bigvee_{-2m+kw < i < kw} P_{i,k} \right. \right. \\
 &\quad \left. \left. \bigvee_{-2m+jw \leq i \leq -2m+(j-1)w+r} P_{i,j} \right) \right) \\
 &\geq H \left(\bigvee_{(j-1)w-r \leq i \leq jw} P_{i,j} \left| \bigvee_{0 \leq k < j} \bigvee_{-2m'+kw \leq i < kw} P_{i,k} \right. \right. \\
 &\quad \left. \left. \bigvee_{-2m'+jw \leq i \leq -2m'+(j-1)w+r} P_{i,j} \right) \right).
 \end{aligned}$$

These observations together with the formula for $H^m(\vec{v})$ above shows $H^{m'}(\vec{v}) \leq H^m(\vec{v})$.

Case 2. $\vec{v} = \eta(1, 0)$ for some real η .

We analogously denote by $H^m(\vec{v})$

$$\lim_{n \rightarrow \infty} \frac{1}{n} H \left(\bigvee_{i=0}^{[n\eta]} \bigvee_{j=-m}^m P_{i,j} \right).$$

We note that

$$\begin{aligned} H^{m'}(\vec{v}) &= \lim_{n \rightarrow \infty} \frac{1}{n} H \left(\bigvee_{i=0}^{[n\eta]} \bigvee_{j=-m}^{m+2(m'-m)} P_{i,j} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left(H \left(\bigvee_{i=0}^{[n\eta]} \bigvee_{j=-m}^m P_{i,j} \right) \right. \\ &\quad \left. + H \left(\bigvee_{i=1}^{[n\eta]} \bigvee_{j=m+1}^{m+2(m'-m)} P_{i,j} \left| \bigvee_{i=0}^{[n\eta]} \bigvee_{j=-m}^m P_{i,j} \right. \right) \right) \\ &\leq H^m(\vec{v}) + \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=m+1}^{m+2(m'-m)} H \left(\bigvee_{i=0}^{[n\eta]} P_{i,j} \left| \bigvee_{i=0}^{[n\eta]} P_{i,j-1} \right. \right) \\ &\leq H^m(\vec{v}) + \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=m+1}^{m+2(m'-m)} H \left(\bigvee_{i=0}^r P_{i,j} \bigvee_{i=[n\eta]-r}^{[n\eta]} P_{i,j} \right) \\ &\leq H^m(\vec{v}) + \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=m+1}^{m+2(m'-m)} 2r\tau \\ &= H^m(\vec{v}) + \lim_{n \rightarrow \infty} \frac{4r\tau(m' - m)}{n} \\ &= H^m(\vec{v}). \end{aligned}$$

Since we have $H^{m'}(\vec{v}) \geq H^m(\vec{v})$ by definition, the proof is complete. □

Corollary 1. *If \vec{v} is not a scalar multiple of $(1, 0)$, then we have*

$$\begin{aligned} &\left| \frac{1}{n} H \left(\bigvee_{j=0}^{n-1} \bigvee_{-m+jw \leq i \leq m+jw} P_{i,j} \right) - \frac{1}{n} H \left(\bigvee_{j=0}^{n-1} \bigvee_{-m'+jw \leq i \leq m'+jw} P_{i,j} \right) \right| \\ &\leq \frac{1}{n} H \left(\bigvee_{m < |i| \leq m'} P_{i,0} \right) \\ &\leq \tau \frac{2(m' - m)}{n}. \end{aligned}$$

Theorem 1. $h^c(\vec{v}) = h(\vec{v})$.

Proof. It is enough to show that $h^c(\vec{v}) - h(\vec{v})$ is small. If $\vec{v} = (x, y)$ where $y \neq 0$, then by rescaling, we may assume that $\vec{v} = (x, 1)$. Given any $\varepsilon > 0$, there exists θ such that if $\kappa \leq \theta$, then

(i)

$$\lim_{n \rightarrow \infty} \frac{1}{n} H \left(\bigvee_{0 \leq j < n} \bigvee_{jx_\kappa \leq i \leq jx - \kappa} P_{i,j} \right) < h^c(\vec{v}) + \varepsilon$$

(ii) $|x_{-\theta} - x_\theta| < \gamma$ where γ satisfies that $\gamma\tau < \varepsilon$. There exists m_0 such that if $m \geq m_0$, then

$$\lim_{n \rightarrow \infty} \frac{1}{n} H \left(\bigvee_{j=0}^{n-1} \bigvee_{-m+jx \leq i \leq m+jx} P_{i,j} \right) = h(\vec{v}).$$

We choose n_o such that if $n \geq n_o$, then we have

(iii)

$$h(\vec{v}) - \varepsilon < \frac{1}{n} H \left(\bigvee_{0 \leq j \leq n-1} \bigvee_{-m_o+jx \leq i \leq m_o+jx} P_{i,j} \right) \leq h(\vec{v}) + \varepsilon,$$

(iv)

$$h^c(\vec{v}) - 2\varepsilon < \frac{1}{n} H \left(\bigvee_{0 \leq j \leq n-1} \bigvee_{jx_\theta \leq i \leq jx - \theta} P_{i,j} \right) \leq h^c(\vec{v}) + 2\varepsilon,$$

(v)

$$\frac{1}{n} H \left(\bigvee_{0 \leq j < K} \bigvee_{-m_o+jx < i < m_o+jx} P_{i,j} \right) < \varepsilon, \text{ where}$$

$$K = \max\{j : j|x_\theta - x| < m_o \text{ and } j|x_{-\theta} - x| < m_o\},$$

and

(vi)

$$\frac{1}{n} H \left(\bigvee_{0 \leq j < n} \bigvee_{jx_\theta \leq i \leq jx - \theta} P_{i,j} \right) \geq \frac{1}{n} H \left(\bigvee_{0 \leq j < n} \bigvee_{-m_o+jx \leq i \leq m_o+jx} P_{i,j} \right).$$

We compute

$$\begin{aligned}
 & |h^c(\vec{v}) - h(\vec{v})| \\
 & \leq \left| \frac{1}{n} H \left(\bigvee_{o \leq j < n} \bigvee_{jx_\theta \leq i \leq jx_{-\theta}} P_{i,j} \right) \right. \\
 & \quad \left. - \frac{1}{n} H \left(\bigvee_{o \leq j < n} \bigvee_{-m_o - jx \leq i < m_o + jx} P_{i,j} \right) \right| + 3\varepsilon \\
 & \leq \left| \frac{1}{n} H \left(\bigvee_{o \leq j < n} \bigvee_{n(x_\theta - x) + jx \leq i \leq n(x_{-\theta} - x) + jx} P_{i,j} \right) \right. \\
 & \quad \left. - \frac{1}{n} H \left(\bigvee_{o \leq j < n} \bigvee_{-m_o + jx \leq i < m_o + jx} P_{i,j} \right) \right| + 3\varepsilon \\
 & \leq \frac{1}{n} H \left(\bigvee_{n(x_\theta - x) \leq i \leq n(x_{-\theta} - x)} P_{i,o} \right) + 3\varepsilon \\
 & \leq \frac{1}{n} \gamma n \tau + 3\varepsilon.
 \end{aligned}$$

Hence we have

$$|h(\vec{v}) - h^c(\vec{v})| < 4\varepsilon.$$

In the case of $\vec{v} = (x, o)$, it is not hard to see that the idea of the second part of the proof of Lemma 1 combined with the idea of the proof above will give the desired result. \square

Theorem 2. *If $\sum_{m=0}^\infty H \left(P_{0,1} \left| \bigvee_{-m \leq i \leq m} P_{i,0} \right. \right)$ is finite, then we have $h^c(\vec{v}) = h(\vec{v})$.*

Proof. We note that if we choose M so that

$$\sum_{m=M}^\infty H \left(P_{0,1} \left| \bigvee_{-m \leq i \leq m} P_{i,0} \right. \right) < \varepsilon,$$

then we get

$$\sum_{k=-m+M}^{m-M} H \left(P_{k,1} \left| \bigvee_{-m \leq i \leq m} P_{i,0} \right. \right) < 2\varepsilon,$$

for all $m > M$. Using this, it is easy to see that if $m_2 \geq m_1 \geq M$, we have that for any n ,

$$\frac{1}{n}H\left(\bigvee_{j=0}^{[ny]} \bigvee_{i=m_2+jw} P_{i,j}\right) < \frac{1}{n}H\left(\bigvee_{j=0}^{[ny]} \bigvee_{i=-m_1+jw} P_{i,j}\right) + 2\varepsilon + \frac{m_2 - m_1}{n}\tau$$

where $\frac{m_2 - m_1}{n}\tau$ comes from the difference between $\frac{1}{n}H\left(\bigvee_{i=-m_1}^{m_1} P_{i,0}\right)$ and $\frac{1}{n}H\left(\bigvee_{i=-m_2}^{m_2} P_{i,0}\right)$.

Hence for a given $\varepsilon > 0$, there exist m_o as in Theorem 1 such that for a sufficiently large n ,

$$\begin{aligned} & |h^c(\vec{v}) - h(\vec{v})| \\ & \leq \left| \frac{1}{n}H\left(\bigvee_{0 \leq j < n} \bigvee_{n(x_\theta - x) + jx \leq i \leq n(x_{-\theta} - x) + jx} P_{i,j}\right) \right. \\ & \quad \left. - \frac{1}{n}H\left(\bigvee_{0 \leq j < n} \bigvee_{-m_o + jx \leq i < m_o + jx} P_{i,j}\right) \right| + 3\varepsilon \\ & \leq \frac{1}{n}\gamma n\tau + 2\varepsilon + 3\varepsilon. \end{aligned}$$

□

Corollary 2. *If V is a finitary code with finite expected code length, then $h^c(\vec{v}) = h(\vec{v})$.*

Proof. It is easy to see that a finitary code with finite expected code length satisfies the condition of Theorem 2. See [Pa3]. □

3. Main Theorem.

Let $\lambda = \mu_1 \times \mu_2$. We denote $\sum_{i=0}^{n-1} \varphi_k(T^i z)$ by $\varphi_k^n(z)$ for $k = 1$ or 2 and $z \in X_1$. Given two partitions, β_1 and β_2 , we write $\beta_1 \leq \beta_2$ if β_2 is a finer partition than β_1 .

Theorem 3. $h(\widehat{T}) = h(T) + h(\vec{v})$ where $\vec{v} = \int \varphi d\mu = (\int \varphi_1 d\mu, \int \varphi_2 d\mu)$.

Proof. Since $\int \varphi d\mu$ is finite, as in the case of a Z -valued skewing function, there exists φ' which is bounded and cohomologous to φ . Hence we may assume that φ is bounded. Let $|\varphi_1(z)| \leq L$ and $|\varphi_2(z)| \leq L$. Suppose $\vec{v} = \int \varphi d\mu = (x, y)$ where $y \neq 0$. We let α denote the generating partition

of the base. Let β denote a partition of X_2 . Both of the partitions α and β can be considered in a natural way to be a partition of $X_1 \times X_2$. For a given $z \in X_1$, we denote the set $\{(z, u) : u \in X_2\}$ by I_z .

Since

$$\frac{1}{n}H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i(\alpha \vee \beta) \right) = \frac{1}{n}H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i \alpha \right) + \frac{1}{n}H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta \middle| \bigvee_{i=0}^{n-1} \widehat{T}^i \alpha \right)$$

and

$$\frac{1}{n}H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta \middle| \bigvee_{i=0}^{n-1} \widehat{T}^i \alpha \right) = \int \frac{1}{n}H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta | I_z \right) d\mu,$$

we have

$$\begin{aligned} \sup_{\beta} h \left(\widehat{T}, \alpha \vee \beta \right) &= \sup_{\beta} \lim_{n \rightarrow \infty} \frac{1}{n}H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i(\alpha \vee \beta) \right) \\ &= h \left(\widehat{T}, \alpha \right) + \sup_{\beta_m} \lim_{n \rightarrow \infty} \int \frac{1}{n}H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_m | I_z \right) d\mu \\ &= h \left(\widehat{T}, \alpha \right) + \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \int \frac{1}{n}H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_m | I_z \right) d\mu, \end{aligned}$$

where β_m denote the partition $\bigvee_{i=-m}^m \bigvee_{j=0}^{L-1} P_{i,j}$.

We denote $\lim_{n \rightarrow \infty} \frac{1}{n}H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_m | I_z \right)$ by $h_z \left(\widehat{T}, \beta_m \right)$.

As in Lemma 1, it is not hard to see that for sufficiently large m and m' , we have

$$h_z \left(\widehat{T}, \beta_m \right) = h_z \left(\widehat{T}, \beta_{m'} \right).$$

We will show that for sufficiently large m ,

$$\frac{1}{n}H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_m | I_z \right) \rightarrow h(\vec{v}) \text{ as } n \rightarrow \infty, \text{ for a.e. } z \in X_1.$$

We denote by x_ℓ the x -intercept of a line in \mathbb{R}^2 passing through $\varphi^\ell(z)$ with the same slope as \vec{v} . Let

$$\begin{aligned} s_n &= \max\{x_1, \dots, x_n\} \text{ and} \\ t_n &= \min\{x_1, \dots, x_n\}. \end{aligned}$$

Given $\varepsilon > 0$, let k_ε be the integer such that if $k \geq k_\varepsilon$, then we have

(i)

$$\left| h(\vec{v}) - \lim_{n \rightarrow \infty} \frac{1}{n} H \left(\bigvee_0^{[ny]} \bigvee_{i=-k+jw}^{k+jw} P_{i,j} \right) \right| < \varepsilon.$$

Given any $\delta > 0$ and $\varepsilon > 0$, there exists n_o such that if $n \geq n_o$, then we have

(ii)

$$\mu E_1 = \mu \left\{ z : \left| \int \varphi d\mu - \frac{1}{n} \varphi^n(z) \right| < \delta \right\} > 1 - \varepsilon,$$

(iii)

$$\left| h(\vec{v}) - \frac{1}{n} H \left(\bigvee_{j=0}^{[ny]} \bigvee_{i=-k_o+jw}^{k_o+jw} P_{i,j} \right) \right| < \varepsilon,$$

(iv)

$$\mu E_2 = \mu \left\{ z : \left| h_z(\hat{T}, \beta_{k_o}) - \frac{1}{n} H \left(\bigvee_{i=0}^{n-1} \hat{T}^i \beta_{k_o} | I_z \right) \right| < \varepsilon \right\} > 1 - \varepsilon,$$

(v) $k_o < \frac{\varepsilon}{2} n_o$,

and

(vi) $|s_n - t_n| < 2n\delta$.

We choose $\delta < \varepsilon^2$ and choose n_o satisfying (ii)-(vi) above. We fix m_o such that $k_o < (\varepsilon/2)n_o < m_o < \varepsilon n_o$. For notational convenience, we write m and n instead of m_o and n_o respectively. We note that

$$\begin{aligned} & \bigvee_{j=0}^{n-1} \hat{T}^j \beta_m \text{ on } I_Z \\ & \leq \beta_m \bigvee F^{\varphi(z)}(\beta_m) \bigvee F^{\varphi^2(z)}(\beta_m) \bigvee \dots \bigvee F^{\varphi^{n-1}(z)}(\beta_m) \text{ on } I_Z \\ & \leq \bigvee_{j=0}^{\varphi_2^{n-1}(z)+L-1} \bigvee_{i=t_n-m+jw}^{s_n+m+jw} P_{i,j} \text{ on } I_Z. \end{aligned}$$

Since t_n and s_n satisfy that

$$|(t_n + m) - (s_n - m)| = |2m + t_n - s_n| > |2m - 2n\delta| > k_o$$

and

$$|(s_n + m) - (t_n - m)| < \varepsilon n,$$

if $z \in E_1$, then by our Corollary and (ii), we have

$$\begin{aligned} & \left| \frac{1}{n} H \left(\bigvee_{j=0}^{\varphi_2^{n-1}(z)+L-1} \bigvee_{i=t_n-m+jw}^{s_n+m+jw} P_{i,j} \right) - \frac{1}{n} H \left(\bigvee_{j=0}^{[ny]} \bigvee_{i=-k_o+jw}^{k_o+jw} P_{i,j} \right) \right| \\ & < \frac{1}{n} H \left(\bigvee_{i=t_n-m}^{s_n+m} P_{i,0} \right) + \frac{1}{n} H \left(\bigvee_{j=q_1}^{q_2} \bigvee_{i=t_n-m+jw}^{s_n+m+jw} P_{i,j} \right) \\ & < \frac{1}{n} \tau \varepsilon n + \frac{1}{n} (q_2 - q_1) \tau (w + 2r) \\ & < \tau (\varepsilon + \delta (w + 2r)), \end{aligned}$$

where $q_1 = \min\{[ny], \varphi_2^{n-1}(z) + L - 1\}$ and $q_2 = \max\{[ny], \varphi_2^{n-1}(z) + L - 1\}$.

Hence we have

$$\begin{aligned} & \left| \frac{1}{n} H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_m | I_z \right) - h(\vec{v}) \right| \\ & \leq \left| \frac{1}{n} H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_m | I_z \right) - \frac{1}{n} H \left(\bigvee_{j=0}^{[ny]} \bigvee_{i=-k_o+jw}^{k_o+jw} P_{i,j} \right) \right| + \varepsilon \\ & \leq \left| \frac{1}{n} H \left(\bigvee_{j=0}^{\varphi_2^{n-1}(z)+L-1} \bigvee_{i=t_n-m+jw}^{s_n+m+jw} P_{i,j} \right) - \frac{1}{n} H \left(\bigvee_{j=0}^{[ny]} \bigvee_{i=-k_o+jw}^{k_o+jw} P_{i,j} \right) \right| + \varepsilon \\ & \leq \tau (\varepsilon + \delta (w + 2r)) + \varepsilon. \end{aligned}$$

Let $E = E_1 \cap E_2$. If $z \in E$, then by our choice of m and Corollary 1, we have

$$\begin{aligned} & \left| \frac{1}{n} H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_m | I_z \right) - h_z(\widehat{T}, \beta_m) \right| \\ & \leq \left| \frac{1}{n} H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_m | I_z \right) - \frac{1}{n} H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_k | I_z \right) \right| \\ & \quad + \left| \frac{1}{n} H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_k | I_z \right) - h_z(\widehat{T}, \beta_k) \right| + \left| h_z(\widehat{T}, \beta_k) - h_z(\widehat{T}, \beta_m) \right| \\ & \leq \varepsilon + \varepsilon + \frac{1}{n} m \tau < \varepsilon (2 + \tau). \end{aligned}$$

Since φ_1 and φ_2 are bounded, it is easy to see that there exists ω such that $\left| h_z(\widehat{T}, \beta) \right| < \omega$ for all β and all z . We may also assume that $h(\vec{v})$ is

bounded above by ω . Now we compute

$$\begin{aligned} & \left| \sup_{\beta} \int h_z(\widehat{T}, \beta) d\mu - h(\vec{v}) \right| \\ & \leq \int_E \left| h_z(\widehat{T}, \beta_m) - h(\vec{v}) \right| d\mu + \sup_{\beta} \int_{E^c} \left| h_z(\widehat{T}, \beta) - h(\vec{v}) \right| d\mu + \varepsilon \\ & \leq \int_E \left| h_z(\widehat{T}, \beta_m) - \frac{1}{n} H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_m | I_z \right) \right| d\mu \\ & \quad + \int_E \left| \frac{1}{n} H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_m | I_z \right) - h(\vec{v}) \right| d\mu \\ & \quad + \sup_{\beta} \int_{E^c} \left| h_z(\widehat{T}, \beta) - h(\vec{v}) \right| d\mu + \varepsilon \\ & \leq \varepsilon(2 + \tau) + \tau(\varepsilon + \delta(w + 2r)) + \varepsilon + 4\omega\varepsilon + \varepsilon \\ & \leq \varepsilon(4 + 2\tau + \tau(w + 2r) + 4\omega). \end{aligned}$$

In the case when $\vec{v} = \int \varphi = \eta(1, 0)$ for some real number η , we need to argue differently. We may assume $\eta > 0$. We construct φ' which is cohomologous to φ as follows. Let $\varphi' = (\varphi'_1, \varphi'_2)$.

- (i) φ'_1 takes the values $[\eta] - 1, [\eta]$ and $[\eta] + 1$
 φ'_2 takes the values $-1, 0, 1$.
- (ii) In an orbit of a point, φ'_2 value, 1 or -1, follows its value 0.
- (iii) We use the ergodic theorem to construct φ'_2 so that it takes the value 0 for all z 's except a set of small measure.

Hence we may assume that φ satisfies these properties.

We let $\beta_m = \bigvee_{i=0}^{[\eta]} \bigvee_{j=-m}^m P_{i,j}$. Recall that r denote the size of the block map.

As in the previous case, we choose m_o so that if $m \geq m_o$, then

- (i) $m_o \geq 10r$,
 - (ii) $|h(\vec{v}) - H^m(\vec{v})| < \varepsilon$,
 - (iii) $\mu \left\{ z : \left| \sup_{\beta} \int h_z(\widehat{T}, \beta) - h_z(\widehat{T}, \beta_m) \right| < \varepsilon \right\} > 1 - \varepsilon$.
 We fix $m \geq m_o$. We choose n_o so that if $n \geq n_o$, then
 - (iv) $\mu \left\{ z : \left| \frac{1}{n} H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_m | I_z \right) - h_z(\widehat{T}, \beta_m) \right| < \varepsilon \right\} > 1 - \varepsilon$,
 - (v) $\mu \left\{ z : \left| \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i(z)) - \int \varphi d\mu \right| < \varepsilon \right\} > 1 - \varepsilon$,
- and
- (vi) $\mu \left\{ z : \left| \frac{1}{n} \sum_{i=0}^k \varphi_2(T^i(z)) \right| < \varepsilon \text{ for all } 0 \leq k < n \right\} > 1 - \varepsilon$.

Let E denote the set satisfying the above conditions, (iii), (iv), (v) and

(vi). We have $\mu E > 1 - 4\varepsilon$. Let $z \in E$.

Let

$$u = \max \left\{ \sum_{i=0}^k \varphi_2(T^i(z)) : k = 0, 1, \dots, n-1 \right\}$$

and

$$v = \min \left\{ \sum_{i=0}^k \varphi_2(T^i(z)) : k = 0, 1, \dots, n-1 \right\}.$$

Since $\eta > 0$, there exists $i_o = \max \{k : \varphi_1^k(z) \leq i\}$ for a.e. $z \in X_1$. We denote by $\Psi_2^i(z)$

$$\max \left\{ \sum_{\zeta=0}^k \varphi_2(T^\zeta(z)) : 0 \leq k \leq i_o, i - [\eta] \leq \varphi_1^k(z) \leq i \right\}.$$

Now we compute

$$\begin{aligned} & \frac{1}{n} H \left(\bigvee_{j=-m}^m \bigvee_{i=0}^{\varphi_1^n(z)} P_{i,j} \right) \\ &= \frac{1}{n} H \left(\bigvee_{j=-m+u}^{m+u} \bigvee_{i=0}^{\varphi_1^n(z)} P_{i,j} \right) \\ &\leq \frac{1}{n} H \left(\bigvee_{i=0}^{\varphi_1^n(z)} \bigvee_{j=-m+\Psi_2^i(z)}^{u+m} P_{i,j} \right) \\ &\leq \frac{1}{n} \left(H \left(\bigvee_{i=0}^{\varphi_1^n(z)} \bigvee_{j=-m+\Psi_2^i(z)}^{m+\Psi_2^i(z)} P_{i,j} \right) + 2 \cdot 2(\varepsilon n) \cdot \tau \cdot r \right) \\ &\leq \frac{1}{n} H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_m | I_z \right) + 4\varepsilon r \tau. \end{aligned}$$

The second to the last inequality is clear because by the condition (i) on m_o , we have

$$\begin{aligned} & H \left(\bigvee_{i=0}^{\varphi_1^n(z)} \bigvee_{j=-m+\psi_2^i(z)}^{u+m} P_{i,j} \middle| \bigvee_{i=0}^{\varphi_1^n(z)} \bigvee_{j=-m+\psi_2^i(z)}^{m+\psi_2^i(z)} P_{i,j} \right) \\ &\leq H \left(\bigvee_{j=m}^{u+m} \bigvee_{i=0}^{r-1} P_{i,j} \bigvee \bigvee_{j=v+m}^{u+m} \bigvee_{i=\varphi_1^n(z)-r+1}^{\varphi_1^n(z)} P_{i,j} \right) \end{aligned}$$

$$\leq u \cdot r \cdot \tau + (u - v) \cdot r \cdot \tau.$$

Since the following inequality is also true

$$\begin{aligned} & \frac{1}{n} H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_m | I_z \right) \\ & \leq \frac{1}{n} H \left(\bigvee_{j=-m+v}^{m+u} \bigvee_{i=0}^{\varphi_1^n(z)} P_{i,j} \right) \\ & = \frac{1}{n} H \left(\bigvee_{j=-m}^{m+u-v} \bigvee_{i=0}^{\varphi_1^n(z)} P_{i,j} \right) \\ & \leq \frac{1}{n} H \left(\bigvee_{j=-m}^m \bigvee_{i=0}^{\varphi_1^n(z)} P_{i,j} \right) + \frac{2}{n} (u - v) r \cdot \tau \\ & = \frac{1}{n} H \left(\bigvee_{j=-m+u}^{m+u} \bigvee_{i=0}^{\varphi_1^n(z)} P_{i,j} \right) + 4\epsilon r \tau, \end{aligned}$$

we have

$$\left| \frac{1}{n} H \left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_m | I_z \right) - \frac{1}{n} H \left(\bigvee_{-m+u}^{m+u} \bigvee_{i=0}^{\varphi_1^n(z)} P_{i,j} \right) \right| < 4\epsilon r \tau.$$

We note that

$$\frac{1}{n} H \left(\bigvee_{-m+u}^{m+u} \bigvee_{i=0}^{\varphi_1^n(z)} P_{i,j} \right) = \frac{\varphi_1^n(z)}{n} \frac{1}{\varphi_1^n(z)} H \left(\bigvee_{-m+u}^{m+u} \bigvee_{i=0}^{\varphi_1^n(z)} P_{i,j} \right)$$

converges to $h(\vec{v})$.

As in the case of $\vec{v} = \int \varphi \, d\mu = (x, y)$ where $y \neq 0$, it is now clear that

$$\left| \sup_{\beta} \int h_z(\widehat{T}, \beta) \, d\mu - h(\vec{v}) \right|$$

can be made arbitrarily small. □

Similarly we can prove the following theorem.

Theorem 4. *If $\sum_{m=0}^{\infty} H \left(P_{0,1} \left| \bigvee_{-m \leq i \leq m} P_{i,0} \right. \right)$ is finite, then we have $h(\widehat{T}) = h(T) + h(\vec{v})$ where $\vec{v} = \int \varphi \, d\mu = (\int \varphi_1 \, d\mu, \int \varphi_2 \, d\mu)$.*

The following Corollaries are also almost immediate from the proof of Theorem 3.

Corollary 3. *If $\sum_{m=0}^{\infty} H \left(P_{0,1} \left| \begin{array}{c} \bigvee \\ -k \leq j \leq k \\ \bigvee \\ -m \leq i \leq m \end{array} P_{i,j} \right. \right)$ is finite for some k , then we have $h(\widehat{T}) = h(T) + h(\vec{v})$ where \vec{v} is given as above.*

Corollary 4. *If a fiber Z^2 -action, F , satisfies the condition of Corollary 3 after a linear transformation by a matrix A in $SL(2, Z)$, that is, $A \circ F$ satisfies the condition, then we have the above formula in Corollary 3 for the entropy.*

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Received February 10, 1993 and revised June 24, 1993. This research has been supported in part by NSF DMS 8902080 and GARC-KOSEF.

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