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## ON NORMALITY OF THE CLOSURE OF A GENERIC TORUS ORBIT IN $G/P$

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In this paper we consider generic orbits for the action of a maximal torus  $T$  in a connected semisimple algebraic group  $G$  on the generalized flag variety  $G/P$ , where  $P$  is a parabolic subgroup of  $G$  containing  $T$ . The union of all generic  $T$ -orbits is an open dense (possibly proper, if  $P$  is not a Borel subgroup) subset of the intersection of the big cells in  $G/P$ . We prove that the closure of a generic  $T$ -orbit in  $G/P$  is a normal equivariant  $T$ -embedding (whose fan we explicitly describe). Moreover, the closures of any two generic  $T$ -orbits are isomorphic as equivariant  $T$ -embeddings.

### 1. Introduction.

Let  $G$  be a connected semisimple algebraic group over an algebraically closed field  $k$  of arbitrary characteristic. As usual, let  $B^+$  denote a fixed Borel subgroup of  $G$ ,  $T$  a maximal torus in  $B^+$ ,  $\Gamma(T)$  the character group of  $T$ ,  $B$  the opposite to  $B^+$ ,  $\Phi$  the corresponding root system in an euclidian space  $(E, (\cdot, \cdot))$ ,  $\Phi_+$  the set of positive roots relative to  $B^+$ ,  $\Delta$  the set of simple roots in  $\Phi_+$ ,  $s_\alpha$  the reflection about the linear subspace of  $E$  perpendicular to root  $\alpha$ ,  $W$  the Weyl group of  $\Phi$  generated by the reflections  $s_\alpha, \alpha \in \Phi_+$  ( $W$  can also be naturally identified with  $N_G(T)/T$ ), and  $R$  the root lattice in  $E$ .

Let  $P$  be a fixed parabolic subgroup containing  $B$ . Let  $\Delta_P$  be the set of simple roots  $\alpha$  such that  $s_\alpha \in W_P = N_P(T)/T$ . Then the map  $P \rightarrow \Delta_P$  is a bijection between the set of all parabolic subgroups containing  $B$  and the power set of  $\Delta$  (see e.g. [B, Proposition 14.18]). We denote by  $S^P$  the subsemigroup of the root lattice generated by all positive roots which are not sums of simple roots in  $\Delta_P$ .

We will be concerned with  $T$ -orbits of points in the projective variety  $G/P$ . Let  $\lambda$  be an integral dominant weight (with respect to  $\Phi_+$ ) whose stabilizer in  $W$  is  $W_P$ . Then  $\lambda$  extends to a character of  $P$  (we will also call it  $\lambda$ ), inducing a line bundle  $\mathcal{L}^\lambda$  on  $G/P$ . We let  $V(\lambda)$  denote the Weyl  $G$ -module

$$H^0(G/P, \mathcal{L}^\lambda) = \{f \in k[G] \mid f(xy) = \lambda^{-1}(y)f(x) \text{ for all } x \in G, y \in P\}$$

of global sections of  $\mathcal{L}^\lambda$  (see e.g. [J, Sec. 5.8, p. 84]).

Let  $\Pi_\lambda$  denote the set of weights of  $V(\lambda)$  for the action of  $T$ . Let  $\mathcal{A}_\lambda$  denote the set of weights of  $V(\lambda)$  listed with multiplicity. For each  $\mu \in \mathcal{A}_\lambda$ , we pick a corresponding weight vector (function)  $f_\mu$  so that  $\{f_\mu | \mu \in \mathcal{A}_\lambda\}$  is a basis of  $V(\lambda)$ . Functions  $f_\mu, \mu \in \mathcal{A}_\lambda$ , are called the Plücker coordinates in  $G/P$ . By abuse of language we use  $f_\mu$  to denote any Plücker coordinate of a given weight  $\mu$ . Let  $x = u.P$  be an element of  $G/P$ . We let  $\Pi_\lambda(x)$  denote the set of weights  $\mu \in \Pi_\lambda$  such that at least one of the Plücker coordinates  $f_\mu$  does not vanish at  $u$ . It is easy to see that  $\Pi_\lambda(x)$  depends on  $x$  and  $\lambda$  only (not on the choice of the Plücker coordinates). It turns out that  $\lambda - \Pi_\lambda \subseteq S^P$ . Hence by  $W$ -invariance of  $\Pi_\lambda$ ,  $\lambda - w\Pi_\lambda(x) \subseteq S^P$ , for any  $x \in G/P$  and  $w \in W$ . Intuitively, a torus orbit  $Tx \subset G/P$  can be called generic if sufficiently many Plücker coordinates of  $x$  do not vanish. The following definition makes this requirement precise.

**Definition 1.1.** Let  $x$  be an element of  $G/P$ . Then the torus orbit  $Tx \subset G/P$  is called *generic* if and only if  $\{w\lambda | w \in W\} \subseteq \Pi_\lambda(x)$ , and for each  $w \in W$ , the semigroup generated by  $\lambda - w\Pi_\lambda(x)$  is  $S^P$  (that is, the maximal semigroup that  $\lambda - w\Pi_\lambda(x)$  can generate).

We will show that this definition does not depend on the choice of  $\lambda$ . It turns out that  $\Pi_\lambda(x) = \Pi_\lambda$  implies  $Tx$  is generic. Therefore generic orbits exist since there are points in  $G/P$  at which all Plücker coordinates do not vanish. We will also prove that in the case of  $G/B$ ,  $Tx$  is generic if and only if  $x$  belongs to  $\bigcap_{w \in W/W_P} wB^+.P$ .

The aim of this note is to prove that the closure of a generic  $T$ -orbit in  $G/P$  is a normal equivariant  $T$ -embedding. We can then use the general theory of equivariant torus embeddings (see e.g. [K, Oda1]) to show that the closures of any two generic orbits are isomorphic (as equivariant  $T$ -embeddings). We prove this by identifying the fan describing the isomorphism class of these  $T$ -embeddings.

**Remark.** We point out that if  $P \neq B$ , the definition of generic  $T$ -orbit given here differs from the one used in [F-H, Remark 1, p. 257]. There, an orbit  $Tx$  is called “generic” if and only if  $x$  belongs to the non-degenerate stratum  $Z = \bigcap_{w \in W/W_P} wB^+.P$  in the stratification of  $G/P$  introduced in [G-S] (note that in [F-H]  $B$  is the “positive” Borel subgroup, while here  $B$  denotes the “negative” Borel subgroup). It is easy to see that the set of all  $x \in G/P$  with  $Tx$  generic in the sense of Definition 1.1 is an open subset of  $Z$ . It is proved in [G-S, Section 5.1, Proposition 1] that if  $k$  is the field of complex numbers then the image under the moment map of the closure of each torus orbit contained in  $Z$  is the convex hull of  $\{w\lambda | w \in W\}$ . In [F-H] the general theory of torus embeddings is used to study the closure of  $Tx$  in

$G/P$  for  $x \in Z$ . It appears however that normality of these varieties, required in the theory, has not been proved (as pointed out in [Oda2, Section 2.6]). Also, contrary to what is claimed in [F-H], two  $T$ -orbits in  $Z$  may have nonisomorphic closures in  $G/P$  (see the example below).

**Example.** Let  $\mathbf{C}$  denote the field of complex numbers. Let  $q$  be a nondegenerate quadratic form on  $V = \mathbf{C}^5$ , and let  $G = SO(q)$  be the subgroup of determinant one linear transformations of  $V$ , preserving  $q$ . Then  $G$  is a connected, semisimple, rank 2 algebraic group over  $\mathbf{C}$ , and  $V$  is an irreducible representation of  $G$ . Let  $L$  be a fixed isotropic line for  $q$  (that is  $q(v) = 0$  for all  $v \in L$ ), and let  $P \subset G$  be the stabilizer of  $L$ . Then  $P$  is a parabolic subgroup of  $G$ , and  $G/P$  is naturally isomorphic to the smooth quadric hypersurface  $Q$  in the complex projective space  $\text{Proj}(V)$  given by the homogeneous equation  $q(x) = 0$ . For brevity, we will equate  $G/P$  with  $Q$ . Let  $\{e_1, e_2, e_3, e_4, e_5\}$  be the standard basis of  $V$  and let  $q(x) = x_1x_3 + x_2x_4 - 2x_5^2$ , where  $[x_1, x_2, x_3, x_4, x_5]$  are the coordinates of  $x \in V$  relative to the standard basis. We let  $L = \mathbf{C}e_1$ . Then the maximal torus contained in  $P$  is  $T = \{\text{diag}(t_1, t_2, 1/t_1, 1/t_2, 1) \mid t_i \in \mathbf{C} \setminus \{0\}, i = 1, 2\}$ . Here, the Plücker coordinates in  $Q = G/P$  are just the standard homogeneous coordinates in  $\text{Proj}(V)$ . Clearly,  $L_1 = \mathbf{C}[1, 1, -1, 1, 0]$  and  $L_2 = \mathbf{C}[1, 1, 1, 1, 1]$  are  $q$ -isotropic. Also, the  $T$ -orbits of  $L_1$  and  $L_2$  are “generic” in the sense of [F-H], but only  $TL_2$  is generic in the sense of the Definition 1.1. Also,  $\Pi(L_1) \neq \Pi(L_2) = \Pi$ , where  $\Pi$  denotes the set of weights of  $V$ . This directly contradicts Lemma 13 in [F-H]. Let  $X_i = \overline{TL_i}$ ,  $i = 1, 2$ , where the closure is taken in  $Q$  (or in  $\text{Proj}(V)$ , since  $Q$  is closed in  $\text{Proj}(V)$ ). It is easy to see that  $X_1$  is isomorphic to  $\mathbf{C}P^1 \times \mathbf{C}P^1$ . On the other hand  $X_2$  is the singular closed subvariety of  $\text{Proj}(V)$  given by homogeneous equations  $x_1x_3 = x_5^2$ ,  $x_2x_4 = x_5^2$  (the singular points of  $X_2$  are  $[1 : 1 : 0 : 0 : 0]$ ,  $[1 : 0 : 0 : 1 : 0]$ ,  $[0 : 1 : 1 : 0 : 0]$ , and  $[0 : 0 : 1 : 1 : 0]$ ). Therefore, the example shows that two  $T$ -orbits “generic” in the sense of [F-H] may not have isomorphic closures in  $G/P$ .

### 2. Weights of Weyl $G$ -modules.

We will need the following notation. For any additive set  $A$  of real numbers and any subset  $Y$  of  $E$ , let  $AY$  denote the set of all linear combinations of elements in  $Y$  with coefficients in  $A$ . By definition, a semigroup  $S$  contained in a lattice  $L$  in  $E$  is saturated in  $L$  if and only if

$$L \cap \mathbf{Q}_+S = S$$

(see [K, Chapter 1, Section 1]). Equivalently,  $S$  is saturated in  $L$  if and only if for any positive integer  $m$ ,  $m\mu \in S$  and  $\mu \in L$  imply  $\mu \in S$ .

**Proposition 2.1.**  $S^P$  is saturated in  $R$ .

*Proof.* Let  $\Phi_+^P$  denote the set of positive roots which are not linear combinations of roots in  $\Delta_P$ . Then  $S^P = \mathbf{Z}_+ \Phi_+^P$ . Suppose that  $S^P$  is not saturated in  $R$ . Let  $\mu \in R$  be an element of minimal height among the elements of  $\mathbf{Q}_+ \Phi_+^P$  which are not elements of  $S^P$ . Then  $\mu = \mu_1 + \mu_2$ , with

$$\mu_1 = \sum_{\beta \in M} m_\beta \beta$$

where  $M \subseteq \Phi_+^P$ ,  $m_\beta$  are positive integers, and

$$\mu_2 = \sum_{\alpha \in N} n_\alpha \alpha$$

where  $N \subseteq \Delta_P$ , and  $n_\alpha$  are positive integers. From the above decompositions of  $\mu$  we choose one with  $\mu_2$  of minimal height. Since the sum of any two roots with negative scalar product is again a root, minimality of  $\mu_2$  implies that

$$(\alpha, \beta) \geq 0,$$

for all  $\alpha \in N, \beta \in M$ . Take any simple root  $\alpha$  in  $N$ , such that  $(\mu_2, \alpha) > 0$ . Consider  $\nu = s_\alpha(\mu) \in R$ . Since elements of  $\Phi_+^P$  are permuted by  $s_\alpha$ ,  $\nu$  belongs to  $\mathbf{Q}_+ \Phi_+^P$  but not to  $S^P$ . This is a contradiction, since  $\text{ht}(\nu) < \text{ht}(\mu)$  and  $\mu$  was assumed to be of minimal height among the root lattice elements in  $\mathbf{Q}_+ \Phi_+^P$ , not in  $S^P$ .  $\square$

Let  $V(\lambda), \lambda, \Pi_\lambda$  be as in the introduction. The following proposition lists some basic properties of  $\Pi_\lambda$ .

**Proposition 2.2.**

- (i)  $\lambda - \Pi_\lambda$  coincides with the set of root lattice points in the convex hull of  $\{\lambda - w\lambda \mid w \in W\}$ .
- (ii)  $S^P$  is generated by  $\lambda - \Pi_\lambda$ . If  $P = B$  and  $\lambda$  is the sum of fundamental weights then  $S^P$  is generated by  $\{\lambda - w\lambda, w \in W\}$ .

**Remark.** Part (i) is well known, but were not able to locate an appropriate reference.

*Proof.* We first observe that the weights of the Weyl module  $V(\lambda)$  ( $\lambda$  integral dominant) are independent of the characteristic of  $k$ . This follows from the fact that character formulas for Weyl modules are the same in each characteristic. Therefore we can assume, that  $\text{char}(k) = 0$ .

Part (i). Let  $C$  denote the convex hull of  $\{w\lambda \mid w \in W\}$  and we let  $\Pi = (\lambda + R) \cap C$ . We have to prove that  $\Pi = \Pi_\lambda$ . It is a known fact that  $\Pi_\lambda \subset \lambda + R$ .

Therefore it is enough to show that  $\Pi_\lambda$  is contained in  $C$ . Suppose that this is not the case, and let  $\mu$  be a weight in  $\Pi_\lambda$ , not in  $C$ . Assume also, that  $\mu$  is a maximal such weight in the usual order in  $E$  relative to  $\Phi_+$ . Since both  $\Pi_\lambda$  and  $C$  are  $W$ -invariant, we must have  $s_\alpha(\mu) \leq \mu$  for all positive roots  $\alpha$ . Hence  $\mu$  is dominant. Since  $\mu$  is not the highest weight  $\lambda$ , there must be a positive root  $\alpha$  and a positive integer  $m$  such that  $\mu_1 = \mu + m\alpha \in \Pi_\lambda$ . Then by maximality of  $\mu$ ,  $\mu_1$  (hence also  $s_\alpha(\mu_1)$ ) is in  $C$ . A straightforward computation shows that  $\mu$  belongs to the line segment connecting  $\mu_1$  and  $s_\alpha(\mu_1)$ . This is a contradiction, since we have assumed that  $\mu$  is not in  $C$ .

We are left with showing that  $\Pi$  is contained in  $\Pi_\lambda$ . An easy argument by induction on the length function in  $W$ , shows that for any  $w \in W$ ,  $\lambda - w\lambda$  is a sum of roots in  $\Phi_+^P$ . Therefore  $\Pi$  is contained in  $\lambda - \mathbf{Z}_+\Phi_+$ . It is proved in [H, Proposition, p. 114] that the elements of  $\Pi_\lambda$  are exactly the weights whose  $W$ -orbit is contained in  $\lambda - \mathbf{Z}_+\Phi_+$ . Hence  $\Pi \subseteq \Pi_\lambda$ , as required.

Part (ii) We have observed in the proof of Part (i) that  $\lambda - C$  is contained in convex cone spanned by  $\Phi_+^P$ . Therefore

$$\lambda - \Pi_\lambda = R \cap (\lambda - C) \subseteq S^P$$

since  $S^P$  is saturated in  $R$ . The opposite inclusion holds since  $\Phi_+^P \subseteq \lambda - \Pi_\lambda$ . This follows from the fact that weights of irreducible  $G$ -representations (in characteristic 0) satisfy the following property: for any positive root  $\alpha$ , and a positive integer  $n$ , if  $\mu$  and  $\mu - n\alpha$  are weights of the representation, so are  $\mu - q\alpha$  for any  $q, 0 \leq q \leq n$  (see e.g. [H, Sec. 21.3, Prop.]). One applies this property to  $\lambda$  and  $s_\alpha(\lambda)$ , where  $\alpha \in \Phi_+^P$ .

The second claim of Part(ii) follows since  $\Delta \subseteq \{\lambda - w\lambda | w \in W\}$  if  $\lambda$  is the sum of fundamental weights. □

### 3. Generic orbits of $T$ in $G/P$ .

Let  $x \in G/P$  and let  $X$  denote the the closure of  $Tx$  in  $G/P$ . For any  $w \in W$ , let

$$Y_w = \{y.P | f_{w\lambda}(y) \neq 0\} = \{y.P | w\lambda \in \Pi_\lambda(y.P)\}$$

and

$$X_w = Y_w \cap X.$$

It is well known that each  $Y_w$  is an affine space which is open in  $G/P$  and whose coordinate ring is generated by functions  $f_\mu/f_{w\lambda}, \mu \in \mathcal{A}_\lambda$ . Moreover, the union of  $Y_w, w \in W$  is  $G/P$ . Let  $T_x = \{t \in T | tx = x\}$  and  $T^x = T/T_x$ . We have the following proposition

**Proposition 3.1.** *Let  $x \in G/P$ .*

- (i)  *$Tx$  is open in  $X$  and it is isomorphic to  $T^x$ . Therefore,  $X$  is an equivariant  $T^x$ -embedding in the sense of [K].*
- (ii)  *$\{X_w | w \in W, w\lambda \in \Pi_\lambda(x)\}$  is a covering of  $X$  by  $T$ -invariant open affine subsets of  $X$ . The coordinate ring of  $X_w, w\lambda \in \Pi_\lambda(x)$ , is the subalgebra of  $k[T^x] = k[\Gamma(T^x)]$  generated by  $\Pi_\lambda(x) - w\lambda$ .*
- (iii) *Let  $w \in W$  be such that  $w\lambda \in \Pi(x)$ . Then*

$$T_x = \{t \in T | \mu(t) = 1 \text{ for all } \mu \in w\lambda - \Pi(x)\},$$

*Proof.* The first part of (i) follows from the fact the map  $t \rightarrow tx$  is a separable morphism from  $T$  onto an open subvariety  $Tx$  of  $X$  whose fibers are the cosets of  $T_x$  in  $T$  (the morphism is separable since it is the composition of the inclusion of  $T$  in  $G$  with the quotient map from  $G$  to  $G/P$ ).

Part (ii) follows, since for each  $w \in W$  such that  $w\lambda \in \Pi_\lambda(x)$ ,  $X_w$  can be viewed as a closed  $T$ -invariant subvariety of the affine space  $Y_w$ . Hence the coordinate ring of  $X_w$  is generated by the restrictions to  $X_w$  of functions  $f_\mu/f_{w\lambda}, \mu \in \mathcal{A}_\lambda$ .

Part(iii). Suppose that  $w \in W$  satisfies  $w\lambda \in \Pi(x)$ . Then  $x \in X_w$ . Clearly,  $t \in T_x$  if and only if  $t$  fixes all elements of  $X_w$  (or equivalently,  $t$  fixes all regular functions on  $X_w$ ). Therefore the desired formula for  $T_x$  follows from the description of the coordinate ring of  $X_w$  given in (ii). □

Before we state a corollary of Proposition 3.1, we need to introduce the following notation. Let  $R^P$  denote the subgroup of the root lattice generated by  $S^P$ . One can show that  $R^P = R$  if  $\Phi$  is an irreducible system. If  $\Phi$  a union of irreducible root systems  $\Phi_j, j \in J$ , then  $R^P$  is the root lattice of the root system

$$\cup\{\Phi_j | \Phi_j \cap S^P \neq \emptyset\}.$$

Let

$$T_P = \bigcap_{\nu \in R^P} \ker(\nu).$$

Note that if  $R^P = R$ , then  $T_P$  is coincides with the center of  $G$ .

**Corollary.** (Suggested by the referee.)

- (i) *The stabilizer of each generic torus orbit is  $T_P$ . Moreover,  $T_P$  it is the smallest subgroup of  $T$  among the  $T$ -stabilizers of elements of  $G/P$ .*
- (ii) *(Partial converse of (i)). If  $x \in G/P$  is such that  $Tx$  is contained in the nondegenerate stratum  $Z, \overline{Tx}$  is normal and  $T_x = T_P$ , then  $Tx$  is generic.*

*Proof.* Part (i) follows from Proposition 3.1 (iii). Suppose that  $Tx$  satisfies the assumptions of (ii). Let  $S^x$  denote the semigroup generated by  $\lambda - \Pi_\lambda(x)$ . We have to show that  $S^x = S^P$ . Since  $T_x = T_P$ , one has

$$\bigcap_{\nu \in R^P} \ker(\nu) = \bigcap_{S^x} \ker(\nu)$$

by Proposition 3.1 (iii). Therefore  $R^P$  is generated by  $S^x$  as a subgroup of  $\Gamma(T)$ . Assumed normality of  $\overline{Tx}$  implies that  $S^x$  is saturated in  $R^P$ . On the other hand  $\{\lambda - w\lambda \mid w \in W\} \subset S^x$  since  $Tx$  is assumed to be generic. Hence  $S^x = S^P$  since both semigroups are saturated in  $R^P$  and  $\mathbf{Q}_+ S^x = \mathbf{Q}_+ S^P$  by Proposition 2.2.  $\square$

From now on we assume for simplicity that  $R^P = R$  (equivalently,  $S^P$  contains at least one root from each irreducible component of  $\Phi$ ). Let  $W^P \subseteq W$  be a fixed set of representatives of  $W/W_P$ . Let  $D$  denote the fundamental chamber  $\{\nu \in E \mid (\nu, \alpha) \geq 0 \text{ for all } \alpha \in \Delta\}$ . We are now ready to state the main result of this paper.

**Theorem 3.2.** *Let  $x \in G/P$  be such that  $Tx \subset G/P$  is generic. Let  $X = \overline{Tx}$ . Then:*

- (i)  *$X$  is a normal variety (hence by [K, Theorem 14, page 52], also Cohen-Macaulay with rational singularities).*
- (ii) *The fan corresponding to  $X$  consists of the cones*

$$C_w = -w \bigcup_{z \in W_P} zD, \quad w \in W^P$$

*together with their faces. In particular, the closures of any two generic orbits in  $G/P$  are isomorphic as  $T$ -equivariant embeddings.*

*Proof.* Part (i). By [K, Theorem 6, p. 24] a general equivariant  $T$ -embedding is a normal variety if and only if it admits a covering by open affine  $T$ -stable subvarieties whose coordinate rings are generated by semigroups saturated in  $\Gamma(T)$ . Hence Part(ii) follows from Propositions 3.1 and 2.1.

Part(ii) follows, since the dual cone of  $S^P$  is  $\bigcup_{z \in W_P} zD$ , and by Proposition 3.1(ii) the coordinate ring of  $X_w$ ,  $w \in W$ , is  $k[-wS^P]$ .  $\square$

The following theorem shows that Definition 1.1 of a generic torus orbit does not depend on the choice of the Weyl module  $V(\lambda)$ .

**Theorem 3.3.** *Let  $x \in G/P$ . The following statements are equivalent.*

- (i) *There exist an integral dominant weight  $\lambda$  whose stabilizer in  $W$  is  $W_P$ , such that for any  $w \in W$ , the semigroup generated by  $\lambda - w\Pi_\lambda(x)$  is  $S^P$ .*

- (ii) For each integral dominant weight  $\lambda$  whose stabilizer in  $W$  is  $W_P$ , and each  $w \in W$ , the semigroup generated by  $\lambda - w\Pi_\lambda(x)$  is  $S^P$ .
- (iii) There exists an integral dominant weight  $\lambda$  whose stabilizer in  $W$  is  $W_P$ , such that  $\Pi_\lambda(x) = \Pi_\lambda$ .

*Proof.* Clearly, (ii) implies (i). Also, by Proposition 2.2, (iii) implies (i). We have to prove that if (i) holds, so do (ii) and (iii). Let  $X = \overline{Tx}$  and let  $X_w, w \in W$  be as in Theorem 3.1. Since the coordinate ring of  $X_w$  does not depend on the choice of a Plücker embedding, Theorem 3.1(ii) implies that (ii) follows from (i).

It remains to prove that (i) implies (iii). Let  $x \in G/P$  and let  $\lambda$  be as in (i). For any integral dominant weight  $\mu$  whose stabilizer in  $W$  is  $W_P$ , let  $\mathcal{L}^\mu$  denote the corresponding line bundle on  $G/P$ . Let  $\mathcal{L}_X^\mu$  denote the pullback of  $\mathcal{L}^\mu$  to  $X = \overline{Tx}$ . Since  $X$  contains an open, dense  $T$ -orbit, every weight of  $H^0(X, \mathcal{L}_X^\mu)$  under the natural  $T$ -action has multiplicity one. Therefore the dimension of the image of the restriction map

$$H^0(G/P, \mathcal{L}^\mu) \rightarrow H^0(X, \mathcal{L}_X^\mu)$$

is  $\sharp(\Pi_\mu(x))$ . We observe that line bundle  $\mathcal{L}_X^\mu$  is ample. This is because the piecewise linear function on  $E$  corresponding to  $\mathcal{L}_X^\mu$  (see [F-H, Theorem 2]) is strictly upper convex. Then the description of the fan of  $X$  given in Theorem 3.2(iii), [Oda1, Theorem 2.13 and Corollary 2.9], and Proposition 2.2 (i) imply that

$$\dim H^0(X, \mathcal{L}_X^\mu) = \sharp(\Pi_\mu).$$

Since  $\mathcal{L}^\lambda$  is ample there exists a positive integer  $q$  such that the restriction map

$$H^0(G/P, \mathcal{L}^{q\lambda}) \rightarrow H^0(X, \mathcal{L}_X^{q\lambda})$$

is surjective. Hence  $\Pi_{q\lambda}(x) = \Pi_{q\lambda}$  as required. □

It is easy to see that Theorem 3.3 and Proposition 2.2 imply:

**Corollary.** *Let  $x \in G/B$ . Then  $Tx$  is generic if and only if  $x \in \bigcap_{w \in W} wB^+.B$  (i.e. it is “generic” in the sense of [F-H]). Moreover, if  $xT$  is generic then  $X = \overline{Tx}$  is smooth.*

**Remark.** Smoothness of the closure of a generic torus orbit in  $G/B$  is well known (we do not know however, to whom this fact should be attributed).

### Final remarks and questions.

1. All results about closures of  $T$ -orbit in  $G/P$  stated in [F-H] hold for generic orbits (in the sense of Definition 1.1) in any characteristic. This is

because the arguments used in [F-H] are valid for normal equivariant  $T$ -embeddings, and we have shown that the closure of a generic orbit is such an embedding. We do not know however, if the results remain valid for all  $T$ -orbits in the nondegenerate stratum if  $P \neq B$ .

2. Let  $X$  denote the closure of a  $T$ -orbit of an element  $x \in G/P$ . It is not difficult to prove that if  $\lambda$  is an integral dominant weight whose stabilizer in  $W$  is  $W_P$ , then the line bundle  $\mathcal{L}_X^\lambda$  is in fact very ample (one can use the criterion for very ampleness given in [F, Lemma, p. 69] or [Oda1, Corollary 2.9]). Then it follows from [F, Exercise, p. 72] that the corresponding embedding of  $X$  in  $\text{Proj}(H^0(X, \mathcal{L}_X^\lambda))$  is projectively normal and Cohen-Macaulay (that is, the homogeneous coordinate ring of  $X$  in  $\text{Proj}(H^0(X, \mathcal{L}_X^\lambda))$  is normal and Cohen-Macaulay). Therefore, the embedding  $X \subset \text{Proj}(H^0(G, \mathcal{L}^\mu))$  is also projectively normal and Cohen-Macaulay, if the restriction map from  $H^0(G/P, \mathcal{L}^\lambda)$  to  $H^0(X, \mathcal{L}_X^\lambda)$  is surjective (equivalently  $\Pi_\lambda(x) = \Pi_\lambda$ ). We do not know if this is so, if  $Tx$  is generic and  $\Pi_\lambda(x) \neq \Pi_\lambda$ .

3. Since the closure of any  $T$ -orbit in an equivariant normal  $T$ -embedding is normal (see [K, Proposition 2, p. 17]),  $X$  is normal if it is contained in the closure of a generic  $T$ -orbit. In this situation, the fan corresponding to  $X$  can be described explicitly in terms of the fan defined in Theorem 3.2 (iii) (see e.g. [Oda2, Section 1.1]). Since there could be non-generic orbits of maximal dimension (see the example in the introduction) not every  $T$ -orbit is contained in the closure of a generic one. The structure of the orbit is not clear. Does it have to be normal? If yes, what is its fan? Suppose that the closures of all  $T$ -orbits in  $G/P$  are indeed normal. Then the Example and the Corollary of Proposition 3.1, suggest the conjecture that the isomorphism type of  $\overline{Tx}$  (as a torus equivariant embedding) is determined by two pieces of data: the stabilizer of  $x$  in  $T$  and the set  $\{w \in W/W_P \mid x \in B^+w.P\}$ .

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