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A relationship between finite dimensional Kac algebras and specified commuting co-commuting squares is discussed. The Majid's bicrossproduct Kac algebra is explained in our context.

1. Introduction.

The theory of Kac algebras (Hopf algebras) has been drawing considerable attention (see [**6**] for the reference), and in fact many intensive studies have been made recently. ([**1**, **18**, **19**, **34**, **35**, **36**], etc.) On the other hand, the announcement by A.Ocneanu ([**20**, **21**]) brought us a new aspect in the theory of Kac algebras : it is his claim (proved in [**4**, **17**] and also [**28**]) that, for an irreducible inclusion of factors $M \supset N$ with finite index and depth $=$ 2, M is described as the crossed product algebra of N by an outer action of a finite dimensional Kac algebra. Hence, we investigate Kac algebras from the Jones index theoretical point of view.

The purpose of this paper is to find a finite dimensional Kac algebra via the index theory : let $L \supset K$ be an irreducible inclusion of factors with finite index. Suppose that, for an intermediate subfactor M, both inclusions $L \supset M$ and $M \supset K$ are of depth 2. Although the inclusion $L \supset K$ does not always satisfy the depth 2 condition, it can be proved that this pair is of depth 2 if these factors L, M, K, and another intermediate subfactor N form a commuting co-commuting square. Details will be explained in §2 after recalling basic facts on commuting co-commuting squares. Another criterion for the inclusion $L \supset K$ to be of depth 2 is also obtained. Examples are given in §3.

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2. Main results.

Let

$$\begin{array}{ccc} L & \supset & M \\ \cup & & \cup \\ N & \supset & K \end{array}$$

be a quadruple of type II_1 factors satisfying $[L : K] < \infty$. (For the standard facts on the index theory, see [**8**, **11**, **23**, **25**, **26**].) It is said to be a commuting square if $E_M^L(N) \subseteq K$, where E_M^L is the conditional expectation from L to M. (See [**8**] for other equivalent conditions.) A quadruple (L, M, N, K) is said to be a co-commuting square if the quadruple

$$\begin{array}{ccc} K' & \supset & M' \\ \cup & & \cup \\ N' & \supset & L', \end{array}$$

or equivalently, that of basic extensions

$$\begin{array}{ccc} \langle L, e_K^L \rangle & \supset & \langle L, e_M^L \rangle \\ \cup & & \cup \\ \langle L, e_N^L \rangle & \supset & L \end{array}$$

on the standard space $L^2(L)$ is a commuting square. Here, e_M^L, e_N^L, and e_K^L are relevant Jones projections (see [**27**, **30**, **31**]). For a commuting co-commuting square (L, M, N, K), we have $K = M \cap N$ and $L = M \vee N$. (In [**27**], a quadruple satisfying these equations is called a quadrilateral and, for a quadrilateral (L, M, N, K), $\mathrm{Ang}(M, N) = Op - \mathrm{ang}(M, N) = \{\frac{\pi}{2}\}$ corresponds to the commuting co-commuting condition.)

For a commuting square, we have characterization of co-commutativity ([**27**, Corollary 7.1] and [**26**, Proposition 1.1.5]).

Proposition 2.1. *Let (L, M, N, K) be a commuting square of type* II_1 *factors satisfying $[L : K] < \infty$. Then the following are equivalent :*

(1) (L, M, N, K) *is co-commuting.*

(2) $L = M \cdot N = \{\sum_{i \in F} m_i n_i ; F$ *is a finite set,* $m_i \in M, n_i \in N\}$.

(3) $[L : M] = [N : K]$.

(4) *A Pimsner-Popa basis for $N \supset K$ is also that for $L \supset M$.*

Remark that, in [**26**], a commuting square satisfying (one of) the above conditions is called "non-degenerate" and (1),(2) of the following proposition are mentioned in [**26**, Proposition 1.1.6] (see also [**9**, Proposition 2.3]). We will see them for the completeness of this article.

Proposition 2.2. *Let (L, M, N, K) be a commuting co-commuting square of type* II_1 *factors satisfying $[L : K] < \infty$. Then,*

(1) *$\langle M, e_K^M\rangle \supset M$ is conjugate to $\langle M, e_N^L\rangle \supset M$.*

(2) *The quadrilateral $(\langle L, e_N^L\rangle, \langle M, e_N^L\rangle, L, M)$ is also commuting co-commuting.*

(3) *$\langle L, e_K^L\rangle$ is identified with the Jones extension for $\langle L, e_N^L\rangle \supset \langle M, e_N^L\rangle$.*

Proof. (1) While the condition $\sum_i a_i e_K^M b_i = 0$ $(a_i, b_i \in M)$ is equivalent to $\sum_i a_i E_K^M(b_i c) = 0$ for $c \in M$ on $L^2(M)$, the condition $\sum_i a_i e_N^L b_i = 0$ $(a_i, b_i \in M)$ means $0 = \sum_i a_i E_N^L(b_i c d) = \sum_i a_i E_N^L(b_i c) d = \sum_i a_i E_K^M(b_i c) d$ for $c \in M, d \in N$ on $L^2(L)$ thanks to Proposition 2.1.(2) and the commuting square condition. Hence, we may consider the map $\phi : \langle M, e_K^M\rangle \to \langle M, e_N^L\rangle$ defined by $\phi(\sum_i a_i e_K^M b_i) = \sum_i a_i e_N^L b_i$. It is easy to see that this map ϕ gives an isomorphism between them and $\phi|_M = \mathrm{id}$.

(2) follows from [**8**, Corollary 4.2.3], [**11**, Proposition 3.1.7], and Proposition 2.1.

(3) Since the commuting square condition means $e_M^L e_N^L = e_K^L$, we have $\langle\langle L, e_N^L\rangle, e_M^L\rangle = \langle L, e_K^L\rangle$. We will show that $\langle L, e_K^L\rangle = \langle\langle L, e_N^L\rangle, e_M^L\rangle$ is the Jones extension for $\langle L, e_N^L\rangle \supset \langle M, e_N^L\rangle$. The commuting square condition implies $[e_M^L, x] = 0$ for $x \in \langle M, e_N^L\rangle$. And for the conditional expectation $E_{\langle L, e_N^L\rangle}^{\langle L, e_K^L\rangle}$, by [**27**, Lemma 7.2], we have $E_{\langle L, e_N^L\rangle}^{\langle L, e_K^L\rangle}(e_M^L) = \frac{1}{[L:M]}$. Therefore, we get the conclusion by [**24**, Proposition 1.2.(2)]. □

Thus, we have extensions of a commuting co-commuting square (L, M, N, K) in compatible ways.

For an irreducible inclusion, we have a refined estimation of the dimension of relative commutant algebras as in [**8**, Theorem 4.6.3] (cf. [**11**, Corollary 2.2.3]). We will see this in terms of sectors ([**10**, **12**, **14**, **15**, **16**]).

Lemma 2.1. *For an irreducible inclusion $M \supset N$ of type* II_1 *factors satisfying $[M : N] < \infty$,*

$$\dim(M_k \cap N') \leq [M : N]^k,$$

where $N \subset M = M_0 \subset M_1 \subset \cdots$ is the Jones tower.

Proof. We only treat the case $k = 2$ since a similar proof will work for any k. We may assume that M and N are properly infinite and isomorphic (by [**16**, Lemma 2.3]) and denote N by $\rho(M)$ for an endomorphism $\rho \in End(M)$. Consider the irreducible decompositions : $\bar\rho\rho = \sum_j m_j \alpha_j, \bar\rho\rho\bar\rho = \sum_{j,k} m_j n_{jk} \beta_k$ $(\alpha_j \bar\rho = \sum_k n_{jk}\beta_k)$, where $\bar\rho$ is the conjugate sector of ρ. By the Frobenius reciprocity, we have $\beta_k\rho \geq \sum_j n_{jk}\alpha_j, \alpha_j\bar{\rho} \geq m_j\bar{\rho}$. Combining these, we get

$$\begin{aligned}[M:N]^2 &= ([M:N]_0^2 =)\, d(\rho)^4 = \sum_{j,k} m_j n_{jk} d(\beta_k\rho) \\ &\geq \sum_{j,k} m_j n_{jk} \sum_{j'} n_{j'k} d(\alpha_{j'}) \\ &\geq \sum_k \left(\sum_j m_j n_{jk}\right)^2 = \dim(M_2 \cap N')\end{aligned}$$

thanks to the additivity and the multiplicativity of the statistical dimension d. □

As a corollary of [**8**, Theorem 4.6.3], we have the following ([**10**, Proposition 4.2]) :

Corollary 2.1. *Let $M \supset N$ be an irreducible inclusion of type* II_1 *factors with finite index. Then the following are equivalent :*

(1) *The inclusion $M \supset N$ is of depth* 2.

(2) $\dim(M_1 \cap N') = [M:N]$.

(3) *$M_2 \cap N'$ is a factor.*

We give another lemma to prove main results.

Lemma 2.2. *Let $(P, Q, R, \mathbf{C})$ be a commuting square of finite dimensional algebras. Then we have*

$$\dim P \geq \dim Q \cdot \dim R.$$

Proof. Let us take a linear basis $\{x_1, x_2, \cdots, x_m\}$ for R and a Pimsner-Popa basis $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ for $Q \supset \mathbf{C}$ with respect to the conditional expectation E from P to R ($m := \dim R, n := \dim Q$). Then $x_i\lambda_j^*(\neq 0)$ are linearly independent ; suppose that $\sum_{i,j} a_{ij}x_i\lambda_j^* = 0$ for $a_{ij} \in \mathbf{C}$. Since $0 = \left(\sum_{i,j} a_{ij}x_i\lambda_j^*\right)\lambda_k = E\left(\sum_{i,j} a_{ij}x_i\lambda_j^*\lambda_k\right) = \sum_{i,j} a_{ij}x_iE\left(\lambda_j^*\lambda_k\right) = \sum_i a_{ik}x_i$ for any k, we have $a_{ik} = 0$. Hence, we get $\dim P \geq nm$. □

For a given commuting co-commuting square, we can get a kind of tiling by double sequences $\{M_{ij}\}_{i,j=0,1,2,\cdots}$ of subfactors (see [**22**, **32**]). By looking at a tiling, we have two criteria for an irreducible inclusion $L \supset K$ to be of depth 2 :

Theorem 2.1. *Let (L, M, N, K) be a commuting co-commuting square of type II_1 factors satisfying $[L : K] < \infty$ and $L \cap K' = \mathbf{C}$. If both inclusions $L \supset M$ and $M \supset K$ are of depth 2, then so is the inclusion $L \supset K$.*

Proof. Let us denote extensions by $\{M_{ij}\}_{i,j=0,1,\cdots}$ such that

$$\begin{aligned}(M_{11}, M_{10}, M_{01}, M_{00}) = (L, M, N, K), M_{22} &= \langle M_{11}, e_{00}^{11}\rangle,\\ M_{21} = \langle M_{11}, e_{01}^{11}\rangle, M_{20} &= \langle M_{10}, e_{01}^{11}\rangle,\\ M_{12} = \langle M_{11}, e_{10}^{11}\rangle, M_{33} &= \langle M_{22}, e_{11}^{22}\rangle,\\ M_{32} = \langle M_{22}, e_{12}^{22}\rangle, M_{31} &= \langle M_{21}, e_{12}^{22}\rangle,\\ M_{30} = \langle M_{20}, e_{12}^{22}\rangle, &\quad \text{and so on.}\end{aligned}$$

Here, e_{kl}^{ij} means the Jones projection for the inclusion $M_{ij} \supset M_{kl}$.

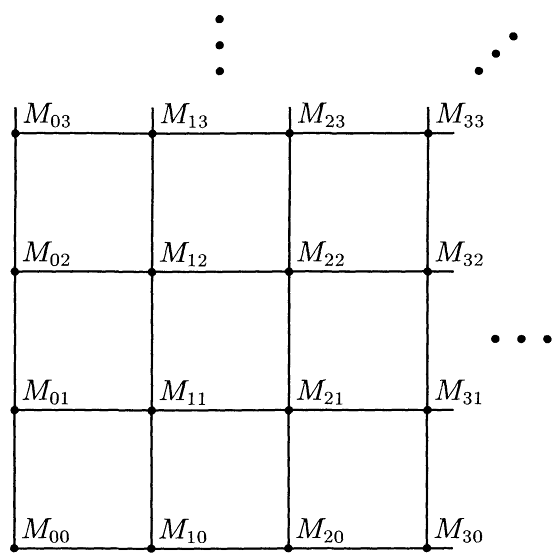

Figure 1.

Clearly, we have $M_{22} \cap M'_{00} \supset M_{20} \cap M'_{00}, M_{12} \cap M'_{10}$. But it can be shown that

$$M_{22} \cap M'_{00} = (M_{20} \cap M'_{00}) \cdot (M_{12} \cap M'_{10}).$$

Let us think of the following commuting square (for the conditional expectations of the restriction of the canonical trace on M_{22}) :

$$\begin{array}{ccc} M_{22} \cap M'_{00} & \supset & M_{12} \cap M'_{10} \\ E\cup & & \cup \\ M_{20} \cap M'_{00} & \supset & \mathbf{C}. \end{array}$$

Here, we remark that $(M_{22}, M_{20}, M_{12}, M_{10})$ forms a commuting square. It follows from Corollary 2.1 that $\dim(M_{20} \cap M'_{00}) = [M : K](=: m)$ and $\dim(M_{12} \cap M'_{10}) = [L : M](=: n)$. Applying Lemma 2.2 to this square, we get $\dim(M_{22} \cap M'_{00}) \geq mn = [L : K]$. Combining this with Lemma 2.1, we have that $\dim(M_{22} \cap M'_{00}) = [L : K]$ and $M_{22} \cap M'_{00} = (M_{20} \cap M'_{00}) \cdot (M_{12} \cap M'_{10})$. Therefore, we get the conclusion by Corollary 2.1. □

Theorem 2.2. *Let (L, M, N, K) be a commuting co-commuting square of type* II_1 *factors satisfying $[L : K] < \infty$ and $L \cap K' = \mathbf{C}$. If both inclusions $M \supset K$ and $N \supset K$(or $L \supset M$ and $L \supset N$) are of depth 2, then so is the inclusion $L \supset K$.*

Proof. Let us keep the same notation as in the proof of Theorem 2.1. It is sufficient to consider the case that $M \supset K$ and $N \supset K$ are of depth 2 since another case can be proved by looking at the extension $(M_{22}, M_{21}, M_{12}, M_{11})$. For the commuting square $(M_{22} \cap M'_{00}, M_{22} \cap M'_{20}, M_{20} \cap M'_{00}, \mathbf{C})$, we remark that

$$M_{22} \cap M'_{20} \cong M_{02} \cap M'_{00}$$

by Proposition 2.2.(3) and Takesaki duality between $M_{21} \supset M_{20}$ and $M_{01} \supset M_{00}$, which follows from a similar argument in [**23**, Proposition 1.5] about a common Pimsner-Popa basis for $M_{10} \supset M_{00}$ and $M_{11} \supset M_{01}$. Applying Lemma 2.2 and 2.1 to the commuting square $(M_{22} \cap M'_{00}, M_{22} \cap M'_{20}, M_{20} \cap M'_{00}, \mathbf{C})$, we get that $M_{22} \cap M'_{00} = (M_{22} \cap M'_{20}) \cdot (M_{20} \cap M'_{00})$, and dim $(M_{22} \cap M'_{00}) = [L : K]$. Therefore, we get the theorem by Corollary 2.1. □

Remark. Let $(L, M, N, K) = (M_{11}, M_{10}, M_{01}, M_{00})$ be a commuting co-commuting square as in Theorem 2.2. The Majid's bicrossproduct method corresponds to looking at the quadruple $(M_{21}, M_{20}, M_{11}, M_{10})$ and the relative commutant algebra $M_{32} \cap M'_{10}$.

3. Examples.

In this section, we will explain two examples. The first one is considered in [**33**, Proposition].

(1) Let G be a finite group with two subgroups A, B satisfying $G = AB$ and $A \cap B = \{e\}$. Let γ be an outer action of G on a type II_1 factor P. Then we have

Proposition 3.1. *The inclusion of crossed product algebras*

$$(L :=)(P \otimes l^{\infty}(G/B)) \rtimes G \supset P \rtimes A(=: K)$$

is irreducible and of depth 2*, where the action of* G *on* $l^\infty(G/B)$ *is induced by the left translation.*

Proof. Let us consider the commuting co-commuting square

$$((P\otimes l^\infty(G/B))\rtimes G,(P\otimes l^\infty(G/B))\rtimes A=:M,P\rtimes G=:N,P\rtimes A).$$

Since $(P\otimes l^\infty(G/B))\rtimes G\cap(P\rtimes A)'=l^\infty(G/B)^A$, the assumption $G=AB$ corresponds to the irreducibility of the inclusion $L\supset K$. Considering Takesaki duality between $L\supset M$ and $P\rtimes B\supset P$ as in the proof of Theorem 2.1, and Proposition 2.2.(1) for $M\supset K(\supset P)$, we also see that $L\supset M$ and $M\supset K$ are of depth 2. Hence, applying Theorem 2.1 to this square (L,M,N,K), we get the conclusion. □

Remark. The Jones tower and the tower of relative commutant algebras can be explicitly written down as in [**3**, **13**, **29**] ; the Jones tower is

$$\begin{gathered}K=P\rtimes A\subset(P\otimes l^\infty(G/B))\rtimes G=L\\ \subset(P\otimes B(l^2(G/B))\otimes l^\infty(G/A))\rtimes G=:L_1\\ \subset(P\otimes B(l^2(G/B))\otimes l^\infty(G/B)\otimes B(l^2(G/A)))\rtimes G=:L_2\\ \cdots.\end{gathered}$$

And the tower of relative commutant algebras is

$$\begin{gathered}\mathbf{C}=K\cap K'\subset L\cap K'=l^\infty(G/B)^A=\mathbf{C}\\ \subset L_1\cap K'=(B(l^2(G/B))\otimes l^\infty(G/A))^A\\ \subset L_2\cap K'=\{B(l^2(G/B))\otimes l^\infty(G/B)\otimes B(l^2(G/A))\}^A\\ \cong B(l^2(G/B))\otimes B(l^2(G/A))\\ \cdots.\end{gathered}$$

Hence, we also see that the depth of $L\supset K$ is 2. Next we recall the matched pair ([**18**, **19**]) ; because of the uniqueness of the decomposition of an element in $G=AB=BA$, we can represent ab for $a\in A,b\in B$ as

$$ab=\alpha_a(b)\beta_{b^{-1}}(a^{-1})^{-1}\in BA.$$

The associative law implies

$$\begin{gathered}\alpha_{aa'}(b)=\alpha_a(\alpha_{a'}(b)),\alpha_a(bb')=\alpha_a(b)\alpha_{\beta_{b^{-1}}(a^{-1})^{-1}}(b'),\\ \beta_{bb'}(a)=\beta_b(\beta_{b'}(a)),\beta_b(aa')=\beta_b(a)\beta_{\alpha_{a^{-1}}(b^{-1})^{-1}}(a')\end{gathered}$$

for $a, a' \in A, b, b' \in B$. Therefore, the matched pair (A, B, α, β) in [**18**, Theorem 2.3] appears. (Here, we remark that if we write $ab = \gamma_a(b^{-1})^{-1}\delta_{b^{-1}}(a)(\in BA)$, then the matched pair of another type (A, B, γ, δ) in [**19**] is obtained, but in this article we would like to treat the former one for our purpose.)

For the matched pair (A, B, α, β), we have a finite dimensional Kac algebra of Majid's type ([**19**]) ; the bicrossproduct Kac algebra consists of the crossed product algebra $Q := l^\infty(B) \rtimes_\alpha A$ on $l^2(B) \otimes l^2(A)$ (and others, see below) generated by $m_f \otimes 1$ (simply denoted by $f \otimes 1 = f$) for $f \in l^\infty(B)$ and $u_a \otimes \lambda_a$ (simply denoted by λ_a) for $a \in A$, where m_f is the pointwise multiplication operator on $l^2(B)$, the action α of A on $l^\infty(B)$ is induced by the action α of A on B ; $\alpha_a(f)(b) = f(\alpha_{a^{-1}}(b))$ for $f \in l^\infty(B)$, u_a is the implementing unitary on $l^2(B)$ such that $(u_a\xi)(b) = \xi(\alpha_{a^{-1}}(b))$ for $\xi \in l^2(B)$, and λ_a is the left regular translation; $(\lambda_a\xi)(a') = \xi(a^{-1}a')$ for $\xi \in l^2(A)$.

We know the Kac algebra structure of this crossed product algebra and its dual Kac algebra ([**19**]) ; for the crossed product algebra $Q = l^\infty(B) \rtimes_\alpha A$, the comultiplication Γ, the antipode κ, and the Haar weight ψ are described by :

$$\begin{aligned}
\Gamma(\chi_b) &= \sum_{b'b''=b} \chi_{b'} \otimes \chi_{b''}, \\
\Gamma(\lambda_a) &= \sum_b \chi_b\lambda_a \otimes \lambda_{\beta_{b^{-1}}(a)}, \\
\kappa(\chi_b) &= \chi_{b^{-1}}, \\
\kappa(\lambda_a) &= \sum_b \chi_b\lambda_{\beta_b(a^{-1})}, \\
\psi\left(\sum_a f_a\lambda_a\right) &= \frac{1}{|B|}\sum_b f_e(b)
\end{aligned}$$

for $f_a \in l^\infty(B)$, and χ_b is the characteristic function on $b \in B$. And we have

$$(l^\infty(B) \rtimes_\alpha A)\hat{} = B_\beta \ltimes l^\infty(A),$$

where the right-hand side is generated by $1 \otimes f (f \in l^\infty(A))$ and $\lambda_b \otimes v_b$ $(b \in B)$ on $l^2(B) \otimes l^2(A)((v_b\xi)(a) = \xi(\beta_{b^{-1}}(a)),\ \xi \in l^2(A))$.

The above Kac algebra $\mathbf{K} = (l^\infty(B) \rtimes_\alpha A(= Q), \Gamma, \kappa, \psi)$ has a left action ([**5**]) on the factor $K = P \rtimes_\gamma A$; let us write two generators $\pi(p)(p \in P)$ and $\lambda'_a(a \in A)$ of $P \rtimes_\gamma A$:

$$(\pi(p)\xi)(a') = \gamma_{a'^{-1}}(p)\xi(a'), (\lambda'_a\xi)(a') = \xi(a^{-1}a')$$

for $\xi \in l^2(A, L^2(P))$.

Lemma 3.1. *The following map $\delta_K : K \to K \otimes Q$ gives a left action of the Kac algebra* $\mathbf{K}$ *on* $K = P \rtimes A$:

$$\delta_K(\pi(p)) = \sum_b \pi(\gamma_{b^{-1}}(p)) \otimes \chi_b,$$
$$\delta_K(\lambda'_a) = \sum_b \lambda'_{\beta_{b^{-1}}(a)} \otimes \chi_b \lambda_a.$$

This lemma follows from direct computation, hence the author leaves its proof to the reader.

So far, we are now ready to give the theorem.

Theorem 3.1. *The factor $(P \otimes l^\infty(G/B)) \rtimes G$ is described as the crossed product algebra of $P \rtimes A$ by the left action δ_K in Lemma* 3.1 *of the Majid's bicrossproduct algebra* $\mathbf{K} = (l^\infty(B) \rtimes A, \Gamma, \kappa, \psi)$.

Proof. We may think that three kinds of generators $\tilde{\pi}(p)$ $(p \in P), \tilde{\pi}(f)$ $(f \in l^\infty(G/B))$, and $\tilde{\lambda}_g$ $(g \in G)$ of the crossed product algebra $(P \otimes l^\infty(G/B)) \rtimes G$ look like

$$(\tilde{\pi}(p)\xi)(aB, g') = \gamma_{g'^{-1}}(p)\xi(aB, g'),$$
$$(\tilde{\pi}(f)\xi)(aB, g') = f(aB)\xi(aB, g'),$$
$$(\tilde{\lambda}_g\xi)(aB, g') = \xi(g^{-1}aB, g^{-1}g')$$

for $\xi \in l^2(G/B \times G, L^2(P))$. Identifing $G/B \times G$ with $A \times B \times A$ by

$$(a'B, g = ba) \leftrightarrow (a, b, a'),$$

we may write these generators acting on $L^2(P) \otimes l^2(A) \otimes l^2(B) \otimes l^2(A)$ such as

$$(\tilde{\pi}\,(p)\,\xi)\,(a,b,a') = \gamma_{(ba)^{-1}}\,(p)\,\xi\,(a,b,a')\,,$$
$$(\tilde{\pi}\,(f)\,\xi)\,(a,b,a') = f\,(a')\,\xi\,(a,b,a')\,,$$
$$\left(\tilde{\lambda}_{\tilde{a}}\xi\right)(a,b,a') = \xi\left(\beta_{b^{-1}}\,(\tilde{a})^{-1}\,a, \alpha_{\tilde{a}^{-1}}\,(b)\,, \tilde{a}^{-1}a'\right),$$
$$\left(\tilde{\lambda}_{\tilde{b}}\xi\right)(a,b,a') = \xi\left(a, \tilde{b}^{-1}b, \beta_{\tilde{b}^{-1}}\,(a')\right)$$

for $f \in l^\infty(A), \tilde{a} \in A, \tilde{b} \in B$ and $\xi \in l^2(A \times B \times A, L^2(P))$. On the other hand, the crossed product algebra of N by the (outer) action δ_K of $l^\infty(B) \rtimes A$ is generated by $\delta_K(K) \vee 1 \otimes (l^\infty(B) \rtimes A)\hat{}= \delta_K(K) \vee 1 \otimes (B \ltimes l^\infty(A))$. (See [**5**].) It is easy to see that $\delta_K(\pi(p)) = \tilde{\pi}(p), \delta_K(\lambda'_a) = \tilde{\lambda}_a, 1 \otimes \lambda_b = \tilde{\lambda}_b$, and $1 \otimes f = \tilde{\pi}(f)$. Therefore, we are done. $\square$

(2) Let $M \supset N$ be an irreducible inclusion of type II_1 factors satisfying $[M : N] < \infty$ and depth 2, and G be a finite group with an outer action γ on both M and N. Moreover, suppose that $(M \rtimes G) \cap N' = \mathbf{C}$. (This condition is equivalent to strong outerness of the action γ for $M \supset N$.) Then we have the depth 2 inclusion $(M \otimes l^\infty(G)) \rtimes G \supset N \rtimes G$. In fact, this inclusion is contained in the commuting co-commutig square $((M \otimes l^\infty(G)) \rtimes G, (N \otimes l^\infty(G)) \rtimes G, M \rtimes G, N \rtimes G)$. Similar argument as in Proposition 3.1 implies that the assumption in Theorem 2.1 for the inclusions $(M \otimes l^\infty(G)) \rtimes G \supset (N \otimes l^\infty(G)) \rtimes G(\cong M \supset N$ by Takesaki duality) and $(N \otimes l^\infty(G)) \rtimes G \supset N \rtimes G$ holds, hence we have the conclusion. (Cf. the orbifold construction [**3**, **7**] and also [**2**].)

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