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**SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS  
WITH FULLY NONLINEAR TWO-POINT BOUNDARY  
CONDITIONS. I**

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## SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH FULLY NONLINEAR TWO POINT BOUNDARY CONDITIONS

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We establish existence results for two point boundary value problems for second order ordinary differential equations of the form  $y'' = f(x, y, y')$ ,  $x \in [0, 1]$ , where  $f$  is continuous and there exist lower and upper solutions. First we consider boundary conditions of the form  $G((y(0), y(1)); (y'(0), y'(1))) = 0$ , where  $G$  is continuous and fully nonlinear. We introduce compatibility conditions between  $G$  and the lower and upper solutions. Assuming these compatibility conditions hold and, in addition,  $f$  satisfies assumptions guaranteeing a priori bounds on the derivatives of solutions we show that solutions exist. In the case the lower and upper solutions are constants one of our results is closely related to a result of Gaines and Mawhin. Secondly we consider boundary conditions of the form  $(y(i), y'(i)) \in \mathcal{J}(i)$ ,  $i = 0, 1$  where the  $\mathcal{J}(i)$  are closed connected subsets of the plane. We introduce various compatibility type conditions relating the  $\mathcal{J}(i)$  and the lower and upper solutions and show each is sufficient to construct a compatible  $G$  which defines these boundary conditions. Thus our existence results apply. Almost all the standard boundary conditions considered in the literature assuming upper and lower solutions are, or can be, defined by compatible  $G$  and their associated existence results follow from ours; in many cases we can improve these results by deleting some of their assumptions.

### 1. Introduction.

In this paper we consider two point boundary value problems for second order ordinary differential equations of the form

$$(1.1) \quad y'' = f(x, y, y'), \text{ for all } x \in [0, 1],$$

where  $f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  continuous. By a solution of (1.1) we mean a twice continuously differentiable function  $y$  satisfying (1.1) everywhere. The first class of boundary conditions we will consider are of the form

$$(1.2) \quad 0 = G((y(0), y(1)); (y'(0), y'(1))),$$

where  $G = (g^0, g^1), g^i : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $i = 1, 2$  are continuous and fully nonlinear. We will refer to boundary conditions of the form (1.2) as fully nonlinear boundary conditions. The second class of boundary conditions we will consider are of the form

$$(1.3) \quad (y(i), y'(i)) \in \mathcal{J}(i) \text{ for } i = 0, 1,$$

where  $\mathcal{J}(i)$  are continua. We will refer to boundary conditions of the form (1.3) as boundary set conditions.

We always assume that lower and upper solutions  $\alpha \leq \beta$ , respectively, exist for (1.1) (see Definition 1 below).

In paragraph 2 we introduce some notation and definitions.

In paragraph 3 we introduce the central notion of compatibility of the boundary conditions  $G$  with the lower and upper solutions. In the literature when lower and upper solutions are assumed to exist and the Picard, Neumann or Periodic boundary conditions are considered the assumptions usually made are equivalent to compatibility. We show by simple examples that if the boundary conditions are not compatible with the lower and upper solutions, then solutions need not exist.

In paragraph 4, we present our main existence results. If the boundary conditions  $G$  are compatible with  $\alpha$  and  $\beta$  and  $f$  satisfies additional assumptions guaranteeing a priori bounds for  $y'$  for solutions  $y$  of (1.1), then there exist solutions  $y$  of (1.1) and (1.2) satisfying  $\alpha \leq y \leq \beta$  on  $[0, 1]$ .

In paragraph 5 we briefly describe the results of Gaines and Mawhin [16] and show their relationship to ours.

In paragraph 6 we consider problem (1.1) and (1.3). We introduce two types of compatibility of the boundary sets  $\mathcal{J}(i)$ ,  $i = 0, 1$  with the lower and upper solutions. These are satisfied by the usual boundary set conditions considered in the literature. Given compatible boundary sets  $\mathcal{J}(i)$ ,  $i = 0, 1$  we show that there exists compatible  $G$  such that (1.2) implies (1.3). Thus our existence results apply to such boundary set conditions.

Lower and upper solutions exist for (1.1) if  $f$  satisfies suitable monotonicity and or growth conditions (see Ako [2, 3], Baxley [6], Gaines [15], Gaines and Mawhin [16], Palamides [30], and Jackson and Palamides [22]).

A priori bounds on  $y'$  follow if, for example,  $f$  satisfies either the Bernstein–Nagumo growth condition with respect to  $y'$  (see Bernstein [12], Nagumo [27]) or its one sided generalisations (see Baxley [6]) or the Scorza Dragoni–Zwirner growth condition with respect to  $(x, y, y')$  or the Nagumo–Knobloch–Ako–Schmitt condition (see Ako [2], Nagumo [28, 29], Knobloch [23, 24], Schmitt [31]) or a Lyapunov condition (see George and Sutton [18]).

To prove existence of solutions we modify the differential equation and turn the modified equation into an integral equation, couple it with the two

equations defining the boundary conditions and regard the resultant as an operator equation defined on  $C^1([0, 1]) \times \mathbb{R}^2$ , where the boundary values are regarded as independent variables with values in  $\mathbb{R}^2$ . We use Schauder degree theory employing homotopies and the reduction theorem to show that the Schauder degree of the operator is the Brouwer degree of a suitable mapping associated with the compatible  $G$  on the domain which contains the boundary values.

Methods used in the literature to establish existence results include shooting with initial values, shooting with boundary values, the Schauder fixed point theorem, and Schauder degree theory. Often these methods are applied to a modified problem whose solutions are solutions of the unmodified problem. Some variants of shooting with initial values use the Kneser–Hukuhara continuum theorem and/or Ważewski’s retract method and their refinements. Initial value shooting has been commonly used to prove existence for fully nonlinear boundary conditions of the form (1.2) and has been the only method used for boundary set conditions of the form (1.3).

Gaines and Mawhin [16] and Guenther Granas and Lee [19] give very general existence theorems via coincidence degree, respectively topological transversality, provided associated one parameter families of boundary value problems have no solutions on the boundary of a suitable domain in a suitable function space. These one parameter families are used to construct homotopies. Gaines and Mawhin and Granas Guenther and Lee go on to show that a substantial number of the earlier existence results follow from their general theorems. Also they go on to give many and substantial new results and a coherent framework for viewing old and new results. Gaines and Mawhin [16], Theorem V.34 establish an existence result for systems of equations which, in the special case of a single equation, is closely related to the special case of our result above of constant  $\alpha$  and  $\beta$ . They apply their result [16], Theorem V.37 to a single equation with non constant  $\alpha$  and  $\beta$  and a restricted class of  $G$  using a modification argument, modifying  $G$ . The interdependence required between the boundary conditions and the lower and upper solutions to guarantee existence of solutions is not clear from their work.

In a forth coming paper we extend our result to systems including [16], Theorem V.34 as a special case. Also, Gaines and Mawhin [16] discuss a priori bounding of solutions from a geometric perspective giving a new insight into the role of many of these conditions.

Ako was one of the first authors to obtain existence results for nonlinear boundary conditions. He used shooting with the boundary values of minimal solutions as he did not assume uniqueness for the Picard problem;  $(y'(0), y'(1))$  is a continuous function of  $(y(0), y(1))$  if solutions of the Picard

problem are unique.

Existence results for boundary set conditions assuming lower and upper solutions have been obtained by a number of authors (see, for example, Jackson and Klassen [21] and Bebernes and Fraker [9], and their references).

The literature on problem (1.1) and (1.2) is vast and for further information we refer the interested reader to the excellent monographs by Bailey, Waltman and Shampine [4], Bernfeld and Lakshmikantham [11], Gaines and Mawhin [16], Guenther, Granas and Lee [19], Hartman [20], and Mawhin [26] and their references.

The contributions this work makes are twofold. First we introduce the compatibility conditions. These conditions are concrete conditions involving the given data which can be easily checked and are satisfied by just about every concrete existence result in the literature. They permit the construction of the one parameter families of boundary value problems used to construct the homotopies in an appropriate function space; both the homotopies and the function space are unusual and clearly demonstrate the role of the compatibility conditions.

Second, we show that the boundary set conditions of the form (1.3) usually considered in the literature are special cases of fully nonlinear boundary conditions.

Most existence results in the literature for (1.1) together with (1.2) or (1.3) which assume lower and upper solutions exist follow as a corollary to our results. In many cases our results can be used to significantly improve upon these results. This is especially true for results concerning fully nonlinear boundary conditions. Some results in the literature concerned both with linear and with nonlinear boundary conditions which assume growth conditions but do not assume explicitly the existence of lower and upper solutions can be obtained from ours by constructing lower and upper solutions compatible with the boundary conditions. In such cases the construction of the lower and upper solutions and the verification of compatibility is usually easier than the given direct proofs of existence. This is true, for example, of some of the results in Baxley [6]. Also the central notion of compatibility extends to Carathéodory  $f$  with  $\alpha$  and  $\beta$  having absolutely continuous first derivatives, to systems with lower and upper solutions, to single equations and systems with lower and upper solutions replaced by other surfaces a priori bounding solutions. We will discuss these extensions of our ideas and further applications of our results and their extensions in forthcoming papers.

### 2. Background Notation and Definitions.

In order to state our results we need some notation.

We denote the closure of a set  $T$  by  $\bar{T}$  and its boundary by  $\partial T$ . As usual,  $C^m(A; B)$  denotes the space of  $m$  times continuously differentiable functions from  $A$  to  $B$  endowed with the maximum norm. In the case of continuous functions we abbreviate this to  $C(A; B)$ . In the case  $B = \mathbb{R}$  we omit the  $B$ . If  $A$  is a bounded open subset of  $\mathbb{R}^n$ ,  $p \in \mathbb{R}^n$ ,  $f \in C(\bar{A}; \mathbb{R}^n)$  and  $p \notin f(\partial A)$  we denote the Brouwer degree of  $f$  on  $A$  at  $p$  by  $d(f, A, p)$ . It is common in the proofs of existence of solutions of two point boundary value problems for (1.1) to modify  $f$ . We will do this making use of the following functions (see [33]).

If  $c \leq d$  are given let  $\pi : \mathbb{R} \rightarrow [c, d]$  be the retraction given by

$$(2.1) \quad \pi(y, c, d) = \max\{\min\{d, y\}, c\}.$$

For each  $\epsilon > 0$ , let  $K \in C(\mathbb{R} \times (0, \infty); [-1, 1])$  satisfy

1.  $K(\cdot, \epsilon)$  is an odd function,
2.  $K(t, \epsilon) = 0$  iff  $t = 0$  and
3.  $K(t, \epsilon) = 1$  for all  $t \geq \epsilon$ .

If  $c \leq d$  and  $\epsilon > 0$  are given, let  $T \in C(\mathbb{R})$  be given by

$$(2.2) \quad T(y, c, d, \epsilon) = K(y - \pi(y, c, d), \epsilon).$$

Let

$$Q(x, t) = \begin{cases} (1-x)t, & \text{for } 0 \leq t \leq x \leq 1 \\ (1-t)x, & \text{for } 0 \leq x \leq t \leq 1, \end{cases}$$

and  $w(y_0, y_1)(x) = y_0(1-x) + y_1x$ . Let  $X = C^1([0, 1]) \times \mathbb{R}^2$  with the usual product norm. Define  $\mathcal{C} : C([0, 1]) \rightarrow C^1([0, 1])$  by

$$\mathcal{C}(\phi)(x) = - \int_0^1 Q(x, t)\phi(t)dt,$$

for all  $\phi \in C([0, 1])$  and  $x \in [0, 1]$ . Clearly  $\mathcal{C}$  is completely continuous.

**Definition 1.** We call  $\alpha$  ( $\beta$ ) a lower (upper) solution for (1.1) if  $\alpha$  ( $\beta$ )  $\in C^2([0, 1])$ , and

$$(2.3) \quad \begin{aligned} \alpha''(x) &\geq f(x, \alpha(x), \alpha'(x)), \text{ for all } x \in [0, 1] \\ (\beta''(x) &\leq f(x, \beta(x), \beta'(x)), \text{ for all } x \in [0, 1]). \end{aligned}$$

If the inequality in (2.3) is strict then we call  $\alpha$  ( $\beta$ ) a strict lower (upper) solution for (1.1). As mentioned earlier we assume that  $\alpha \leq \beta$  and set

$\beta_M = \max\{\beta(x) : x \in [0, 1]\}$  and  $\alpha_m = \min\{\alpha(x) : x \in [0, 1]\}$ . We will call the pair non-degenerate if  $\Delta = (\alpha(0), \beta(0)) \times (\alpha(1), \beta(1))$  is nonempty. We set

$$(2.4) \quad \bar{\omega} = \{(x, y) \in [0, 1] \times \mathbb{R} : \alpha(x) \leq y \leq \beta(x)\}.$$

We will discuss the degenerate case  $\Delta$  empty later.

Lower solutions are used with maximum principle arguments to obtain a priori bounds on solutions. For a discussion of the relationship between them and subfunctions and some indication of how the smoothness conditions can be relaxed see Thompson [35]. As mentioned earlier our central idea leads to existence results for those  $f$  for which there are a priori bounds on  $y'$  for solutions  $y$  satisfying  $\alpha \leq y \leq \beta$ . There are two well known conditions which we employ in our existence results either of which guarantee a priori bounds on  $y'$  for solutions. The first is the Bernstein–Nagumo condition.

**Definition 2.** Let  $\alpha \leq \beta$  be lower and upper solutions for (1.1) on  $[0, 1]$ . We say  $f$  satisfies the Bernstein–Nagumo condition if there exists  $h \in C([0, \infty); (0, \infty))$  and  $N > 0$  such that

$$(2.5) \quad |f(x, y, p)| \leq h(|p|), \text{ for all } (x, y) \in [0, 1] \times [\alpha(x), \beta(x)] \text{ and}$$

$$(2.6) \quad \int_{\sigma}^N \frac{sd s}{h(s)} > \beta_M - \alpha_m$$

where  $\sigma = \max\{|\beta(1) - \alpha(0)|, |\beta(0) - \alpha(1)|\}$ . We say  $f$  satisfies the strengthened Bernstein–Nagumo condition if (2.6) is replaced by

$$(2.7) \quad \int^{\infty} \frac{sd s}{h(s)} = \infty.$$

The second condition for guaranteeing a priori bounds on  $y'$  for solutions we call the Nagumo–Knobloch–Schmitt condition.

**Definition 3.** Let  $\alpha \leq \beta$  be lower and upper solutions for (1.1) on  $[0, 1]$ . We say  $f$  satisfies the Nagumo–Knobloch–Schmitt condition relative to  $\alpha$  and  $\beta$  if there exists  $\Phi < \Upsilon \in C^1([0, 1] \times \mathbb{R})$  such that

$$(2.8) \quad \begin{aligned} f(x, y, \Phi(x, y)) &\geq \Phi_x(x, y) + \Phi_y(x, y)\Phi(x, y) \text{ and} \\ f(x, y, \Upsilon(x, y)) &\leq \Upsilon_x(x, y) + \Upsilon_y(x, y)\Upsilon(x, y), \end{aligned}$$

for all  $(x, y) \in \bar{\omega}$ .

See Gaines and Mawhin [16] for some discussion of the relationship between these conditions.

### 3. Nonlinear Boundary Conditions and Compatibility.

**Definition 4.** We call the vector field  $\Psi = (\psi^0, \psi^1) \in C(\bar{\Delta}; \mathbb{R}^2)$  strongly inwardly pointing on  $\Delta$  if for all  $(C, D) \in \partial\Delta$

$$(3.1) \quad \begin{aligned} \psi^0(\alpha(0), D) &> \alpha'(0), \quad \psi^0(\beta(0), D) < \beta'(0) \text{ and} \\ \psi^1(C, \alpha(1)) &< \alpha'(1), \quad \psi^1(C, \beta(1)) > \beta'(1). \end{aligned}$$

We call  $\Psi$  inwardly pointing if the strict inequalities are replaced by weak inequalities.

**Definition 5.** Let  $G \in C(\bar{\Delta} \times \mathbb{R}^2; \mathbb{R}^2)$ . We say  $G$  is strongly compatible with  $\alpha$  and  $\beta$  if for all strongly inwardly pointing  $\Psi$  on  $\bar{\Delta}$

$$(3.2) \quad \mathcal{G}(C, D) \neq 0 \text{ for all } (C, D) \in \partial\Delta \text{ and}$$

$$(3.3) \quad d(\mathcal{G}, \Delta, 0) \neq 0,$$

where

$$(3.4) \quad \mathcal{G}(C, D) = G((C, D); \Psi(C, D)) \text{ for all } (C, D) \in \bar{\Delta}.$$

We say  $G$  is compatible with  $\alpha$  and  $\beta$  if there is a sequence  $G_i \in C(\bar{\Delta} \times \mathbb{R}^2; \mathbb{R}^2)$  strongly compatible with  $\alpha$  and  $\beta$  and converging uniformly to  $G$  on compact subsets of  $\bar{\Delta} \times \mathbb{R}^2$ .

In what follows where there is a strongly inwardly pointing vector field clearly defined from the context  $\mathcal{G}$  will denote the vector field defined by (3.4).

**Remark 6.** If  $G$  is (strongly) compatible with  $\alpha$  and  $\beta$ , then the Brouwer degree (3.3) is independent of the strongly inwardly pointing vector field  $\Psi$ . To see this for strongly compatible  $G$  let  $\Psi_i, i = 1, 2$  be two such vector fields. Then setting  $\Psi(C, D, \theta) = \theta\Psi_1(C, D) + (1 - \theta)\Psi_2(C, D)$  and  $\mathcal{H}(C, D, \theta) = G((C, D); \Psi(C, D, \theta))$  on  $\bar{\Delta} \times [0, 1]$ ,  $\mathcal{H}$  is a homotopy for the Brouwer degree (3.3).

It is not difficult to see that strongly inwardly pointing vector fields always exist. Moreover, if (3.2) holds for all inwardly pointing vector fields we may choose

$$\begin{aligned} \psi^0(C, D) &= \beta'(0) \frac{C - \alpha(0)}{\beta(0) - \alpha(0)} + \alpha'(0) \frac{\beta(0) - C}{\beta(0) - \alpha(0)} \text{ and} \\ \psi^1(C, D) &= \beta'(1) \frac{D - \alpha(1)}{\beta(1) - \alpha(1)} + \alpha'(1) \frac{\beta(1) - D}{\beta(1) - \alpha(1)}, \end{aligned}$$

when computing the Brouwer degree (3.3).

For the Picard (also called Dirichlet) boundary conditions

$$(3.5) \quad \begin{aligned} g^0((y(0), y(1)); (y'(0), y'(1))) &= y(0) - A = 0 \text{ and} \\ g^1((y(0), y(1)); (y'(0), y'(1))) &= y(1) - B = 0, \end{aligned}$$

while for the Neumann boundary conditions

$$(3.6) \quad \begin{aligned} g^0((y(0), y(1)); (y'(0), y'(1))) &= y'(0) - A = 0 \text{ and} \\ g^1((y(0), y(1)); (y'(0), y'(1))) &= y'(1) - B = 0, \end{aligned}$$

and for the Periodic boundary conditions

$$(3.7) \quad \begin{aligned} g^0((y(0), y(1)); (y'(0), y'(1))) &= y(0) - y(1) = 0 \text{ and} \\ g^1((y(0), y(1)); (y'(0), y'(1))) &= y'(1) - y'(0) = 0. \end{aligned}$$

For these boundary conditions the compatibility conditions become the familiar ones usually assumed in the presence of lower and upper solutions; that is,

$$(3.8) \quad \alpha(0) \leq A \leq \beta(0), \quad \alpha(1) \leq B \leq \beta(1),$$

$$(3.9) \quad \alpha'(0) \geq A, \quad \beta'(0) \leq A, \quad \alpha(1) \leq B, \quad \beta(1) \geq B,$$

and

$$(3.10) \quad \alpha(0) = \alpha(1), \quad \beta(0) = \beta(1), \quad \alpha'(0) \geq \alpha'(1), \quad \beta'(0) \leq \beta'(1),$$

respectively. We prove this in the case of periodic boundary conditions.

**Lemma 8.** *The periodic boundary conditions are (strongly) compatible iff (3.10) holds.*

*Proof.* Let  $\Psi$  be an strongly inwardly pointing vector field on  $\bar{\Delta}$ .

Assume that (3.10) is satisfied, let  $G = (g^0, g^1)$  be given by (3.7) and  $(C, D) \in \partial\Delta$ . If  $C = D = \alpha(0)$  then  $\mathcal{G}^1 = \psi^1(C, D) - \psi^0(C, D) < \alpha'(1) - \alpha'(0) \leq 0$ . If  $C = \alpha(0) < D$  then  $\mathcal{G}^0 = C - D < 0$ . Similar inequalities hold for the other cases  $(C, D) \in \partial\Delta$ . Thus  $\mathcal{G} \neq 0$  for  $(C, D) \in \partial\Delta$ . Let  $\gamma(x) = (\alpha(x) + \beta(x))/2$ ,  $\mathcal{H}(C, D, \theta) = (1 - 2\theta)\mathcal{G}(C, D) + 2\theta(\mathcal{G}^0(C, D), D - \gamma(1)/2)$ , for  $\theta \in [0, 1/2]$  and  $\mathcal{H}(C, D, \theta) = (2 - 2\theta)(\mathcal{G}^0(C, D), D - \gamma(1)/2) + (2\theta - 1)(C - \gamma(0)/2, D - \gamma(1)/2)$ , for  $\theta \in [1/2, 1]$ . Since  $\mathcal{H}$  is a homotopy for Brouwer degree  $d(\mathcal{G}(\cdot), \Delta, 0) = d(\mathcal{H}(\cdot, 0), \Delta, 0) = d(\mathcal{H}(\cdot, 1), \Delta, 0) = 1 \neq 0$ . Thus  $G$  is strongly compatible and hence compatible.

Assume now that  $G$  is given by (3.7) and that  $G$  is strongly compatible with  $\alpha$  and  $\beta$ . We show that (3.10) is satisfied.

We may assume that  $[\alpha(0), \beta(0)] \cap [\alpha(1), \beta(1)] \neq \emptyset$  otherwise for any strongly inwardly pointing vector field  $\Psi$  setting  $\mathcal{G} = G((C, D)); \Psi(C, D)$ ,  $\mathcal{G}(C, D) \neq 0$  for all  $(C, D) \in \bar{\Delta}$  and  $d(\mathcal{G}, \Delta, 0) = 0$ , a contradiction. Assume that  $\alpha(0) \neq \alpha(1)$  and in particular that  $\alpha(0) < \alpha(1) = D = C \leq \beta(0)$ . Thus we may choose a strongly inwardly pointing vector field  $\Psi$  as follows. First choose  $\psi^1$  then define  $\psi^0$  such that  $\Psi$  is strongly inwardly pointing and  $\psi^0(C, D) < \min\{\beta'(0), \psi^1(C, D)\}$  for all  $C > \gamma(0)/2$ . Thus  $\mathcal{G}(C, D) \neq 0$  for all  $(C, D) \in \bar{\Delta}$  and again  $d(\mathcal{G}, \Delta, 0) = 0$ , a contradiction. The other cases that  $\alpha(0) \neq \alpha(1)$  and  $\beta(0) \neq \beta(1)$  follow by similar arguments.

Assume now that  $\alpha'(1) > \alpha'(0)$ . Define a strongly inwardly pointing vector field  $\Psi$  as follows. Let  $\alpha'(1) > \psi^1(C, \alpha(1)) > \alpha'(0)$  for all  $C \in [\alpha(0), \beta(0)]$  and  $\psi^1(C, \beta(1)) > \beta'(1)$  for all  $C \in [\alpha(0), \beta(0)]$ . Using the Teitze extension theorem (see Dugundji [13]) we may extend  $\psi^1$  as a continuous function to  $\bar{\Delta}$ . By continuity and compactness we may choose  $\epsilon > 0$  such that  $\psi^1(C, \alpha(1)) - \epsilon > \alpha'(0)$  for all  $C \in [\alpha(0), \beta(0)]$ . Let  $\Xi = (\xi^0, \xi^1)$  be a strongly inwardly pointing vector field. Set  $\psi^0(C, D) = \min\{\xi^0(C, D), \psi^1(D, C) - \epsilon\}$ . Thus  $\Psi$  is strongly inwardly pointing and  $\mathcal{G}^1 \neq 0$  on  $\bar{\Delta}$ , where  $\mathcal{G}$  is defined by (3.4). Thus  $d(\mathcal{G}(\cdot), \Delta, 0) = 0$ , a contradiction. The other case  $\beta'(0) > \beta'(1)$  leads similarly to a contradiction. Thus (3.10) holds. If  $G$  is compatible then there exist strongly compatible  $G_i$  and the result follows.  $\square$

The proof that the Picard, respectively Neumann, boundary conditions are compatible iff (3.8), respectively (3.9), is satisfied is simpler, follows similar lines and hence is omitted.

**Remark 8.** In the following two examples  $G$  is not strongly compatible since condition (3.3) fails. In the first there are solutions of (1.1) and (1.2) lying between  $\alpha$  and  $\beta$  and in the second there are no such solutions.

We choose  $R > 0$  and

$$G \in C([-R, R]^2 \times \mathbb{R}^2; \mathbb{R}^2)$$

such that

$$G((C, D); (P, Q)) = G((C, D); (S, T))$$

for all

$$((C, D); (P, Q)), ((C, D); (S, T)) \in [-R, R]^2 \times \mathbb{R}^2, \quad \mathcal{G} \neq 0$$

on  $\partial(-R, R)^2$  and  $d(\mathcal{G}, (-R, R)^2, 0) = 0$ ;  $\mathcal{G}$  is independent of the choice of strongly inwardly pointing vector field  $\Psi$ . Let  $f$  be identically zero and  $-\alpha = R = \beta$ .

For the first example we choose  $G$  so that  $G((C, D); (P, Q)) = 0$ , for some  $(C, D) \in (-R, R)^2$ .

For the second example we choose  $G$  so that  $G \neq 0$  for any  $(C, D) \in (-R, R)^2$ .

#### 4. Existence of Solutions.

**Theorem 1.** *Assume that there exist non-degenerate lower and upper solutions  $\alpha \leq \beta$  for (1.1), that  $f$  satisfies the Bernstein–Nagumo condition and that  $G \in C(\bar{\Delta} \times \mathbb{R}^2; \mathbb{R}^2)$  is compatible with  $\alpha$  and  $\beta$ , then problem (1.1) and (1.2) has a solution  $y$  lying between  $\alpha$  and  $\beta$ .*

*Proof.* Assume first that  $G$  is strongly compatible with  $\alpha$  and  $\beta$ .

We modify the differential equation for  $y$  not between  $\alpha$  and  $\beta$  to obtain a second pair of lower and upper solutions. We reformulate the problem as a coupled system of integral and boundary condition equations and show that a solution of the modified problem lies in the region where  $f$  is unmodified and hence is the required solution. We use Schauder degree theory to prove existence for the modified problem and compute the degree using a homotopy; the modification is chosen to facilitate the construction of a suitable homotopy.

Choose  $L, \epsilon > 0$  such that

$$(4.1) \quad \int_{\sigma}^L \frac{sd s}{h(s) + \epsilon} > \beta_M - \alpha_m + 2\epsilon$$

where  $L > \max\{|\alpha'(x)|, |\beta'(x)| : x \in [0, 1]\}$ . Let

$$\begin{aligned} j(x, y, p) &= f(x, \pi(y, \alpha(x), \beta(x)), \pi(p - L, L)), \text{ and} \\ k(x, y, p) &= (1 - |T(y, \alpha(x), \beta(x), \epsilon)|)j(x, y, p) + \\ &\quad T(y, \alpha(x), \beta(x), \epsilon)(|j(x, y, p)| + \epsilon), \end{aligned}$$

where  $\pi$  and  $T$  are given by (2.1) and (2.2), respectively. Thus  $k$  is a bounded continuous function on  $[0, 1] \times \mathbb{R}^2$  and satisfies

$$|k(x, y, p)| \leq h(|p|) + \epsilon,$$

for all  $p$  with  $|p| \leq L$ .

Consider

$$(4.2) \quad y'' = k(x, y, y'), \text{ for all } x \in [0, 1]$$

together with (1.2). It suffices to show that problem (4.2) and (1.2) has a solution  $y$  satisfying  $\alpha \leq y \leq \beta$  and  $|y'| \leq L$  on  $[0, 1]$  since  $f$  and  $k$  coincide in this region.

Let

$$\begin{aligned} \alpha_\epsilon &= \alpha_m - \epsilon \\ \beta_\epsilon &= \beta_M + \epsilon \end{aligned}$$

Now

$$\begin{aligned} \alpha_\epsilon'' &= 0 > -(|j(x, \alpha_\epsilon, 0)| + \epsilon) \\ &= k(x, \alpha_\epsilon, \alpha_\epsilon'), \text{ for all } x \in [0, 1] \end{aligned}$$

so  $\alpha_\epsilon$  is a lower solution for (4.2). Similarly  $\beta_\epsilon$  is an upper solution for (4.2).

Also, by (4.1),

$$(4.3) \quad \int_\sigma^L \frac{s ds}{h(s) + \epsilon} > \beta_\epsilon - \alpha_\epsilon.$$

Suppose that  $y$  is a solution of (4.2) and  $(y(0), y(1)) \in \bar{\Delta}$ . We show that  $y$  is a solution of (1.1). We show that  $\alpha \leq y \leq \beta$  on  $[0, 1]$ . Suppose for example that  $y(t) < \alpha(t)$  for some  $t \in [0, 1]$ . From the boundary conditions and continuity we may assume that  $\alpha - y$  attains its positive maximum at  $t \in (0, 1)$ . Thus  $\alpha'(t) = y'(t)$  so that  $|y'(t)| < L$  and  $\alpha''(t) \leq y''(t)$ . From the definition of  $k$  we have

$$\begin{aligned} y''(t) &= k(t, y(t), y'(t)) \\ &< f(t, \alpha(t), \alpha'(t)) \leq \alpha''(t) \end{aligned}$$

a contradiction. Similarly  $y \leq \beta$  on  $[0, 1]$ . Now  $|y'| < L$  on  $[0, 1]$  by the standard argument and  $y$  is the required solution.

Let  $\Omega_\epsilon = \{y \in C^1([0, 1]) : \alpha_\epsilon < y < \beta_\epsilon, |y'| < L, \text{ on } [0, 1]\}$  and  $\Gamma_\epsilon = \Omega_\epsilon \times \Delta$ .

Define  $\mathcal{K} : C^1([0, 1]) \rightarrow C([0, 1])$  at  $x \in [0, 1]$  by

$$\mathcal{K}(\phi)(x) = k(x, \phi(x), \phi'(x)).$$

Define  $\mathcal{H} : \bar{\Gamma}_\epsilon \times [0, 1] \rightarrow X$  by

$$\mathcal{H}(\phi, C, D, \theta) = (\phi + C\mathcal{K}(\phi) - w(C, D), \mathcal{S}(\phi, C, D, \theta))$$

for  $\frac{2}{3} \leq \theta \leq 1$ ,

$$\mathcal{H}(\phi, C, D, \theta) = (\phi + C3(\theta - 1/3)\mathcal{K}(\phi) - w(C, D), \mathcal{G}(C, D))$$

for  $\frac{1}{3} \leq \theta \leq \frac{2}{3}$ , and

$$\mathcal{H}(\phi, C, D, \theta) = (\phi - 3\theta w(C, D) - (1 - 3\theta)(\alpha_\epsilon + \beta_\epsilon)/2, \mathcal{G}(C, D))$$

for  $0 \leq \theta \leq \frac{1}{3}$ , where

$$\mathcal{S}(\phi, C, D, \theta) = G((C, D); (3(\theta - 2/3)(\phi'(0), \phi'(1)) + 3(1 - \theta)\Psi(C, D))).$$

Clearly  $\mathcal{H}$  is completely continuous. It is easy to see that  $y$  is a solution of (4.2) and (1.2) with  $(y, y(0), y(1)) \in \bar{\Gamma}_\epsilon$  iff  $\mathcal{H}(y, y(0), y(1), 1) = 0$ . If there is a solution with  $(y, y(0), y(1)) \in \partial\Gamma_\epsilon$  then we are through. Suppose there is no solution in  $\partial\Gamma_\epsilon$ . We show that  $\mathcal{H}$  is a homotopy for Schauder degree on  $\Gamma_\epsilon$  at 0. To see this assume there are solutions of  $\mathcal{H}(y, C, D, \theta) = 0$  with  $\theta \in [0, 1]$  and  $(y, C, D) \in \partial\Gamma_\epsilon$ . We consider the cases  $\theta \in [2/3, 1]$  and  $[1/3, 2/3]$ ; the case  $\theta \in [0, 1/3]$  is trivial.

Consider the case  $\theta \in [2/3, 1]$ . By assumption there is no solution with  $\theta = 1$ . Assume there is a solution  $(y, C, D)$  with  $\theta \in [2/3, 1)$ . As before  $\alpha(x) \leq y(x) \leq \beta(x)$  on  $[0, 1]$ ,  $y(0) = C$  and  $y(1) = D$ .

Assume that  $(y(0), y(1)) \in \partial\Delta$ . If  $y(0) = \alpha(0)$ , then  $y'(0) \geq \alpha'(0)$ . Thus  $3(\theta - 2/3)y'(0) + 3(1 - \theta)\psi^0(y(0), y(1)) > \alpha'(0)$  and  $\mathcal{S}(y, y(0), y(1), \theta) \neq 0$ , a contradiction. Similarly the other cases  $(y(0), y(1)) \in \partial\Delta$  lead to a contradiction.

Assume that  $y \in \partial\Omega_\epsilon$ . Again, by a standard argument,  $|y'| < L$  on  $[0, 1]$ . Assume that  $y(t) = \alpha_\epsilon(t)$  for some  $t \in [0, 1]$ . From the boundary conditions we see that  $t \in (0, 1)$  and thus  $y'(t) = \alpha'_\epsilon(t) = 0$  while  $y''(t) \geq \alpha''_\epsilon(t) = 0$ . From the definition of  $k$  we have

$$\begin{aligned} y''(t) &= k(t, y(t), y'(t)) \\ &= -(|f(t, \alpha(t), 0)| + \epsilon) < 0, \end{aligned}$$

a contradiction. Similarly the assumption  $y(t) = \beta_\epsilon(t)$  for some  $t \in [0, 1]$  leads to a contradiction. Thus there are no solutions of  $\mathcal{H}(y, C, D, \theta) = 0$  with  $\theta \in [2/3, 1]$  and  $(y, C, D) \in \partial\Gamma_\epsilon$ .

Assume that  $\theta \in [1/3, 2/3]$ . Since  $\Psi$  is strongly inwardly pointing and  $G$  is strongly compatible, by (3.2) there are no solutions  $(y, C, D)$  with  $(C, D) \in \partial\Delta$ . The proof that the case  $y \in \partial\Omega_\epsilon$  leads to a contradiction is similar to that for  $\theta \in [2/3, 1)$ .

Thus  $\mathcal{H}$  is a homotopy for the Schauder degree and since  $\mathcal{H}(\cdot, 0) = (I - c, \mathcal{G})$  where  $I$  is the identity on  $C^1([0, 1])$  and  $c \in \Omega_\epsilon$  is a constant it follows that

$$\begin{aligned} d(\mathcal{H}(\cdot, 1), \Gamma_\epsilon, 0) &= d(\mathcal{H}(\cdot, 0), \Gamma_\epsilon, 0) \\ &= d(\mathcal{G}, \Delta, 0) \neq 0. \end{aligned}$$

Suppose now that  $G$  is compatible with  $\alpha$  and  $\beta$ . Then there is a sequence  $\{G_i\}_{i=1}^\infty$  strongly compatible with  $\alpha$  and  $\beta$  and converging uniformly to  $G$  on compact subsets of  $\mathbb{R}^2 \times \mathbb{R}^2$  to  $G$ . Let  $y_i$  be the corresponding solutions. By compactness there is a subsequence of the  $y_i$  converging in  $C^2([0, 1])$  to the desired solution. □

**Remark 9.** The Bernstein–Nagumo growth condition can be generalised to: There exist  $h \in C([0, \infty); (0, \infty))$ ,  $\bar{h} \in C([\alpha_m, \beta_M]; (0, \infty))$  and  $r \in$

$C([0, 1]; (0, \infty))$  such that

$$|f(x, y, p)| \leq h(|p|)\bar{h}(y) + r(x), \text{ for all } (x, y) \in [0, 1] \times [\alpha(x), \beta(x)] \text{ and}$$

$$\int_{\sigma}^{\infty} \frac{sd s}{h(s)} > \int_{\alpha_m}^{\beta_M} \bar{h}(s)ds + K \int_0^1 r(x)dx,$$

where  $K = \sup\{s/h(s) : s \in [\sigma, L]\}$ .

See Scorza Dragoni [32], Zwirner [36] and Thompson [34].

**Remark 10.** In the case  $\Delta$  is degenerate we have to modify the result. Suppose, for example, that  $\alpha(0) = \beta(0)$ . Then we set  $\Delta = (\alpha(1), \beta(1))$  and change the other conditions as follows.

The vector field  $\Psi \in C(\bar{\Delta})$  is said to be strongly inwardly pointing on  $\Delta$  if

$$\Psi(\alpha(1)) < \alpha'(1), \Psi(\beta(1)) > \alpha'(1).$$

Let  $G \in C(\bar{\Delta} \times \mathbb{R})$  and  $\alpha \leq \beta$  be lower and upper solutions for (1.1), respectively. We say  $G$  is strongly compatible with  $\alpha$  and  $\beta$  if for all strongly inwardly pointing  $\Psi$  on  $\Delta$

$$\mathcal{G}(D) \neq 0 \text{ for all } D \in \partial\Delta \text{ and}$$

$$d(\mathcal{G}, \Delta, 0) \neq 0,$$

where  $\mathcal{G}(D) = G(D, \Psi(D))$ . We define compatible as before. Theorem 1 and its proof are modified in the obvious way.

Our results do not apply to the case  $\alpha(0) = \beta(0)$  and  $\alpha(1) = \beta(1)$  since there are no solutions if our compatibility conditions are extended in the natural way.

As mentioned earlier our central idea leads to existence results for those  $f$  for which there are a priori bounds on  $y'$  for solutions  $y$  satisfying  $\alpha \leq y \leq \beta$ . We now discuss the case where  $f$  satisfies the Nagumo–Knobloch–Schmitt condition.

**Theorem 2.** *Assume that there exist nondegenerate lower and upper solutions  $\alpha \leq \beta$  for (1.1), that  $f$  satisfies the Nagumo–Knobloch–Schmitt condition, that  $G \in C(\bar{\Delta} \times \mathbb{R}^2; \mathbb{R}^2)$  is compatible with  $\alpha$  and  $\beta$ , that  $\alpha'(x) \geq \Phi(x, \alpha(x))$  and  $\beta'(x) \leq \Upsilon(x, \beta(x))$  on  $[0, 1]$  and that  $G((C, D); (E, F)) = 0$  only if  $E \in [\Phi(0, C), \Upsilon(0, C)]$ . Then problem (1.1) and (1.2) has a solution  $y$  lying between  $\alpha$  and  $\beta$ .*

*Proof.* Again we modify  $f$ . First choose

$$L > \max\{|\Phi(x, y)|, |\Upsilon(x, y)|, |\alpha'(x)|, |\beta'(x)| : (x, y) \in \bar{\omega}\},$$

where  $\bar{\omega} = \{(x, y) \in [0, 1] \times \mathbb{R} : \alpha(x) \leq y \leq \beta(x), x \in [0, 1]\}$ . Let

$$l(x, y, p) = \begin{cases} \max\{f(x, y, \Phi(x, y)) + (\Phi(x, y) - p), f(x, y, p)\}, & \text{for } p \leq \Phi(x, y) \\ \min\{f(x, y, \Upsilon(x, y)) + (\Upsilon(x, y) - p), f(x, y, p)\}, & \text{for } p \geq \Upsilon(x, y) \\ f(x, y, p), & \text{otherwise} \end{cases}$$

and

$$m(x, y, p) = l(x, y, \pi(p, -L, L)).$$

Thus  $\alpha$  and  $\beta$  are lower and upper solutions for

$$(4.4) \quad y'' = m(x, y, y') \text{ for all } x \in [0, 1].$$

It is easy to see that  $m$  satisfies the conditions of Theorem 1 and thus there is a solution  $y$  of problem (4.4) and (1.2) satisfying  $\alpha \leq y \leq \beta$ . To show that this is a solution of our problem it suffices to show that  $\Phi(x, y) \leq y' \leq \Upsilon(x, y)$ . From the boundary conditions there are no solutions for  $y'(0) \notin [\Phi(0, y(0)), \Upsilon(0, y(0))]$ . Suppose that  $y'(t) < \Phi(t, y(t))$  for some  $t \in (0, 1]$ . By continuity and the definition of  $L$  we may choose  $t$  and  $u \in (0, t)$  such that  $-L < y'(x) < \Phi(x, y(x))$  for all  $x \in (u, t]$  and  $y'(u) = \Phi(u, y(u))$ . Now

$$\begin{aligned} (y'(x) - \Phi(x, y(x)))' &= m(x, y(x), y'(x)) \\ &\quad - \Phi_x(x, y(x)) - \Phi_y(x, y(x))\Phi(x, y(x)) \\ &> f(x, y(x), \Phi(x, y(x))) \\ &\quad - \Phi_x(x, y(x)) - \Phi_y(x, y(x))\Phi(x, y(x)) \\ &\geq 0, \end{aligned}$$

a contradiction. Thus  $\Phi(x, y) \leq y'$ . Similarly the  $y' \leq \Upsilon(x, y)$  and the result follows. □

**Remark 11.** The conditions  $G((C, D); (E, F)) = 0$  only if

$$E \in [\Phi(0, C), \Upsilon(0, C)],$$

(2.8) guarantees the solution  $y$  satisfies  $\Phi(x, y(x)) \leq y'(x) \leq \Upsilon(x, y(x))$ . There are other ways to guarantee this as for example in the case of periodic boundary conditions where we may replace the inequality signs in (2.8) by not equals to signs (see for example Schmitt [31]).

**5. Comparison with Gaines and Mawhin.**

Gaines and Mawhin consider problem (1.1) and (1.2), where  $f : [0, 1] \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ ,  $G : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  and  $y, y' \in \mathbb{R}^n$ , making the following definitions.

**Definition** [16, Definitions V.2 and V.18]. A set  $\tilde{G} \subseteq \mathbb{R}^n$  is called autonomous curvature bounded with respect to (1.1) if for each  $y_0 \in \partial\tilde{G}$  there exists  $V(y) = V(y_0; y)$  such that  $V \in C^2(\mathbb{R}^n)$ ,  $\tilde{G} \subseteq \{y \in \mathbb{R}^n : V(y) < 0\}$ ,  $V(y_0) = 0$  and

$$p^t V_{yy}(y_0)p + V_y(y_0)f(x, y_0, p) > 0$$

for all  $p \in \mathbb{R}^n$  satisfying  $V_y(y_0)p = 0$  and  $x \in (0, 1)$ .

**Theorem** [16, Theorem V.34]. Let  $\tilde{G}$  be a convex autonomous curvature bounded set relative (1.1) such that  $0 \in \tilde{G}$  and for each  $y_0 \in \partial\tilde{G}$ ,  $V_{yy}(y_0)$  is positive semi-definite. Assume that there exists nondecreasing  $h \in C^1([0, \infty); (0, \infty))$  such that

$$\lim_{t \rightarrow \infty} \frac{t^2}{h(t)} = \infty \text{ and } |f(x, y, p)| \leq h(|p|) \text{ for all } (x, y) \in G_1,$$

where  $G_1 = [0, 1] \times \tilde{G}$ . Let  $T = \sup\{|y(x)| : x \in [0, 1]\}$  and  $M \geq 8T$  be chosen such that  $\frac{t^2}{h(t)} > 4T$  for all  $t > M$ . Assume that if  $y$  is a solution of

$$(5.1) \quad \begin{aligned} y'' &= \lambda f(x, y, y') \text{ for all } x \in [0, 1] \\ 0 &= G((y(0), y(1)); (y'(0), y'(1))) \end{aligned}$$

with  $\lambda \in (0, 1)$ ,  $y(x) \in \tilde{G}$  for  $x \in [0, 1]$  and  $|y'(x)| \leq M$ , then  $y(0), y(1) \notin \partial\tilde{G}$ . Moreover assume that

$$(5.2) \quad \begin{aligned} \mathcal{G}_1(C, D) &\neq 0 \text{ for all } (D, C) \in \partial\Delta_0 \\ d(\mathcal{G}_1, \Delta_0, 0) &\neq 0 \end{aligned}$$

where  $\mathcal{G}_0(C, D) = G((C, D + C); (D, D))$  for all  $(C, D) \in \tilde{\Delta}_0$  and  $\Delta_0 = \{(C, D) \in \mathbb{R}^{2n} : (C, D + C) \in \tilde{G}\}$ . Then there exists a solution  $y$  of (1.1) and (1.2) with  $(x, y) \in G_1$ .

The assumptions on  $h$  are standard for systems where it is well known that the Bernstein–Nagumo condition is no longer sufficient to guarantee a priori bounds on the derivative of solutions. For some more recent variants see Fabry [14], for example, and the references cited there.

The essential problems with this remarkable result are concerned with how to extend it to the case  $\tilde{G}_1$  is not autonomous but varies with  $x$ . What is the

appropriate replacement for  $\Delta_0$  bearing in mind the requirements imposed by the method of proof? In such an extension how would one check that (5.1) is satisfied; this is not difficult in the autonomous case? Such an extension is necessary if the result is to be applied to the range of problems currently considered in the literature as we show in a paper forshaddowed above. In the context of systems as opposed to single equations such an extension may not have seemed so important since other technicalities involved in proving such results have hitherto required the autonomous bounding set.

There is a close relationship between conditions (5.2) and the compatibility condition as the following theorem and example demonstrate.

**Theorem 3.** *Let  $G$  be as in [16], Theorem V.34,*

$$(5.3) \quad \Psi(C, D) = (D - C, D - C) \text{ for all } (C, D) \in \bar{\Delta}$$

where  $\Delta = \tilde{G}^2$  and let  $\mathcal{G}(C, D)$  be given by (3.4). Then

$$d(\mathcal{G}_0, \Delta_0, 0) = d(\mathcal{G}, \Delta, 0).$$

*Proof.* This follows by the Leray Multiplication Theorem (see Lloyd [25, Theorem 2.3.1]) letting the homeomorphism  $\iota : \mathbb{R}^{2n} \rightarrow \bar{\Delta}$  be given by  $\iota(C, D) = (C, D - C)$ .  $\square$

In the following examples compatibility fails however there exist strongly inwardly pointing vector fields for which (3.2) holds. In the first example there exists solutions and in the second there are no solutions. Moreover the first example highlights a difference between our assumptions and those of Gaines and Mawhin. Let

$$(5.4) \quad y'' = y,$$

$\beta = 1 = -\alpha$ . There is a solution  $y$  with

$$(y(0), y'(0)) = (-1, a), \quad (y(1), y'(1)) = (1, b) \text{ and } a, b > 2.$$

Choose

$$g^0((-1, D); (P, Q)) = -1 \text{ if } P \leq 1 + a/2, \quad g^0((1, D); (P, Q)) = 1,$$

$$g^1((C, 1); (P, Q)) = 1 \text{ if } Q \leq 1 + b/2, \quad g^1((C, -1); (P, Q)) = -1$$

and  $g^i((-1, 1); (a, b)) = 0$ ,  $i = 0, 1$ ;  $G$  can be extended to all of  $\bar{\Delta} \times \mathbb{R}^2$  as continuous functions using Teitze's Theorem. It is easy to see from Lemma 14 below that (3.2) holds. It is easy to see that Theorem V.34 applies

and there are solutions. Moreover it is easy to see that there is a strongly inwardly pointing vector field such that (3.2) holds but (3.3) does not hold for all strongly inwardly pointing vector fields.

To produce an example with no solutions is easy. Just let

$$W = \{((y(0), y(1)); (y'(0), y'(1))) :$$

there is a solution of (5.4) with these values }.

Set  $g^0((C, D); (P, Q)) = 1$  on  $W$ . Let  $g^0((-1, D); (P, Q)) = -1$  for  $P \leq b/2$  and  $g^0((1, D); (P, Q)) = 1$  for  $P \geq -b/2$ , where  $b = (\cosh 1 - 1)/(\sinh 1)$ . Again  $G$  can be extended to all of  $\bar{\Delta} \times \mathbb{R}^2$  as continuous functions using Teitze's Theorem. Again, by construction there is a strongly inwardly pointing vector field  $\Psi$  such that (3.2) holds and (3.3) does not hold for all strongly inwardly pointing vector fields.

It is easy to see from the above that our compatibility condition is a strengthening of (5.2) and the trade off is that we do not require that (5.1) holds. Apart from the fact that [16], Theorem V.34 applies to systems whereas Theorem 1 applies to a single equation this is the essential difference between these results.

For the convenience of the reader and to highlight the difficulty in applying Theorem V.34 to nonconstant  $\alpha$  and  $\beta$  we state [16], Theorem V.37 which is the result in [16] closest to our Theorem 1.

**Theorem** [16, Theorem V.37]. *Assume that there exist strict lower and upper solutions  $\alpha < \beta$  for (1.1),  $g^0$  is independent of  $D, Q$ , nondecreasing in  $P$ ,  $g^1$  is independent of  $C, P$ , nondecreasing in  $Q$  and  $G$  satisfies*

$$g^0((\beta(0), D); (\beta'(0), Q)) < 0, g^0((\alpha(0), D); (\alpha'(0), Q)) > 0$$

$$g^1((C, \beta(1)); (P, \beta'(1))) > 0, g^1((C, \alpha(1)); (P, \alpha'(1))) < 0.$$

*Assume that  $f$  satisfies the strengthened Bernstein–Nagumo condition, where  $h \in C^1([0, \infty); (0, \infty))$ . Then (1.1) and (1.2) has a solution.*

The interested reader is referred to [16] and [19] for other results of a similar nature.

### 6. Compatibility for Boundary Set Conditions.

In this section we consider problem (1.1) and (1.3) again assuming that there exist lower and upper solutions  $\alpha \leq \beta$ , respectively and look for solutions  $y$  lying between  $\alpha$  and  $\beta$ .

Problems of the form (1.1) together with (1.2) and with (1.3) have been considered by many authors. Shooting methods have been used combined

variously with the maximum principle, with the Jordan separation theorem, the Kneser–Hukuhara continuum theorem and/or the Ważewski retraction theorem. Often these have been refined in the process. See Baxley [6], Baxley and Brown [5], Bebernes and Fraker [9], Bebernes and Wilhelmson [7], Bernfeld and Palamides [10], Jackson and Klassen [21], Jackson and Palamides [22], Palamides [30] and their references.

We show that the results of Bebernes and Fraker [9] can be derived from our results. In a forthcoming paper we systematically show how all these results involving set valued boundary conditions can be obtained from our results. At first sight this may seem surprising since our results are derived from Schauder degree theory while the others are derived from variants of shooting. The key idea in shooting is a generalised Jordan separation theorem while the key idea in the retract method is that the boundary of the sphere is not a retract of the sphere. Both of these can be derived from degree theory thus a connection of this nature is not surprising but to be expected.

In order to state our results we need some notation (see Bebernes and Fraker [9]). For  $x \in [0, 1]$  let  $C(x) = \{(x, y, p) \in \bar{\omega} \times \mathbb{R}\}$ ,  $S_\alpha(x) = \{(x, y, p) \in C(x) : y = \alpha(x)\}$ , and  $S_\beta(x) = \{(x, y, p) \in C(x) : y = \beta(x)\}$ .

**Definition 12.** We say the pair of sets  $\{\mathcal{J}(0), \mathcal{J}(1)\} \subset \mathbb{R}^2$  is strongly compatible, respectively compatible, for (1.1),  $\alpha$  and  $\beta$  if there exists  $G \in C(\bar{\Delta} \times \mathbb{R}^2; \mathbb{R}^2)$  which is strongly compatible, respectively compatible, for (1.1),  $\alpha$  and  $\beta$  and such that  $G((C, D); (E, F)) \neq 0$  for all  $(C, E, D, F) \notin \mathcal{J}(0) \times \mathcal{J}(1)$ .

**Definition 13.** Let  $\mathcal{J}(i) \subset \mathbb{R}^2$ ,  $i = 0$  or  $1$  be a closed connected subset of  $[\alpha(i), \beta(i)] \times \mathbb{R}$ . We say it is of compatible type 1 if there is  $(\alpha(i), u(i)) \in \mathcal{J}(i)$ , where  $(-1)^i(\alpha'(i) - u(i)) \geq 0$ , and there is  $(\beta(i), u(i)) \in \mathcal{J}(i)$ , where  $(-1)^i(u(i) - \beta'(i)) \geq 0$ . We say it is of compatible type 2 if for every  $p \in \mathbb{R}$  there is  $y \in [\alpha(i), \beta(i)]$  such that  $(y, p) \in \mathcal{J}(i)$ . If it is of compatible type 1 or 2 we say simply it is of compatible type.

**Theorem 4.** *Let the sets  $\mathcal{J}(i) \subset \mathbb{R}^2$   $i = 0, 1$  be of compatible type, then the pair  $\{\mathcal{J}(0), \mathcal{J}(1)\}$  is compatible for (1.1),  $\alpha$  and  $\beta$ .*

We will need the following Lemma in the proof of Theorem 4.

**Lemma 14.** *Let  $M = (m^0, m^1) \in C(\bar{\Delta}; \mathbb{R}^2)$  satisfy  $m^0(\alpha(0), D) \leq 0$ ,  $m^0(\beta(0), D) \geq 0$  and  $m^1(C, \alpha(1)) \leq 0$ ,  $m^1(C, \beta(1)) \geq 0$  for all  $(C, D) \in \bar{\Delta}$  and  $M(C, D) \neq 0$  for all  $(C, D) \in \partial\Delta$ , then  $d(M, \Delta, 0) \neq 0$ .*

This follows since  $\mathcal{S} = \theta M + (1 - \theta)(I - p)$  is a homotopy of  $M$  with  $I - p$  where  $I$  is the identity on  $\mathbb{R}^2$  and  $p \in \Delta$  is any point.

*Proof of Theorem 4.* Let  $\mathcal{J}(i)$   $i = 0, 1$  be of compatible type. We define  $G = (g^0, g^1)$  as follows.

Let  $\mathcal{O} = [\alpha(0), \beta(0)] \times \mathbb{R} \setminus \mathcal{J}(0)$ . Then  $\mathcal{O}$  is a relatively open subset of  $[\alpha(0), \beta(0)] \times \mathbb{R}$ . Since  $\mathcal{J}(0)$  is of compatible type, by the generalised Jordan curve theorem, (see Lloyd [25])  $\mathcal{O} = U \cup V \cup W$  where  $U$  is the union of all components of  $\mathcal{O}$  which intersect  $L_0 = \{\alpha(0)\} \times [\alpha'(0), \infty)$ ,  $V$  is the union of all components of  $\mathcal{O}$  which intersect  $L_1 = \{\beta(0)\} \times (-\infty, \beta'(0)]$  and  $W = \mathcal{O} \setminus \{U \cup V\}$ . Now  $U \neq \emptyset \neq V$ . Set

$$g^0((C, D); (E, F)) = \begin{cases} \text{dist}((C, E), \mathcal{J}(0)), & \text{for all } (C, E) \in V \cup W \\ -\text{dist}((C, E), \mathcal{J}(0)), & \text{for all } (C, E) \in U \cup \mathcal{J}(0). \end{cases}$$

It is easy to see that  $g^0$  is continuous,  $g^0((\alpha(0), D); (E, F)) \leq 0$  for all  $E \geq \alpha'(0)$  and that  $g^0((\beta(0), D); (E, F)) \geq 0$  for all  $F \geq \beta'(0)$ . Similarly we define  $g^1$  using  $\mathcal{J}(1)$ . To see that  $G$  is compatible we set  $g_i^0((C, D); (E, F)) = g^0((C, D); (E, F)) + (C - (\alpha(0) + \beta(0))/2)/i$  and similarly approximate  $g^1$ . Let  $\Psi$  be a strongly inwardly pointing vector field on  $\bar{\Delta}$ . Thus  $\mathcal{G}_i(C, D) = g_i((C, D); \Psi(C, D))$  satisfies  $\mathcal{G}_i^0(\alpha(0), D) < 0$ ,  $\mathcal{G}_i^0(\beta(0), D) > 0$ ,  $\mathcal{G}_i^1(C, \alpha(1)) < 0$  and  $\mathcal{G}_i^1(C, \beta(1)) > 0$  for all  $(C, D) \in \bar{\Delta}$ . The result follows by Lemma 14. □

The next result is an immediate consequence of Theorems 1 and 4.

**Corollary 15** [9, Theorem 2]. *Assume that there exist lower and upper solutions,  $\alpha \leq \beta$ , respectively, for (1.1), that  $f$  satisfies a Bernstein–Nagumo condition and that the sets  $\mathcal{J}(i)$ ,  $i = 0, 1$  are of compatible type. Then there is a solution of (1.1) and (1.3) lying between  $\alpha$  and  $\beta$ .*

We show now how the results of Bebernes and Fraker follow from ours.

**Corollary 16** [9, Theorem 1]. *Assume that there exist lower and upper solutions  $\alpha \leq \beta$  for (1.1) and that for any  $B_2 \geq 0$  and  $t_0 \in (0, 1]$  there is  $N(B_2) > 0$  such that any solution of (1.1) with  $|y'(0)| \leq B_2$  and  $\alpha(x) \leq y(x) \leq \beta(x)$  for all  $x \in [0, t_0]$  satisfies  $|y'(x)| \leq N(B_2)$  for all  $x \in [0, t_0]$ . If  $\mathcal{J}(0)$  is compact and of compatible type 1 and  $\mathcal{J}(1)$  is of compatible type 2, then problem (1.1) and (1.3) has a solution lying between  $\alpha$  and  $\beta$ .*

*Proof.* By compactness there is  $B_2 > 0$  such that  $(y(0), y'(0)) \in \mathcal{J}(0)$  implies that  $|y'(0)| \leq B_2$ . By assumption we may choose  $N$  such that  $|y'| < N$  for all solutions  $y$  of (1.1) with  $(x, y) \in \bar{\omega}$  on  $[0, 1]$  and  $L$  such that  $L > \max\{|\alpha'(x)|, |\beta'(x)|, N : x \in [0, 1]\}$ . Let  $j(x, y, p) = f(x, y, \pi(p; -L, L))$  for all  $(x, y, p) \in [0, 1] \times \mathbb{R}^2$ . Consider

$$(6.1) \quad y'' = j(x, y, y') \quad \text{for all } x \in [0, 1]$$

together with (1.3). Now  $\alpha$  and  $\beta$  are lower and upper solutions for (6.1) so by Corollary 15 there is a solution  $y$  for this problem between  $\alpha$  and  $\beta$ . Assume that  $|y'| \geq L$  for some  $x \in [0, 1]$ . Set  $t = \min\{x \in [0, 1] : |y'(x)| \leq L\}$ . By continuity and the choice of  $L$ , we have  $t > 0$  and  $|y'(x)| \leq L$  for all  $x \in [0, t]$ . Thus  $y$  is a solution of (1.1) on  $[0, t]$  lying between  $\alpha$  and  $\beta$  but  $|y'(t)| > N$  a contradiction. Thus  $|y'| < L$  on  $[0, 1]$  and  $y$  is the required solution.  $\square$

In the above result Bebernes and Fraker assume that there exist strict lower and upper solutions.

The above result can be sharpened as follows.

**Definition 17.** Set

(6.3)

$$S_0 = \{(y, -N) : \alpha(0) \leq y \leq \beta(0)\} \cup \{(\alpha(0), p) : \alpha'(0) \geq p \geq -N\},$$

(6.4)

$$S_2 = \{(y, N) : \alpha(0) \leq y \leq \beta(0)\} \cup \{(\beta(0), p) : \beta'(0) \leq p \leq N\},$$

(6.5)

$$S_1 = \{(y, -M) : \alpha(1) \leq y \leq \beta(1)\} \cup \{(\beta(1), p) : \beta'(1) \geq p \geq -M\} \text{ and}$$

(6.6)

$$S_3 = \{(y, M) : \alpha(1) \leq y \leq \beta(1)\} \cup \{(\alpha(1), p) : \alpha'(1) \leq p \leq M\}.$$

**Corollary 18** [9, Theorem 3]. *Assume that there exist lower and upper solutions  $\alpha \leq \beta$  for (1.1) and that  $f$  satisfies a Bernstein–Nagumo condition. Further assume that  $\mathcal{J}(i) \subseteq C(i)$ ,  $i = 0, 1$  are closed connected sets satisfying  $\mathcal{J}(0) \cap \{S_0 \cup S_2\} \neq \emptyset$  and  $\mathcal{J}(1) \cap \{S_1 \cup S_3\} \neq \emptyset$ , where the  $S_i$  are given by (6.3) to (6.6) and  $M = N$  is given by (2.5). Then there is a solution  $y$  lying between  $\alpha$  and  $\beta$ .*

*Proof.* This follows since either  $(\beta(0), u) \in \mathcal{J}(0)$  for some  $u \geq \beta'(0)$  or  $(y, N) \in \mathcal{J}(0)$  for some  $y \in [\alpha(0), \beta(0))$  and we add the straight line segment joining  $(y, N)$  to  $(\beta(0), N)$  to  $\mathcal{J}(0)$ . Similarly, either  $(\alpha(0), u) \in \mathcal{J}(0)$  for some  $u \leq \alpha'(0)$  or  $(y, -N) \in \mathcal{J}(0)$  for some  $y \in (\alpha(0), \beta(0)]$  and we add the straight line segment joining  $(y, -N)$  to  $(\alpha(0), -N)$  to  $\mathcal{J}(0)$ . Similarly we modify  $\mathcal{J}(1)$  as above. Thus the modified  $\mathcal{J}(i)$  are of compatible type and, by Corollary 15, there exists a solution for (1.1) and (1.3). This solution satisfies  $|y'| < N$  and hence is the required solution.  $\square$

There is a typographical error in Bebernes and Fraker [9], Theorem 3 and a counterexample can be constructed to the result as they have stated it. To explain the error we need the following notation. Let  $f$  satisfy the

strengthened Bernstein–Nagumo condition, and  $\lambda = \max\{\sigma, |\alpha'(x)|, |\beta'(x)| : x \in [0, 1]\}$ . Define  $N(t)$  by

$$\int_{\lambda}^{N(t)} \frac{s}{\phi(s)} ds = \max\{\beta(u) : u \in [0, t]\} - \min\{\alpha(u) : u \in [0, t]\},$$

and  $N$  by

$$(6.7) \quad N = \min\{N(t) : t \in [0, 1]\}.$$

Bebernes and Fraker take  $N$  from (6.7) and  $M = N(1)$ . The min should be max in (6.7).

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