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ON THE FAILURE CYCLES FOR THE QUADRATIC NORMALITY OF A PROJECTIVE VARIETY

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Let X be a smooth projective surface and L a very ample line bundle on X which is not quadratically normal; set $r+1 = h^0(X, L)$. Here we give numerical conditions on X and L which imply the existence of a finite subscheme T of X with length$(T) \geq 2s+2$ and contained in a dimension $s \leq r-2$ linear subspace of $P(H^0(X, L))$ and such that $L \mid T$ is not quadratically normal.

Introduction.

It is very classical the following problem (with several variations). Suppose that a curve $C \subset \mathbf{P}^r$ has some bad property, e.g. it is not projectively normal. Show the existence of a finite subscheme S of C contained in a smaller linear subspace such that S explains the failure of C to be projectively norrnal. In modern times there is the important paper [**4**]. Here we consider the corresponding problem when the scheme C has $\dim(C) > 1$. We were also motivated from the notion of k-ampleness and k-very ampleness introduced in [**2**]. By definition these conditions fail for a scheme C if and only if there is a zero dimensional subscheme S of C with a bad property. We were interested (see e.g. [**1**]) in showing that under suitable conditions there are many such subschemes. A natural question was if there is some bad positive dimensional proper subscheme Y containing all of them for a natural reason (for example if it were the union of them) or if there was some bad "free" zero dimensional subscheme. Here we consider the condition of quadratic normality and give a positive answer if $\dim(C) = 2$ under suitable numerical conditions. These numerical conditions are strange, far from optimal and just come from the proof. We will state them below as Theorem 0.2. But first and most important: the proofs are essentially technical variations on an alternative proof ([**5**, §2.5]) of a theorem in [**4**]; hence the idea originates ultimately with Robert Lazarsfeld. After the present results were proven, we checked the references and found that exactly that subsection was deleted in the printed version [**6**] of [**5**]. After a while we decided to rewrite a little bit the paper, but to write it anyway.

We fix an integral variety X and a very ample line bundle L on X; set $r+1 := h^0(X,L)$ and $\mathbf{O} := \mathbf{O}_X$; let $\phi_L : X \to \mathbf{P}^r$ be the embedding associated to $H^0(X,L)$ into a projective space. Recall that a subvariety U of $\mathbf{P}(V)$ is called quadratically normal if the restriction map $V \otimes V \to H_0(U, \mathbf{O}_U(2))$ is surjective. The pair (X,L) (or just L) is called quadratically normal if $\phi_L(X)$ is quadratically normal.

Definition 0.1. If L is not quadratically normal, we will call *amount of failure of quadratic normality* the integer $\dim(\mathrm{coker}(H^0(L) \otimes H^0(L) \to H^0(L^2))$.

Let $G = G(r+1-\dim(X), r+1)$ be the Grassmannian of codimension $\dim(X)$ linear subspaces of $\mathbf{P}^r$; set

$$B := \{U \in G : X \cap U \text{is not zero dimensional}\}.$$

Here is the main result proven in this paper.

Theorem 0.2. *Assume* $\dim(X) = 2$ *and that* L *is not quadratically normal. Let* $\mathbf{f} > 0$ *be the amount of failure of the quadratic normality of* X. *If* $h^1(\mathbf{O}_X) < \mathbf{f} + \mathrm{codim}(B) - 1$, *then there is a codimension 2 linear subspace* $[U] \in G \backslash B$ *such that the scheme* $X \cap U$ *is 0-dimensional and is not quadratically normal with respect to* $L \mid (X \cap U)$. *Furthermore, there is a an integer* $s \leq r-2$, *a linear subspace* V *of* U *with* $\dim(V) = s$ *and a subscheme* T *of* $U \cap X$ *contained in* V *with length*$(T) = 2s+2$ *such that* T *is not quadratically normal with respect to* $L \mid T$.

In particular Theorem 0.2 applies to all linearly normal but not quadratically normal embedded surfaces with $h^1(\mathbf{O}_X) = 0$.

For other related l results proven within the same framework, see 2.2 and 2.3. In §1 (after fixing the notations) we will give the framework and the main ingredients for the proofs of all the results of this paper. In §2 we will prove Theorem 0.2.

The author owes a huge debt to the referee for essential constructive criticism and for fundamental mathematical contributions which improved the original statement of 0.2.

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1. Preliminaries and general set up.

We work over an algebraically closed base field. We fix an integral variety X and a very ample line bundle L on X; set $r+1 := h^0(X,L)$; let $\phi_L : X \to \mathbf{P}^r$ the embedding associated to $H^0(X,L)$. If A is a sheaf on X, we will often write $H^i(A)$ or $h^i(A)$ for $H^i(X,A)$ or $h^i(A)$. Set $Y := \phi_L(X)$. Let Ω be the cotangent sheaf of $\mathbf{P}^r$. Set $M_L := \phi_L{}^*(\Omega(1))$. By the dual of the Euler sequence of $T\mathbf{P}^r$ and the completeness of the embedding of X we obtain the following exact sequence on X:

$$(1) \qquad 0 \to M_L \to H^0(X,L) \otimes \mathbf{O}_X \to L \to 0$$

which contains a lot of informations on the cohomology of I_Y.

Now we generalize the Remark in [**6**] given at page 510 (between the statement of [**6**], Prop. 1.3.3, and its proof).

Lemma 1.1. *With the notations X, L, ϕ_L, M_L, and so on introduced at the beginning, we have:*

(i) *Fix an integer $k > 0$ and assume $H^1(L^k) = 0$; the multiplication map $H^0(L) \otimes H^0(L^k) \to H^0(L^{k+1})$ is surjective if and only if $H^1(M_L \otimes L^k) = 0$. In particular if $h^1(L^s) = 0$ for every $s > 0$, then L is normally generated if and only if $H^1(M_L \otimes L^t) = 0$ for every $t > 0$.*

(ii) *The amount of failure for the quadratic normality of L is*

$$\dim(\ker(H^1(M_L \otimes L) \to H^0(L) \otimes H^1(L))).$$

(iii) *If $H^1(L^2) = 0$ the amount of failure of quadratic normality is*

$$h^1(M_L \otimes L) - h^0(L) \cdot h^1(L).$$

Proof. Just use a twist of the exact sequence (1).

Let $G := G(r - x + 1, r + 1)$ be the Grassmannian of codimension x linear subspaces of $\mathbf{P}(H^0(X,L))$ and $F := \{(y,U) \in X \times G : y \in U\} \subset X \times G$ be the incidence variety. On G we have the exact sequence

$$(2) \qquad 0 \to S \to H^0(X,L) \otimes \mathbf{O}_G \to Q \to 0$$

with Q tautological quotient bundle and S tautological rank x subbundle. Let $f : X \times G \to G$ and $p : X \times G \to X$ be the projections. The incidence variety F is defined by the vanishing of the induced morphism $s : f^*S \to p^*L$ i.e., its ideal sheaf $\mathbf{I}$ in $X \times G$ is the image of the associated map $f^*S \otimes p^*L^* \to \mathbf{O}_{X \times G}$. Note that this ideal sheaf $\mathbf{I}$ has a resolution:

$$(3) \qquad \begin{aligned} \cdots f^*\Lambda^{t+1}S \otimes p^*L^{*t} \to f^*\Lambda^t S \otimes p^*L^{*(t-1)} \to f^*\Lambda^{(t-1)}S \otimes p^*L^{(t-2)} \cdots \\ \cdots \to f^*\Lambda^2 S \otimes p^*L^* \to f^*S \to \mathbf{I} \otimes p^*L \to 0. \end{aligned}$$

On $X \times G$ there is an important commutative diagram. First, we will write it as formula (4) in the particular case $x = \mathrm{rank}(S) = 2$ needed in the proof of 0.2.

$$
(4)\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & f^*\Lambda^2 S \otimes p^*L^* & \longrightarrow & p^*M_L & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & f^*S & \longrightarrow & H^0(L)\otimes \mathbf{O}_{X\times G} & \longrightarrow & f^*Q & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & p^*L & \longrightarrow & p^*L\otimes \mathbf{O}_F & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

□

In the general case this commutative diagram has 3 columns. The first column of this diagram is the resolution (3) of $\mathbf{I}$ (without $\mathbf{I}$). The second column of the diagram is the pull-back p^* of the exact sequence (1) and the third column is just the tautological surjection $f^*Q \to p^*L \otimes \mathbf{O}_F$. These columns are connected so that the only long row in the diagram is the pull-back by f^* of the exact sequence (1); just above this exact sequence there is a map $f^*\Lambda^2 S \otimes p^*L^* \to p^*M_L$ and just below the exact sequence there is the surjection $p^*L \to p^*L \otimes \mathbf{O}_F$ coming from the surjection $\mathbf{O}_{X\times G} \to \mathbf{O}_F$. Follow the first column of the diagram till the term $f^*\Lambda^2 S \otimes p^*L^*$; then go on the right one step and find f^*Q; then go down one step and find $f^*\Lambda^2 S \otimes p^*L^*$. In this way from this diagram we obtain an exact sequence obtained from the exact sequence (3) substituting the last part $f^*S \to \mathbf{I} \otimes p^*L \to 0$ with

$$p^*M_L \to f^*Q \to p^*L \otimes \mathbf{O}_F \to 0.$$

Call (§§)(k) the exact sequence obtained twisting by p^*L_k the sequence just described. If $x = 2$ the complex (§§)(1) is the following exact sequence:

$$(5)\qquad 0 \to f^*\Lambda^2 S \to p^*(M_L \otimes L) \to f^*Q \otimes p^*L \to p^*L^2 \otimes \mathbf{O}_F \to 0.$$

Now we push-forward the complex (§§)(1) to the Grassmannian; since (§§)(1) is exact, its higher pushforwards vanish and we obtain a spectral sequence (call it (#)) converging to zero.

$$(\#)\qquad E_1^{qt} = R^t f_*(C^q) \Rightarrow 0$$

where

$$0 \to C^{2-x} \to \cdots \to C^0 \to C^1 \to C^2 \to C^3 \to 0$$

is the complex

$$0 \to f^*\Lambda^x S \otimes p^* L^{*(x-2)} \to \cdots \to f^*\Lambda^2 S \to p^*(M_L \otimes L) \to f^* Q \otimes p^* L \to p^* L^2 \otimes \mathbf{O}_F \to 0.$$

In Section 2 we will write the E_1-part of (#) as formulas (7), (8) and (9) in the case $\dim(X) = x = \operatorname{rank}(S) = 2$ we need for the proof of 0.2. Use the projection formula $R^i f_*(f^*A' \otimes p^*A) = H^i(X, A) \otimes A'$ for all locally free sheaves A on X. We normalize the indices of the complex (§§)(1) in such a way that the term E_1^{00} of the spectral sequence (#) is $H^0(X, \mathbf{O}_X) \otimes \Lambda^2 S$. With this normalization the term E_1^{qt} of (#) is 0 if either $t < 0$ or $q < 2 - x$, it is $H^t(X, \mathbf{O}_X) \otimes \Lambda^{-q+2} S$ for $2 - x \leq q \leq 0, H^t(X, M_L \otimes L) \otimes \mathbf{O}_G$ for $q = 1, H^t(X, L) \otimes Q$ for $q = 2$ and $R^t f_*(p^*L^2 \otimes \mathbf{O}_F)$ for $q = 3$.

Remark 1.2. Note that over $G \backslash B$ we have $R^i f_*(p^*L^j \otimes \mathbf{O}_F) = 0$ for every j and every $i > x-2$ because the fibers of f over $G \backslash B$ have dimension $\leq x-2$. Fix a point $[U] \in G \backslash B$ corresponding to a codimension x linear subspace U of $\mathbf{P}^r$. Then for every integer k the fiber of the sheaf $f_*(p^*L^{k+1} \otimes \mathbf{O}_F)$ at $[U]$ is canonically isomorphic to the vector space $H^0(U, \mathbf{O}_{U \cap X}(k+1))$ and fiber over $[U]$ of the homomorphism $u := d_1^{20} : H^0(X, L) \otimes Q \to f_*(p^*L^2 \otimes \mathbf{O}_F)$ in the E_1-part of the spectral sequence (#) is identified at $[U]$ with the natural multiplication map

(6) $$H^0(X, L) \otimes H^0(U, \mathbf{O}_U(1)) \to H^0(U \mathbf{O}_{X \cap U}(2)).$$

2. Proof of Theorem 0.2.

Now we specialize the situation of §1 to the situation of Theorem 0.2, whose proof will be given now.

Proof of Theorem 0.2. First, note that the "Furthermore part" of the statement of 0.2 follows from the first part and [**6**, Lemma 2.4.4].

Now we will prove the first part of 0.2. We write as formulas (7), (8) and (9) the 3 non trivial lines of the E_1-term of the spectral sequence (#) under the assumptions of 0.2; in particular we have $x = 2, \operatorname{rank}(S) = 2, \Lambda^2 S \cong \mathbf{O}_G(-1), \dim(X) = 2$.

(7) $$H^2(\mathbf{O}_X) \otimes \mathbf{O}_G(-1) \to H^2(M_L \otimes L) \otimes \mathbf{O}_G \to H^2(L) \otimes Q$$

(8)
$$H^1(\mathbf{O}_X) \otimes \mathbf{O}_G(-1) \xrightarrow{\alpha} H^1(M_L \otimes L) \otimes \mathbf{O}_G \xrightarrow{\beta} H^1(L) \otimes Q \to R^1 f_*(p^*L^2 \otimes \mathbf{O}_F)$$

(9) $$H^0(\mathbf{O}_X) \otimes \mathbf{O}_G(-1) \to H^0(M_L \otimes L) \otimes \mathbf{O}_G \to H^0(L) \otimes Q \xrightarrow{u} R^0 f_*(p^*L^2 \otimes \mathbf{O}_F).$$

Let $\alpha := d_1^{01}, \beta := d_1^{11}, u := d_1^{20}$ be the maps indicated above. By Remark 1.2 to prove 0.2 it is sufficient to prove that the map u is not surjective on $G \backslash B$. We use that the spectral sequence (#) converges to 0 because the complex (§§)(1) is exact. We have $\operatorname{coker}(u) = E_2^{30}$. We divide the proof into two parts.

(A) Here we assume $h^1(L) = 0$, hence $\beta = 0$ and $\operatorname{coker}(a) = E_2^{11}$. Since the spectral sequence (#) abuts to 0, we have

$$0 = E_\infty^{11} = E_3^{11} = \ker(d_2^{11} : E_2^{11} \to E_2^{30}).$$

Hence $\operatorname{coker}(\alpha)$ injects onto $\operatorname{coker}(u)$. Hence it is sufficient to prove that the codimension of the support of $\operatorname{coker}(\alpha)$ is at most $h^1(\mathbf{O}_X) - \mathbf{f} + 1$. Since $\alpha : H^1(\mathbf{O}_X) \otimes \mathbf{O}_G(-1) \to \mathbf{O}_G^{\mathbf{f}}$ and $\mathbf{O}_G(1)$ is ample, this follows from [**3**, Th. 1.1(a)].

(B) Now we make no assumption on $H^1(L)$. As in the corresponding case of [**5**], the exact sequence (1) gives a homomorphism

$$c : H^1(M_L \otimes L) \to H^1(L) \otimes H^0(L)$$

and $\dim(\ker(c))$ is the amount of failure $\mathbf{f}$ of quadratic normality of L by Lemma 1.1 (ii). On G there is an inclusion of sheaves $(\ker(c)) \otimes \mathbf{O}_G \to \ker(\beta)$. Since $\ker(\beta)$ is a subsheaf of a trivial sheaf, this inclusion is an isomorphism of $(\ker(c)) \otimes \mathbf{O}_G$ onto a direct summand of $\ker(\beta)$. Hence projecting $\ker(\beta)$ onto this summand we obtain a surjection from $E_2^{11} = \ker(\beta)/\operatorname{im}(\alpha)$ onto $\operatorname{coker}(H^1(\mathbf{O}_X) \otimes \mathbf{O}_G(-1) \to \mathbf{O}_G^{\mathbf{f}})$. We conclude as in part (A).

The proof of 0.2 is over. □

Remark 2.1. The proof of 0.2 depends only on $\dim(B)$. If we want to exclude a bigger subset of G, then we obtain a corresponding result in a suitable range. Viceversa, if we may control a dense part of B the corresponding result is true in a larger range. The proof of Theorem 0.2 gives with no change the following result.

Proposition 2.2. *Fix an integer $k \geq 1$. Assume $\dim(X) = 2$. Assume the surjectivity of the restriction map $H^0(\mathbf{P}^r, \mathbf{O}_\mathbf{P}(k)) \to H^0(\phi_L(X), \mathbf{O}(k)) \cong H^0(L^k)$. Let $\mathbf{f}(k)$ be the dimension of the cokernel of the multiplication map $H^0(L) \otimes H^0(L^k) \to H^0(L^{k+1})$. Assume $\mathbf{f}(k) > 0$ and*

$$h^1(L^{k-1}) < \mathbf{f}(k) + \operatorname{codim}(B) - 1.$$

Then there is a codimension 2 linear subspace $[U] \in G \backslash B$ such that the scheme $X \cap U$ is 0-dimensional and the multiplication map

$$H^0(L) \otimes H^0(X \cap U, (L \mid (X \cap U))^k) \to H^0(X \cap U, (L \mid (X \cap U))^{k+1})$$

is not surjective.

Remark 2.3. Note that for a complete but not projectively normal embedding the machine can start (and give informations on (X, L)) using the proposition just given exactly at the first step, say the $(k+1)^{\text{th}}$ step, at which the embedding is not $(k+1)$-normal. However, it can also be used at an intermediate step with large $h^0(L^k)$, obtaining a result of Castelnuovo - Mumford type.

If we look at the proof of Theorem 0.2 when X is a smooth curve with $H^1(L) \leq 1$, we find exactly the proof of [**5**, §2.5]. In the statement we have the small precision about the amount of failure of quadratic normality of L.

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ON THE MINIMAL FREE RESOLUTION OF GENERAL EMBEDDINGS OF CURVES

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Here we study the minimal free resolution of general embeddings in $\mathbf{P}^n$ of genus g curves with general moduli. We prove that if p is an integer with, roughly, $g \leq n^2/(2p+2)$, then the embedding has the property N_p, i.e., the first p pieces of the resolution are as simple as possible.

We work over an algebraically closed field. Let C be a smooth curve embedded in $\mathbf{P}^n$. We are interested in the minimal free resolution of C. Here we will consider the case in which the curve has general moduli and the embedding is general. Recall the following definition ([**5**], [**6**]).

Definition 0.1. Let $C \subset \mathbf{P}^n$ be a reduced curve; fix an integer $p \geq 1$; C satisfies the property N_p if C is arithmetically Cohen - Macaulay and for every integer i with $1 \leq i \leq p$ the i^{th}-sheaf appearing in the minimal free resolution of the homogeneus ideal of C is the direct sum of line bundles of degree $-i-1$.

For instance if we say that N_0 means "C is arithmetically Cohen-Macaulay", then N_1 means that the curve C is N_0 and its homogeneous ideal is generated by quadrics. Furthermore, if $p > 0$, then N_p implies N_{p-1}.

In this paper, using degeneration techniques, we will prove the following results (Theorems 0.2 and 0.3).

Theorem 0.2. *Fix an integer $p \geq 1$. For every integer u, set:*

$$\alpha_p(u) := (u^2)/(2p+2) - (u/2). \tag{1}$$

Fix an integer $n \geq 3$ with $n \geq p+1$, and set:

$$G_p(n) := \alpha_p((p+1)[n/(p+1)]) \tag{2}$$

where $[y]$ is the greatest integer $\leq y$. Then for every integer $g \leq G_p(n)$ the general linearly normal non special curve $C \subset \mathbf{P}^n$ with $p_a(C) = g$ and $\deg(C) = g+n$ satisfies the property N_p.

Note that $G_p(n)$ has order $(n^2)/(2p+2)$ and hence $d := g+n$ is usually much smaller than $2g+p$ if n is much larger than p.

In the case of special linearly normal embeddings we have the following "conditional" result.

Theorem 0.3. *Fix an integer $p \geq 1$ and an integer $s > 2p$. Assume that a general canonical curve of genus $s+1$ in $\mathbf{P}^s$ has the property N_p. Fix an integer $n > p+s$; write $n = s + a(p+1) + b$ with a, b integers and $0 \leq b \leq p$. Set:*

$$S_{p,s}(n) := (a+1)s + 1 + a(a-1)(p+1)/2. \tag{3}$$

Then for every integer g with $s+1 \leq g \leq S_{p,s}(n)$ a general linearly normal curve $C \subset \mathbf{P}^n$ with $p_a(C) = g$ and $h^1(C, \mathbf{O}_C(1)) = 1$ (hence of degree $g+n-1$) has the property N_p.

Quoting existing references on N_p for the canonical model of a curve, C, with general moduli (e.g. [**7**], [**3**]), one obtains corresponding statements for special embeddings of C. We stress that the proof of Theorems 0.2 and 0.3, being a kind of induction on n using as inductive tool Lemma 1.2, may be used to obtain many other cases not covered by the statements of 0.2 and 0.3; the proof of 0.2 should be helpful to the reader interested in other cases.

I want to thank the referee for several suggestions which improved very much the readability of the paper;

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1. In this section we will prove Theorems 0.2 and 0.3. A key quotation for the proofs here is the criterion for condition N_p given in [**6**, Prop. 1.3.3]; the base field for all [**6**] was the complex number field, but the statement of the quoted criterion 1.3.3 works in arbitrary characteristic because each of the steps of its proof either works verbatim in positive characteristic or it is known to hold in general; furthermore, [**6**, Prop. 1.3.3], although stated only for smooth curves, works with the same proof for all reduced curves. It is the use of this criterion for N_p which gives the condition of linear normality in the statements of 0.2, 0.3 and 1.3.

Fix an integer $n \geq 3$; following the notations of [**6**], M_n will denote the rank n vector bundle on $\mathbf{P}^n$ with $M_n(-1)$ isomorphic to the cotangent bundle.

The following result is well known (see e.g. [**1**, Lemma 1.3]):

Lemma 1.1. *Let $D \subset \mathbf{P}^n$ be a rational normal curve. Then $M \mid D$ is the direct sum of n line bundles of degree -1.*

Lemma 1.2. *Fix integers n, p, t, j and m with $n \geq t \geq j > p > 1$ and $m \geq 0$. Let $C \subset \mathbf{P}^{n-j}$ be a reduced curve satisfying condition N_p. See $\mathbf{P}^{n-j}$*

as a linear subspace V of $\mathbf{P}^n$ and let D be a smooth rational curve of degree t in $\mathbf{P}^n$, D spanning a linear subspace W of dimension t, with $\dim(V \cap W) = t-j$ *and* $\operatorname{card}(C \cap D) = t-j+1$, *$D$ intersecting quasi transversally C, (hence $C \cup D$ spanning $\mathbf{P}^n$); assume that $H^1(C, (\Lambda^{(p+1)}M_{n-j}(1))(m) \mid C) = 0$; then $H^1(C \cup D, (\Lambda^{(p+1)}M_n(1))(m) \mid (C \cup D)) = 0$.*

Proof. Consider the following Mayer-Vietoris exact sequence:

$$\begin{aligned} 0 \to & (\Lambda^{(p+1)}M_n(1))(m) \mid (C \cup D) \\ \to & (\Lambda^{(p+1)}M_n(1))(m) \mid C \oplus (\Lambda^{(p+1)}M_n(1))(m) \mid D \\ \to & (\Lambda^{(p+1)}M_n(1))(m) \mid (C \cap D) \to 0. \end{aligned} \tag{4}$$

Note that $M_n \mid C \cong (M_{n-j} \mid C) \oplus \mathbf{O}_C^j$. Hence $\Lambda^{(p+1)}M_n \mid C$ is a direct sum of trivial factors and factors isomorphic to $\Lambda^u M_{n-j} \mid C$ for some integer $u \leq p$. Since N_p implies N_u for every $u < p$ we have

$$H^1(C, (\Lambda^{(p+1)}M_n(1))(m) \mid C) = 0.$$

Note that $M_n \mid D$ is the direct sum of t line bundles of degree -1 and $n-t$ copies of $\mathbf{O}_D$. Hence $(\Lambda^{(p+1)}M_n(1))(m) \mid D$ is a direct sum of line bundles of degree at least $mt + t - p - 1$. To conclude it is sufficient to use (4) and to check (see below) the surjectivity of the restriction map $p : H^0((\Lambda^{(p+1)}M_n(1))(m) \mid D) \to H^0((\Lambda^{(p+1)}M_n(1))(m) \mid (C \cap D))$. For the surjectivity of p, note that, since $\deg(\mathbf{O}_D(-(C \cap D))) = -\operatorname{card}(C \cap D)$ and $j > p$, we have $H^1(D, E(-(C \cap D))) = 0$ for every line bundle E on D with $\deg(E) \geq t - p - 1$. □

Note that $p_a(C \cup D) = p_a(C) + t - j$ and that $C \cup D$ spans $\mathbf{P}^n$. An easy Mayer-Vietoris exact sequence gives $h^1(C \cup D, \mathbf{O}_{C \cup D}(1)) = h^1(C, \mathbf{O}_C(1))$ and $h^0(C \cup D, \mathbf{O}_{C \cup D}(1)) = n + 1$. Hence any smoothing of $C \cup D$ will give linearly normal smooth curves "near" $C \cup D$.

Proof of 0.2. We fix the integer p. First we will prove N_p for the genus $G_p(n)$ and every integer n. Set $n = a(p+1) + r$. As a starting point we assume the property N_p for the rational normal curve of $\mathbf{P}^{p+1}$. Of course, better results and other example can be obtained using other curves in $\mathbf{P}^{p+1}$ with property N_p, e.g. the ones given by an important theorem of M. Green (proven in any characteristic in [**5**, Prop. 3.2]) saying that a linearly normal embedding of degree at least $2k + 1 + p$ of any smooth curve of genus k has the property N_p. The main property of the function G_p is the property $G_p(m + p + 1) = G_p(m) + (p+1)[m/(p+1)]$; its normalization $G_p(p+1) = 0$ comes from the choice we made for the starting point of the induction. Then we apply $(a-1)$ times Lemma 1.2, always with $j = p + 1$ and at each step with the maximal possible t; in the k^{th}-step we pass from a curve of genus $G_p((k+1)(p+1))$ in $\mathbf{P}^{(k+1)(p+1)}$ to a curve of arithmetic genus $G_p((k+2)(p+1))$ in $\mathbf{P}^{(k+2)(p+1)}$. Then we apply Lemma 1.2 for the integers n, t, j with $t = n$ and $j = r$, concluding the case $g = G_p(n)$. Now we will check N_p in $\mathbf{P}^n$ for any non negative integer $g < G_p(n)$ There is an integer $x \leq n - p - 1, x$ divisible by $p+1$, with $G_p(x) \leq g < G_p(x+1)$, say $x = m(p+1)$. We take a curve, C, in $\mathbf{P}^X$ with N_p and genus $G_p(x)$ and we apply Lemma 1.2 for the integers n, t, j, with $j = n - x$ and $t - j = g - G_p(x)$.

At each step of the induction the possibility of deforming the reducible curve to a smooth linearly normal curve (i.e., a smooth curve, T, with the correct $h^1(T, \mathbf{O}_T(1)))$ is proven (in a much stronger form than needed here and for 0.3) independently in several papers: see for instance [**BE**, Lemma 1.2, (1)], or [**4**, Th.4.1], or [**8**, §5]; one can also see a discussion of the way the smoothing concerns the moduli spaces in [**8**] and [**2**, §1, §2, §3]. □

Note that the bound (2) on the genus is just a byproduct of the inductive proof.

Proof of 0.3. The proof of 0.2 works with the following modifications. Instead of quoting [**6**, Prop. 1.3.3], use Remark (2) after the proof of [**6**, Prop. 1.3.3]. The starting point of the induction is a general canonical curve in $\mathbf{P}^s$ which in the statement of 0.3 is assumed to have property N_p. To check the smoothability of reduced curves, use for instance [**2**, Lemma 1.2 (1)]. To check the condition "$h^1(T, \mathbf{O}_T(1)) = 1$", first (as remarked before the proof of 0.2) use a Maver - Vietoris exact sequence to prove it for the reducible curves; then use semicontinuity to obtain that $h^1(T, \mathbf{O}_T(1)) \leq 1$ if T is general, while $h^1(T, \mathbf{O}_T(1)) \geq 1$ by Riemann - Roch. □

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ON NORMALITY OF THE CLOSURE OF A GENERIC TORUS ORBIT IN G/P

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In this paper we consider generic orbits for the action of a maximal torus T in a connected semisimple algebraic group G on the generalized flag variety G/P, where P is a parabolic subgroup of G containing T. The union of all generic T-orbits is an open dense (possibly proper, if P is not a Borel subgroup) subset of the intersection of the big cells in G/P. We prove that the closure of a generic T-orbit in G/P is a normal equivariant T-embedding (whose fan we explicitely describe). Moreover, the closures of any two generic T-orbits are isomorphic as equivariant T-embeddings.

1. Introduction.

Let G be a connected semisimple algebraic group over an algebraically closed field k of arbitrary characteristic. As usual, let B^+ denote a fixed Borel subgroup of G, T a maximal torus in B^+, $\Gamma(T)$ the character group of T, B the opposite to B^+, Φ the corresponding root system in an euclidian space $(E, (\ ,\ ))$, Φ_+ the set of positive roots relative to B^+, Δ the set of simple roots in Φ_+, s_α the reflection about the linear subspace of E perpendicular to root α, W the Weyl group of Φ generated by the reflections $s_\alpha, \alpha \in \Phi_+$ (W can also be naturally identified with $N_G(T)/T$), and R the root lattice in E.

Let P be a fixed parabolic subgroup containing B. Let Δ_P be the set of simple roots α such that $s_\alpha \in W_P = N_P(T)/T$. Then the map $P \to \Delta_P$ is a bijection between the set of all parabolic subgroups containing B and the power set of Δ (see e.g. [**B**, Proposition 14.18]). We denote by S^P the subsemigroup of the root lattice generated by all positive roots which are not sums of simple roots in Δ_P.

We will be concerned with T-orbits of points in the projective variety G/P. Let λ be an integral dominant weight (with respect to Φ_+) whose stabilizer in W is W_P Then λ extends to a character of P (we will also call it λ), inducing a line bundle $\mathcal{L}^\lambda$ on G/P. We let $V(\lambda)$ denote the Weyl G-module

$$H^0(G/P, \mathcal{L}^\lambda) = \{f \in k[G] | f(xy) = \lambda^{-1}(y) f(x) \text{ for all } x \in G, y \in P\}$$

of global sections of $\mathcal{L}^\lambda$ (see e.g. [**J**, Sec. 5.8, p. 84]).

Let Π_λ denote the set of weights of $V(\lambda)$ for the action of T. Let $\mathcal{A}_\lambda$ denote the set of weights of $V(\lambda)$ listed with multiplicity. For each $\mu \in \mathcal{A}_\lambda$, we pick a corresponding weight vector (function) f_μ so that $\{f_\mu | \mu \in \mathcal{A}_\lambda\}$ is a basis of $V(\lambda)$. Functions $f_\mu, \mu \in \mathcal{A}_\lambda$, are called the Plücker coordinates in G/P. By abuse of language we use f_μ to denote any Plücker coordinate of a given weight μ. Let $x = u.P$ be an element of G/P. We let $\Pi_\lambda(x)$ denote the set of weights $\mu \in \Pi_\lambda$ such that at least one of the Plücker coordinates f_μ does not vanish at u. It is easy to see that $\Pi_\lambda(x)$ depends on x and λ only (not on the choice of the Plücker coordinates). It turns out that $\lambda - \Pi_\lambda \subseteq S^P$. Hence by W-invariance of Π_λ, $\lambda - w\Pi_\lambda(x) \subseteq S^P$, for any $x \in G/P$ and $w \in W$. Intuitively, a torus orbit $Tx \subset G/P$ can be called generic if sufficiently many Plücker coordinates of x do not vanish. The following definition makes this requirement precise.

Definition 1.1. Let x be an element of G/P. Then the torus orbit $Tx \subset G/P$ is called *generic* if and only if $\{w\lambda | w \in W\} \subseteq \Pi_\lambda(x)$, and for each $w \in W$, the semigroup generated by $\lambda - w\Pi_\lambda(x)$ is S^P (that is, the maximal semigroup that $\lambda - w\Pi_\lambda(x)$ can generate).

We will show that this definition does not depend on the choice of λ. It turns out that $\Pi_\lambda(x) = \Pi_\lambda$ implies Tx is generic. Therefore generic orbits exist since there are points in G/P at which all Plücker coordinates do not vanish. We will also prove that in the case of G/B, Tx is generic if and only if x belongs to $\bigcap_{w \in W/W_P} wB^+.P$.

The aim of this note is to prove that the closure of a generic T-orbit in G/P is a normal equivariant T-embedding. We can then use the general theory of equivariant torus embeddings (see e.g. [**K**, **Oda1**]) to show that the closures of any two generic orbits are isomorphic (as equivariant T-embeddings). We prove this by identifying the fan describing the isomorphism class of these T-embeddings.

Remark. We point out that if $P \neq B$, the definition of generic T-orbit given here differs from the one used in [**F-H**, Remark 1, p. 257]. There, an orbit Tx is called "generic" if and only if x belongs to the non-degenerate stratum $Z = \bigcap_{w \in W/W_P} wB^+.P$ in the stratification of G/P introduced in [**G-S**] (note that in [**F-H**] B is the "positive" Borel subgroup, while here B denotes the "negative" Borel subgroup). It is easy to see that the set of all $x \in G/P$ with Tx generic in the sense of Definition 1.1 is an open subset of Z. It is proved in [**G-S**, Section 5.1, Proposition 1] that if k is the field of complex numbers then the image under the moment map of the closure of each torus orbit contained in Z is the convex hull of $\{w\lambda | w \in W\}$. In [**F-H**] the general theory of torus embeddings is used to study the closure of Tx in

G/P for $x \in Z$. It appears however that normality of these varieties, required in the theory, has not been proved (as pointed out in [**Oda2**, Section 2.6]). Also, contrary to what is claimed in [**F-H**], two T-orbits in Z may have nonisomorphic closures in G/P (see the example below).

Example. Let $\mathbf{C}$ denote the field of complex numbers. Let q be a nondegenerate quadratic form on $V = \mathbf{C}^5$, and let $G = SO(q)$ be the subgroup of determinant one linear transformations of V, preserving q. Then G is a connected, semisimple, rank 2 algebraic group over $\mathbf{C}$, and V is an irreducible representation of G. Let L be a fixed isotropic line for q (that is $q(v) = 0$ for all $v \in L$), and let $P \subset G$ be the stabilizer of L. Then P is a parabolic subgroup of G, and G/P is naturally isomorphic to the smooth quadric hypersurface Q in the complex projective space $\mathrm{Proj}(V)$ given by the homogeneous equation $q(x) = 0$. For brevity, we will equate G/P with Q. Let $\{e_1, e_2, e_3, e_4, e_4\}$ be the standard basis of V and let $q(x) = x_1x_3 + x_2x_4 - 2x_5^2$, where $[x_1, x_2, x_3, x_4, x_5]$ are the coordinates of $x \in V$ relative to the standard basis. We let $L = \mathbf{C}e_1$. Then the maximal torus contained in P is $T = \{\mathrm{diag}(t_1, t_2, 1/t_1, 1/t_2, 1) | t_i \in \mathbf{C} \setminus \{0\}, i = 1, 2\}$. Here, the Plücker coordinates in $Q = G/P$ are just the standard homogeneous coordinates in $\mathrm{Proj}(V)$. Clearly, $L_1 = \mathbf{C}[1, 1, -1, 1, 0]$ and $L_2 = \mathbf{C}[1, 1, 1, 1, 1]$ are q-isotropic. Also, the T-orbits of L_1 and L_2 are "generic" in the sense of [**F-H**], but only TL_2 is generic in the sense of the Definition 1.1. Also, $\Pi(L_1) \neq \Pi(L_2) = \Pi$, where Π denotes the set of weights of V. This directly contradicts Lemma 13 in [**F-H**]. Let $X_i = \overline{TL_i}, \quad i = 1, 2$, where the closure is taken in Q (or in $\mathrm{Proj}(V)$, since Q is closed in $\mathrm{Proj}(V)$). It is easy to see that X_1 is isomorphic to $\mathbf{C}P^1 \times \mathbf{C}P^1$. On the other hand X_2 is the singular closed subvariety of $\mathrm{Proj}(V)$ given by homogeneous equations $x_1x_3 = x_5^2$, $x_2x_4 = x_5^2$ (the singular points of X_2 are $[1:1:0:0:0]$, $[1:0:0:1:0]$, $[0:1:1:0:0]$, and $[0:0:1:1:0]$). Therefore, the example shows that two T-orbits "generic" in the sense of [**F-H**] may not have isomorphic closures in G/P.

2. Weights of Weyl G-modules.

We will need the following notation. For any additive set A of real numbers and any subset Y of E, let AY denote the set of all linear combinations of elements in Y with coefficients in A. By definition, a semigroup S contained in a lattice L in E is saturated in L if and only if

$$L \cap \mathbf{Q}_+ S = S$$

(see [**K**, Chapter 1, Section 1]). Equivalently, S is saturated in L if and only if for any positive integer m, $m\mu \in S$ and $\mu \in L$ imply $\mu \in S$.

Proposition 2.1. *S^P is saturated in R.*

Proof. Let Φ_+^P denote the set of positive roots which are not linear combinations of roots in Δ_P. Then $S^P = \mathbf{Z}_+\Phi_+^P$. Suppose that S^P is not saturated in R. Let $\mu \in R$ be an element of minimal height among the elements of $\mathbf{Q}_+\Phi_+^P$ which are not elements of S^P. Then $\mu = \mu_1 + \mu_2$, with

$$\mu_1 = \sum_{\beta \in M} m_\beta \beta$$

where $M \subseteq \Phi_+^P$, m_β are positive integers, and

$$\mu_2 = \sum_{\alpha \in N} n_\alpha \alpha$$

where $N \subseteq \Delta_P$, and n_α are positive integers. From the above decompositions of μ we choose one with μ_2 of minimal height. Since the sum of any two roots with negative scalar product is again a root, minimality of μ_2 implies that

$$(\alpha, \beta) \geq 0,$$

for all $\alpha \in N, \beta \in M$. Take any simple root α in N, such that $(\mu_2, \alpha) > 0$. Consider $\nu = s_\alpha(\mu) \in R$. Since elements of Φ_+^P are permuted by s_α, ν belongs to $\mathbf{Q}_+\Phi_+^P$ but not to S^P. This is a contradiction, since $\mathrm{ht}(\nu) < \mathrm{ht}(\mu)$ and μ was assumed to be of minimal height among the root lattice elements in $\mathbf{Q}_+\Phi_+^P$, not in S^P. □

Let $V(\lambda), \lambda, \Pi_\lambda$ be as in the introduction. The following proposition lists some basic properties of Π_λ.

Proposition 2.2.

(i) *$\lambda - \Pi_\lambda$ coincides with the set of root lattice points in the convex hull of $\{\lambda - w\lambda | w \in W\}$.*

(ii) *S^P is generated by $\lambda - \Pi_\lambda$. If $P = B$ and λ is the sum of fundamental weights then S^P is generated by $\{\lambda - w\lambda, w \in W\}$.*

Remark. Part (i) is well known, but were not able to locate an appropriate reference.

Proof. We first observe that the weights of the Weyl module $V(\lambda)$ (λ integral dominant) are independent of the characteristic of k. This follows from the fact that character formulas for Weyl modules are the same in each characteristic. Therefore we can assume, that $\mathrm{char}(k) = 0$.

Part (i). Let C denote the convex hull of $\{w\lambda | w \in W\}$ and we let $\Pi = (\lambda + R) \cap C$. We have to prove that $\Pi = \Pi_\lambda$. It is a known fact that $\Pi_\lambda \subset \lambda + R$.

Therefore it is enough to show that Π_λ is contained in C. Suppose that this is not the case, and let μ be a weight in Π_λ, not in C. Assume also, that μ is a maximal such weight in the usual order in E relative to Φ_+. Since both Π_λ and C are W−invariant, we must have $s_\alpha(\mu) \leq \mu$ for all positive roots α. Hence μ is dominant. Since μ is not the highest weight λ, there must be a positive root α and a positive integer m such that $\mu_1 = \mu + m\alpha \in \Pi_\lambda$. Then by maximality of μ, μ_1 (hence also $s_\alpha(\mu_1)$) is in C. A straightforward computation shows that μ belongs to the line segment connecting μ_1 and $s_\alpha(\mu_1)$. This is a contradiction, since we have assumed that μ is not in C.

We are left with showing that Π is contained in Π_λ. An easy argument by induction on the length function in W, shows that for any $w \in W$, $\lambda - w\lambda$ is a sum of roots in Φ_+^P. Therefore Π is contained in $\lambda - \mathbf{Z}_+\Phi_+$. It is proved in [**H**, Proposition, p. 114] that the elements of Π_λ are exactly the weights whose W-orbit is contained in $\lambda - \mathbf{Z}_+\Phi_+$. Hence $\Pi \subseteq \Pi_\lambda$, as required.

Part (ii) We have observed in the proof of Part (i) that $\lambda - C$ is contained in convex cone spanned by Φ_+^P. Therefore

$$\lambda - \Pi_\lambda = R \cap (\lambda - C) \subseteq S^P$$

since S^P is saturated in R. The opposite inclusion holds since $\Phi_+^P \subseteq \lambda - \Pi_\lambda$. This follows from the fact that weigths of irreducible G−representations (in characteristic 0) satisfy the following property: for any positive root α, and a positive integer n, if μ and $\mu - n\alpha$ are weights of the representation, so are $\mu - q\alpha$ for any $q, 0 \leq q \leq n$ (see e.g. [**H**, Sec. 21.3, Prop.]). One applies this property to λ and $s_\alpha(\lambda)$, where $\alpha \in \Phi_+^P$.

The second claim of Part(ii) follows since $\Delta \subseteq \{\lambda - w\lambda | w \in W\}$ if λ is the sum of fundamental weights. □

3. Generic orbits of T in G/P.

Let $x \in G/P$ and let X denote the the closure of Tx in G/P. For any $w \in W$, let

$$Y_w = \{y.P | f_{w\lambda}(y) \neq 0\} = \{y.P | w\lambda \in \Pi_\lambda(y.P)\}$$

and

$$X_w = Y_w \cap X.$$

It is well known that each Y_w is an affine space which is open in G/P and whose coordinate ring is generated by functions $f_\mu/f_{w\lambda}, \mu \in \mathcal{A}_\lambda$. Moreover, the union of $Y_w, w \in W$ is G/P. Let $T_x = \{t \in T | tx = x\}$ and $T^x = T/T_x$. We have the following proposition

Proposition 3.1. *Let $x \in G/P$.*

(i) *Tx is open in X and it is isomorphic to T^x. Therefore, X is an equivariant T^x–embedding in the sense of* [**K**].

(ii) *$\{X_w | w \in W, w\lambda \in \Pi_\lambda(x)\}$ is a covering of X by T-invariant open affine subsets of X. The coordinate ring of $X_w, w\lambda \in \Pi_\lambda(x)$, is the subalgebra of $k[T^x] = k[\Gamma(T^x)]$ generated by $\Pi_\lambda(x) - w\lambda$.*

(iii) *Let $w \in W$ be such that $w\lambda \in \Pi(x)$. Then*

$$T_x = \{t \in T | \mu(t) = 1 \text{ for all } \mu \in w\lambda - \Pi(x)\},$$

Proof. The fist part of (i) follows from the fact the map $t \to tx$ is a separable morphism from T onto an open subvariety Tx of X whose fibers are the cosets of T_x in T (the morphism is separable since it is the composition of the inclusion of T in G with the quotient map from G to G/P).

Part (ii) follows, since for each $w \in W$ such that $w\lambda \in \Pi_\lambda(x)$, X_w can be viewed as a closed T-invariant subvariety of the affine space Y_w. Hence the coordinate ring of X_w is generated by the restrictions to X_w of functions $f_\mu / f_{w\lambda}, \mu \in \mathcal{A}_\lambda$.

Part(iii). Suppose that $w \in W$ satisfies $w\lambda \in \Pi(x)$. Then $x \in X_w$. Clearly, $t \in T_x$ if and only if t fixes all elements of X_w (or equivalently, t fixes all regular functions on X_w). Therefore the desired formula for T_x follows from the description of the coordinate ring of X_w given in (ii). □

Before we state a corollary of Proposition 3.1, we need to introduce the following notation. Let R^P denote the subgroup of the root lattice generated by S^P. One can show that $R^P = R$ if Φ is an irreducible system. If Φ a union of irreducible root systems $\Phi_j, j \in J$, then R^P is the root lattice of the root system

$$\cup\{\Phi_j | \Phi_j \cap S^P \neq \emptyset\}.$$

Let

$$T_P = \bigcap_{\nu \in R^P} \ker(\nu).$$

Note that if $R^P = R$, then T_P is coincides with the center of G.

Corollary. (Suggested by the referee.)

(i) *The stabilizer of each generic torus orbit is T_P. Moreover, T_P it is the smallest subgroup of T among the T-stabilizers of elements of G/P.*

(ii) *(Partial converse of* (i)*). If $x \in G/P$ is such that Tx is contained in the nondegenerate stratum Z, $\overline{Tx}$ is normal and $T_x = T_P$, then Tx is generic.*

Proof. Part (i) follows from Proposition 3.1 (iii). Suppose that Tx satisfies the assumptions of (ii). Let S^x denote the semigroup generated by $\lambda - \Pi_\lambda(x)$. We have to show that $S^x = S^P$. Since $T_x = T_P$, one has

$$\bigcap_{\nu \in R^P} \ker(\nu) = \bigcap_{S^x} \ker(\nu)$$

by Proposition 3.1 (iii). Therefore R^P is generated by S^x as a subgroup of $\Gamma(T)$. Assumed normality of $\overline{Tx}$ implies that S^x is saturated in R^P. On the other hand $\{\lambda - w\lambda | w \in W\} \subset S^x$ since Tx is assumed to be generic. Hence $S^x = S^P$ since both semigroups are saturated in R^P and $\mathbf{Q}_+ S^x = \mathbf{Q}_+ S^P$ by Proposition 2.2. □

From now on we assume for simplicity that $R^P = R$ (equivalently, S^P contains at least one root from each irreducible component of Φ). Let $W^P \subseteq W$ be a fixed set of representatives of W/W_P. Let D denote the fundamental chamber $\{\nu \in E | (\mu, \alpha) \geq 0 \text{ for all } \alpha \in \Delta\}$. We are now ready to state the main result of this paper.

Theorem 3.2. *Let $x \in G/P$ be such that $Tx \subset G/P$ is generic. Let $X = \overline{Tx}$. Then:*

(i) *X is a normal variety (hence by* [**K**, Theorem 14, page 52], *also Cohen-Macaulay with rational singularities).*

(ii) *The fan corresponding to X consists of the cones*

$$C_w = -w \bigcup_{z \in W_P} zD, \quad w \in W^P$$

together with their faces. In particular, the closures of any two generic orbits in G/P are isomorphic as T-equivariant embeddings.

Proof. Part (i). By [**K**, Theorem 6, p. 24] a general equivariant T-embedding is a normal variety if and only if it admits a covering by open affine T-stable subvarieties whose coordinate rings are generated by semigroups saturated in $\Gamma(T)$. Hence Part(ii) follows from Propositions 3.1 and 2.1.

Part(ii) follows, since the dual cone of S^P is $\bigcup_{z \in W_P} zD$, and by Proposition 3.1(ii) the coordinate ring of X_w, $w \in W$, is $k[-wS^P]$. □

The following theorem shows that Definition 1.1 of a generic torus orbit does not depend on the choice of the Weyl module $V(\lambda)$.

Theorem 3.3. *Let $x \in G/P$. The following statements are equivalent.*

(i) *There exist an integral dominant weight λ whose stabilizer in W is W_P, such that for any $w \in W$, the semigroup generated by $\lambda - w\Pi_\lambda(x)$ is S^P.*

(ii) *For each integral dominant weight λ whose stabilizer in W is W_P, and each $w \in W$, the semigroup generated by $\lambda - w\Pi_\lambda(x)$ is S^P.*

(iii) *There exists an integral dominant weight λ whose stabilizer in W is W_P, such that $\Pi_\lambda(x) = \Pi_\lambda$.*

Proof. Clearly, (ii) implies (i). Also, by Proposition 2.2, (iii) implies (i). We have to prove that if (i) holds, so do (ii) and (iii). Let $X = \overline{Tx}$ and let $X_w, w \in W$ be as in Theorem 3.1. Since the coordinate ring of X_w does not depend on the choice of a Plücker embedding, Theorem 3.1(ii) implies that (ii) follows from (i).

It remains to prove that (i) implies (iii). Let $x \in G/P$ and let λ be as in (i). For any integral dominant weight μ whose stabilizer in W is W_P, let $\mathcal{L}^\mu$ denote the corresponding line bundle on G/P. Let $\mathcal{L}^\mu_X$ denote the pullback of $\mathcal{L}^\mu$ to $X = \overline{Tx}$. Since X contains an open, dense T-orbit, every weight of $H^0(X, \mathcal{L}^\mu_X)$ under the natural T-action has multiplicity one. Therefore the dimension of the image of the restriction map

$$H^0(G/P, \mathcal{L}^\mu) \to H^0(X, \mathcal{L}^\mu_X)$$

is $\sharp(\Pi_\mu(x))$. We observe that line bundle $\mathcal{L}^\mu_X$ is ample. This is because the piecewise linear function on E corresponding to $\mathcal{L}^\mu_X$ (see [**F-H**, Theorem 2]) is strictly upper convex. Then the description of the fan of X given in Theorem 3.2(iii), [**Oda1**, Theorem 2.13 and Corollary 2.9], and Proposition 2.2 (i) imply that

$$\dim \quad H^0(X, \mathcal{L}^\mu_X) = \sharp(\Pi_\mu).$$

Since $\mathcal{L}^\lambda$ is ample there exists a positive integer q such that the restriction map

$$H^0(G/P, \mathcal{L}^{q\lambda}) \to H^0(X, \mathcal{L}^{q\lambda}_X)$$

is surjective. Hence $\Pi_{q\lambda}(x) = \Pi_{q\lambda}$ as required. □

It is easy to see that Theorem 3.3 and Proposition 2.2 imply:

Corollary. *Let $x \in G/B$. Then Tx is generic if and only if $x \in \bigcap_{w \in W} wB^+.B$ (i.e. it is "generic" in the sense of* [**F-H**]). *Moreover, if xT is generic then $X = \overline{Tx}$ is smooth.*

Remark. Smoothness of the closure of a generic torus orbit in G/B is well known (we do not know however, to whom this fact should be attributed).

Final remarks and questions.

1. All results about closures of T-orbit in G/P stated in [**F-H**] hold for generic orbits (in the sense of Definition 1.1) in any characteristic. This is because the arguments used in [**F-H**] are valid for normal equivariant T-embeddings, and we have shown that the closure of a generic orbit is such an embedding. We do not know however, if the results remain valid for all T-orbits in the nondegenerate stratum if $P \neq B$.

2. Let X denote the closure of a T-orbit of an element $x \in G/P$. It is not difficult to prove that if λ is an integral dominant weight whose stabilizer in W is W_P, then the line bundle $\mathcal{L}_X^\lambda$ is in fact very ample (one can use the criterion for very ampleness given in [**F**, Lemma, p. 69] or [**Oda1**, Corollary 2.9]). Then it follows from [**F**, Exercise, p. 72] that the corresponding embedding of X in $\mathrm{Proj}(H^0(X, \mathcal{L}_X^\lambda))$ is projectively normal and Cohen-Macaulay (that is, the homogeneous coordinate ring of X in $\mathrm{Proj}(H^0(X, \mathcal{L}_X^\lambda))$ is normal and Cohen-Maculay). Therefore, the embedding $X \subset \mathrm{Proj}(H^0(G, \mathcal{L}^\mu))$ is also projectively normal and Cohen-Macaulay, if the restriction map from $H^0(G/P, \mathcal{L}^\lambda)$ to $H^0(X, \mathcal{L}_X^\lambda)$ is surjective (equivalently $\Pi_\lambda(x) = \Pi_\lambda$). We do not know if this is so, if Tx is generic and $\Pi_\lambda(x) \neq \Pi_\lambda$.

3. Since the closure of any T-orbit in an equivariant normal T-embedding is normal (see [**K**, Proposition 2, p. 17]), X is normal if it is contained in the closure of a generic T-orbit. In this situation, the fan corresponding to X can be described explicitly in terms of the fan defined in Theorem 3.2 (iii) (see e.g. [**Oda2**, Section 1.1]). Since there could be non-generic orbits of maximal dimension (see the example in the introduction) not every T-orbit is contained in the closure of a generic one. The structure of the orbit is not clear. Does it have to be normal? If yes, what is its fan? Suppose that the closures of all T-orbits in G/P are indeed normal. Then the Example and the Corollary of Proposition 3.1, suggest the conjecture that the isomorphism type of $\overline{Tx}$ (as a torus equivariant embedding) is determined by two pieces of data: the stabilizer of x in T and the set $\{w \in W/W_P | x \in B^+w.P\}$.

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PARAGROUPE D'ADRIAN OCNEANU ET ALGEBRE DE KAC

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Dans ces quelques pages, nous reprenons l'essentiel des notes manuscrites d'Adrian Ocneanu intitulées "A Galois theory for operator algebras"(1986). Nous en précisons les définitions et démontrons les théorèmes essentiels: les propriétés fondamentales du paragroupe, le résultat de classification qui est un corollaire du théorème de classification de S. Popa et la caractérisation de l'inclusion d'un facteur dans son produit croisé par une algèbre de Kac de dimension finie. Il nous a paru important que le paragroupe soit explicitement défini et que ces résultats admis par tous et souvent cités par F. Goodman, P. de la Harpe, V. Jones dans leur livre [**GHJ**] et par S. Popa dans ses article de classification [**Popa**, **1** et **2**] reçoivent enfin une demonstraion exhaustive. Je me suis attachée à rédiger les démonstrations qui auraient pu être données à l'époque à deux exception près:

Le caractère d'invariant complet du paragroupe est démontré grâce aux carrés commutatifs de S. Popa.

La coassociativité du coproduit de l'algèbre de Kac (§5) était vérifiée directement dans ma première version (Publications de l'Université Paris-Sud #93-06). Claire Anantharaman a attiré mon attention sur l'article de W. Szymanski [**S**], je l'en remercie: la dualité qu'il définit me permet de donner une démonstration plus algébrique.

Parmi les développements de la théorie qui pourraient fournir d'autres démonstrations à ces résultats, on peut citer, par exemple, la théorie des bimodules d'A. Ocneanu [**O**], la théorie des secteurs [**L1**, **L2**], [**I1**, **I2**]...

Je remercie particulièrement Vaughan Jones qui m'a encouragée à entreprendre ce travail et m'a guidée lors de nombreuses discussions. Je remercie aussi Michel Enock pour ses conseils qui m'ont aidée à achever cet article.

0. Introduction.

Soit N un sous-facteur d'indice fini dans M, un facteur de type II_1 dont tr est la trace finie fidèle normalisée. Soit M_1 l'algèbre oblenue par construction de base: M_1 est l'algèbre de von Neumann sur $L^2(M, \mathrm{tr})$ engendrée par M et e_N la projection sur $L^2(N, \mathrm{tr})$ [**VJ1**]. Si J est l'involution standard de $L^2(M, \mathrm{tr})$,

M_1 est égal à $JN'J$. J permet donc de définir un anti-automorphisme γ_0 de $N' \cap M_1$:

$$\gamma_0(x) = Jx^*J \qquad (x \in N' \cap M_1).$$

Plus généralement, comme M. Pimsner et S. Popa ont montré que

$$N \subset M_n \subset M_{2n+1}$$

est isomorphe à la construction de base, on sera tenté de définir, pour x élément de $N' \cap M_{2n+1}$, $\gamma_n(x)$ par $J_n x^* J_n$ où J_n est l'involution standard de $L^2(M_n, \mathrm{tr})$. La première partie de cet article contient les vérifications nécessaires à une définition cohérente de γ_n.

La deuxième partie donne la définition et les propriétés fondamentales des γ_n.

La troisième partie contient la démonstration de ces propriétés et une expression de $\gamma_n(y)$ quand y est un élément de $N' \cap M_{2n+1}$.

La quatrième partie montre que le paragroupe (la tour dérivée munie des anti-automorphismes) est un invariant complet pour l'inclusion d'un sous-facteur de profondeur finie dans le facteur hyperfini de type II_1 équivalent à l'invariant défini par S. Popa dans [**Popa1**], à savoir le carré commutatif canonique.

On rappelle que *l'inclusion $N \subset M$ est de profondeur finie* si le graphe principal est fini [**GHJ**, 4.1]; on obtient le *graphe principal* de $N \subset M$ en effaçant dans le diagramme de Bratteli de la tour dérivée ce qui s'obtient par réflexion de l'étage précédent.

La cinquième partie donne une caractérisation de l'inclusion d'un facteur de type II_1 dans son produit croisé par une algèbre de Kac de dimension finie:

Soient M un facteur de type II_1, tr sa trace normale finie fidèle normalisée et N un sous-facteur d'indice fini dans M. Les proposition suivantes sont équivalentes:

(a) N est de profondeur au plus 2 dans M et $N' \cap M$ est égal à $\mathbb{C}$.

(b) M est le produit croisé de N par une action extérieure d'une algèbre de Kac de dimension finie $\mathbb{K}$.

(c) N est la sous-algèbre des points fixes de M sous une action extérieure d'une algèbre de Kac de dimension finie $\widehat{\mathbb{K}}$.

Une démonstration de ce résultat utilisant la méthode des secteurs se trouve dans [**L2**] (voir aussi [**I2**]). Dans [**I1**], on trouvera une caractérisation d'une inclusion irréductible de profondeur 2 de facteurs proprement infinis.

Un résultat semblable dans le cas où l'indice est infini est montré dans [**EN**]. D'autre part, si $N' \cap M_1$ est commutatif, l'algèbre de Kac est un groupe fini.

1. Représentations des algèbres de la tour obtenue en itérant la construction de base.

1.1. Définitions [**VJ1**]. Soient M un facteur de type II_1, tr sa trace normale finie fidèle normalisée et N un sous-facteur d'indice fini dans M. On regarde M dans sa représentation standard π_0 sur $L^2(M,\mathrm{tr})$. L'espérance conditionelle E_N de M sur N définit le projecteur e_N de $L^2(M,\mathrm{tr})$ sur $L^2(N,\mathrm{tr})$:

Si ξ est le vecteur cyclique canonique donné par la trace, si x appartient à M, on a:

$$e_N(x\xi) = E_N(x)\xi.$$

La construction de base sur $N \subset M$ est la définition de l'algèbre de von Neumann M_1 sur $L^2(M,\mathrm{tr})$ engendrée par M et e_N. On connaît donc M_1 par sa représentation fidèle π_0 sur $L^2(M,\mathrm{tr})$ qui prolonge la représentation π_0 de M par multiplication à gauche, c'est-à-dire pour tout a de M et tout x de M, on a:

$$\pi_0(a)(x\xi) = (ax)\xi \text{ et } \pi_0(e_N)(x\xi) = E_N(x)\xi.$$

D'après [**VJ1**, 3.1.7], la trace canonique Tr sur M_1 est une $([M:N]^{-1}, M)$ trace, c'est-à-dire Tr étend tr et $\mathrm{Tr}(e_N x)$ est égal à $[M:N]^{-1}\,\mathrm{tr}(x)$ pour tout x de M. On notera alors la trace sur M_1 comme la trace sur M par tr.

1.2. La tour. D'après [**VJ1**, 3.1.7], on peut recommencer la construction de base à partir de l'inclusion $M \subset M_1$ et on obtient une algèbre de von Neumann M_2 que l'on connait par sa représentation π_1 sur $L^2(M_1,\mathrm{tr})$. Les restrictions de π_1 à M_1 ou M sont les représentation de ces algèbres qui prolongent leur action par multiplication à gauche sur $M_1\xi_1$, où ξ_1 est le vecteur cyclique canonique.

La trace sur M_2 prolonge celle de M_1 et vérifie la propriété de Markov, on la notera encore tr et ainsi pour la trace de chaque algèbre construite par construction de base. En effet, en répétant la construction de base, on obtient la tour d'algèbres:

$$N \subset M \overset{e_0}{\subset} M_1 \overset{e_1}{\subset} M_2 \overset{e_2}{\subset} \cdots M_n \overset{e_n}{\subset} M_{n+1} \cdots$$

On connaît M_{n+1} par sa représentation π_n sur $L^2(M_n,\mathrm{tr})$, M_{n+1} est l'algèbre de von Neumann engendrée par M_n et e_n, la projection de $L^2(M_n,\mathrm{tr})$ sur $L^2(M_{n-1},\mathrm{tr})$.

1.3. La construction de base $N \subset M_n \subset M_{2n+1}$. Dans [**PiPo2**], M. Pimsner et S. Popa remarquent qu'on peut définir abstraitement l'algèbre de la construction de base sur $N \subset M$ comme l'unique (à un isomorphisme près) facteur fini M_1, muni d'une trace τ, qui contienne M et une projection e et vérifie

$$\begin{aligned}
&[M_1 : M] = [M : N]\\
&[e, y] = 0 \quad (y \in N)\\
&exe = E_N(x)e \quad (x \in M)\\
&\tau(ex) = [M_1 : M]^{-1}\, \mathrm{tr}(x) \quad (x \in M).
\end{aligned}$$

Ils montrent alors que l'algèbre M_{2n+1} est isomorphe à l'algèbre obtenue par la construction de base sur $N \subset M_n$. Il existe donc une représentation fidèle π_n de M_{2n+1} sur $L^2(M_n, \mathrm{tr})$. Le projecteur de la construction de base est alors [**PiPo2**, 2.6].:

$$f_n^{-1} = [M : N]^{\frac{n(n+1)}{2}} (e_n e_{n-1} \dots e_0)(e_{n+1} e_n \dots e_1) \dots (e_{2n} e_{2n-1} \dots e_n)$$

M_{2n+1} est donc le facteur engendré par M_n et f_n^{-1}.

Si ξ_n est le vecteur cyclique canonique de $L^2(M_n, \mathrm{tr})$ et x_n un élément de M_n, on a:

$$\pi_n(f_n^{-1})(x_n \xi_n) = E_N(x_n)\xi_n$$

et la restriction de π_n à M_n est la representation standard de M_n sur $L^2(M_n, \mathrm{tr})$.

1.4. D'autres représentations. On pourrait aussi regarder la construction de base sur $M_p \subset M_n$, $p \le n$, qui nous donne une représentation π_n^p de M_{2n-p} sur $L^2(M_n, \mathrm{tr})$; posons alors

$$f_n^p = [M : N]^{\frac{(n-p)(n-p-1)}{2}} (e_n e_{n-1} \dots e_{p+1})(e_{n+1} e_n \dots e_{p+2}) \dots (e_{2n-p-1} \dots e_n).$$

M_{2n-p} est le facteur engendré par M_n et f_n^p et, pour x_n dans M_n, $\pi_n^p(f_n^p)(x_n \xi_n)$ vaut $E_{M_p}(x_n)\xi_n$.

La restriction de π_n (définie en 3) à M_{2n-p} nous donne aussi une représentation de M_{2n-p} sur $L^2(M_n, \mathrm{tr}) \dots$ Nous allons voir dans le paragraphe suivant que π_n^p et π_n coïncident sur M_{2n-p}.

1.5. Compatibilité des représentations obtenues à partir de différentes constructions de base. Nous commençons par fixer les notations et rappeler les règles de calcul dans les algèbres de la tour.

1.5.1. Notations.

a) $\alpha = [M : N]$, a est la partie entière de α et $\alpha(n, p) = \alpha^{\frac{(n-p)(n-p-1)}{2}}$.

b) a_n est la partie entière de α^n, l'indice de M_n dans M_{2n}.

c) $g_n^k = e_n e_{n-1} \dots e_{k+1} e_k \qquad n \in \mathbb{N}, k \in \mathbb{N}, n \ge k.$

d) $g_n^k = e_n e_{n+1} \dots e_{k-1} e_k \qquad n \in \mathbb{N}, k \in \mathbb{N}, n \leq k$
remarquons que $g_k^n = \left(g_n^k\right)^*$ pour tous n et k.

e) $f_n^p = \alpha(n,p) g_n^{p+1} g_{n+1}^{p+2} \dots g_{2n-p-1}^n \qquad n \in \mathbb{N}, p \in \mathbb{N}, n \geq p \geq -1$.

1.5.2. Bases de Pimsner-Popa [**PiPo1**, 1.3]**.**
Il existe une famille $\{\lambda_j, 1 \leq j \leq a+1\}$ d'éléments de M, appelée base de Pimsner-Popa de M sur N, telle que:

a) $E_N(\lambda_j^* \lambda_k) = 0$ si $j \neq k$.

b) $E_N(\lambda_j^* \lambda_j) = p_j$ où p_j est un projecteur de N de trace $\alpha - a$ si $j = a+1$ et est l'identité sinon.

Une telle famille vérifie de plus:

c) $\lambda_j e_N$ est une isométrie partielle pour $1 \leq j \leq a+1$.

d) $\sum_{j=1}^{a+1} \lambda_j e_N \lambda_j^* = 1$.

e) $\sum_{j=1}^{a+1} \lambda_j \lambda_j^* = \alpha$.

f) Tout y de M admet une unique décomposition $y = \sum_{j=1}^{a+1} \lambda_j y_j$ où y_j est un élément de $p_j N$ égal à $E_N(\lambda_j^* y)$. De même $y = \sum_{j=1}^{a+1} E_N(y\lambda_j)\lambda_j^*$.

g) La famille $\{\alpha^{1/2}\lambda_i e_0, 1 \leq i \leq a+1\}$ est une base de Pimsner-Popa de M_1 sur M. Plus généralement, la famille $\{\alpha^{(p+1)/2}\lambda_i g_0^p, 1 \leq i \leq a+1\}$ est une base de Pimsner-Popa de M_{p+1} sur M_p. (Ceci résulte de la démonstration de la proposition 1.5 de [**PiPo1**].)

1.5.3. Règles de calcul. On rappelle que

a) e_k appartient à M_{k+1}.

b) e_k commute avec les éléments de M_{k-1}.

c) e_k commute avec e_h si $|k-h| \geq 2$.

d) $e_k e_{k+1} e_k = \alpha^{-1} e_k$ et $e_{k+1} e_k e_{k+1} = \alpha^{-1} e_{k+1}$.

e) $e_{k+1} x e_{k+1} = E_{M_{k-1}}(x) e_{k+1}$ $(x \in M_k)$.

f) $E_{M_k}(e_k) = \alpha^{-1}$.

g) f_n^p appartient à M_{2n-p} et commute avec les éléments de M_p.

h) $\pi_n(f_n^p)(x_n \xi_n) = E_{M_p}(x_n)\xi_n$.

Lemme 1.5.4. $f_n^p f_n^{-1} = f_n^{-1}$.

Démonstration. Pour montrer cette égalité, nous allons faire disparaître un à un les produits du type g_{n+k}^{p+k} de f_n^p.

Voici un procédé pour faire disparaître un projecteur:

Lemme. $g_h^{n+k} g_{n+k}^k g_{n+k+1}^{k+1} = \alpha^{-1} g_{n+k-1}^k g_h^{n+k+1} g_{n+k+1}^{k+1}$ *si* $h \geq n+k+1$ *et*

$k \geq 0$.

Démonstration.

$$\begin{aligned}
g_h^{n+k} g_{n+k}^k g_{n+k+1}^{k+1} &= g_h^{n+k+2}\, e_{n+k+1}\, e_{n+k}\, g_{n+k}^k\, g_{n+k+1}^{k+1} \qquad (1.5.1c)\\
&= g_h^{n+k+2}\, e_{n+k+1}\, e_{n+k}\, g_{n+k-1}^k\, e_{n+k+1}\, g_{n+k+1}^{k+1} \qquad (1.5.1c)\\
&= g_h^{n+k+2}\, e_{n+k+1}\, e_{n+k}\, e_{n+k+1}\, g_{n+k-1}^k\, g_{n+k+1}^{k+1} \qquad (1.5.2b)\\
&= \alpha^{-1} g_h^{n+k+2}\, e_{n+k+1}\, g_{n+k-1}^k\, g_{n+k+1}^{k+1} \qquad (1.5.3d)\\
&= \alpha^{-1} g_{n+k-1}^k\, g_h^{n+k+2}\, e_{n+k+1}\, g_{n+k+1}^{k+1} \qquad (1.5.1c) \text{ et } (1.5.3c)\\
&= \alpha^{-1} g_{n+k-1}^k\, g_h^{n+k+1}\, g_{n+k+1}^{k+1} \qquad (1.5.1c).
\end{aligned}$$

Suite de la démonstration de 1.5.2.
Ce procédé va nous premettre de faire disparaître un à un les projecteurs de g_{2n-p-1}^n.

$$g_{2n-p-1}^n f_n^{-1} = \alpha(n,-1)\left(g_{2n-p-1}^n\, g_n^0 g_{n+1}^1 \cdots g_{2n-p-2}^{n-p-2}\right) g_{2n-p-1}^{n-p-1} \cdots g_{2n}^n \quad (1.5.1e).$$

En appliquant le lemme une fois, nous obtenons d'abord:

$$g_{2n-p-1}^n f_n^{-1} = \alpha(n,-1)\left(\alpha^{-1} g_{n-1}^0\, g_{2n-p-1}^{n+1}\, g_{n+1}^1\, g_{n+2}^2 \cdots g_{2n-p-2}^{n-p-2}\right) g_{2n-p-1}^{n-p-1} \cdots g_{2n}^n.$$

En appliquant ce résultat $(n-p-2)$ fois de plus, nous faisons disparaître g_{2n-p-1}^n.

$$\begin{aligned}
g_{2n-p-1}^n f_n^{-1} &= \alpha(n,-1)\Big(\alpha^{-(n-p-2)} g_{n-1}^0\, g_n^1 g_{n+1}^2 \cdots \\
&\qquad g_{2n-p-4}^{n-p-3}\, g_{2n-p-1}^{2n-p-2}\, g_{2n-p-2}^{n-p-2}\Big)\, g_{2n-p-1}^{n-p-1} \cdots g_{2n}^n\\
&= \alpha(n,-1)\left(\alpha^{-(n-p-1)} g_{n-1}^0\, g_n^1 \cdots g_{2n-p-3}^{n-p-3}\right) g_{2n-p-1}^{2n-p-1}\, g_{2n-p-1}^{n-p-1} \cdots g_{2n}^n\\
&= \alpha(n,-1)\left(\alpha^{-(n-p-1)} g_{n-1}^o\, g_n^1 \cdots g_{2n-p-3}^{n-p-2}\right) g_{2n-p-1}^{n-p-1} \cdots g_{2n}^n.
\end{aligned}$$

Multiplions maintenant par g_{2n-p-2}^{n-1}:

$$\begin{aligned}
g_{2n-p-2}^{n-1} g_{2n-p-1}^n f_n^{-1} &= \alpha(n,-1)\alpha^{-(n-p-1)}\\
&\qquad \left(g_{2n-p-2}^{n-1}\, g_{n-1}^0\, g_n^1 \cdots g_{2n-p-3}^{n-p-2}\right) g_{2n-p-1}^{n-p-1} \cdots g_{2n}^n.
\end{aligned}$$

Par le procédé précédent appliqué $(n-p-2)$ fois, nous avons:

$$\begin{aligned}
g_{2n-p-2}^{n-1} g_{2n-p-1}^n f_n^{-1} &= \alpha(n,-1)\alpha^{-(n-p-1)}\\
&\qquad \left(\alpha^{-(n-p-2)} g_{n-2}^0 \cdots g_{2n-p-5}^{n-p-3}\, g_{2n-p-2}^{2n-p-3}\, g_{2n-p-3}^{n-p-2}\right) g_{2n-p-1}^{n-p-1} \cdots g_{2n}^n.\\
g_{2n-p-2}^{n-1} g_{2n-p-1}^n f_n^{-1} &= \alpha(n,-1)\alpha^{-(n-p-1)-(n-p-2)}\\
&\qquad \left(g_{n-2}^0 g_{n-1}^1 \cdots g_{2n-p-5}^{n-p-3}\right) g_{2n-p-2}^{n-p-2}\, g_{2n-p-1}^{n-p-1} \cdots g_{2n}^n.
\end{aligned}$$

Après la disparition de g_{n+1}^{p+2}, c'est-à-dire après $(n-p-1)$ absorptions, nous obtenons

$$f_n^p f_n^{-1} = \alpha(n,p)\alpha(n,-1)\alpha^{-(n-p-1)-(n-p-2)\dots-1} g_n^{p+1}\left(g_{p+1}^0\right) g_{n+1}^1 \cdots g_{2n}^n.$$

Comme $(n-p-1)+(n-p-2)\dots+1 = \frac{(n-p)(n-p-1)}{2}$ et $g_n^{p+1} g_{p+1}^0 = g_n^0$, l'égalité est démontrée.

Nous allons vérifier maintenant que les représentations π_n, π_n^p, π_n^k coïncident sur M_{2n-p}.

Proposition 1.5.5. *Avec les notations précédentes, $\pi_{n/M_{2n-p}} = \pi_n^p$.*

Corollaire 1.5.6. *Toutes représentations de la même algèbre sur le même espace obtenues à partir de différentes constructions de base sont les mêmes. Plus précisement, si $p \le k \le n$, alors les représentations π_n, π_n^p, π_n^k coïncident sur M_{2n-p}.*

Démonstration de la proposition. On sait que π_n^p et π_n ont même restriction à M_n, c'est la représentation standard de M_n sur $L^2(M_n, \mathrm{tr})$; il reste donc à comparer $\pi_n^p(f_n^p)$ et $\pi_n(f_n^p)$.

Si x, y et z sont des éléments quelconques de M_n, $xE_N(yz)$ appartient M_n et on peut écrire:

$$\pi_n^p(f_n^p)\pi_n(xf_n^{-1}y)(z\xi_n) = \pi_n^p(f_n^p)(xE_N(yz)\xi_n) = E_{M_p}(xE_N(yz))\xi_n.$$

D'où:

$$\pi_n^p(f_n^p)\pi_n(xf_n^{-1}y)(z\xi_n) = E_{M_p}(x)E_N(yz)\xi_n = \pi_n(E_{M_p}(x)f_n^{-1}y)(z\xi_n).$$

Nous avons donc obtenu:

$$\pi_n^p(f_n^p)\pi_n(xf_n^{-1}y) = \pi_n(E_{M_p}(x)f_n^{-1}y) \quad (x \in M_n, y \in M_n).$$

D'autre part, calculons $f_n^p x f_n^{-1} y$:

$$f_n^p x f_n^{-1} y = f_n^p x f_n^p f_n^{-1} y = E_{M_p}(x) f_n^p f_n^{-1} y = E_{M_p}(x) f_n^{-1} y \qquad (1.5.4).$$

De ces deux calculs, nous déduisons:

$$(*) \qquad \pi_n^p(f_n^p)\pi_n(xf_n^{-1}y) = \pi_n(f_n^p x f_n^{-1} y) \quad (x \in M_n, y \in M_n).$$

L'égalité (*) nous permet d'écrire:

$$\pi_n^p(f_n^p)\pi_n\left(\sum_{j=1}^{a_n+1} \lambda_j f_n^{-1}\lambda_j^*\right) = \pi_n\left(f_n^p \sum_{j=1}^{a_n+1} \lambda_j f_n^{-1}\lambda_j^*\right).$$

L'égalité des éléments $\pi_n^p(f_n^p)$ et $\pi_n(f_n^p)$ découle alors de la propriété de la base de Pimsner-Popa (1.5.2d) qui donne l'expression de l'identité de M_n:

$$\sum_{j-1}^{a_n+1} \lambda_j f_n^{-1} \lambda_j^* = 1.$$

La compatibilité des différentes représentations nous permet de définir sans ambiguïté les anti-automorphismes associés à la tour derivée.

2. Anti-automorphismes associés à la tour dérivée.

2.1. Définitions et notations. On reprend ici [**VJ1**, 3] et on rappelle que J_n est l'involution de l'espace de Hilbert $L^2(M_n, \mathrm{tr})$ définie par $J_n(x_n\xi_n) = x_n^*\xi_n$ si x_n est un élément de M_n.

On notera Γ_n l'anti-automorphisme de l'algèbre $\mathcal{B}_n$ des opérateurs bornés de $L^2(M_n, \mathrm{tr})$ défini par:

$$\Gamma_n(v)(x\xi_n) = J_n v^* J_n(x\xi_n) \qquad (v \in \mathcal{B}_n, x \in M_n).$$

On sait que $\Gamma_n(M_n) = M_n'$ et $\Gamma_n(\pi_n(N)') = \pi_n(M_{2n+1})$ puisque $N \subset M_n \subset M_{2n+1}$ est la construction de base; plus généralement, considérant $M_p \subset M_n \subset M_{2n-p}$, on obtient $\Gamma_n(\pi_n(M_p)') = \pi_n(M_{2n-p})$.

Définition. Soit $A_k = N' \cap M_k$, on note encore tr la restriction à A_k de la trace de M_k. Γ_n envoie $\pi_n(A_{2n+1})$ sur $\pi_n(A_{2n+1})$. On appellera γ_n l'anti-automorphisme de A_{2n+1} défini par:

$$\pi_n(\gamma_n(x)) = \Gamma_n(\pi_n(x)) \qquad (x \in A_{2n+1}).$$

Si $0 \leq p \leq n$, γ_n coïncide sur $M_p' \cap M_{2n-p}$ avec l'anti-automorphisme construit à partir de la tour $M_p \subset M_n \subset M_{2n-p}$ (1.5.6).

2.2. Propriétés fondamentales.

Théorème 2.2.1. *Pour tout n entier naturel, les anti-automorphismes γ_n satisfont les relations suivantes:*

a) $\gamma_{n+2}\gamma_{n+1}|_{A_{2n+1}} = \gamma_{n+1}\gamma_n$

b) $f_n^{-1}\gamma_n(x) = f_n^{-1}x \quad (x \in A_n)$

c) $\gamma_n(e_k) = e_{2n-k} \quad (0 \leq k \leq n)$.

La démonstation de ce théorème est l'objet du paragraphe 3. Au cours de cette démonstration, nous donnerons une formule pour $\gamma_n(y)$ quand y appartient à A_{2n+1}.

Proposition 2.2.2 ([**Popa**]). *Si N est un sous-facteur de profondeur finie dans un facteur M de type* II_1*, les anti-automorphismes γ_n conservent la trace* tr *de A_{2n+1}.*

Démonstration. Soit $\ldots N_{p+1} \subset N_p \subset \ldots \subset N_1 \subset N \subset M$ un tunnel dans $N \subset M$ [**GHJ**, 4.7e]; en considérant la construction de base $N_{p+1} \subset M \subset M_{p+2}$, on peut définir γ_0 un anti-automorphisme de $N'_{p+1} \cap M_{p+2}$ par:

$$\text{si } x \in N'_{p+1} \cap M_{p+2} \qquad \pi_0(\gamma_0(x)) = J_0(\pi_0(x))J_0$$

où J_0 est l'involution canonique de $L^2(M, \mathrm{Tr})$.

Comme N est un sous-facteur de profondeur finie dans un facteur M de type II_1, on sait que

$$E_{N' \cap M}(e_0) = [M : N]^{-1}1.$$

Grâce à 4.5 de [**PiPo1**], 3.1 et 3.2.ii de [**PiPo2**], cette dernière propriété est suffisante pour affirmer que γ_0 conserve la trace de $N'_{p+1} \cap M$ sur $M' \cap M_{p+2}$.

La proposition annonce un résultat plus fort. S. Popa affirme que γ_0 conserve la trace de $N'_{p+1} \cap M_{p+2}$. Pour démontrer ce résultat, il utilise l'hypotèse de la profondeur finie. En effet, on va plonger $N'_{p+1} \cap M_{p+2}$ dans $N'_{k+1} \cap M_{k+2}$ tel que k soit supérieur à la profondeur du graphe principal. γ_0 s'étend à $N'_{k+1} \cap M_{k+2}$ et conserve la trace de $N'_{k+1} \cap M$ sur $M' \cap M_{k+2}$. Ce choix de k permet d'affirmer que $N'_{k+1} \cap M_1$ est l'espace vectoriel engendré par $N'_{k+1} \cap M e_0 N'_{k+1} \cap M$ donc γ_0 conserve la trace de $N'_{k+1} \cap M_1$ sur $N' \cap M_{p+2}$; par récurrence, on montre ainsi que γ_0 conserve la trace de $N'_{p+1} \cap M_{p+2}$.

De la même façon, γ_n conserve la trace sur A_{2n+1}, il suffit, pour le démontrer, d'opérer une translation des indices sur la suite d'algèbres formée du tunnel et de la tour dérivée.

Remarque. La relation (a) du théorème 2.2.1 permet de définir un isomorphisme T de la tour dérivée de l'inclusion $N \subset M$ sur la tour dérivée de $M_1 \subset M_3$:

Soient $A_p = N' \cap M_p$ et $B_p = M'_1 \cap M_p$,

T est égal à $\gamma_{n+1}\gamma_n$ sur A_{2n+1} et $T(A_{2n+1}) = B_{2n+1}$.

D'après (2.2.1a), T est bien défini, c'est un isomorphisme des tours dérivées conservant la trace (2.2.2) et les anti-isomorphismes, c'est un isomorphisme de paragroupe (voir 4) qui opère une translation de 2 sur les indices des projecteurs de V. Jones.

3. Démonstration du théorème 2.2.1.

3.1. Lemmes.

Lemme 3.1.1.
a)

$$f_n^p = \alpha^{(n-p-1)} g_n^{p+1} f_{n+1}^{p+2} (g_n^{p+1})^* \text{ et}$$
$$\pi_{n+1}(f_n^p) = \alpha^{(n-p-1)} g_n^{p+1} \pi_{n+1}(f_{n+1}^{p+2})(g_n^{p+1})^*$$

(g_n^{p+1}, élément de M_{n+1}, agit par multiplication à gauche sur $L^2(M_{n+1}, \mathrm{tr})$).

b) $f_{n+1}^{-1} = \alpha^{n+1} g_{n+1}^{2n+2} f_n^{-1} g_{2n+2}^{n+1}$.

Démonstration.
a)

$$\begin{aligned} f_n^p &= \alpha(n,p) g_n^{p+1} g_{n+1}^{p+2} \cdots g_{2n-p-1}^{n} \\ &= \alpha(n,p) g_n^{p+1} e_{p+1} g_{n+1}^{p+3} e_{p+2} \cdots g_{n+k-1}^{p+k+1} e_{p+k} \cdots g_{2n-p-1}^{n+1} e_n. \end{aligned}$$

Grâce aux règles de commutation des projecteurs (1.5.3c), on obtient:

$$f_n^p = \alpha(n,p) g_n^{p+1} g_{n+1}^{p+3} \cdots g_{n+k-1}^{p+k+1} \cdots g_{2n-p-1}^{n+1} e_{p+1} e_{p+2} \cdots e_{p+k} \cdots e_n.$$

Comme $f_{n+1}^{p+2} = \alpha(n+1, p+2) g_{n+1}^{p+3} \cdots g_{n+k-1}^{p+k+1} \cdots g_{2n-p-1}^{n+1}$ et $\alpha(p,n) = \alpha^{(n-p-1)} \alpha(n+1, p+2)$, f_n^p vaut $\alpha^{(n-p-1)} g_n^{p+1} f_{n+1}^{p+2} (g_n^{p+1})^*$ et (a) est démontré.

b)

$$\begin{aligned} f_{n+1}^{-1} &= \alpha(n+1,-1) g_{n+1}^0\, g_{n+2}^1 \cdots g_{2n+2}^{n+1} \\ &= \alpha(n+1,-1) e_{n+1} g_n^0\, e_{n+2}\, g_{n+1}^1 \cdots e_{2n+1}\, g_{2n}^n e_{2n+2}\, g_{2n+2}^{n+1} && (1.5.1c) \\ &= \alpha(n+1,-1) g_{n+1}^{2n+2}\, g_n^0\, g_{n+1}^1 \cdots g_{2n}^n\, g_{2n+2}^{n+1} && (1.5.3b) \\ &= \alpha(n,-1) \alpha^{n+1} g_{n+1}^{2n+2} \left(g_n^0\, g_{n+1}^1 \cdots g_{2n}^n\right) g_{2n+2}^{n+1} && (1.5.1a) \\ &= \alpha^{n+1} g_{n+1}^{2n+2} f_n^{-1} g_{2n+2}^{n+1} && (1.5.1e). \end{aligned}$$

Lemme 3.1.2. (*dit du tour de passe-passe*)

$$g_{k+3}^{k+1}\, e_{k+3} = e_{k+1}\, g_{k+3}^{k+1}.$$

Démonstration.

$$\begin{aligned} g_{k+3}^{k+1}\, e_{k+3} &= (e_{k+3}e_{k+2}e_{k+1})e_{k+3} \\ &= (e_{k+3}e_{k+2}e_{k+3})e_{k+1} \qquad (1.5.3c) \\ &= e_{k+3}(\alpha e_{k+1}) \qquad (1.5.3e) \\ &= e_{k+3}(e_{k+1}e_{k+2}e_{k+1}) \qquad (1.5.3e) \\ &= e_{k+1}(e_{k+3}e_{k+2}e_{k+1}) \qquad (1.5.3c). \end{aligned}$$

Lemme 3.1.3. *Si* $0 \leq k \leq n$, $e_{2n-k}f_n^{-1} = e_k f_n^{-1}$.

Démonstration.

$$\begin{aligned} e_{2n-k}f_n^{-1} &= e_{2n-k}\alpha(n,-1)g_n^0 g_{n+1}^1 \cdots g_{2n}^n \\ &= \alpha(n,-1)g_n^0 g_{n+1}^1 \cdots g_{2n-k-2}^{n-k-2} e_{2n-k} g_{2n-k-1}^{n-k-1} \cdots g_{2n}^n \qquad (1.5.3c). \end{aligned}$$

Nous avons e_{2n-k} devant g_{2n-k-1}^{n-k-1} et nous voudrions e_k devant g_n^0.

1ère *étape:* transformons e_{2n-k} en e_{2n-k-2} devant g_{2n-k-1}^{n-k-1}. D'abord nous faisons disparaître e_{2n-k}:

$$e_{2n-k}g_{2n-k-1}^{n-k-1}g_{2n-k}^{n-k} = e_{2n-k}e_{2n-k-1}g_{2n-k-2}^{n-k-1}e_{2n-k}g_{2n-k-1}^{n-k} \qquad (1.5.1c).$$

Comme e_{2n-k} et g_{2n-k-2}^{n-k-1} commutent [(1.5.3a) et (1.5.3b)],

$$\begin{aligned} e_{2n-k}\, g_{2n-k-1}^{n-k-1}\, g_{2n-k}^{n-k} &= e_{2n-k}\, e_{2n-k-1}\, e_{2n-k}\, g_{2n-k-2}^{n-k-1}\, g_{2n-k-1}^{n-k} \\ &= \alpha g_{2n-k-2}^{n-k-1}\, e_{2n-k}\, g_{2n-k-1}^{n-k} \qquad (1.5.3d) \\ &= \alpha g_{2n-k-2}^{n-k-1}\, g_{2n-k}^{n-k}. \end{aligned}$$

Ensuite nous faisons apparaître e_{2n-k-2}:

$$\begin{aligned} e_{2n-k}\, g_{2n-k-1}^{n-k-1}\, g_{2n-k}^{n-k} &= \alpha e_{2n-k-2}\, g_{2n-k-3}^{n-k-1}\, g_{2n-k}^{n-k} \\ &= e_{2n-k-2}\, e_{2n-k-1}\, e_{2n-k-2}\, g_{2n-k-3}^{n-k-1}\, g_{2n-k}^{n-k} \qquad (1.5.3d) \\ &= e_{2n-k-2}\, g_{2n-k-1}^{n-k-1}\, g_{2n-k}^{n-k} \qquad (1.5.1c). \end{aligned}$$

2ème *étape:* nous avons maintenant e_{2n-k-2} entre g_{2n-k-2}^{n-k-2} et g_{2n-k-1}^{n-k-1}, grâce à $n-k-1$ tours de passe-passe (3.1.2), nous allons obtenir e_k devant g_n^0:

1$^{\text{er}}$ tour de passe-passe:

$$\begin{aligned} g_{2n-k-2}^{n-k-2}\, e_{2n-k-2} &= e_{2n-k-2}\, e_{2n-k-3}\, e_{2n-k-4}\, e_{2n-k-2}\, g_{2n-k-5}^{n-k-2} \qquad (1.5.3c) \\ &= e_{2n-k-4}\, e_{2n-k-2}\, e_{2n-k-3}\, e_{2n-k-4}\, g_{2n-k-5}^{n-k-2} \qquad (3.1.2) \\ &= e_{2n-k-4}\, g_{2n-k-2}^{n-k-2}. \end{aligned}$$

Après p tours de passe-passe, nous avons $e_{2n-k-2(p+1)}$ devant $g_{2n-k-(p+1)}^{n-k-(p+1)}$ et après $n-k-1$ tours de passe-passe, nous obtenons e_k devant g_n^0 et la relation annoncée est vérifiée.

Lemme 3.1.4. *Si* $x \in A_n, x_n \in M_n, \pi_n(\gamma_n(x))(x_n\xi_n) = x_n x \xi_n$.

Démonstration.

$$\pi_n(\gamma_n(x))(x_n\xi_n) = J_n\pi_n(x)^* J_n(x_n\xi_n) = J_n\pi_n(x)^* \left(x_n^*\xi_n\right).$$

Comme x est un élément de M_n, $\pi_n(\gamma_n(x))(x_n\xi_n)$ est donc égal à $J_n(x^* x_n^* \xi_n)$, soit $x_n x \xi_n$. Ainsi $\gamma_n(x)$, élément de M_n', agit par multiplication à droite sur $L^2(M, \mathrm{tr})$.

Lemme 3.1.5. *Si la famille* $\{m_j, 1 \leq j \leq a_n + 1\}$ *est une base de Pimsner-Popa de* M_{n-1} *sur* N*, alors la famille* $\{\alpha^{n/2}\lambda_i g_0^{n-1} m_j, 1 \leq j \leq a_n + 1, 1 \leq i \leq a + 1\}$ *est une base de Pimsner-Popa de* M_n *sur* N.

Démonstration. Vérifions (a) est (b) de (1.5.2):

$$\begin{aligned} E_N\left((\alpha^{n/2} m_h^* g_{n-1}^0 \lambda_k^*)(\alpha^{n/2}\lambda_i g_0^{n-1} m_j)\right) &= \alpha^n E_N(m_h^* g_{n-1}^1 e_0 (\lambda_k^*\lambda_i) e_0 g_1^{n-1} m_j) \\ &= \alpha^n \delta_{i,k} E_N(m_h^* g_{n-1}^1 e_0 g_1^{n-1} m_j) \quad \text{(1.5.1c,d et 3d)} \\ &= \alpha^1 \delta_{i,k} E_N(m_h^* E_{M_{n-1}}(e_{n-1}) m_j) = \delta_{i,k}\delta_{j,h} \text{(1.5.2a)}. \end{aligned}$$

Quand l'indice α de N dans M est entier, on peut facilement donner une base de Pimsner-Popa de M_n sur N:

Proposition 3.1.6. *On suppose que l'indice* α *de* N *dans* M *est entier.*

On note $\mathcal{J}_n$ *le produit de* n *copies de* $\{0, 1, 2, \ldots, \alpha\}$ *et si* $I = (i_0, i_1, i_2, \ldots, i_n)$ *est un élément de* $\mathcal{J}_{n+1}$*, on pose*

$$m_I = \alpha^{n(n+1)/4} \lambda_{i_n} g_0^{n-1} \lambda_{i_{n-1}} g_0^{n-2} \ldots \lambda_{i_2} g_0^1 \lambda_{i_1} e_0 \lambda_{i_0}.$$

Alors la famille $\{m_I, I \in \mathcal{J}_{n+1}\}$ *est une base de Pimsner-Popa de* M_n *sur* N.

Ceci résulte par récurrence de (1.5.2g) et du Lemme 3.1.5.

3.2. Une formule pour $\gamma_n(y)$ quand y appartient à A_{2n+1}. Nous allons maintenant donner l'expression de $\gamma_n(y)$ quand y appartient à A_{2n+1}. Cette formule nous permettra d'établir 2.2.1(a) et nous sera très utile dans la partie 5. Quand l'indice est entier, elle coïncide, grâce à la proposition 3.1.6, avec celle annoncée par A. Ocneanu.

Proposition 3.2.1. *Si $\{m_i, 1 \leq i \leq a_{n+1}+1\}$ est une base de Pimsner-Popa de M_n sur N, alors pour tout y de A_{2n+1}, on a:*

$$\gamma_n(y) = \alpha^{n+1} \sum_{k=1}^{a_{n+1}+1} E_{M_n}\,(f_n^{-1} m_k y)\, f_n^{-1} m_k^*.$$

Démonstration. D'après 1.5.2, la famille $\{\alpha^{(n+1)/2} m_i f_n^{-1}, 1 \leq i \leq a_{n+1}+1\}$ est une base de Pimsner-Popa de M_{2n+1} sur M_n et

$$y = \alpha^{2n+1} \sum_{i=1}^{a_{n+1}+1} E_{M_n}(y m_i f_n^{-1}) f_n^{-1} m_i^*.$$

Si $E_N(m_h^* m_h) = q_h$, tout élément de M_n s'écrit $\sum_{k+1}^{a_{n+1}+1} z_h m_h^*$ où z_h est un élément de $N q_h$, il suffit donc de connaître $\pi_n(\gamma_n(y))$ sur les vecteurs de $L^2(M_n, \mathrm{tr})$ de la forme $z m_h^* \xi_n$ où z appartient à N_{q_h}. J'omets π_n quand l'élément de M_n agit par multiplication à gauche.

$$\begin{aligned}\pi_n(y^*) J_n(z m_h^* \xi_n) &= \alpha^{n+1} \sum_{i=1}^{a_{n+1}+1} E_{M_n}(y^* m_i f_n^{-1}) \pi_n(f_n^{-1})(m_i^* m_h z^* \xi_n) \\ &= \alpha^{n+1} \sum_{i=1}^{a_{n+1}+1} E_{M_n}(y^* m_i f_n^{-1}) E_N(m_i^* m_h)(z^* \xi_n). \qquad (1.5.3\mathrm{h})\end{aligned}$$

Or

$$E_N(m_i^* m_h) z^* = \delta_{i,h} q_h z^* = \delta_{i,h} z^* \qquad (1.5.2\mathrm{a,b}),$$

d'où

$$\pi_n(y^*) J_n(z m_h^* \xi_n) = \alpha^{n+1} E_{M_n}(y^* m_h f_n^{-1})(z^* \xi_n)$$

et

$$\pi_n(\gamma_n(y))(z m_h^* \xi_n) = \alpha^{n+1} z E_{M_n}(f_n^{-1} m_h^* y) \xi_n.$$

Comme z commute à E_{M_n} et f_n^{-1} et que $z m_h^*$ se décompose en

$$\sum_{k=1}^{a_{n+1}+1} m_k E_N(m_k^* z m_h^*) \qquad (1.5.2\mathrm{f}),$$

on obtient:

$$\pi_n(\gamma_n(y))(z m_h^* \xi_n) = \alpha^{n+1} \sum_{k=1}^{a_{n+1}+1} E_{M_n}(f_n^{-1} m_k E_N(m_k^* z m_h^*) y) \xi_n.$$

Comme l'élément y de A_{2n+1} commute à $E_N(m_k^* z m_h^*)$, on obtient:

$$\pi_n(\gamma_n(y))(zm_h^*\xi_n) = \alpha^{n+1} \sum_{k=1}^{a_{n+1}+1} E_{M_n}(f_n^{-1}m_k y)E_N(m_k^* z m_h^*)\xi_n$$

$$= \alpha^{n+1} \sum_{k=1}^{a_{n+1}+1} E_{M_n}(f_n^{-1}m_k y)\pi_n(f_n^{-1})m_k^*(zm_h^*\xi_n). \qquad (1.5.3h)$$

Et la formule est démontrée.

3.3. $\gamma_{n+2}\gamma_{n+1}|_{A_{2n+1}} = \gamma_{n+1}\gamma_n$.

Lemme 3.3.1. *Si $x \in M_n$, $E_{M_1}(g_1^n x g_n^1) = \alpha^{-n}E_M(x)$.*

Démonstration.

$$E_{M_1}(g_1^n x g_n^1) = E_{M_1}(e_1e_2\dots e_n x e_n \dots e_2e_1)$$
$$= E_{M_1}(e_1e_2\dots e_{n-1}E_{M_{n-1}}(x)e_n e_{n-1}\dots e_2e_1).$$

Comme E_{M_1} est égal à $E_{M_1}E_{M_n}$, l'égalité devient:

$$E_{M_1}(g_1^n x g_n^1) = \alpha^{-1}E_{M_1}(e_1e_2\dots e_{n-1}E_{M_{n-1}}(x)e_{n-1}\dots e_2e_1).$$

Après $n-1$ manœuvres de ce type, on obtient:

$$E_{M_1}(g_1^n x g_n^1) = \alpha^{-(n-1)}E_{M_1}(e_1E_{M_1}(x)e_1) = \alpha^{-n}E_M(x).$$

Proposition 3.3.2. *Si, pour $1 \le k \le a+1$, $\mu_k = \alpha^{(2n+2)/2}\lambda_k g_0^{2n+1}$ et si y est un élément de A_{2n+1}, alors $\gamma_{n+1}(y)$ est l'élément $\sum_{k=1}^{a+1}\mu_k\gamma_n(y)e_{2n+2}\mu_k^*$ de $M_1' \cap M_{2n+3}$.*

Démonstration. Comme $\{\alpha^{(n+1)/2}\lambda_i g_0^n, i = 1,\dots,n\}$ est une base de Pimsner-Popa de M_{n+1} sur M_n (1.5.2), il suffit de calculer $\pi_{n+1}(\gamma_{n+1}(y))$ sur les vecteurs de $L^2(M_{n+1}, \mathrm{tr})$ de la forme $\lambda_j g_0^n z\xi_{n+1}$ où p_j est le projecteur $E_N(\lambda_j^*\lambda_j)$ et z un élément de p_jM_n.

$$\pi_{n+1}(y^*)J_{n+1}(\lambda_j g_0^n z\xi_{n+1})$$
$$= \alpha^{2n+1}\sum_{i=1}^{a_{n+1}+1} E_{M_n}(y^* m_i f_n^{-1})\pi_{n+1}(f_n^{-1})(m_i^* z^* g_n^0 \lambda_j^* \xi_{n+1}).$$

Comme $f_n^{-1} = \alpha^n g_n^0 f_{n+1}^1 (g_n)^*$ et $\pi_{n+1}(f_{n+1}^1) = E_{M_1}$ (1.5.3h), on a:

$$\pi_{n+1}(y^*)J_{n+1}(\lambda_j g_0^n z\xi_{n+1})$$
$$= \alpha^{2n+1}\sum_{i=1}^{a_{n+1}+1} E_{M_n}(y^* m_i f_n^{-1})g_n^0 E_{M_1}\left(g_0^n m_i^* z^* g_n^0 \lambda_j^*\right)\xi_{n+1}.$$

Donc

$$\begin{aligned}
&\pi_{n+1}(\gamma_{n+1}(y))(\lambda_j g_0^n z\xi_{n+1})\\
&\qquad = \alpha^{2n+1}\sum_{i=1}^{a_{n+1}+1} E_{M_1}\left(\lambda_j g_0^n z m_i g_n^0\right) g_0^n E_{M_n}\left(f_n^{-1} m_i^* y\right)\xi_{n+1}\\
&\qquad = \alpha^{2n+1}\sum_{i=1}^{a_{n+1}+1} \lambda_j e_0 E_{M_1}\left(g_1^n z m_i g_n^1\right) g_0^n E_{M_n}\left(f_n^{-1} m_i^* y\right)\xi_{n+1}.
\end{aligned}$$

Appliquons le lemme 3.3.1 à zm_i et nous obtenons:

$$\begin{aligned}
&\pi_{n+1}(\gamma_{n+1}(y))(\lambda_j g_0^n z\xi_{n+1})\\
&\qquad = \alpha^{n+1}\sum_{i=1}^{a_{n+1}+1} \lambda_j e_0 E_M(zm_i) e_0 g_o^n E_{M_n}\left(f_n^{-1} m_i^* y\right)\xi_{n+1}\\
&\qquad = \alpha^{n+1}\sum_{i=1}^{a_{n+1}+1} \lambda_j e_0 E_N(zm_i) g_0^n E_{M_n}\left(f_n^{-1} m_i^* y\right)\xi_{n+1}.
\end{aligned}$$

Comme $E_N(zm_i)$ commute à g_0^n, à E_{M_n} et à f_n^{-1} et que $\sum_{i=1}^{a_{n+1}+1} E_N(zm_i)m_i^*$ vaut z, on peut écrire:

$$\pi_{n+1}(\gamma_{n+1}(y))(\lambda_j g_0^n z\xi_{n+1}) = \alpha^{n+1}\lambda_j g_0^n E_{M_n}\left(f_n^{-1} z y\right)\xi_{n+1}.$$

On utilise maintenant la décomposition de ξ sur la base m_i (1.5.2f):

$$z = \sum_{i=1}^{a_{n+1}+1} m_i E_N(m_i^* z)$$

pour faire commuter z avec y, élément de A_{2n+1} et le sortir de E_{M_n}:

$$\begin{aligned}
\pi_{n+1}(\gamma_{n+1}(y))(\lambda_j g_0^n z\xi_{n+1}) &= \alpha^{n+1}\sum_{i=1}^{a_n+1} \lambda_j g_0^n E_{M_n}\left(f_n^{-1} m_i y\right) E_N(m_i^* z)\xi_{n+1}\\
&= \alpha^{n+1}\sum_{i=1,k=1}^{a_{n+1}+1,a+1} \delta_{k,j}\lambda_k g_o^n E_{M_n}\left(f_n^{-1} m_i y\right) E_N(m_i^* p_j z)\xi_{n+1}.
\end{aligned}$$

Comme $\delta_{k,j}p_j = \alpha^{(n+1)}E_{M_n}(g_n^0\lambda_k^*\lambda_j g_0^n)$, $\pi_{n+1}(\gamma_{n+1}(y))(\lambda_j g_0^n z\xi_{n+1})$ est égal à

$$\alpha^{2(n+1)}\sum_{i=1,k=1}^{a_{n+1}+1,a+1} \lambda_k g_0^n E_{M_n}\left(f_n^{-1} m_i y\right) E_N\left(m_i^* E_{M_n}(g_n^0\lambda_k^*\lambda_j g_0^n) z\right)\xi_{n+1}.$$

$$\begin{aligned}
&\pi_{n+1}(\gamma_{n+1}(y))(\lambda_j g_0^n z\xi_{n+1})\\
&\qquad = \alpha^{2(n+1)}\sum_{i=1,k=1}^{a_{n+1}+1,a+1} \lambda_k g_0^n E_{M_n}\left(f_n^{-1} m_i y\right) E_N\left(m_i^* g_n^0\lambda_k^*\lambda_j g_0^n z\right)\xi_{n+1}.
\end{aligned}$$

Or $\pi_{n+1}(f_{n+1}^{-1})(x)\xi_{n+1} = E_N(x)\xi_{n+1}$ si $x \in M_{n+1}$ (1.5.3h), donc

$$\gamma_{n+1}(y) = \alpha^{2(n+1)} \sum_{i=1,k=1}^{a_n+1,a+1} \lambda_k g_0^n E_{M_n}\left(f_n^{-1} m_i y\right) f_{n+1}^{-1} m_i^* g_n^0 \lambda_k^*.$$

Comme f_{n+1}^{-1} est égal à $\alpha^{n+1} g_{n+1}^{2n+2} f_n^{-1} g_{2n+2}^{n+1}$ (3.1.1), en utilisant les règles de commutation, on écrit:

$$\gamma_{n+1}(y) = \alpha^{3(n+1)} \sum_{k=1}^{a+1} \lambda_k g_0^{2n+2} \left[\sum_{i=1}^{a_{n+1}+1} E_{M_n}\left(f_n^{-1} m_i y\right) f_n^{-1} m_i^* \right] g_{2n+2}^0 \lambda_k^*.$$

D'où $\gamma_{n+1}(y) = \alpha^{2(n+1)} \sum_{k=1}^{a+1} \lambda_k g_0^{2n+2} \gamma_n(y) g_{2n+2}^0 \lambda_k^*$.

Si, pour $1 \leq k \leq a+1$, on note μ_k l'élément $\alpha^{2(n+1)/2} \lambda_k g_0^{n+1}$ de la base de M_{2n+2} sur M_{2n+1}, ce résultat s'écrit:

$$\gamma_{n+1}(y) = \sum_{k=1}^{a+1} \mu_k \gamma_n(y) e_{2n+2} \mu_k^*.$$

Corollaire 3.3.3. *Si, pour* $1 \leq k \leq a+1$, $\mu_k = \alpha^{(2n+2)/2} \lambda_k g_0^{2n+1}$ *et si* y *est un élément de* A_{2n+1}, *alors*

$$\gamma_{n+1}(\gamma_n(y)) = \sum_{k=1}^{a+1} \mu_k y e_{2n+2} \mu_k^*.$$

Proposition 3.3.4. $\gamma_{n+2}\gamma_{n+1}|_{A_{2n+1}} = \gamma_{n+1}\gamma_n \quad (n \in \mathbb{N})$.

Démonstration. Ecrivons la formule du corollaire 3.3.3 pour $n+1$: Si y est un élément de A_{2n+3},

$$\gamma_{n+2}(\gamma_{n+1}(y)) = \alpha^2 \sum_{k=1}^{a+1} \mu_k e_{2n+2} e_{2n+3} y e_{2n+4} e_{2n+3} e_{2n+2} \mu_k^*.$$

Mais si, de plus, y est dans A_{2n+1}, la formule se simplifie car

$$\begin{aligned} \alpha^2 e_{2n+2} e_{2n+3} y e_{2n+4} e_{2n+3} e_{2n+2} &= \alpha^2 y e_{2n+2} (e_{2n+3} e_{2n+4} e_{2n+3}) e_{2n+2} \\ &= \alpha y (e_{2n+2} e_{2n+3} e_{2n+2}) = y. \end{aligned}$$

On obtient alors:

$$\gamma_{n+2}(\gamma_{n+1}(y)) = \gamma_{n+1}(\gamma_n(y)).$$

3.4. $f_n^{-1}\gamma_n(x)$ et f_n^{-1} sont égaux si x est un élément de A_n.

Proposition 3.4. *Pour tout x de A_n, $f_n^{-1}\gamma_n(x) = f_n^{-1}x$.*

Démonstration. Il suffit de démontrer la relation pour les unitaires de A_n. Soient u un unitaire de A_n et z un élément de M_n,

$$\pi_n(f_n^{-1}\gamma_n(u))(z\xi_n) = \pi_n(f_n^{-1})(zu\xi_n) \qquad (3.1.4).$$

Alors par définition de f_n^{-1}, $\pi_n(f_n^{-1}\gamma_n(u))(z\xi_n) = E_N(zu)\xi_n$. Comme u normalise N, pour tout x de M_n, $E_N(uxu^*) = uE_N(x)u^*$, on obtient donc:

$$\pi_n(f_n^{-1}\gamma_n(u))(z\xi_n) = E_N(zu)\xi_n = u^*E_N(uz)u\xi_n = E_N(uz)\xi_n$$

car u commute à N. Cela s'écrit aussi:

$$\pi_n(f_n^{-1}\gamma_n(u))(z\xi_n) = \pi_n(f_n^{-1}u)(z\xi_n).$$

La relation est démontrée.

3.5. Si $0 \leq k \leq n$, l'image de e_k par l'anti-automorphisme γ_n est e_{2n-k}.

Proposition 3.5. $\gamma_n(e_k) = e_{2n-k} \qquad (0 \leq k \leq n)$.

Démonstration.
1^{er} *cas:* $k = n$.
e_n est la projection de $L^2(M_n, \mathrm{tr})$ sur $L^2(M_{n-1}, \mathrm{tr})$, aussi J_n et e_n commutent et $\gamma_n(e_n) = e_n$.
$2^{ème}$ *cas:* $0 \leq k < n$.

Soient x, y, z des éléments de M_n. Calculons d'abord $\gamma_n(e_k)xf_n^{-1}y$:

$$\pi_n(\gamma_n(e_k)xf_n^{-1}y)(z\xi_n) = \pi_n(\gamma_n(e_k))\,(xE_N(yz)\xi_n) = xE_N(yz)e_k\xi_n \qquad (3.1.4).$$

Comme e_k commute à M_{k-1} donc à N,

$$\pi_n(\gamma_n(e_k)xf_n^{-1}y)(z\xi_n) = xe_kE_N(yz)\xi_n.$$

On obtient donc $\gamma_n(e_k)xf_n^{-1}y = xe_kf_n^{-1}y$.

Calculons maintenant $e_{2n-k}xf_n^{-1}y$:
Comme $k < n$, e_{2n-k} commute à M_n, donc

$$e_{2n-k}xf_n^{-1}y = xe_{2n-k}f_n^{-1}y = xe_kf_n^{-1}y \qquad (3.1.3).$$

Soient a_{n+1} la partie entière de α^{n+1}, l'indice de N dans M_n et

$$\{m_j, j = 1, \ldots a_{n+1} + 1\}$$

une base de Pimsner-Popa de M_n sur N. Comme $\sum_{j=1}^{a_{n+1}+1} m_j f_n^{-1} m_j^*$ est l'identité (1.5.2d) et, pour tous x et y éléments de M_n, $\gamma_n(e_k) x f_n^{-1} y$ et $e_{2n-k} x f_n^{-1} y$ coïncident, on peut écrire:

$$\gamma_n(e_k) \left(\sum_{j=1}^{a_{n+1}+1} m_j f_n^{-1} m_j^* \right) = e_{2n-k} \left(\sum_{j=1}^{a_{n+1}+1} m_j f_n^{-1} m_j^* \right),$$

c'est-à-dire $\gamma_n(e_k) = e_{2n-k}$.

4. Le paragroupe, invariant complet pour l'inclusion d'un sous-facteur de profondeur finie dans le facteur hyperfini de type II_1.

4.1. Paragroupe ou carré commutatif. Popa a montré que l'inclusion d'un sous-facteur de profondeur finie dans le facteur hyperfini de type II_1 est determinée par son carré commutatif canonique [**Popa1**, 6.6]. Le paragroupe est une autre version de cet invariant.

Définition. Le paragroupe de l'inclusion $N \subset M$ est la tour dérivée $(A_k)_{k \geq 0}$ de $N \subset M$ munie de ses anti-automorphismes canoniques γ_k.

Remarque. La donnée du graphe principal équivaut à celle de la tour dérivée [**GHJ**, 4.6.5].

Théorème 4.1.1. *Soient N (resp. $\widetilde{N}$) un sous-facteur de profondeur finie dans le facteur hyperfini de type* II_1 *M (resp. $\widetilde{M}$). Si les couples $N \subset M$ et $\widetilde{N} \subset \widetilde{M}$ ont même paragroupe, ils sont isomorphes.*

Démonstration. On suppose qu'il existe un isomorphisme β des tours dérivées conservant la trace tel que $\beta(A_k) = \widetilde{A}_k$ et $\gamma_k = \beta^{-1} \widetilde{\gamma}_k \beta$.

Si $N \subset M$ est de profondeur finie p, $N_1 \subset N$ est de profondeur finie p_1, alors si $2j$ est supérieur à $p-1$ et p_1, le carré commutatif

$$(\mathcal{C}) \qquad \begin{array}{ccc} M' \cap M_{2j} & \subset & N' \cap M_{2j} \\ \cap & & \cap \\ M' \cap M_{2j+1} & \subset & N' \cap M_{2j+1} \end{array}$$

est le carré canonique de $N_1 \subset N$. Pour montrer que les deux couples sont isomorphes, il suffit de montrer que $N_1 \subset N$ et $\widetilde{N}_1 \subset \widetilde{N}$ ont même carré canonique [**Popa1**, 6.6].

Montrons que $(\mathcal{C})$ et $(\widetilde{\mathcal{C}})$ sont isomorphes:
Nous savons déjà que l'isomorphime β envoie $N' \cap M_{2j}$ sur $\widetilde{N}' \cap \widetilde{M_{2j}}$ et $N' \cap M_{2j+1}$ sur $\widetilde{N}' \cap \widetilde{M_{2j+1}}$ en conservant la trace.

Comme $M' \cap M_{2j}$ est l'intersection de $(N' \cap M_{2j})$ et $(M' \cap M_{2j+1})$, il nous suffit de démontrer que

$$\beta(M' \cap M_{2j+1}) = \widetilde{M}' \cap \widetilde{M}_{2j+1}.$$

Or

$$M' \cap M_{2j+1} = \gamma_j(N' \cap M_{2j}),$$

donc

$$\widetilde{M}' \cap \widetilde{M}_{2j+1} = \widetilde{\gamma}_j(\widetilde{N}' \cap \widetilde{M}_{2j}) = \widetilde{\gamma}_j\beta(N' \cap M_{2j})$$

et comme $\gamma_j = \beta^{-1}\widetilde{\gamma}_j\beta$,

$$\widetilde{M}' \cap \widetilde{M}_{2j+1} = \beta\gamma_j(N' \cap M_{2j}) = \beta(M' \cap M_{2j+1}).$$

Les deux carrés sont isomorphes.

5. Produit croisé par une algèbre de Kac de dimension finie.

Dans cette partie, nous donnons une caractérisation de l'inclusion d'un facteur de type II_1 dans son produit croisé par une algèbre de Kac de dimension finie:

Théorème 5.0. *Soient M un facteur de type* II_1, tr *sa trace normale finie fidèle normalisée et N un sous-facteur d'indice fini dans M. Les propositions suivantes sont équivalentes:*

(a) *N est de profondeur au plus 2 dans M et $N' \cap M$ est égal à $\mathbb{C}$.*

(b) *M est le produit croisé de N par une action extérieure d'une algèbre de Kac de dimension finie $\mathbb{K}$.*

(c) *N est la sous-algèbre des point fixes de M sous une action extérieure d'une algèbre de Kac de dimension finie $\widehat{\mathbb{K}}$.*

L'équivalence entre (b) est (c) est un résultat de M. Enock et J.-M. Schwartz [**ES2**]; dans [**EN**], on trouvera la démonstration de (b)$\Rightarrow$(a) dans le cas le plus général. Nous nous attacherons ici à construire une algèbre de Kac de dimension finie à partir de l'inclusion d'un sous-facteur dans un facteur de type II_1, c'est-à-dire à montrer (a)$\Rightarrow$(c).

Soient M un facteur de type II_1, tr sa trace normale finie fidèle normalisée et N un sous-facteur d'indice fini n et de profondeur 2 dans M. Par construction de base, on obtient la tour de facteurs

$$N \subset M \overset{e_0}{\subset} M_1 \overset{e_1}{\subset} M_2 \overset{e_2}{\subset} \ldots M_p \overset{e_p}{\subset} M_{p+1} \ldots,$$

la suite des projecteurs de V. Jones et les anti-automorphismes γ_p définie en 2.2.1. On s'intéressera plus particulièrement à la tour dérivée de l'inclusion $M \subset M_1$, aussi on notera $B_p = M' \cap M_p$. On supposera, de plus, que $N' \cap M = \mathbb{C}$.

5.1. La tour dérivée de l'inclusion $M \subset M_1$**.** Comme $N' \cap M = \mathbb{C}$, $B_1 = \gamma_0(N' \cap M) = \mathbb{C}$.

Comme l'inclusion de N dans M est de profondeur 2, $N' \cap M_2$ est obtenue par construction de base sur l'inclusion $N' \cap M \subset N' \cap M_1$ [**GHJ**, 2.4.1 ou 4.6.3], donc $N' \cap M_2$ est un facteur de dimension finie puisque $[M : N]$ est fini, soit $M_n(\mathbb{C})$ où $n = [M : N]$ qui est donc entier. Alors $B_3 = \gamma_1(N' \cap M_2)$ est aussi isomorphe au facteur $M_n(\mathbb{C})$, plus précisément c'est le facteur $\mathcal{B}(H_\varphi)$, si $(B_2, H_\varphi, J_\varphi, \varphi)$ est la forme standard de B_2.

Alors B_2 est une algèbre de dimension finie n, soit $B_2 = \oplus_{i \in I} M_{n_i}(\mathbb{C})$ où I est un ensemble fini et $\sum_{i \in I} n_i^2 = n$. On notera $M_{n_1}(\mathbb{C})$ le sous-facteur $\mathbb{C}e_1$ de B_2 (donc $n_1 = 1$) et p_i le projecteur central de B_2 tel que $B_2 p_i \cong M_{n_i}(\mathbb{C})$. La trace de Markov normalisée sur B_2, notée φ, est la restriction de la trace de M_2, sa valeur sur le projecteur minimal du facteur $B_2 p_i$ est $\frac{n_i}{n}$.

Nous allons montrer que $(B_2, \gamma_2\gamma_1, \gamma_1, n\varphi)$ est une algèbre de Kac qui agit sur M en laissant fixes les éléments de N.

Nous utiliserons seulement deux réflexions de la tour dérivée:

i) γ_φ, l'anti-automorphisme de l'algèbre B_3 défini par

$$\gamma_\varphi(a) = J_\varphi a^* J_\varphi \qquad (a \in B_3)$$

γ_φ envoie B_2 dans B_2'.

ii) γ_{H_φ} défini de la même façon à partir de J_{H_φ} l'involution standard de B_3, c'est un anti-automorphisme de l'algèbre $B_2' \cap B_4$ qui envoie $B_2' \cap B_3$ sur $B_3' \cap B_4$.

Comme nous ne considérons que des constructions de base à un étage, il n'y a aucun problème de compatibilité de répresentations, de plus les formules de la partie 3 sont encore valables; on peut vérifier directement que ces réflexions conservent le trace de Markov de la tour dérivée.

5.2. Bases de Pimsner-Popa et unités matricielles. Pour appliquer les formules de 3, nous allons choisir des bases de Pimsner-Popa particulières. La proposition suivante motive ce choix.

Proposition 5.2.1. *Soit N un sous-facteur d'indice fini n du facteur M,* tr *la trace normale finie fidèle normalisée sur M et $N \subset M \overset{e_0}{\subset} M_1$ la construction de base.*

Si N est de profondeur au plus 2 dans M et que $N' \cap M$ est égal à $\mathbb{C}$, on pose:

$$N' \cap M_1 = \oplus_{i \in I} M_{n_i}(\mathbb{C}) \quad \text{et} \quad \mathcal{J} = \{K = (k, k_1, k_2), k \in I, 1 \leq k_1, k_2 \leq n_k\}.$$

Soit $\left\{f^k_{k_1,k_2} = f_K, K \in \mathcal{J}\right\}$ une famille d'unités matricielles de $N' \cap M_1$, où $f^0_{1,1} = f_0 = e_0$, alors la famille $\left\{\sqrt{\frac{n}{n_k}} f_K, K \in \mathcal{J}\right\}$ est une base de Pimsner-

Popa de M_1 sur M ainsi qu'une base de Pimsner-Popa de $N' \cap M_1$ sur $\mathbb{C}$; de même pour $\left\{\sqrt{\frac{n}{n_k}}(f_K)^*, K \in \mathcal{J}\right\}$.

Démonstration. On vérifie facilement les propriétés de 3.1.5 car, comme $N' \cap M = \mathbb{C}$, sur $N' \cap M_1$, l'espérance conditionelle sur M est la trace.

Définitions.

1. On choisira pour base de Pimsner-Popa de M_1 sur M la base de Pimsner-Popa associée à une famille d'unités matricielles de $N' \cap M_1$ comme dans la proposition 5.2.1 et on la notera pour simplifier $\{\lambda_s, 1 \leq s \leq n\}$, où $\lambda_1 = e_0$. On ne souviendra que $\sum_{r=1}^{n} \operatorname{tr}(\lambda_r)\lambda_r^* = 1$ et que $\{\lambda_s^*, 1 \leq s \leq n\}$ est aussi une base de Pimsner-Popa de M_1 sur M.

On rappelle que $\{n^{1/2}\lambda_s e_1, 1 \leq s \leq n\}$ est alors une base de M_2 sur M_1 (1.5.2g).

2. Si $B_2 = \oplus_{i \in I} M_{n_i}(\mathbb{C})$ et $\mathcal{J} = \{K = (k, k_1, k_2), k \in I, 1 \leq k_1, k_2 \leq n_k\}$, on choisit une famille d'unités matricielles de B_2, $\left\{f^k_{k_1,k_2} = f_K, K \in \mathcal{J}\right\}$ où $f^0_{1,1} = e_1$. La proposition 5.2.1 permet alors d'affirmer que la famille $\left\{\sqrt{\frac{n}{n_k}} f_K, K \in \mathcal{J}\right\}$ est une base de Pimsner-Popa de M_2 sur $\mathbb{C}$ ainsi qui une base de Pimsner-Popa de B_2 sur $\mathbb{C}$; de même pour la famille $\left\{\sqrt{\frac{n}{n_k}}(f_K)^*, K \in \mathcal{J}\right\}$.

On voudrait voir $\gamma_2\gamma_1$ comme un co-produit sur B_2, or $\gamma_2\gamma_1$ est un isomorphisme de B_2 sur $M_2' \cap M_4$ qui est contenu dans $B_2' \cap B_4$; il reste à mettre l'algèbre $B_2' \cap B_4$ dans le produit tensoriel $B_2 \otimes B_2$, c'est l'objet de la proposition suivante qui fixe les notations.

Proposition et Définitions 5.2.2.

a) *L'application $\gamma_\varphi\gamma_1$ est un isomorphisme de B_2 sur $B_2' \cap B_3$ qui conserve la trace.*

Posons $f'_K = \gamma_\varphi\gamma_1(f_K)$.

La famille $\{f'_K, K \in \mathcal{J}\}$ est une famille d'unités matricielles de $B_2' \cap B_3$.

b) *L'application γ_{H_φ} est un isomorphisme de B_2 sur $B_3' \cap B_4$ qui conserve la trace.*

Posons $F_K = \gamma_{H_\varphi}\gamma_\varphi(f_K)$.

La famille $\{F_K, K \in \mathcal{J}\}$ est une famille d'unités matricielles de $B_3' \cap B_4$.

c) *La famille $\{f'_H F_K, H \in \mathcal{J}, K \in J\}$ est une base de $B_2' \cap B_4$. L'algèbre $B_2' \cap B_4$ est isomorphe à $B_2 \otimes B_2$ par l'isomorphisme θ:*

$$\theta(f'_H F_K) = f_H \otimes f_K.$$

d) *L'application $\gamma_2\gamma_1$ est un isomorphisme de B_2 sur $M_2' \cap M_4$ qui envoie B_2 dans $B_2' \cap B_4$ en conservant la trace.*

Posons $g_K = \gamma_2\gamma_1(f_K)$.

Alors $g_K = \sum_{P,Q\in\mathcal{J}} x^K_{P,Q} f'_P F_Q$ *où les* $x^K_{P.Q}$ *sont des nombres complexes. On posera* $\Gamma = \theta\gamma_2\gamma_1$.

La démonstration de cette proposition est laissée au lecteur.

On peut appliquer les résultats du 3 et obtenir les formules suivantes:

Proposition 5.2.3.

$$\forall K \in \mathcal{J}, \qquad \gamma_1(f_K) = n\sum_{s=1}^{n} E_{M_1}(e_1\lambda_s f_K)e_1\lambda_s^*.$$

$$f'_K = \sum_{P\in\mathcal{J}} \sqrt{\frac{n}{n_p}} f_P\gamma_1(f_K)e_2\sqrt{\frac{n}{n_p}} f_P^*.$$

$$F_K = n\sum_{P\in\mathcal{J}} \sqrt{\frac{n}{n_p}} f_P e_2 f_K e_3 e_2 \sqrt{\frac{n}{n_p}} f_P^*.$$

$$g_K = n^2\sum_{s=1}^{n} \lambda_s e_1 e_2 f_K e_3 e_2 e_1 \lambda_s^*.$$

Démonstration.

a) C'est la formule (3.2.1) pour γ_1 l'anti-automorphisme de B_2.

b) C'est la formule (3.2.1) pour γ_φ l'anti-automorphisme de B_2 qui se simplifie:

$$f'_K = n\sum_{P\in\mathcal{J}} E_{B_2}\left(e_2\sqrt{\frac{n}{n_p}} f_P\gamma_1(f_K)\right) e_2\sqrt{\frac{n}{n_p}} f_P^*$$
$$= \sum_{P\in\mathcal{J}} \sqrt{\frac{n}{n_p}} f_P\gamma_1(f_K)e_2\sqrt{\frac{n}{n_p}} f_P^*.$$

(c) et (d) cf. formule (3.3.3).

5.3. Définition d'un coproduit sur B_2. Nous allons préciser les composantes de g_K sur la base $\{f'_H F_K, H \in \mathcal{J}, K \in \mathcal{J}\}$, connaisant l'expression de $\Gamma(f_K)$, nous pourrons vérifier que Γ est un co-produit co-associatif.

Proposition 5.3.1. $g_K = \sum_{P,Q\in\mathcal{J}} x^K_{P.Q} f'_P F_Q$, *c'est-à-dire*

$$\Gamma(f_K) = \sum_{P,Q\in\mathcal{J}} x^K_{P,Q} f_P \otimes f_Q$$

où

$$x^K_{P,Q} = \frac{n^2}{n_p n_q}\operatorname{tr}\left(\sum_{r=1}^{n} E_{M_1}\left(e_1\lambda_r f_P^*\right) f_K \lambda_r^* f_Q^*\right).$$

En particulier $e_3 = \Gamma(e_1) = \sum_{P\in\mathcal{J}} \frac{1}{n_p}\gamma_1(f_P^*) \otimes f_P$.

Démonstration.

Si $P = (p; p_1, p_2)$ et $Q = (q; q_1, q_2)$,

$$g_K f_P^* F_Q^* = x_{P,Q}^K f'^{p}_{p_1 p_1} F^q_{q_1 q_1}$$

et en passant à la trace, on obtient $x_{P,Q}^K = \frac{n^2}{n_p n_q} \operatorname{tr}\left(g_K f_P'^* F_Q^*\right)$.

Calculons donc $\operatorname{tr}(g_K f_P' F_Q)$.

$$\begin{aligned} f_P' F_Q &= n \sum_{B,C\in\mathcal{J}} \sqrt{\frac{n}{n_b}} f_B \gamma_1(f_P) e_2 \left(\sqrt{\frac{n}{n_b}} f_B^* \sqrt{\frac{n}{n_c}} f_C\right) e_2 f_Q e_3 e_2 \sqrt{\frac{n}{n_c}} f_C^* \\ &= n \sum_{B\in\mathcal{J}} \sqrt{\frac{n}{n_b}} f_B \gamma_1(f_P) e_2 f_Q e_3 e_2 \sqrt{\frac{n}{n_b}} f_B^* \end{aligned}$$

car $\left\{\sqrt{\frac{n}{n_c}} f_C, C \in \mathcal{J}\right\}$ est une base de Pimsner-Popa [**1.5.2a**]. Donc

$$\begin{aligned} &\operatorname{tr}(g_K f_P' F_Q) = \\ &= n^3 \operatorname{tr}\left(\sum_{B\in\mathcal{J},s=1}^{n} \lambda_s e_1 e_2 f_K e_3 e_2 \left(e_1 \lambda_s^* \sqrt{\frac{n}{n_b}} f_B \gamma_1(f_P)\right) e_2 f_Q e_3 e_2 \sqrt{\frac{n}{n_b}} f_B^*\right) \\ &= n^3 \operatorname{tr}\left(\sum_{B\in\mathcal{J},s=1}^{n} \lambda_s e_1 e_2 f_K e_3 E_{M_1} \left(e_1 \lambda_s^* \sqrt{\frac{n}{n_b}} f_B \gamma_1(f_P)\right) e_2 f_Q e_3 e_2 \sqrt{\frac{n}{n_b}} f_B^*\right) \\ &= n^2 \operatorname{tr}\left(\sum_{B\in\mathcal{J},s=1}^{n} \lambda_s e_1 e_2 f_K E_{M_1} \left(e_1 \lambda_s^* \sqrt{\frac{n}{n_b}} f_B \gamma_1(f_P)\right) f_Q e_3 e_2 \sqrt{\frac{n}{n_b}} f_B^*\right). \end{aligned}$$

Utilisons la propriété de commutation de la trace et sa propriété de Markov [**VJ1**, 3.1.7]: on a

$$\begin{aligned} &\operatorname{tr}(g_K f_P' F_Q) = \\ &= n \operatorname{tr}\left(\sum_{B\in\mathcal{J},s=1}^{n} e_2 \left(\sqrt{\frac{n}{n_b}} f_B^* \lambda_s e_1\right) e_2 f_K E_{M_1} \left(e_1 \lambda_s^* \sqrt{\frac{n}{n_b}} f_B \gamma_1(f_P)\right) f_Q\right) \\ &= n \operatorname{tr}\left(\sum_{B\in\mathcal{J},s=1}^{n} E_{M_1} \left(\sqrt{\frac{n}{n_b}} f_B^* \lambda_s e_1\right) e_2 f_K E_{M_1} \left(e_1 \lambda_s^* \sqrt{\frac{n}{n_b}} f_B \gamma_1(f_P)\right) f_Q\right) \\ &= \operatorname{tr}\left(\sum_{B\in\mathcal{J},s=1}^{n} E_{M_1} \left(\sqrt{\frac{n}{n_b}} f_B^* \lambda_s e_1\right) f_K E_{M_1} \left(e_1 \lambda_s^* \sqrt{\frac{n}{n_b}} f_B \gamma_1(f_P)\right) f_Q\right). \end{aligned}$$

Ecrivons $E_{M_1}\left(e_1\lambda_s^*\sqrt{\frac{n}{n_b}}f_B\gamma_1(f_P)\right)$ sur la base $\{\lambda_r, r=1 \text{ à } n\}$:

$$E_{M_1}\left(e_1\lambda_s^*\sqrt{\frac{n}{n_b}}f_B\gamma_1(f_P)\right)=\sum_{r=1}^{n}E_M\left(E_{M_1}\left(e_1\lambda_s^*\sqrt{\frac{n}{n_b}}f_B\gamma_1(f_P)\right)\lambda_r\right)\lambda_r^*$$

$$E_{M_1}\left(e_1\lambda_s^*\sqrt{\frac{n}{n_b}}f_B\gamma_1(f_P)\right)=\sum_{r=1}^{n}E_M\left(e_1\lambda_s^*\sqrt{\frac{n}{n_b}}f_B\gamma_1(f_P)\lambda_r\right)\lambda_r^*.$$

Comme f_K commute à M, on en déduit:

$$\begin{aligned}&\operatorname{tr}(g_Kf_P'F_Q)=\\&=\operatorname{tr}\left(\sum_{B\in\mathcal{J},r,s=1}^{n}E_{M_1}\left(\sqrt{\frac{n}{n_b}}f_B^*\lambda_se_1\right)E_M\left(e_1\lambda_s^*\sqrt{\frac{n}{n_b}}f_B\gamma_1(f_P)\lambda_r\right)f_K\lambda_r^*f_Q\right)\\&=\operatorname{tr}\left(\sum_{B\in\mathcal{J},r,s=1}^{n}E_{M_1}\left(\sqrt{\frac{n}{n_b}}f_B^*\lambda_se_1E_M\left(e_1\lambda_s^*\sqrt{\frac{n}{n_b}}f_B\gamma_1(f_P)\lambda_r\right)\right)f_K\lambda_r^*f_Q\right)\\&=\operatorname{tr}\left(\sum_{B\in\mathcal{J},r,s=1}^{n}E_{M_1}\left(\sqrt{\frac{n}{n_b}}f_B^*\lambda_se_1E_{M_1}\left(e_1\lambda_s^*\sqrt{\frac{n}{n_b}}f_B\gamma_1(f_P)\lambda_r\right)e_1\right)f_K\lambda_r^*f_Q\right)\end{aligned}$$

Comme on a:

$$n\sum_{s+1}^{n}\lambda_se_1E_{M_1}\left(e_1\lambda_s^*\sqrt{\frac{n}{n_b}}f_B\gamma_1(f_P)\lambda_r\right)=\sqrt{\frac{n}{n_b}}f_B\gamma_1(f_P)\lambda_r \quad (1.5.2\text{f}),$$

on en déduit:

$$\operatorname{tr}(g_Kf_P'F_Q)=\frac{1}{n}\operatorname{tr}\left(\sum_{B\in\mathcal{J},r=1}^{n}E_{M_1}\left(\sqrt{\frac{n}{n_b}}f_B^*\sqrt{\frac{n}{n_b}}f_B\gamma_1(f_P)\lambda_re_1\right)f_K\lambda_r^*f_Q\right).$$

Et comme $\sum_B\sqrt{\frac{n}{n_b}}f_B^*\sqrt{\frac{n}{n_b}}f_B=n$ (1.5.3e), on peut écrire:

$$\operatorname{tr}(g_Kf_P'F_Q)=\operatorname{tr}\left(\sum_{r=1}^{n}E_M\left(\gamma_1(f_P)\lambda_re_1\right)f_K\lambda_r^*f_Q\right).$$

Simplifions $E_{M_1}(\gamma_1(f_P)\lambda_re_1)$ en remplaçant $\gamma_1(f_P)$ par

$$n\sum_{s=1}^{n}E_{M_1}(e_1\lambda_sf_P)e_1\lambda_s^*.$$

On obtient alors

$$x_{P,Q}^K=\frac{n^2}{n_pn_q}\operatorname{tr}\left(g_Kf_P'^*F_Q^*\right)=\frac{n^2}{n_pn_q}\operatorname{tr}\left(\sum_{r=1}^{n}E_{M_1}\left(e_1\lambda_rf_P^*\right)f_K\lambda_r^*f_Q^*\right).$$

On notera $\gamma_K(f_P^*) = n\sum_{r=1}^n E_{M_1}(e_1\lambda_r f_P^*)\, f_K\lambda_r^*$. (Cette notation est cohérente puisque $\gamma_1(f_P) = n\sum_{s=1}^n E_{M_1}(e_1\lambda_s f_P)e_1\lambda_s^*$). On peut alors écrire

$$x_{P,Q}^K = \frac{n}{n_p n_q}\,\mathrm{tr}\left(\gamma_K(f_P^*)f_Q^*\right).$$

En particulier:

$$\begin{aligned} e_3 &= \gamma_2\gamma_1(e_1) \\ &= \sum_{P,Q\in\mathcal{J}} \frac{n}{n_p n_q}\,\mathrm{tr}\left(f_Q^*\gamma_1(f_P^*)\right) f_P' F_Q \\ &= \sum_{P,Q\in\mathcal{J}} \frac{n}{n_p n_q}\,\mathrm{tr}\left(f_Q^*\gamma_1(f_P^*)\right) f_P \otimes f_Q. \end{aligned}$$

Comme $\left\{\sqrt{\frac{n}{n_b}}f_B, B\in\mathcal{J}\right\}$ est une base de Pimsner-Popa de B_2 sur $\mathbb{C}$ et que γ_1 conserve la trace, on peut écrire

$$e_3 = \sum_{P\in\mathcal{J}} \frac{1}{n_p} f_P \otimes \gamma_1(f_P^*) = \sum_{P\in\mathcal{J}} \frac{1}{n_p}\gamma_1(f_P^*)\otimes f_P.$$

5.4. Dualité entre A_1 et B_2. Dans [**S**], W. Szymanski définit une dualité entre les espaces vectoriels A_1 etB_2.

Définition et Proposition 5.4.1. (W. Szymanski) *La forme linéaire définie sur $A_1\times B_2$ par*

$$(a,b) = n^2\,\mathrm{tr}(ae_1e_0b) \qquad (a\in A_1, b\in B_2)$$

établit une dualité entre A_1 et B_2.

Démonstration.

Rappelons d'abord que $A_1e_0 = \mathbb{C}e_0$ et $e_1B_2 = \mathbb{C}e_1$. Alors si, pour b donné dans B_2, $n^2\,\mathrm{tr}(ae_1e_0b)$ est nul pour tout a de A_1, $n^2\,\mathrm{tr}(ae_1a'e_0b)$ est nul pour tous a et a' de A_1.

Comme $A_2 = A_1e_1A_1$ et que e_0b est un élément de A_2, nous concluons à la nullité de $\mathrm{tr}(e_0bb^*e_0)$.

Or comme $E_{M_1}(bb^*)$ appartient à l'algèbre $M'\cap M_1$ qui vaut $\mathbb{C}$, on peut écrire:

$$0 = \mathrm{tr}(e_0bb^*e_0) = \mathrm{tr}(bb^*e_0) = \mathrm{tr}\left(E_{M_1}(bb^*)e_0\right) = n^{-1}\,\mathrm{tr}(bb^*).$$

La fidélité de la trace permet de conclure à la nullité de b.

Nous démontrons de même que, pour a donné dans A_1, la nullité de (a,b) pour tout b de B_2 implique la nullité de a.

La proposition suivante nous assure la co-associativité de Γ.

Proposition 5.4.2. Γ, *le coproduit de B_2, est le dual du produit de l'algèbre A_1.*

Démonstration.
L'algèbre A_1 (resp. B_2) est engendrée par $\{\lambda_s, 1 \leq s \leq n\}$ (resp. $\{f_K, K \in \mathcal{J}\}$). Nous allons donc calculer $(\lambda_h \otimes \lambda_s, , \Gamma(f_K))$. D'après 5.4.1 et 5.3.1, nous avons

$$(\lambda_h \otimes \lambda_s, , \Gamma(f_K)) = n^4 \sum_{P,Q \in \mathcal{J}} x^K_{P,Q} \operatorname{tr}(\lambda_h e_1 e_0 f_P) \operatorname{tr}(\lambda_s e_1 e_0 f_Q).$$

Lemme 1.

$$e_0 E_{M_1}(e_1 \lambda_r f^*_P) = n e_0 E_M(e_0 e_1 \lambda_r f^*_P) = n e_0 \operatorname{tr}(e_0 e_1 \lambda_r f^*_P).$$

Démonstration.
La première égalité est une application directe de [**PiPo1**, lemma 1.2].
Comme $e_0 e_1 \lambda_r f^*_P$ appartient à A_2, $E_M(e_0 e_1 \lambda_r f^*_P)$ appartient à $N' \cap M$ donc vaut $\operatorname{tr}(e_0 e_1 \lambda_r f^*_P)$.

Lemme 2. *L'élément $\gamma_K(f^*_P) = n \sum_{r=1}^n E_{M_1}(e_1 \lambda_r f^*_P) f_K \lambda^*_r$ appartient à B_2.*

Démonstration.
Soit y un élément de M, $\lambda^*_r y = \sum_{r=1}^n E_M(\lambda^*_r y \lambda_s)\lambda^*_s$, alors comme f_K, f^*_P et e_1 commutent à M, on peut écrire:

$$\begin{aligned}
\gamma_K(f^*_P) y &= n \sum_{r=1}^n E_{M_1}(e_1 \lambda_r f^*_P) f_K \lambda^*_r y \\
&= n \sum_{r=1}^n E_{M_1}(e_1 \lambda_r f^*_P) f_K E_M(\lambda^*_r y \lambda_s)\lambda^*_s \\
&= n \sum_{r=1}^n E_{M_1}(e_1 \lambda_r E_M(\lambda^*_r y \lambda_s) f^*_P) f_K \lambda^*_s \\
&= n \sum_{r=1}^n E_{M_1}(e_1 y \lambda_s f^*_P) f_K \lambda^*_s \\
&= y \gamma_K(f^*_P)
\end{aligned}$$

donc $\gamma_K(f^*_P) \in M' \cap M_2$.

Lemme 3.

$$\sum_{Q\in\mathcal{J}} x^K_{P,Q}\,\mathrm{tr}(\lambda_s e_1 e_0 f_Q) = \frac{1}{n_p}\sum_{r=1}^{n}(\lambda_r^*\lambda_s, f_K)\,\mathrm{tr}(e_0 e_1 \lambda_r f_P^*).$$

Démonstration.

$$\sum_{Q\in\mathcal{J}} x^K_{P,Q}\,\mathrm{tr}(\lambda_s e_1 e_0 f_Q) = \frac{n}{n_p n_q}\sum_{Q\in\mathcal{J}}\mathrm{tr}\left[\mathrm{tr}\left(\gamma_K(f_P^*)f_Q^*\right) f_Q\lambda_s e_1 e_0\right].$$

Puisque $\left\{\sqrt{\frac{n}{n_q}}f_Q^*\right\}$ est une base de B_2 sur $\mathbb{C}$, le Lemme 2 nous permet d'affirmer:

$$\sum_{Q\in\mathcal{J}}\frac{n}{n_q}\,\mathrm{tr}\left(\gamma_K(f_P^*)f_Q^*\right) f_Q = \gamma_K(f_P^*).$$

On simplifie alors l'expression:

$$\sum_{Q\in\mathcal{J}} x^K_{P,Q}\,\mathrm{tr}(\lambda_s e_1 e_0 f_Q) = \frac{1}{n_p}\,\mathrm{tr}\left(e_0\gamma_K(f_P^*)\lambda_s e_1\right).$$

En remplaçant $\gamma_K(f_P^*)$ par son expression, on obtient:

$$\sum_{Q\in\mathcal{J}} x^K_{P,Q}\,\mathrm{tr}(\lambda_s e_1 e_0 f_Q) = \frac{n}{n_p}\sum_{r=1}^{n}\mathrm{tr}\left(e_0 E_{M_1}\left(e_1\lambda_r f_P^*\right) f_K\lambda_r^*\lambda_s e_1\right).$$

Le Lemme 1 nous permet d'écrire:

$$\sum_{Q\in\mathcal{J}} x^K_{P,Q}\,\mathrm{tr}(\lambda_s e_1 e_0 f_Q) = \frac{n^2}{n_p}\sum_{r=1}^{n}\mathrm{tr}\left(e_0\,\mathrm{tr}(e_0 e_1\lambda_r f_P^*) f_K\lambda_r^*\lambda_s e_1\right).$$

On en déduit facilement le résultat annoncé.

Suite de démonstration de la proposition. D'après le Lemme 3, on peut écrire:

$$\begin{aligned}
(\lambda_h\otimes\lambda_s, \Gamma(f_K)) &= \sum_{r=1}^{n}(\lambda_r^*\lambda_s, f_K)\sum_{P\in\mathcal{J}}\frac{n^4}{n_p}\mathrm{tr}(e_0 e_1\lambda_r f_P^*)\,\mathrm{tr}(f_P\lambda_h e_1 e_0)\\
&= \sum_{r=1}^{n}(\lambda_r^*\lambda_s, f_K)\sum_{P\in\mathcal{J}}\frac{n^3}{n_p}\mathrm{tr}\left(n e_0\,\mathrm{tr}(e_0 e_1\lambda_r f_P^*) f_P\lambda_h e_1\right)\\
&= \sum_{r=1}^{n}(\lambda_r^*\lambda_s, f_K)\sum_{P\in\mathcal{J}}\frac{n^3}{n_p}\mathrm{tr}\left(e_0 E_{M_1}(e_1\lambda_r f_P^*) f_P\lambda_h e_1\right).
\end{aligned}$$

Comme $\left\{\sqrt{\frac{n}{n_p}} f_P, P \in \mathcal{J}\right\}$ est une base de Pimsner-Popa de M_2 sur M_1 (5.2), on peut simplifier l'expression:

$$(\lambda_h \otimes \lambda_s, \Gamma(f_K)) = n^2 \sum_{r=1}^{n} (\lambda_r^* \lambda_s, f_K) \operatorname{tr}(e_0 e_1 \lambda_r \lambda_h e_1).$$

Grâce à 1.5.2a et b et à définition de λ_s (5,2), on arrive au résultat espéré.

$$(\lambda_h \otimes \lambda_s, \Gamma(f_K)) = n^2 (\lambda_h \lambda_s, f_K) \operatorname{tr}(e_0 e_1) = (\lambda_h \lambda_s, f_K).$$

Corollaire 5.4.3. *$\Gamma = \theta\gamma_2\gamma_1$ est un coproduit co-associatif sur B_2.*

5.5. γ_1 est une co-involution sur (B_2, Γ). γ_1 est, par définition, une involution sur B_2, pour montrer qu'avec Γ, elle munit B_2 d'une structure d'algèbre de Hopf-Von Neumann co-involutive, nous avons besoin d'en savoir plus sur γ_{H_φ}.

Lemme 5.5.1.

a)

$$\forall P \in \mathcal{J}, \forall Q \in \mathcal{J}, \theta\gamma_{H_\varphi}\theta^{-1}(f_P \otimes f_Q) = \gamma_1(f_Q) \otimes \gamma_1(f_P),$$

c'est-à-dire que modulo l'identification θ entre $B_2' \cap B_4$ et $B_2 \otimes B_2$,

$$\gamma_{H_\varphi} = (\gamma_1 \otimes \gamma_1)\sigma$$

où σ est l'automorphisme de $B_2 \otimes B_2$ défini par $\sigma(x \otimes y) = \sigma(y \otimes x)$ [**ES1**, 1.2.5].

b) $\gamma_3(a) = \gamma_{H_\varphi}(a) \qquad (a \in B_2' \cap B_4)$.

Démonstration.

a) $\gamma_{H_\varphi}(f_P' F_Q) =$
$= \gamma_{H_\varphi}(\gamma_{H_\varphi}\gamma_\varphi(f_Q))\,\gamma_{H_\varphi}(\gamma_\varphi\gamma_1(f_P)) = \gamma_\varphi\gamma_1(\gamma_1(f_Q))\,\gamma_{H_\varphi}\gamma_\varphi(\gamma_1(f_P))$.

b) D'après la formule 3.2.1, $\left\{\frac{n}{\sqrt{n_p}} f_P\, e_2, P \in \mathcal{J}\right\}$ étant une base de Pimsner-Popa de M_3 sur M_2, si $a \in B_2' \cap B_4$,

$$\gamma_3(a) = n \sum_{P \in \mathcal{J}} E_{M_3}\left(e_3 \frac{n}{\sqrt{n_p}} f_P\, e_2 a\right) e_3 e_2 \frac{n}{\sqrt{n_p}} f_P^*.$$

Comme $\left(e_3 \frac{n}{\sqrt{n_p}} f_P e_2\, a\right)$ appartient à B_4, $E_{M_3}\left(e_3 \frac{n}{\sqrt{n_p}} f_P e_2\, a\right)$ commute à M donc vaut $E_{B_3}\left(e_3 \frac{n}{\sqrt{n_p}} f_P e_2 a\right)$, on a alors

$$\gamma_3(a) = n \sum_{P \in \mathcal{J}} E_{B_3}\left(e_3 \frac{n}{\sqrt{n_p}} f_P e_2\, a\right) e_3 e_2 \frac{n}{\sqrt{n_p}} f_P^*.$$

C'est la formule 3.2.1 pour γ_{H_φ} défini sur $B_2' \cap B_4$, ainsi γ_3 et γ_{H_φ} coïncident sur $B_2' \cap B_4$.

Proposition 5.5.2. *Le triplet (B_2, Γ, γ_1) est une algèbre de Hopf-Von Neumann co-involutive.*

Démonstration.

D'après 5.5.1a, $(\gamma_1 \otimes \gamma_1)\sigma\Gamma\gamma_1 = \theta\gamma_{H_\varphi}\gamma_2|_{B_2}$. Comme $\gamma_2(B_2)$ est contenu dans $B_2' \cap B_4$, d'après 5.5.1b, $\gamma_{H_\varphi}\gamma_2|_{B_2} = \gamma_3\gamma_2|_{B_2}$. La propriété (2.2.1a) des anti-automorphismes nous permet alors d'écrire:

$$(\gamma_1 \otimes \gamma_1)\sigma\Gamma\gamma_1 = \theta\gamma_2\gamma_1 = \Gamma.$$

Le triplet (B_2, Γ, γ_1) est une algèbre de Hopf-Von Neumann co-involutive [**ES1**, 1.2.5].

Remarque. On peut vérifier facilement que γ_1 est la co-involution définie par la dualité entre A_1 et B_2.

Proposition 5.5.3. *γ_1, la co-involution sur B_2, est le dual de l'involution de A_1.*

Démonstration.

Pour λ_r dans A_1 et f_K dans B_2, on a

$$\begin{aligned}(\lambda_r, \gamma_1(f_K)) &= n^3 \sum_{s=1}^{n} \operatorname{tr}(\lambda_r e_1 e_0 E_{M_1}(e_1\lambda_s f_K) e_1 \lambda_s^*) \\ &= n^3 \sum_{s=1}^{n} \operatorname{tr}(\lambda_r E_M(e_0 E_{M_1}(e_1\lambda_s f_K)) e_1 \lambda_s^*) .\end{aligned}$$

Comme $e_0 E_{M_1}(e_1\lambda_s f_K)$ appartient à $N' \cap M_1$, on a

$$\begin{aligned}(\lambda_r, \gamma_1(b)) &= n^3 \sum_{s=1}^{n} \operatorname{tr}(\lambda_r \operatorname{tr}(e_0 e_1 \lambda_s f_K) e_1 \lambda_s^*) \\ &= n^3 \sum_{s=1}^{n} \operatorname{tr}(\lambda_r e_1 \operatorname{tr}(f_K e_0 e_1 \lambda_s) \lambda_s^*) .\end{aligned}$$

Comme $\{\lambda_s, 1 \le s \le n\}$ est une base de Pimsner-Popa de A_1 sur $\mathbb{C}$, on peut écrire:

$$(\lambda_r, \gamma_1(b)) = n^3 \operatorname{tr}(\lambda_r e_1 E_{M_1}(f_K e_0 e_1)) = n^2 \operatorname{tr}(\lambda_r E_{M_1}(f_K e_0 e_1)) .$$

On en déduit que

$$(\lambda_r, \gamma_1(b)) = n^2 \operatorname{tr}(f_K e_0 e_1 \lambda_r) = \overline{(\lambda_r^*, f_K^*)}.$$

5.6. $(B_2, \Gamma, \gamma_1, n\varphi)$ **est une algèbre de Kac.** Nous allons montrer que $n\varphi$ est un poids de Haar sur (B_2, Γ, γ_1) en utilisant le théorème 6.3.5 de [**ES2**].

Lemme 5.6.1. *$\forall K \in \mathcal{J}$, $g_K e_1 = e_1 F_K$, c'est-à-dire*

$$\forall a \in B_2, \qquad \Gamma(a)(e_1 \otimes 1) = e_1 \otimes a.$$

Démonstration.
Nous utilisons la formule de la Proposition 5.3.1 et la notation déjà utilisée $\gamma_K(f_P^*) = n \sum_{r=1}^{n} E_{M_1}(e_1 \lambda_r f_P^*) f_K \lambda_r^*$, en particulier

$$\gamma_K(e_1) = \sum_{r=1}^{n} \mathrm{tr}(\lambda_r) f_K \lambda_r^* = f_K.$$

Comme $g_K = \sum_{P \in \mathcal{J}} \frac{1}{n_p} f_P \otimes \gamma_K(f_P^*)$, $g_K e_1 = e_1 \otimes \gamma_K(e_1) = e_1 \otimes f_K$.

Lemme 5.6.2. *$\forall K \in \mathcal{J}$, $\theta(g_K)(1 \otimes e_1) = f_K \otimes e_1$, c'est-à-dire*

$$\forall a \in B_2, \qquad \Gamma(a)(1 \otimes e_1) = a \otimes e_1.$$

Démonstration.
Comme $\theta(g_K) = \sum_{P,Q \in \mathcal{J}} \frac{n^2}{n_p n_q} \mathrm{tr}\left(\sum_{r=1}^{n} E_{M_1}(e_1 \lambda_r f_P^*) f_K \lambda_r^* f_Q^*\right) f_P \otimes f_Q$, on a:

$$\theta(g_K)(1 \otimes e_1) = \sum_{P \in \mathcal{J}} \frac{n}{n_p} \mathrm{tr}\left(n \sum_{r=1}^{n} E_{M_1}(f_K \lambda_r^* e_1) e_1 \lambda_r f_P^*\right) f_P \otimes e_1.$$

Or, puisque $\{n^{1/2} \lambda_r e_1, 1 \leq r \leq n\}$ est une base de Pimsner-Popa de M_2 sur M_1,

$$n \sum_{r=1}^{n} E_{M_1}(f_K \lambda_r^* e_1) e_1 \lambda_r = n \sum_{r=1}^{n} E_{M_1}(f_K \lambda_r e_1) e_1 \lambda_r^* = f_K.$$

Alors,

$$\sum_{P \in \mathcal{J}} \frac{n}{n_p} \mathrm{tr}\left(n \sum_{r=1}^{n} E_{M_1}(f_K \lambda_r^* e_1) e_1 \lambda_r f_P^*\right) f_P = \sum_{P \in \mathcal{J}} \frac{n}{n_p} \mathrm{tr}(f_K f_P^*) f_P = f_K$$

et l'égalité est démontrée.

Proposition 5.6.3. *$(B_2, \Gamma, \gamma_1, n\varphi)$ est une algèbre de Kac de dimension finie.*

Démonstration.
On a vu dans la Proposition 5.3.1 que le projecteur central de B_2, e_1, vérifie:

$$\Gamma(e_1) = \sum_{P=(p;p_1,p_2)\in\mathcal{J}} \frac{1}{n_p}\gamma_1(f^p_{p_2,p_1}) \otimes f^p_{p_1,p_2}.$$

Le Théorème 6.3.5. de [**ES2**] et les Lemmes 5.6.1 et 2 permettent de conclure.

Remarque. L'algèbre A_1 munie du co-produit dual du produit de l'algèbre B_2, de la co-involution duale de l'involution de B_2 et du poids $n\,\mathrm{tr}$ est l'algèbre de Kac duale de l'algèbre $(B_2, \Gamma, \gamma_1, n\varphi)$ [**ES1**, 6.9.9].

5.7. Une action de $(B_2, \Gamma, \gamma_1, n\varphi)$ **sur** M**.** Il nous reste à faire agir l'algèbre de Kac sur M.

Proposition 5.7.1.

a) *Si N_1 est la première algèbre d'un tunnel construit dans $N \subset M$, c'est-à-dire que $N_1 \subset N \subset M$ est la construction de base, soit $\nu = \gamma_0\gamma_1$ l'isomorphisme de B_2 sur $N_1' \cap M$, $\{\nu(f_K), K \in \mathcal{J}\}$ est une famille d'unités matricielles de $N_1' \cap M$ et, à une constante multiplicative près, une base de Pimsner-Popa de M sur N.*

b) *Soit β le morphisme de M dans $M \otimes B_2$ défini par:*

$$\textit{si } y \in N \textit{ et } K \in \mathcal{J}, \qquad \beta(y\,\nu(f_K)) = (y \otimes 1)(\nu \otimes i)\Gamma(f_K)$$

β est une action de B_2 sur M dont l'algèbre des point fixes est N.

c) *ν se prolonge en un morphisme normal de B_2 dans M qui vérifie:*

$$\nu(1) = 1 \textit{ et } \beta\nu = (\nu \otimes i)\Gamma.$$

Démonstration.

a) C'est la Proposition 5.2.1.

b) β est une action car c'est un morphisme injective de M dans $M \otimes B_2$ qui vérifie

$$\beta(1) = 1 \qquad \text{et} \qquad (\beta \otimes i)\beta = (i \otimes \Gamma)\beta. \qquad [\mathbf{E1}, 1.1]$$

En effet si $y \in N$,

$$\begin{aligned}(i \otimes \Gamma)\beta(y\nu(f_K)) &= (y \otimes 1 \otimes 1)(i \otimes \Gamma)(\nu \otimes i)\Gamma(f_K) \\ &= (y \otimes 1 \otimes 1)(\nu \otimes i \otimes i)(i \otimes \Gamma)\Gamma(f_K) \\ &= (y \otimes 1 \otimes 1)(\nu \otimes i \otimes i)(\Gamma \otimes i)\Gamma(f_K)\end{aligned}$$

$$\begin{aligned} &= \sum_{P,Q\in\mathcal{J}} x^K_{P,Q}\,[(y\otimes 1)(\nu\otimes i)\Gamma(f_P)]\otimes f_Q \\ &= (\beta\otimes i)\,[(y\otimes 1)(\nu\otimes i)\Gamma(f_K)] \\ &= (\beta\otimes i)\beta\,(y\nu(f_K))\,. \end{aligned}$$

Comme $1 = \sum_{K\in\mathcal{J}} f_K$, β laisse fixe les éléments de N. D'autre part, comme $[M:N] = n = [M:N^\beta]$, l'inclusion $N^\beta \subset N$ implique l'égalité.

c) résulte de la définition de ν et β.

La Proposition 5.7.1 et le Théorème 5.2 de [**ES2**] permettent de conclure:

Théorème 5.7.2. *Soient M un facteur de type* II_1, tr *sa trace normale finie fidèle normalisée et N un sous-facteur d'indice fini dans M. Si N est de profondeur au plus 2 dans M et $N'\cap M$ est égal à $\mathbb{C}$, N est la sous-algèbre des points fixes de M sous l'action extérieure β de l'algèbre de Kac de dimension finie $(M'\cap M_2, \theta\gamma_2\gamma_1, \gamma_1, n\varphi)$.*

L'action β est extérieure puisque $N'\cap M = \mathbb{C}$.

5.8. Remarque: cas d'un groupe fini. Le cas où $N'\cap M_1$ est abélien peut se traiter directement, en effet le groupe G apparaît comme le quotient du normalisateur de N par le groupe unitaire de N. Un résultat de Sutherland repris dans la thèse de V. Jones permet de conclure [**thèse VJ**, 4.1.7].

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IRREDUCIBILITY AND DIMENSION THEOREMS FOR FAMILIES OF HEIGHT 3 GORENSTEIN ALGEBRAS

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We show that the family of graded Gorenstein Artin algebras of height 3 with a fixed Hilbert function is irreducible, and we prove some dimension theorems about these families.

0. Introduction.

In Chapter 1 we show that when a set $D = (Q, P)$ of generator and relation degrees is given, $Q = \{q_1, \dots, q_u\}$ and $P = \{p_1, \dots, p_u\}$, the family Gor_D of Gorenstein algebras $A = R/J$ with J having this set of generator and relation degrees is irreducible. We show that Gor_D is the image of an algebraic map from a dense open set in a product of affine spaces. This depends on Buchsbaum and Eisenbud's structure theorem for height 3 Gorenstein ideals [**BE1**], which is discussed in Chapter 2.

In Chapter 2 we show that when T is fixed, the family Gor_T of all Gorenstein algebras with Hilbert function T is irreducible. We show this by giving an explicit deformation of an ideal with degree set D to an ideal with a smaller degree set D' consistent with T. The minimal set D_{min} of generator and relation degrees given T is unique, and we conclude in Theorem 2.7 that $\overline{Gor_D} \supset Gor_{D'}$ for $D' \supset D$, and therefore $Gor_T = \overline{Gor_{D_{min}}}$, the Zariski closure of $Gor_{D_{min}}$ inside Gor_T. We give a method for determining the alternating matrix whose pfaffians generate the ideal with the smaller degree set, and show that it is Gorenstein of height 3. We conclude that whenever an ideal J determining T is generated by more than the minimum number needed for T, it can be deformed to an ideal with fewer generators.

We again work from the perspective of a fixed Hilbert function T in Chapter 3 to determine the maximum number of generators an ideal determining T may have. This uses the combinatorial data described in [**BE1**] and [**St1**], specifically the conditions on a sequence $\{r_1, r_2, \dots, r_u\}$ of integers, which we call *diagonal degrees*, that can occur as the differences in a degree set D defining a Gorenstein algebra with Hilbert function T. We determine the maximum number of generators possible for a ideal determining a given Hilbert function, and we give an explicit example of a matrix whose pfaffians generate this number for any given T. There is a lattice structure we can give to the degree sets that determine T, where a vertex of the lattice is a sequence of diagonal degrees that is consistent with T, and where two vertices are connected if the sequence at one vertex is a subsequence of the other. There is a unique minimum vertex, corresponding to the difference sequence of D_{min} and a unique maximum vertex, corresponding to the saturated sequence of diagonal degrees defining T. We also prove in Proposition 3.12 that there is a one to one correspondence between permissible Hilbert functions T of socle degree n and order $k(T)$ and self-complementary subpartitions of rectangular blocks of size $2k$ by $n-2k+2$.

Chapter 4 investigates various methods for determining the dimension of Gor_T by studying the ranks of the Catalecticant matrices associated to a dual form f. For certain Hilbert functions of socle degree n, order $k(T)=d$, and bounded by t, we make a conjecture for the dimension of the family Gor_T, and we show the conjecture to be true for approximately two thirds of the possible cases. We determine the dimension of $Gor(T)$ for certain other T.

0.1. Notation and definitions. We will use the following notation and definitions throughout this paper.

- R is the ring $k[x,y,z]$, where k is an algebraically closed field. The maximal ideal of R is $m=(x,y,z)$.
- A is a graded, height 3 Gorenstein algebra quotient of $k[x,y,z]$.
- R_i is the space of forms of homogeneous degree i in R. R is a graded ring and can be expressed as $\oplus_{i\geq 0}R_i$, with $R_0=k$.
- $R_d(g_i)$ is the subspace $(x^d g_i, x^{d-1}yg_i, \dots, z^d g_i)$ of forms of degree $d+\deg g_i$ in R generated by the homogeneous polynomial g_i.
- We let $T=T(A)=(h_0,h_1,\dots,h_n,0,\dots)$ be the Hilbert function of a Gorenstein Artin algebra $A=R/J$, where J is a homogeneous ideal in $k[x,y,z]$. J therefore has a grading $J=\oplus_{i\geq 0}J_i$. The nonnegative integer h_i is the dimension of R_i/J_i as a k-vector space.
- The socle of R/J in the set $\{a\in R/J\,|\,a\cdot h\in J\ \forall h\in m\}$. We call R/J a *Gorenstein ring* and J a *Gorenstein ideal* if the dimension of the socle of R/J as a k-vector space is 1.
- $D=(\{q_i\},\{p_i\})$ is a set of generator degrees and relation degrees for an ideal J in R corresponding to a given Hilbert function T.
- $\mathcal{R}=\mathrm{Hom}_k(R,k)$ is the ring dual to R. R acts on $\mathcal{R}$ by contraction; if $X^aY^{c+d}Z^e\in\mathcal{R}$, then $x^ay^d\circ X^aY^{c+d}Z^e=Y^cZ^e$.
- We denote by f a homogeneous polynomial in $\mathcal{R}$ whose annihilator in R is J.
- $I_t(M)$ is the ideal of $t+1$ by $t+1$ minors of an n by m matrix M. If $t\geq\min\{m,n\}$, then $I_t(M)$ is defined to be the zero ideal (0).

1. Variety structure on Gor_D.

We assume in this chapter that k is an algebraically closed field, $R = k[x, y, z]$ is a ring, and J is a homogeneous ideal in R.

Let $A = R/J$ be a Gorenstein algebra. The set $D = D(J) = (\{q_i, p_i\})$, $i = 1 \dots u$ of generator and relation degrees of J determines the Hilbert function of R/J. Buchsbaum and Eisenbud's structure theorem for height 3 Gorenstein ideals proves that all such ideals can be obtained as pfaffians of a suitable alternating matrix M with entries in R. The degree matrix E_M of the entries of M is determined by D (see Chapter 2). Denote by $E = E(D)$ the set of entry degrees in E_M. The degree matrix determines the number of ways of filling in M with entries chosen generally from R. This number $h(E_M)$ is a polynomial in the entry degrees E.

Let π be the map from the family $\mathbb{A}^{h(E_M)}$ of all alternating matrices with degree matrix equal to E_M to the family of algebras $A = R/J$ having the set $D(J)$ of generator and relation degrees, determined by E_M. We will show that when we restrict to a single D and E_M, the matrices whose pfaffians form a height 3 ideal is a nonempty dense open set U_{E_M} in $\mathbb{A}^{h(E_M)}$. We say a degree matrix E_M is permissible if U_{E_M} is nonempty. We discuss the conditions for U_{E_M} to be nonempty in Chapter 3.

1.1. Definition and parametrization of Gor_D. Let Gor_D be the family of Gorenstein algebras having the set D of generator and relation degrees. D determines the Hilbert function of any Artin algebra R/J with $D = D(J)$. We define Gor_T to be the union of all families of Gorenstein algebras $Gor_{D,}$ associated to Hilbert function T. Gor_T is a locally closed subset of G_T, the family of all graded algebras R/I with Hilbert function equal to T. The ideal I has a grading $I = \oplus_{t \geq 0} I_t$. G_T is embedded in a product of Grassmannians $\prod Grass(d_t, R_t)$, with $d_t = |I_t|$, the size of I_t as a k-vector space. We give Gor_T the reduced subscheme structure coming from this product.

Define π to be the map from an open set U_{E_M} in $\mathbb{A}^{h(E_M)}$ to the Gorenstein algebra Gor_D whose degree set D is determined by the degree matrix E_M.

Theorem 1.1. *Gor_D is the image of U_{E_M} under the algebraic map π, and is therefore irreducible.*

Proof. Let D be given and let M be an alternating u by u matrix with degree matrix E_M such that the set of pfaffians of M generates a height 3 Gorenstein ideal J. Let M_i denote the submatrix of M obtained by eliminating row i and column i. The image $\pi(M)$ is the algebra R/J where $J = (g_1, \dots, g_u)$. A generator g_i of J is the square root of the determinant of M_i. This generator can also be computed by the formula

$$\sqrt{\det M_i} = Pf(M_i) = \sum_{k=1}^{u-1}(-1)^k \cdot m_{rk} \cdot Pf(M_{ikr}), \tag{1.1}$$

where r is a row by which to expand, m_{rk} is the (r,k) entry of M_i, M_{ikr} is the submatrix of M with rows and columns i, k and r eliminated [**Sa**, p. 71]. This expresses the generators $(g_1, g_2, \ldots, g_u)$ of J as polynomials in the entries of M, so π is an map from U_{E_M} to Gor_D. U_{E_M} is an open set in $\mathbb{A}^{h(E_M)}$ since an ideal has height 3 when the determinant of a certain matrix does not vanish (see proof of Theorem 2.3 in Section 2.3). Consequently U_{E_M} is irreducible, and so is its image Gor_D. □

Remark. The fiber π^{-1} over a point p_J parametrizing J includes a product of general linear groups parametrizing different choices of generators for J. We get an upper bound for the dimension of Gor_D by subtracting the dimension of this product from the dimension of $\mathbb{A}^{h(E_M)}$. We will use this fact in Chapter 4 in the proof of Theorem 4.4.

If we look at all degree sets $D_1, D_2, \ldots$ of ideals $J_1, J_2, \ldots$ in R such that the Hilbert function of R/J_i equals T for each i, then each Gor_{D_i} is irreducible by Theorem 1.1. There are a finite number of different degree sets D_i for a given Hilbert function, a result of the structure theorem, and we will show in Chapter 2 that the entire family Gor_T of all algebras $A = R/J$ with Hilbert function T is irreducible.

We parametrize the family Gor_D by the product $\prod Grass(t_d, R_d)$ of Grassmannians, which embeds Gor_D as a subspace of a product of projective spaces $\prod \mathbb{P}$ whose coordinates depend polynomially on $(g_1, \ldots, g_u)$.

2. Irreducibility of Gor_T.

As a result of Theorem 1.1 in the previous chapter, we know that given T and a set of generator and relation degrees D, the family Gor_D is irreducible. However, T may have several different degree sets that correspond to the same Hilbert function.

2.1. Definition of Gor_T. We have defined Gor_T to be the family of all algebras with Hilbert function T. Gor_T is the finite union $\bigcup_i Gor_{D_i}$ over all degree sets $D_i = D(J_i)$ consistent with T.

2.2. Structure theorem for Gorenstein ideals of height 3. The following is the statement of Buchsbaum and Eisenbud's structure theorem for Gorenstein ideals of height 3 in a local noetherian ring [**BE1**, p. 456].

Theorem 2.1. *Let R be a noetherian local ring with maximal ideal m.*

1) *Let $n \geq 3$ be an odd integer, and let F be a free R-module of rank n. Let $f : F^* \mapsto F$ be an alternating map whose image is contained in mF. Suppose $Pf_{n-1}(f)$ has grade 3. Then $Pf_{n-1}(f)$ is a Gorenstein ideal, minimally generated by n elements.*

2) *Every Gorenstein ideal of grade 3 arises as in 1).*

Buchsbaum and Eisenbud develop the machinery in [**BE1**] to prove the above theorem. If J is a homogeneous Gorenstein ideal of grade 3 in $R = k[x_0 \dots x_m]$, then a free resolution of R/J has the form

$$(2.1) \qquad E:\ 0 \longrightarrow R(s) \xrightarrow{g^*} \sum_{i=1}^{n} R(p_i) \xrightarrow{f} \sum_{i=1}^{n} R(q_i) \xrightarrow{g} R(0)$$

where the maps f and g are homogeneous of degree 0, the socle of R/J is in degree $s-3$, the integers $\{q_i\}$ are the degrees of the generators of J and integers $\{p_i\}$ are the degrees of the relations among the generators.

Buchsbaum and Eisenbud prove that the matrix representing f in the resolution of R/J will be skew-symmetric, and the matrix representing g (resp. g^*) will be the column (resp. row) matrix of pfaffians of the matrix representing f. The degree of the (i,j) entry of this middle matrix is $p_j - q_i$. We will always consider the sequence $\{q_i\}$ to be nondecreasing and the sequence $\{p_i\}$ to be nonincreasing. This defines a new sequence $\{r_i\}$, where $r_i = p_i - q_i$. With this ordering on the degrees, $p_i + q_i = s$. The integers $\{r_i\}$ are all even or all odd, since the degree of the (i,j) entry of the alternating matrix representing f is also expressed as $(r_i + r_j)/2$. We have the further relation that $s = \sum r_i$. Therefore the sequence of integers $\{r_i\}$ completely determines the socle degree, generator and relation degrees, and the Hilbert function. We will discuss which sequences of $\{r_i\}$ can occur in Chapter 3. It follows from (2.1) that the Hilbert function $(h_0, h_1, \dots)$ of R/J equals

$$(2.2) \quad h_t = \binom{m+t}{m} - \sum_{i=1}^{n} \binom{m+t-q_i}{m} + \sum_{i=1}^{n} \binom{m+t-p_i}{m} - \binom{m+t-s}{m}.$$

Here the binomial coefficient $\binom{a}{b}$ equals zero if a is less than b.

Example 2.2. Let $D = (\{3,5,6\},\{11,9,8\})$ be the set of generator and relation degrees of a Gorenstein ideal. The $\{r_i\}$ are equal to $\{8,4,2\}$. We get $s = 8+4+2 = 14$, so the socle degree is 11 and the Hilbert function is

$$(1,3,6,9,12,14,14,12,9,6,3,1,0,\dots).$$

This Hilbert function can also be determined by the sequence

$$D' = (\{3,5,6,6,8\},\{11,9,8,8,6\}).$$

2.3. Deformation theorem. We will now show that when the ideal of pfaffians of the matrix representing f has height 3 and is Gorenstein, and the matrix contains non-diagonal degree zero entries, we can allow a pair of these entries to be nonzero constants $c, -c$. The resulting ideal of pfaffians will be Gorenstein, height 3, minimally generated by $n-2$ elements. We start with an existing resolution of a Gorenstein ideal which satisfies all the conditions of the structure theorem. Since R is a polynomial ring it is Cohen-Macaulay, so depth(I) =height(I) for every ideal $I \subset R$. A homogeneous ideal I in R is also *perfect*, so all height 3 ideals have projective resolutions of length 3.

Let J be a Gorenstein ideal of height 3 generated by $v+2$ elements, v odd, with the Hilbert function of R/J equal to T. Assume a minimal free resolution of R/J has the form in (2.1). Let the alternating matrix representing the map f be

$$M = \begin{pmatrix} 0 & m_{1,2} & m_{1,3} & \dots & m_{1,v+2} \\ -m_{1,2} & 0 & m_{2,3} & \dots & m_{2,v+2} \\ \vdots & \vdots & \vdots & & \vdots \\ -m_{1,v+2} & -m_{2,v+2} & -m_{3,v+2} & \dots & 0 \end{pmatrix}.$$

Assume that the set $D(J)$ requires that M contain two non-diagonal degree zero entries, which we may assume to be the $(v+1, v+2)$ and $(v+2, v+1)$ entries of M. Under the conditions of the structure theorem these must equal zero in order for the image of f to be contained in mR^{v+2}. Because the degrees of entries of M are determined by the values of the diagonal degrees $\{r_i\}$, we have $r_{v+1} = -r_{v+2}$.

Let $M'(c)$ be the matrix obtained from M by letting $m'_{v+1,v+2}$ and $m'_{v+2,v+1}$ be nonzero elements c and $-c \in k$ in a neighborhood of zero, and all other entries $m'_{i,j}$ of $M'(c)$ equal to $m_{i,j}$. Let $\pi(M'(c)) = J'(c)$ be the ideal of pfaffians of $M'(c)$.

Theorem 2.3. *Let J be a Gorenstein ideal of height 3 with Hilbert function T and minimal free resolution*

$$E:\ 0 \longrightarrow R(s) \xrightarrow{g^*} \sum_{i=1}^{v+2} R(p_i) \xrightarrow{f} \sum_{i=1}^{v+2} R(q_i) \xrightarrow{g} R(0)$$

where f is represented by the $v+2$ by $v+2$ alternating matrix M, $v+2$ odd. Let $M'(c)$ and $J'(c)$ be the matrix and ideal of pfaffians described above. Then:

(i) *$J'(c)$ has height 3 for all but finitely many c, and is generated by v elements;*

(ii) *$R/J'(c)$ has a resolution E' obtained from E by replacing M by $M'(c)$, g by g' and g^* by g'^*;*

(iii) *$J'(c)$ is Gorenstein, and $R/J'(c)$ has a minimal resolution explicitly determined by M and c;*

(iv) *the Hilbert function of $R/J'(c)$ is T.*

We will need the following lemma, which we use without proof:

Lemma 2.4. *Let $T = (h_0, h_1, \dots, h_n, 0, \dots)$ be the Hilbert function of a Gorenstein algebra $A = R/J$ and let $Q = \{q_i\}$ and $P = \{p_i\}$, $i = 1 \dots u$, be the sequences of generator and relation degrees of J satisfying*

$$\binom{m+t}{m} - \sum_{i=1}^{u} \binom{m+t-q_i}{m} + \sum_{i=1}^{u} \binom{m+t-p_i}{m} - \binom{m+t-s}{m} = h_t.$$

Let I be another ideal which defines the same Hilbert function T, and $Q' = \{q_j\}$ and $P' = \{p_j\}$ its generator and relation degrees, $j = 1 \dots u+d$, such that P' contains P and Q' contains Q as subsequences. If we let P'' and Q'' be the sequences $P' \setminus P$ and $Q' \setminus Q$, both arranged in increasing order, then $p_k = q_k$ for $p_k \in P''$ and $q_k \in Q''$, $k = 1 \dots d$.

Proof of Theorem 2.3. Let $\{g_1 \dots g_{v+2}\}$ be the pfaffians of M. These are a minimal set of generators for J under the assumptions of Theorem 2.2. The ideal $J'(c)$ will be generated by homogeneous polynomials $\{g'_1 \dots g'_{v+2}\}$, where $g'_i = g_i + c \cdot h_i(x, y, z)$. Let $q_i = \deg g_i = \deg g'_i$.

i). Assume J has height 3. We will show that the vector space J'_{n+1} of forms of degree $n+1$ in $J'(c)$ has dimension $\binom{n+3}{2}$, and therefore contains everything in degree $n+1$.

Since J is a Gorenstein ideal whose socle is in degree n, the vector space J_{n+1} has dimension $\binom{n+3}{2}$. The set of forms in the vector space

$$R_{n+1-q_1}(g_1), \dots, R_{n+1-q_{v+2}}(g_{v+2})$$

span J_{n+1} and form the row space of a matrix N of size G by $\binom{n+2}{2}$, where $G \geq \binom{n+3}{2}$ is the sum of the dimensions of the vector spaces $R_{n+1-q_i}(g_i)$. Each element of J_{n+1} is expressed in terms of the standard basis of R_{n+1} of monomials $\{x^{n+1}, x^n y, \dots, z^{n+1}\}$. N has entries in k, and the ideal of maximal minors of N must contain at least one nonzero constant $\delta \in k$.

In the same way we take the generators $\{g'_i\}$ of $J'(c)$ and look at the matrix $N(c)$ whose rows are spanned by the forms in the vector spaces $R_{n+1-q_i}(g'_i)$. Since each $g'_i = g_i + c \cdot h_i(x, y, z)$, $N(0) = N$, and the ideal of maximal minors of $N(c)$ contains an element $\delta(c)$ such that $\delta(0) = \delta$.

Since $\delta(c)$ is a polynomial function of the entries of $N(c)$, there are finitely many values of c for which $\delta(c)$ equals zero. Since k is algebraically closed and therefore infinite, we can choose a Zariski open set U containing zero so that when $c \in U$, $\delta(c)$ is nonzero. Since $N(c)$ contains at least one nonzero maximal minor, it has rank $\binom{n+3}{2}$, and therefore the dimension of J'_{n+1} is $\binom{n+3}{2}$.

To see that $J'(c)$ is minimally generated by v elements, note that when we multiply row i of $M'(c)$ by the column matrix g' we get the sum $\sum_{j=1}^{v+2} M'(c)_{ij} \cdot g'_j = 0$. If $i = v+2$, this becomes $\sum_{j=1}^{v+1} M'(c)_{v+2,j} \cdot g'_j = -cg'_{v+2}$, which for nonzero c allows g'_{v+2} to be expressed in terms of previous entries of g'. We can express g_{v+1} in the same way in terms of previous generators.

ii). The following is a resolution for $J'(c)$:

$$E' : 0 \longrightarrow R(s) \xrightarrow{g'^*} \sum_{i=1}^{v+2} R(p_i) \xrightarrow{f'} \sum_{i=1}^{v+2} R(q_i) \xrightarrow{g'} R(0)$$

where f' is represented by $M'(c)$ and the pfaffian map g' is represented by $J'(c)$. Note that E' is not a minimal resolution, because the image of f' is no longer contained in mR^{v+2}.

To show that E' is a resolution of $J'(c)$, we must show that E' is a complex and that it is exact. Since the matrices representing the maps g'^* and g' are 1 by $v+2$ and $v+2$ by 1 matrices of pfaffians of $M'(c)$, the compositions $g'^* \cdot f'$ and $f' \cdot g' = 0$, so E' is a complex. For any complex of free R-modules

$$A : 0 \longrightarrow F_n \xrightarrow{\phi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0$$

let $J(\phi_k)$ the ideal of minors of size k of the matrix representing ϕ_k, where k is the size of the largest nonvanishing minor. To show that $\mathbb{A}$ is exact, it is enough to show the following [**BE2**]:

a) rank $\phi_{k+1}+$ rank $\phi_k =$ rank F_k;

b) grade $J(\phi_k) \geq k$ or $J(\phi_k) = R$.

a). We need to show $\operatorname{rank} g' + \operatorname{rank} f' = \operatorname{rank} g'^* + \operatorname{rank} f' = v+2$. We know $\operatorname{rank} g' = \operatorname{rank} g'^* = 1$, so we need to show $\operatorname{rank} f' = v+1$. Since E is a resolution, we know $\operatorname{rank} f = v+1$, so M contains a $v+1$ by $v+1$ submatrix whose determinant is nonzero; in particular, the submatrix obtained by eliminating the last row and column has nonzero determinant, since its square root is one of the $v+2$ generators of J. This same submatrix occurs in $M'(c)$, so $f' \geq v+1$. On the other hand, $M'(c)$ is skew-symmetric and $v+2$ odd, implying that the determinant of $M'(c)$ equals zero. Therefore rank $f' = v+1$.

b). We already know height $J(g'^*) =$ height $J(g') = 3$. The ideal of $v+1$ by $v+1$ minors of $M'(c)$ contains $J(g')$, therefore $J(M'(c))$ has height ≥ 3.

iii). To show $J'(c)$ is Gorenstein we will exhibit a *minimal* resolution for $J'(c)$ of the form

$$(2.3) \qquad 0 \longrightarrow R \xrightarrow{g'^*} R^v \xrightarrow{\psi} R^v \xrightarrow{g'} R$$

where ψ is represented by a v by v alternating matrix Y_ψ whose pfaffians generate $J'(c)$.

Let W be the upper triangular $v+2$ by $v+2$ matrix

$$W = \begin{pmatrix} c\,0\,0\,0 \dots 0 & -m_{1,v+2} & m_{1,v+1} \\ 0\,c\,0\,0 \dots 0 & -m_{2,v+2} & m_{2,v+1} \\ \vdots\;\vdots\;\vdots\;\vdots \quad \vdots & \vdots & \vdots \\ 0\,0\,0\,0 \dots c & -m_{v,v+2} & m_{v,v+1} \\ 0\,0\,0\,0 \dots 0 & 1 & 0 \\ 0\,0\,0\,0 \dots 0 & 0 & 1 \end{pmatrix}.$$

The product $\frac{1}{c}(W \cdot M'(c) \cdot W^T)$ is the matrix $Y =$

$$\begin{pmatrix} 0 & cm_{1,2} - D_{12} & cm_{1,3} - D_{13} & \dots & cm_{1,n} - D_{1n} & 0 & 0 \\ -cm_{1,2} + D_{12} & 0 & cm_{2,3} - D_{23} & \dots & cm_{2,n} - D_{2n} & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ -cm_{1,n} + D_{1n} & cm_{2,n} - D_{2n} & cm_{3,n} - D_{3n} & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 \end{pmatrix}$$

where the term D_{ij} is equal to $m_{i,v+1} \cdot m_{j,v+2} - m_{i,v+2} \cdot m_{j,n+1}$. The pfaffians of this product generate $J'(c)$, and in particular the minors obtained by omitting rows and columns $v+1$ or $v+2$ are equal to zero.

We define the matrix Y_ψ representing ψ in (2.3) to be Y with rows and columns $v+1$ and $v+2$ removed. When $j \leq v$, $Pf(Y_j) = Pf((Y_\psi)_j)$, so the pfaffians of Y_ψ generate $J'(c)$. The map ψ now satisfies the condition of the structure theorem that its image is contained in mR^v. Since $J'(c)$ has height 3 it is Gorenstein, minimally generated by v elements.

iv). The complex given in (2.3) is exact, since it satisfies the criteria a) and b), so the Hilbert function of $R/J'(c)$ can be computed by (2.2); by Lemma 2.4, this will be the same as the Hilbert function of R/J. This completes the proof. □

When we work in 3 variables, the third difference sequence $(d_0, d_1, \dots, d_t, \dots)$ of T gives the net difference between the number of relations and the number of generators in each degree t. We denote by the odd number

$\mu(T)$ the minimum number of generators determined by the third difference sequence.

Corollary 2.6. *For a Hilbert function $T = (h_0, h_1, \ldots, h_n, 0, \ldots)$ of a height 3 Gorenstein algebra R/J, the integer*

$$\mu(T) = 2\left\lceil \frac{-\sum_{d_i<0} d_i}{2} \right\rceil - 1$$

can be realized as the minimum number of generators for J.

Proof. This follows from repeated application of Theorem 2.3, since if an ideal defining T has more generators than $\mu(T)$, then it has generators and relations in the same degree, and its alternating matrix has degree zero entries occuring in pairs. Each pair of entries may be deformed to nonzero constants, causing the number of generators in the ideal to drop by 2. □

Remark. When the socle degree n is odd, adding up the negative terms in the third difference sequence may not indicate a minimum number of generators. For example, $T = (1, 3, 6, 6, 3, 1)$ has for its third differences the sequence

$$1, 0, 0, -4, 0, 4, 0, 0, -1,$$

which indicates at least four generators in degree 3 for an ideal which determines T. Since $\mu(T)$ must be odd, we need at least 5 generators. T can be generated by the ideal $(x^3, y^3, xz^2, yz^2, x^2y^2 - z^4)$, and it is Gorenstein, with dual polynomial equal to $x^2y^2z + z^5$. The extra generator and relation must be in degree $\frac{n+3}{2}$ to preserve the symmetry of the sums $p_i + q_i = n + 3$. The degree matrix determined by these generator and relation degrees will have no nonzero degree 0 entries, since the only entry whose degree is 0 is on the diagonal, in which case it must be equal to zero.

Remark. The entry degrees $E(D)$ of the degree matrix determined by the degree set D need not all be positive. The Hilbert function $(1, 3, 6, 6, 6, 3, 1)$ has for its third differences the sequence

$$1, 0, 0, -4, 3, -3, 4, 0, 0, -1,$$

indicating row degrees $\{3, 3, 3, 3, 5, 5, 5\}$ and column degree $\{6, 6, 6, 6, 4, 4, 4\}$. This means the 7 by 7 degree matrix will have a 3 by 3 block of degree -1 entries, which must be zeros.

Theorem 2.3 and Corollary 2.6 give us the result we want:

Theorem 2.7. *Gor_T is irreducible.*

Proof. The family of Gorenstein algebras Gor_T with Hilbert function equal to T is equal to the closure $\overline{Gor_{D_{min}}} = \cup Gor_{D'}$ for all $D' \supset D$ by Theorem 2.3, since ideals with degree sets D' can be deformed to the minimal degree set D_{min}. Since $Gor_{D_{min}}$ is irreducible by Theorem 1.1, Gor_T is irreducible. □

3. Number of generators of height 3 Gorenstein ideals.

We have seen in Corollary 2.6 that when we fix the Hilbert function T of a Gorenstein algebra $A = R/J$ we can determine the minimum number of generators needed for J by taking the third differences of the sequence of integers in T. We proved in Theorem 2.3 in the previous chapter that if J has more than $\mu(T)$ generators, we can deform the entries of an alternating matrix whose pfaffians generate J so that J needs two fewer generators.

The degrees of the generators and relations of a height 3 Gorenstein ideal J can be described completely by the integers $\{r_i\}$ defined in Section 2.2 as the differences $p_i - q_i$ of relation and generator degrees of J when arranged in decreasing and increasing order, respectively. These integers can be used to determine the maximum number of generators possible for an ideal J defining a given Hilbert function T.

3.1. Saturated sequences of integers $\{r_i\}$. Recall that an alternating matrix M can be assigned row degrees $\{q_i\}$ and column degrees $\{p_i\}$ such that the $\{q_i\}$ are nondecreasing and the $\{p_i\}$ are nonincreasing, with integers $\{r_i\}$ defined by $r_i = p_i - q_i$. In order for M to have pfaffians that satisfy Theorem 2.1, the diagonal degrees $\{r_i\}$ of M must satisfy the following conditions:

Proposition 3.1. *Let M be an u by u alternating matrix with generic entries, u odd, whose diagonal degrees $\{r_i\}$ are arranged in nonincreasing order. A necessary and sufficient condition for M to have u nonzero pfaffians is*

the integers r_i are all even or all odd;

$$r_1 > 0; \tag{3.1}$$

$$r_i + r_{u-i+2} > 0 \text{ for } i = 2 \dots \frac{(u+1)}{2}.$$

Proof. If the condition $r_i + r_{u-i+2} > 0$ is not satisfied, M will contain zeros in all entries (i, j) with $i, j \geq \frac{u+1}{2}$. The submatrix obtained by eliminating row and column l not passing through this block of zeros will have the shape

$$M_l = \begin{pmatrix} \square & \square & A \\ \square & \square & \text{zeros} \\ A^T & \text{zeros} & \text{zeros} \end{pmatrix}$$

where □ indicates a block containing nonzero entries. A will be size $i-2$ by $i-1$, so the determinant of M_l will be zero, contradicting the conditions of the structure theorem that M have u independent pfaffians.

To show that these conditions are sufficient, we exhibit a u by u matrix whose pfaffians generate a Gorenstein ideal with u generators:

$$(3.2)\qquad \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & y^{\gamma_1} & z^{\beta_1} \\ 0 & 0 & \dots & 0 & y^{\gamma_2} & z^{\beta_2} & x^{\alpha_1} \\ 0 & 0 & \dots & y^{\gamma_3} & z^{\beta_3} & x^{\alpha_2} & 0 \\ \vdots & \vdots & & & & & \vdots \\ 0 & -y^{\gamma_2} & -z^{\beta_3} & -x^{\alpha_3} & 0 & \dots & 0 \\ -y^{\gamma_1} & -z^{\beta_2} & -x^{\alpha_2} & 0 & 0 & \dots & 0 \\ -z^{\beta_1} & -x^{\alpha_1} & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

The integers α_i, β_i and γ_i are defined as

$$\alpha_i = (r_{i+1} + r_{u+1-i})/2,\ \beta_i = (r_i + r_{u+1-i})/2,\ \gamma_i = (r_i + r_{u-i})/2.$$

If the diagonal degrees satisfy (3.1), the ideal of pfaffians of this matrix contains $x^{\sum \alpha_i}, y^{\sum \gamma_i}$ and $z^{\sum \beta_i}+$ other terms, so it clearly contains pure powers of x, y, and z and therefore has height 3. Finally, since the ideal of pfaffians satisfies the conditions of Theorem 2.1 it is Gorenstein, minimally generated by u elements. □

We have learned that J. Herzog, N.V. Trung and G. Valla have arrived independently at the conditions of Proposition 3.1 and give matrix (3.2) as an example for the sufficiency of the conditions.

We say a sequence of integers $R = \{r_i\}$ occuring as the sequence of diagonal degrees of an alternating matrix is *saturated* if it satisfies (3.1) and it is impossible to lengthen the sequence by adding a pair of integers $d, -d$ to R and still satisfy (3.1) without changing the Hilbert function determined by R. A sequence R has a unique saturation if whenever R' and R'' are two saturations of R with lengths v and w respectively, we have $v = w$ and $r'_i = r''_i$ for all i.

Theorem 3.2.

(i) *A sequence $R = \{r_i\}$, $i = 1 \dots u$ is saturated if*

$$r_1 > 0,$$

$$(3.3)\qquad r_i \geq r_{i+1} \text{ for } i = 1 \dots u-1,$$

$$\text{and } r_i + r_{u-i+2} = 2 \text{ for all } i = 2 \dots \frac{u+1}{2}.$$

(ii) *Every sequence of integers $\{r_i\}$ arising from the generator and relation degrees of a Gorenstein ideal has a unique saturation.*

Proof. Let $R = \{r_1, r_2, \ldots, r_u\}$ be given and let R' be a saturation of R. Denote by Q, P and Q', P' the ordered sequences of generator and relation degrees whose differences are the two sequences R and R'. Let n be the socle degree of all ideals defined by these sequences. We will show R' is unique.

As we saw in Chapter 2, the minimum number of generators of a Gorenstein ideal J defining T is $\mu(T)$. Any other ideal J' with a larger number of generators determining the same Hilbert function as J must have additional generators and relations occuring in the same degrees. This implies that $\sum r_i = n+3$ is constant, since adding d_1 and d_2 to the sequences $\{q_i\}$ and $\{p_i\}$ will add $d_1 - d_2$ and $d_2 - d_1$ to the sequence $\{r_i\}$, leaving $\sum r_i$ unchanged.

The sequences must satisfy $p_i + q_i = p'_i + q'_i = n+3$. Since the sequences determine the same Hilbert function, the smallest generator degree in the ideals they define must be the same; therefore $q_1 = q'_1$ and $p_1 = p'_1$, so $r_1 = r'_1$.

Proof of i). To saturate R, we begin by adding a pair $d, -d$ to R. Clearly d must be less than r_1, otherwise we would be adding a generator in degree q_1 to the ideal, which would change the Hilbert function. We reorder $R \cup \{d, -d\}$ so that the new sequence is nonincreasing, and continue adding pairs until $r'_i + r'_{v-i+2} = 2$ for $i \geq 2$ in a larger sequence $R' = \{r'_1, \ldots, r'_v\}$. R' will be of the form

$$\left(r_1, \ldots \text{ positive integers }, \{1's\}, \text{ negative integers} \right)$$

when the integers are all odd, or

$$\left(r_1, \ldots \text{ positive integers }, \{2's\}, \{0's\}, \text{ negative integers} \right)$$

when the integers are all even. Once we reach the point where $r'_i + r'_{v-i+2} = 2$ we cannot lengthen R', since the insertion of $d, -d$ into a sequence of length v forces a sum $r'_i + r'_{v+2-i+2} = 0$, where $r'_i = d$ and $r'_{v+2-i+2} = -d$.

If not all sums in R' satisfy $r'_i + r'_{v-i+2} = 2$, there must exist a sum $r'_k + r'_{v+2-k} \geq 4$. We can insert the pair $d, -d$ where $d = r'_k - 2$ to get a new sequence $R' \cup \{d, -d\}$ of the form

$$\left(r_1, \ldots, r'_k, d, \ldots, r'_{v-k+2}, -d, \ldots \right).$$

This new sequence satisfies (3.1).

Proof of ii). Now assume R has two different saturations, R' and R'', of lengths v and w respectively. Since $\sum r'_i = r'_1 + 2 \cdot \frac{v-1}{2}$ must equal $\sum r''_i = r''_1 + 2 \cdot \frac{w-1}{2}$ and $r'_1 = r''_1$, it follows that v is equal to w.

Let k be first position for which $r'_k \neq r''_k$; we may assume $r'_k < r''_k$. Therefore there is at least one more occurrence of the pair $r''_k, -r''_k$ in R''. Since R'' and R' agree in position 1 through $k-1$ and they are both saturated sequences, they must also agree in position $u-k+3$ through u. Suppose $-r''_k$ occurs in position p in R''. We must have $p > v-k+2$ to satisfy (3.1). But now p is in the range where both saturations agree, contradicting the fact that R' doesn't contain this occurrence of $-r''_k$. Therefore the two saturations must agree everywhere. □

3.2. Maximum number of generators of a Gorenstein ideal. Let $T = (h_0, h_1, \ldots, h_n, 0, \ldots)$ be given, where T is the Hilbert function of R/J. Let k be the first position in which $h_k < \binom{k+2}{2}$. We call $k = k(T)$ the *order of* T, and it is equal to the smallest degree of the generators of J for any graded ideal J which determines T. Clearly, k depends only on T.

Theorem 3.3. *Let a Hilbert function T of order $k(T)$ be given. The maximum number of generators of a Gorenstein ideal J which determines T is $2 \cdot k(T) + 1$, and it occurs if and only if the sequence of diagonal degrees determined by J is saturated.*

Proof. Let $R = \{r_1, r_2, \ldots, r_u\}$ be a saturated sequence of diagonal degrees which satisfies (3.3). Since $\sum_{i=1}^{u} r_i = n+3 = s$, and we assume $r_i + r_{u+2-i} = 2$ for $i = 2 \ldots (u+1)/2$, we know $s = r_1 + u - 1$, and from the definitions of s and $\{r_i\}$, the smallest generator degree equals $k = (s - r_1)/2 = (u-1)/2$, so $u = 2k+1$.

Conversely, if the smallest degree is k and R has length $2k+1$, we get $2k = s - r_1 = \sum_{i=2}^{2k} r_i = (r_2 + r_{2k+1}) + \cdots (r_{(2k+1+1)/2} + r_{(2k+1+3)/2})$, which must all be positive. Therefore they must all be equal to 2, so R is saturated. □

If we set all α_i equal to 1 in the matrix (3.2), we are working with a saturated sequence of $\{r_i\}$. The smallest degree generator is $x^{\sum \alpha_i} = x^k$, and $u = 2k+1$.

Corollary 3.4. *Given a permissible Hilbert function T, all values between $\mu(T)$ and $2 \cdot k(T) + 1$ can occur as the number of generators of an ideal defining T.*

Proof. This follows from Theorem 2.3 and Proposition 3.1. □

Stanley has shown [**St1**] that a sequence $T = (h_0, h_1, \ldots, h_n)$ of non-negative integers with $h_1 \leq 3$ occurs as the Hilbert function of a Gorenstein algebra if and only if it is symmetric and the first difference sequence

$(h_0, h_1 - h_0, h_2 - h_1, \dots, h_s - h_{s-1})$ is a permissible Hilbert function, where $s = \lfloor n/2 \rfloor$.

Suppose T is a permissible Hilbert function of some graded Gorenstein quotient of R. We will consider the family Gor_T of all graded Gorenstein algebra quotients $A = R/J$ having Hilbert function T.

Example 3.5. $T = (1, 3, 4, 4, \dots, 4, 4, 3, 1)$. The third differences of T are

$$1, 0, -2, 0, 1, 0, 0, \dots, 0, 0, -1, 0, 2, 0, -1.$$

The ideals having this Hilbert function have at least two generators of degree 2 and one generator of degree $n - 1$.

The third difference sequence of T determines diagonal degrees $\{n-1, n-1, 5-n\}$. The sequence is not saturated, so we can lengthen it by adding at least one pair of integers. Since $k(T) = 2$, the only pair we can add is $n-3, 3-n$, getting a new sequence $\{n-1, n-1, n-3, 5-n, 3-n\}$. This sequence is saturated, so the maximum number of generators of an ideal determining T is 5. T can be generated by three generators in degrees $2, 2$ and $n-1$, for example $J = (x^2, y^2, z^{n-1})$, or five generators in degrees $2, 2, 3, n-1$ and n, for example $J = (x^2, xy, yz^2, y^{n-1}, z^n)$.

Example 3.6. Let $T = (1, 3, 6, 7, 6, 3, 1)$. The third differences of T indicate at least 3 generators of degree 3. Since $k(T) = 3$, the maximum number of generators an ideal J determining T can have is 7. A 5-generator ideal having Hilbert function T is $J_5 = (x^3, x^2z, xy^2 - z^3, y^3z, y^5)$, corresponding to $R = \{3, 3, 3, 1, -1\}$, and a 3-generator ideal is $J_3 = (x^3, y^3, z^3)$, corresponding to $\{3, 3, 3\}$. The 7-generator ideal

$$J_7 = (x^2y, x^2z, xyz, y^3z - xz^3, xy^3 - yz^3, x^5 - z^5, y^5 - z^5)$$

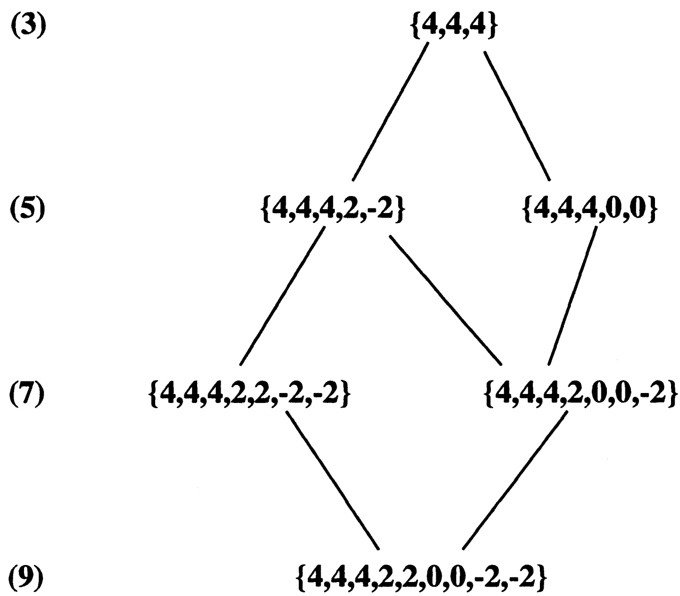
corresponding to the saturated sequence $\{3, 3, 3, 1, 1, -1, -1\}$, determines the same Hilbert function. The Hilbert function of R/J_7 was computed using *Macaulay*.

In summary, if a Hilbert function T with socle in degree n is given, and an ideal J determining T has k for its smallest generator degree, then the upper bound on the number of generators that can generate J is $2k + 1$. This upper bound can be achieved for all permissible Hilbert functions in 3 variables.

3.3. The lattice $\mathcal{L}(T)$ of T. Define $\mathcal{L}(T)$ to be a lattice associated to the Hilbert function T. Its vertices are all the sequences of integers $\{r_i\}$ satisfying (3.1) which determine T. Two vertices are connected if the sequence at one vertex is a subsequence of the vertex below it. We call the vertex corresponding to the smallest subset the *minimal* vertex.

Example 3.7. Let $T = (1,3,6,10,12,12,10,6,3,1)$. The minimum and maximum number of generators for ideals which determine T are 3 and 9, respectively.

Number of generators



The sequence at the bottom level is saturated.

We have seen that if Hilbert function T is fixed, we can exhibit ideals which achieve the minimum and maximum number of generators possible among all ideals which determine T. By Theorem 2.3, we can deform an ideal with $v+2$ generators into one with v generators without changing T as long as v is at least $\mu(T)$. We are able to deform an ideal with $v+2$ generators into one with v generators only when the matrix M contains degree zero entries, which happens only when the sequence of integers $\{r_i\}$ for M contains a pair r_i and r_j, $i \neq j$ such that $r_i + r_j = 0$. Therefore there is a one to one correspondence between degree sets $D = (\{Q_i\}, \{P_i\})$ for ideals defining T and the sequence of diagonal degrees for the corresponding alternating matrix.

The lattice structure associated to T illustrates the irreducibility of certain subfamilies of Gor_T. A vertex $\mathcal{V}$ in the lattice represents a family Gor_D, with the degree sets D specified by $\mathcal{V}$. $\mathcal{L}(T)$ is a geometric lattice, since any two vertices $\mathcal{V}_1$ and $\mathcal{V}_2$ representing Gor_{D_1} and Gor_{D_2} determine unique vertices corresponding to $D_1 \cap D_2$ and $D_1 \cup D_2$. $Gor_{D_{min}}$ corresponds to the minimal vertex of $\mathcal{L}(T)$.

Theorem 3.8. *The closure $\overline{Gor_D}$ is equal to $\bigcup Gor_{D'}$ for all $D' \supset D$ and is irreducible.*

Proof. As seen in Chapter 2, ideals with degree sets D' can be deformed to an ideal with degree set D when $D' \supset D$. This shows in the lattice representation as $\overline{\mathcal{V}}$ = sublattice descending from $\mathcal{V}$. The minimal vertex is unique as a result of the unique third difference sequence of T. Thus $Gor_T = \overline{Gor_{D_{min}}}$, where D_{min} is the degree set specified by the minimal vertex. □

3.4. Saturated sequences of $\{r_i\}$ and partitions. There is a convenient pairing between saturated sequences of $\{r_i\}$ and partitions as follows. Let the socle degree n of a Gorenstein ideal J and the order $k(T)$ of the Hilbert function it defines be given. We will look at all possible sequences $\{r_i\}$ of diagonal degrees for J.

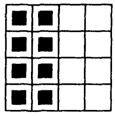
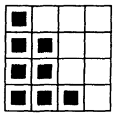
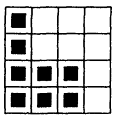
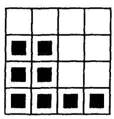
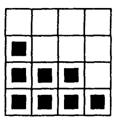
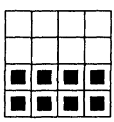
The maximum number of generators for J is $2 \cdot k(T) + 1$. Construct a partition of a rectangle of size $2k + 1$ by $s = n + 3$ by dividing the s blocks of row i into q_i and p_i blocks, such that $p_i - q_i = r_i$. Eliminate the first row, k columns from the left and $k + 1$ columns from the right to get a $2k$ by $n - 2k + 2$ rectangle. The resulting partition of this rectangle will be self-complementary, and it retains the original information needed to reconstruct the ideal and the Hilbert function it determines.

Proposition 3.9. *When the socle degree n of a height 3 Gorenstein ideal J is fixed and the order $k = k(T)$ of the Hilbert function defined by J is given, there is a one to one correspondence between Hilbert functions T and self-complementary partitions of $2k$ by $n - 2k + 2$ blocks.*

Proof. The partition constructed above for any given Hilbert function will be self-complementary, since the $n - 2k + 2$ blocks of row i are partitioned into $q_{i+1} - k$ and $p_{i+1} - k - 1$ blocks, whose difference is $r_{i+1} - 1$. If $j = 2k + 1 - i$, the blocks of row j are partitioned into two parts whose difference is $r_{j+1} - 1$. Since $r_{i+1} + r_{j+1} = 2$, we know $r_{i+1} = -r_{j+1}$, so rows i and j are complementary.

If we are given a self-complementary partition of a $2k$ by $n - 2k + 2$ rectangle, where row i is divided into a_i and b_i blocks, $a_i + b_i = n - 2k + 2$, then we can recover the saturated sequence of diagonal degrees from the partition by letting $r_1 = n + 3 - 2k$ and $r_{i+1} = b_i - a_i + 1$ for $i = 1 \dots 2k$. Since a partition defines a unique sequence of diagonal degrees, it defines a unique Hilbert function. □

Example 3.10. Let $n = 6$ and $k(T) = 2$. There are 6 self-complementary partitions of a 4 by 4 rectangle, corresponding to the 6 different Hilbert functions with order 2 and socle degree 6:

T	*partition*	$\{r_i\}$	*generator degrees*	*colength*
1 3 5 7 5 3 1		5 1 1 1 1	2 4 4 4 4	25
1 3 5 6 5 3 1		5 3 1 1 -1	2 3 4 4 5	24
1 3 5 5 5 3 1		5 3 3 -1 -1	2 3 3 5 5	23
1 3 4 5 4 3 1		5 5 1 1 -3	2 2 4 4 6	21
1 3 4 4 4 3 1		5 5 3 -1 -3	2 2 3 5 6	20
1 3 3 3 3 3 1		5 5 5 -3 -3	2 2 2 6 6	17

Obviously T is no longer fixed; the constants are now the socle degree and the order k. By counting the number of self-complementary partitions of a given size we are counting the number of Hilbert functions with the given socle degree and order. Since the partition is self-complementary, it is determined by the partition of a subrectangle with k rows and $\lfloor n/2 \rfloor - k + 1$ columns into nondecreasing rows. There are $\binom{\lfloor n/2 \rfloor + 1}{k}$ such partitions.

Therefore the number of permissible Hilbert functions of a given socle degree equals

$$\sum_{k=0}^{\lfloor n/2\rfloor+1} \binom{\lfloor n/2\rfloor+1}{k} = 2^{\lfloor n/2\rfloor+1}.$$

See [**St2**] for a more general discussion of generating functions for plane partitions with varying degrees of symmetry.

4. Dimension of Gor_T.

We define a different parametrization in Section 4.2 for Gor_T than that used in earlier chapters. With this parametrization the closure $\overline{Gor_T}$ includes Gorenstein algebras with different Hilbert functions.

4.1. Matlis Duality and the dual polynomial f. Let k be an algebraically closed field of characteristic zero, $R = k[x,y,z]$ with maximal ideal $m = (x,y,z)$. Emsalem [**Em**] states that the dual $\mathcal{A} = \mathrm{Hom}_R(A,k)$ of a Gorenstein algebra $A = R/J$ with socle in degree n can be obtained by the procedure of taking the vector space generated by a homogeneous degree n polynomial f and its partial derivatives of all orders. We use the divided powers of the derivatives of f and write $f = \sum_{\alpha=1}^{\binom{n+2}{2}} b_\alpha x^\alpha$, $b_\alpha \in k$, where the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ satisfies $|\alpha| = n$.

An isomorphism exists between R-closed subspaces

$$\mathcal{J} \subset \mathcal{R} \qquad \text{and} \qquad \mathrm{Hom}_R(R/J, E),$$

where J is an ideal of R and E is the injective envelope of R/m. This isomorphism is shown in the theorem proved by Matlis [**Ma**] and discussed by Miri [**Mi**]. We assume R is a commutative, Noetherian, complete local ring, with $E = \mathcal{R}$. From the exact sequence

$$0 \longrightarrow J \longrightarrow R \longrightarrow R/J \longrightarrow 0$$

we derive the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(R/J,E) & \longrightarrow & \mathrm{Hom}_R(R,E) & \longrightarrow & \mathrm{Hom}_R(J,E) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\cong} & & & & \\ 0 & \longrightarrow & \mathcal{J} & \longrightarrow & E & \longrightarrow & E/\mathcal{J} & \longrightarrow & 0. \end{array}$$

Since E is injective, the top row is exact. The bottom row is clearly exact, and the vertical map ϕ is an isomorphism defined as follows [**Ma**, p. 526]:

Let R/J be generated by the element e, so that $g \in J$ if and only if $ge = 0$. Let $h \in \mathrm{Hom}_R(R/J, E)$ and define $\phi : \mathrm{Hom}_R(R/J,E) \to \mathcal{J}$ by $\phi(h) = (h \cdot e)$. Clearly ϕ is a well-defined R-homomorphism. If $\phi(h) = 0$, then $he = 0$, so $h = 0$. This shows ϕ is one-to-one. If we let $x \in \mathcal{J}$, define $h : R/J \to E$ by $he = x$. Then $\phi(h) = x$, so ϕ is onto, and therefore $\mathrm{Hom}_R(R/J, E) \cong \mathcal{J}$. Since the first two vertical maps are isomorphisms, so is the third, so $\mathrm{Hom}_R(J, E) \cong E/\mathcal{J}$.

4.2. Catalecticant matrices associated to f. We parametrize Gor_T by using the coefficients of the dual polynomial f described by Emsalem up to nonzero constant multiple. To specify a Hilbert function $(h_0, h_1, \dots, h_n)$, we require that a degree n polynomial f have h_d linearly independent partial derivatives of order d. The permissible Hilbert functions are those for which such a polynomial exists. This is the intersection of an open and closed condition on the $\binom{n+2}{2}$ coefficients of f.

When f is a homogeneous polynomial in 3 variables, the rth partial derivatives of f form the row space of a $\binom{r+2}{2}$ by $\binom{n-r+2}{2}$ matrix. We denote this matrix $M_{r,n-r}(f)$, called the rth *Catalecticant* matrix associated to f. When $n = 2d$ is even, the Catalecticant $M_{d,d}(f)$ is square and symmetric.

Let $\mathcal{A} = k[\{b_\alpha\}][x,y,z]/J$, where J is the annihilator of f in the Matlis duality. Let $I_t(M_{d,d}(f))$ be the ideal in $k[\{b_\alpha\}] = k[B]$ of all determinantal minors of size $t+1$ of $M_{d,d}(f)$. When $t = \binom{d+2}{2}$ we set $I_t(M_{d,d}(f)) = (0)$, the zero ideal. For each t from 0 to $\binom{d+2}{2}$ we get a different Hilbert function T, equal to $(1,3,6,\dots,t,t,\dots,t,\dots,6,3,1)$, the largest possible given t. The codimension in $k[B]$ of $I_t(M_{d,d}(f))$ will be the dimension of $k[B]/I_t(M_{d,d}(f))$.

4.3. Codimension of $I_t(M_{2,2}(f))$.

Example 4.1. The Hilbert functions of $k[B]/I_t(M_{2,2}(f))$ were computed for values of t from 0 to 6 using the commutative algebra computer program *Macaulay.* In the case $n = 4$ the results are summarized below, where H denotes the Hilbert function of a minimal reduction of $k[B]/I_t(M_{2,2}(f))$.

t	*codimension*	*dimension*	H	*degree*
0	15	0	1	1
1	12	3	1 12 3	16
2	9	6	1 9 45 17 3	75
3	6	9	1 6 21 56 21 6 1	112
4	3	12	1 3 6 10 15	35
5	1	14	1 1 1 1 1 1	6
6	0	15	0	1

This information was computed using the commutative algebra computer program Macaulay.

All values of t can occur in the 3-variable case [**BE1**], and in the general case for any number of variables [**I1**].

The pattern of codimensions of $I_t(M_{2,2}(f))$ at first exhibits behavior similar to that of a generic symmetric matrix: the codimensions follow the pattern $1, 3, 6$, for corank $1, 2, 3$; after that corank they jump by 3's. This suggests examining other even values of n to see if this pattern is sustained. When $n = 3$, $M_{3,3}(f)$ will be size 10 by 10. If the same pattern evolves, we expect to see the values in the following table.

Table 4.2

t	*codimension* of $I_t(M_{3,3}(f))$	*dimension* of $I_t(M_{3,3}(f))$
1	25	3
2	22	6
3	(19)	$\geq$9
4	(16)	$\geq$12
5	(13)	$\geq$15
6	(10)	$\geq$18
7	6	22
8	3	25
9	1	27
10	0	28

Numbers in () have not been verified. Dimension and codimension for $t = 7, 8$ and 9 follow from Theorem 4.4. Dimension for $t = 1$ and 10 have been shown independently. The dimension for $t = 2$ has been verified by Macaulay. The lower bounds of dimension of $I_t(M_{3,3}(f))$ for $3 \leq t \leq 6$ follow from Lemmas 4.8 and 4.9 below.

4.4. Codimension of $I_t(M_{d,d}(f))$. Let $f = \sum b_\alpha x^\alpha$, $|\alpha| = 2d = n$. The dimension of $k[B]$ equals $\binom{n+2}{2}$. The size of the matrix $M_{d,d}(f)$ is $\binom{d+2}{2}$ by $\binom{d+2}{2}$. Let t equal the rank of $M_{d,d}(f)$, determined by the vanishing of the minors of size $t+1$ of $M_{d,d}(f)$, and let a equal $\binom{d+2}{2} - t$, the corank of $M_{d,d}(f)$.

Conjecture 4.3. *The codimension of $I_t(M_{d,d}(f))$ in $k[B]$ is $\binom{a+1}{2}$ if $t \geq \binom{d+1}{2}$, or $\binom{d+2}{2} - 3t$ if $t \leq \binom{d+1}{2}$.*

Remark. This pattern is numerically consistent. Suppose the sequence of codimensions of $I_t(M_{d,d}(f))$ is $0, 1, 3, 6, 10, \ldots$ for $a = 0, 1, 2, \ldots$, and persists to the jth term. We find that the equation $\binom{n+2}{2} = \binom{j+1}{2} + 3\left(\binom{d+2}{2} - j\right)$ has solution $j = d + 1$ for each d.

If $X = \{x_{ij}\}$ is a symmetric $\binom{d+2}{2}$ by $\binom{d+2}{2}$ matrix of indeterminants in $S = k[x_{ij}]$, then $I_t(X)$ will have codimension $\binom{d+2}{2} - \binom{t+1}{2}$ in S. The resolution structure for ideals $I_t(X)$ is given in [**JPW**]. We obtain $I_t(M_{d,d}(f))$ by a change of rings, defining $\phi : S \to R$ by $\phi(x_{ij}) =$ the (i, j) entry of $M_{d,d}(f)$. Since the ideals $I_t(X)$ are generically perfect [**EN**], if $\mathbb{F}$ is a resolution of

$I_t(X)$, then $\mathbb{F} \otimes_S R$ will be a resolution of $I_t(M_{d,d}(f))$. It follows from [**EN**] that the codimension of $I_t(M_{d,d}(f))$ in $k[B]$ is less than or equal to the codimension of $I_t(X)$ in S. Conjecture 4.3 says that this is an equality for $\binom{d+1}{2} \leq t \leq \binom{d+2}{2}$, or equivalently, for $0 \leq a \leq d+1$.

To verify Conjecture 4.3 for large values of t, we will use Theorem 2.1 to determine an upper bound for the dimension of the family of algebras having a given Hilbert function. In part 2 of the proof of the theorem, Buchsbaum and Eisenbud show that if a Gorenstein ideal I has a resolution

$$\mathbb{F} : 0 \longrightarrow F_3 \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} R,$$

then f_2 may be chosen to be an alternating map for any appropriate map f_1, so that all sets of generators for I occur as pfaffians of an alternating matrix.

In the discussion that follows we add 1 to the computation of the projective dimension of $\mathbb{A}^{h(E_M)}$ in order to compare it with the affine count for the dimension of Gor_D.

Let f be a degree $n = 2d$ homogeneous polynomial in x, y and z. Let $I_t(M_{d,d}(f))$be the ideal of size $t+1$ minors of $M_{d,d}(f)$, imposing the condition that the rank of $M_{d,d}(f)$ is less than or equal to $t = \binom{d+2}{2} - a$. If we restrict a so that $a \leq d+1$, then we determine the Hilbert function

$$T = 1, 3, 6, 10, \dots, t, \dots, 10, 6, 3, 1, \tag{4.1}$$

where all the matrices $M_{r,n-r}(f)$ have maximal rank except $M_{d,d}(f)$. The third difference sequence of T is

$$1, 0, 0, \dots, 0, -a, 3a-3-n, -3a+3+n, a, 0, \dots, 0, 0, -1.$$

Theorem 4.4. *If* $3a \leq n+3$, *then there is an irreducible component* $Gor(T)$ *of* $I_t(M_{d,d}(f))$ *with codimension equal to* $\binom{n+2}{2} - \binom{a+1}{2}$.

Proof. We can break up the proof into 2 cases:
Case 1. $3a = n+3$. The sequence of third differences is

$$1, 0, 0, \dots, 0, -a, 0, 0, -a, 0, \dots, 0, 0, -1.$$

We get a generators in degree d and a relations in degree $d+3$ in the ideal J with the smallest possible number of generators defining a Gorenstein algebra A with Hilbert function T. All entries in the a by a alternating matrix M will have degree 3, meaning a choice of 10 coefficients possible for each, since we are choosing generic entries, so $h(E_M) = 10\binom{a}{2}$. For J, there are a generators to choose, each of which is a linear combination of a fixed set of a forms in J of degree d, giving dimension a^2. We subtract this from the number of parameters we found for M, since each of these ideals determines the same algebra. Therefore, the maximum dimension of the family equals

$$10\binom{a}{2} + 1 - a^2 = 4a^2 - 5a + 1.$$

Since the codimension of the family cannot be larger than that for a generic symmetric matrix, this dimension is also a minimum, so it is equal to the dimension given in the conjecture.

Case 2. $3a < n + 3$. This gives a generators in degree d and $-3a + 3 + n$ generators in degree $d+1$, $-3a+3+n$ relations in degree $d+2$ and a relations in degree $d + 3$. The alternating matrix M will have this shape:

$$\begin{pmatrix} 0 & \text{Degree 3 entries} & \text{Degree 2 entries} \\ \text{Degree 3 entries} & 0 & \text{Degree 1 entries} \\ \text{Degree 2 entries} & \text{Degree 1 entries} & 0 \end{pmatrix}.$$

The maximum affine dimension of the family equals

$$\begin{aligned} 10\binom{a}{2} &+ 6a(-3a+3+n) + 3\binom{-3a+3+n}{2} + 1 \\ &- (a^2 + (2n+3)(-3a+3+n)) \\ &= \left(\frac{a^2}{2} - 6da + 6d^2 - 19\frac{a}{2} + 15d + 10\right) \\ &\quad - (a^2 - 6ad + 12d + 4d^2 - 9a + 9) \\ &= -\frac{a^2}{2} + 2d^2 - \frac{a}{2} + 3d + 1, \end{aligned}$$

and this is equal to $\binom{n+2}{2} - \binom{a+1}{2}$ as claimed. □

Remark. This $Gor(T)$ is not in the closure of another, larger $Gor(T')$, because T is maximal given t, and by Theorem 3.8.

Remark. When $3a > n + 3$, the third differences of (4.1) show that an ideal with this Hilbert function needs a minimum of a generators in degree d and $3a - 3 - n$ generators in degree $3, 1$, and -1 in the following shape:

$$\begin{pmatrix} 0 & \text{Degree 3 entries} & \text{Degree 1 entries} \\ \text{Degree 3 entries} & 0 & \text{Degree -1 entries} \\ \text{Degree 1 entries} & \text{Degree -1 entries} & 0 \end{pmatrix}.$$

If we assume the degree -1 entries are all zeros, we get

$$10\binom{a}{2} + 3a(3a-3-n) - (a^2 + (4d+6)(3a-3-n))$$
$$= 13a^2 - 32a + 4da + 24d + 8d^2 + 19$$

for the maximum dimension of Gor_T and

$$2d^2 + 3d + 1 - \frac{a^2}{2} - \frac{a}{2}$$

for the conjectured dimension. The difference between these is

$$3\binom{3a-3-n}{2},$$

which is three times the number of -1's in the degree matrix. This suggests that the fiber π^{-1} over an ideal is larger than we have accounted for. It remains an interesting problem to justify subtracting $3\binom{3a-3-n}{2}$ from our count of the dimension of U_{E_M} by explaining this difference.

Since we have shown the result for $a \leq 2d/3 + 1$, and the only values of a that are possible are $0 \leq a \leq d+1$, we have proven the conjecture true for roughly two thirds of the range where we expect codimension $I_t(M_{d,d}(f))$ to be the same as codimension $I_t(X)$.

Example 4.5. Let us look at the Hilbert function $T = (1,3,4,3,1)$. We get the sequence of third differences $1, 0, -2, -1, 1, 2, 0, -1$, indicating a smallest possible minimal resolution of 2 generators in degree 2, one generator in degree 3, one relation in degree 4, and two relations in degree 5. This translates to the following alternating matrix pattern:

$$\begin{pmatrix} - & 3 & 2 \\ 3 & - & 2 \\ 2 & 2 & - \end{pmatrix}.$$

When we count dimensions for the entries we get $1\cdot 10 + 2\cdot 6 + 1 = 22 + 1 = 23$. There are $2\cdot 2 + 1\cdot 7 = 11$ choice of generators for the ideal. Therefore the maximum dimension of the family is $23 - 11 = 12$. The conjecture predicts 12 as well, and this information is shown in Example 4.1.

One such matrix M is

$$\begin{pmatrix} 0 & z^3 & y^2 \\ -z^3 & 0 & x^2 \\ -y^2 & -x^2 & 0 \end{pmatrix}.$$

It can be easily checked that the ideal of pfaffians (z^3, y^2, x^2) determines the Hilbert function $(1,3,4,3,1)$.

Remark. We can also use this same method of counting to find the dimension of a variety in some cases when the Hilbert function of an algebra is determined by a rank condition on one of the nonsquare Catalecticant matrices. The result is that when a corank 1 condition is imposed on $M_{r,n-r}(f)$, the codimension of $I_t(M_{r,n-r}(f))$ for $t = \binom{r+2}{2} - 1$ is $\frac{1}{2}(n+3)(n-2r)+2$, the same as the codimension of $I_t(G)$ when G is a generic $\binom{r+2}{2}$ by $\binom{n-r+2}{2}$ matrix. This is the case no matter how "nonsquare" $M_{r,n-r}(f)$ is.

4.5. A lower bound for the dimension of Gor_T. We can determine a lower bound on the dimension of $M_{d,d}(f)$ by looking at sums of powers of linear forms. Let $l_1, \ldots, l_s$ be linear forms in the variables x, y and z. We look at the tangent space of the image of the map $P : (k^3)^s$ to k^N

$$P : l_1, \ldots, l_s \mapsto l_1^n + \cdots + l_s^n,$$

which we denote $\mathcal{T}_n(P)$.

Example 4.6. Take $l_1 = ax + by + cz$ and $l_2 = dx + ey + fz$, $n = 4$.

$$\begin{aligned} l_1^4 + l_2^4 = {} & (ax)^4 + 4(ax)^3by + \cdots + (cz)^4 \\ & + (dx)^4 + 4(dx)^3ey + \cdots + (fz)^4, \end{aligned}$$

so the points (a,b,c) and (d,e,f) get mapped to

$$(a^4, 4a^3b, 4a^3c, \ldots, c^4) + (d^4, 4d^3e, 4d^3f, \ldots, f^4),$$

a 15-dimensional space. If we let (a', b', c') and (d', e', f') be tangent vectors at the points (a,b,c) and (d,e,f), then $\mathcal{T}_4(P)$ is

$$\begin{aligned} & ((a+a')^4, 4(a+a')^3(b+b'), \ldots, (c+c')^4) \\ & \quad - (a^4, 4a^3b, \ldots, c^4) \\ & + ((d+d')^4, 4(d+d')^3(e+e'), \ldots, (f+f')^4) \\ & \quad - (d^4, 4a^3e, \ldots, d^4). \end{aligned}$$

If we choose $a', \ldots, f'$ small, then the quadratic terms and those of higher degree are approximately zero, so we only need to look at the linear terms in $a', \ldots, f'$.

The dimension of $\mathcal{T}_4(P)$ will be equal to the rank of the following 15 by 6 matrix:

$$\begin{pmatrix} 4a^3 & 0 & 0 & 4d^3 & 0 & 0 \\ 12a^2b & 4a^3 & 0 & 12d^2e & 4d^3 & 0 \\ 12a^2c & 0 & 4a^3 & 12d^2f & 0 & 4d^3 \\ 12ab^2 & 12a^2b & 0 & 12de^2 & 12d^2e & 0 \\ 24abc & 12a^2c & 12a^2b & 24def & 12d^2f & 12d^2e \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 4c^3 & 0 & 0 & 4f^3 \end{pmatrix}.$$

This matrix has rank 6, computed by *Macaulay*.

When we add more linear forms we add columns to this matrix. Let $l_3 = gx + hy + zi$, $l_4 = jx + ky + lz$, $l_5 = mx + ny + oz$. The dimension of $\mathcal{T}_4(P)$ is equal to the rank of the 15 by 15 matrix

$$\begin{pmatrix} 4a^3 & 0 & 0 & \dots & 4m^3 & 0 & 0 \\ 12a^2b & 4a^3 & 0 & \dots & 12m^2n & 4m^3 & 0 \\ 12a^2c & 0 & 4a^3 & \dots & 12m^2o & 0 & 4m^3 \\ 12ab^2 & 12a^2b & 0 & \dots & 12mn^2 & 12m^2n & 0 \\ 24abc & 12a^2c & 12a^2b & \dots & 24mno & 12m^2o & 12m^2n \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 4c^3 & \dots & 0 & 0 & 4o^3 \end{pmatrix}$$

which has rank 14, not 15 as we might expect. This information is contained in the table in Example 4.1, and is a classical result by Sylvester [**El**, pp. 293-295].

We would like to be able to say that when $n > 4$ we can find r linear forms so that the dimension of $\mathcal{T}_n(P)$ is $3r$, providing $3r \leq \binom{n+2}{2}$. Since the dimension of the tangent space is given by the condition that a certain matrix has maximal rank, which is an open condition, we will be assured of being able to find r linear forms whenever the matrix has maximal rank.

Conjecture 4.7. *When $n > 4$, there exist $s = \left\lfloor \frac{\binom{n+2}{2}}{3} \right\rfloor$ linear forms $l_1, \dots, l_s$ in x, y, z which the map P injective (that is, for which the tangent map $T_n(P)$ has rank $3s$).*

Lemma 4.8. *If $l_1, \dots, l_s \mapsto l_1^n + \cdots + l_s^n$ is injective, then $l_1, \dots, l_s \mapsto l_1^m + \cdots + l_s^m$ is injective for $m \geq n$.*

Proof. Assume $l_1, \dots, l_s \mapsto l_1^m + \cdots + l_s^m$ is not injective; then there exist coefficients $c_1, \dots, c_s$ in R not all equal to zero such that $\sum c_i l_i^{m-1} = 0$. If we differentiate this sum $m - n$ times, we will get a nontrivial linear relation $\sum c_i' l_i^{n-1} = 0$. But this gives a nontrivial linear relation among the l_i^n, which means the map $l_1, \dots, l_s \mapsto l_1^n + \cdots + l_s^n$ is not injective, a contradiction. □

Lemma 4.9. *Conjecture* 4.7 *is true for $n = 5$.*

Proof. Choose linear forms $l_1 = x$, $l_2 = y$, $l_3 = z$, $l_4 = ax+y$, $l_5 = dx+z$, $l_6 = y + cz$, and $l_7 = x + y + z$, and look at the tangent space of the image of these forms under the map P at the tangent vectors

$$(a_1, b_1, c_1),\ (a_2, b_2, c_2), \dots, (a_7, b_7, c_7).$$

The dimension of $\mathcal{T}_5(P)$ is the rank of this 21 by 21 matrix:

$$\begin{pmatrix}
5 & 0 & 0 & \dots & 5d^4 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\
0 & 5 & 0 & \dots & 0 & 5d^4 & 0 & 0 & 0 & 0 & 20 & 5 & 0 \\
0 & 0 & 5 & \dots & 0 & 0 & 5d^4 & 0 & 0 & 0 & 20 & 0 & 5 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 20c^3 & 30c^2 & 0 & 20 & 30 \\
0 & 0 & 0 & \dots & 0 & 20 & 0 & 0 & 5c^4 & 20c^3 & 0 & 5 & 20 \\
0 & 0 & 0 & \dots & 0 & 0 & 5 & 0 & 0 & 5c^4 & 0 & 0 & 5
\end{pmatrix}$$

The determinant was computed to be nonzero using *Macaulay*. □

The following table shows the degree n, value of $\left\lfloor \frac{\binom{n+2}{2}}{3} \right\rfloor$, and number s of forms such that the dimension of $\mathcal{T}_n(P) = 3s$. Unverified values are in parentheses.

Table 4.10.

n	$\binom{n+2}{2}$	$s = \left\lfloor \frac{\binom{n+2}{2}}{3} \right\rfloor$
1	3	1
2	6	1
3	10	3
4	15	4
5	21	7
6	28	(9)
7	36	(12)
8	45	(15)
9	55	(18)
$\vdots$	$\vdots$	$\vdots$

Verifying Conjecture 4.7 for $n > 5$ requires finding s linear forms so that the dimension of $\mathcal{T}_n(P)$ is $3s$. Once we have s linear forms where the dimension of $T_n(P)$ is $3s$, by Lemma 4.8 those forms will still "spread out" to fill up dimension $3s$ when we increase n. Since computing determinants of large matrices is cumbersome, it would be nice to be able to choose the s simplest linear forms and show that they give dimension $\mathcal{T}_n(P) = 3s$, but this is not always possible. For the proof of Lemma 4.9 we could not have chosen $l_7 = ex + fz$ instead of $l_7 = x + y + z$. The linear forms x, y and z will have nonzero coefficients on the 9 monomials $x^n, x^{n-1}y, x^{n-1}z, xy^{n-1}, y^n, y^{n-1}z, xz^{n-1}$, yz^{n-1}, and z^n. The image of $ax+y$ will be nonzero on monomials with power of z at most one. The image of

$dx+z$ will have nonzero coefficients on $2n+1$ monomials whose power of y is at most one; the same for $ex+fz$. However, when $n=5$ this means the 4 linear forms $x, z, ax+y$, and $dx+z$ are nonzero on the 11 monomials whose power of y is at most one; therefore the dimension of $\mathcal{T}_5(P) \leq 20$ for this choice of 7 forms.

4.6. Dimension of a family of complete intersections. A height 3 Gorenstein ideal I defines a complete intersection when it can be generated by 3 elements. For example, the Hilbert function

$$(1,3,6,7,6,3,1)$$

can be determined by 3 generators in degree 3.

If an ideal I defines a complete intersection with the Hilbert function of R/I equal to $T=(h_0,h_1,\ldots,h_t,\ldots)$, the dimension of $Gor_T=\sum e_i h_i$, where e_i is the number of generators in degree i in a minimal generating set for I [**I2**]. We can also give the projective dimension of a complete intersection ideal strictly in terms of the generator degrees by using the pfaffian method of Section 4.4. There are several cases to consider when all generator degrees are less than all relation degrees:

Case 1. $q_1 \neq q_2 \neq q_3$. In degree q_1 the size of the remaining space is 1, in degree q_2 it is $1+\binom{q_2-q_1+2}{2}$, and in degree q_3 it is $1+\binom{q_3-q_1+2}{2}+\binom{q_3-q_2+2}{2}$. The sum equals the number of ways of choosing the generators for J. By subtracting the sum from the number of choices for M, we find the dimension of the complete intersection to be

$$\frac{9q_1}{2}+\frac{3q_2}{2}-\frac{3q_3}{2}+q_1q_2+q_1q_3+q_2q_3-\frac{q_1^2}{2}-\frac{q_2^2}{2}-\frac{q_3^2}{2}-3.$$

Case 2. $q_1=q_2\neq q_3$. The dimension of the complete intersection will be

$$6q_1-\frac{3q_3}{2}+2q_1q_3-\frac{q_3^2}{2}-4.$$

Case 3. $q_1\neq q_2=q_3$. The dimension is

$$\frac{9q_1}{2}+2q_1q_2-\frac{q_1^2}{2}-4.$$

Case 4. $q_1=q_2=q_3$. The size of the remaining space is $\binom{q_1+2}{2}-3$, so the dimension of the complete intersection is

$$\frac{q_1^2+3q_1+2}{2}-9.$$

Note that the projective dimension of the complete intersection above is computed to be 21 using this formula, which agrees with the affine dimension 22 shown in Table 4.2.

When a relation occurs between the two generators of degrees q_1 and q_2 before the generator in degree q_3, the size of the remaining space in degree q_3 is independent of q_3, so it doesn't appear in the formula for the dimension.

Case 5. $q_1 \neq q_2, p_3 < q_3$. The dimension is

$$3q_1 + 2q_1q_2 - 2.$$

Case 6. $q_1 = q_2, p_3 < q_3$. We get

$$2q_1^2 + 3q_1 - 3.$$

4.7. Dimensions of $Gor_{T(1,k,n)}, Gor_{T(2,k,n)}, Gor_{T(3,k,n)}$.

Let $T(2,k,n)$ denote the symmetric Hilbert function with socle in degree n which follows this pattern:

$$1,3,6,\ldots,\binom{k+1}{2},\binom{k+2}{2}-2,\binom{k+3}{2}-5,\ldots,$$
$$\binom{k+3}{2}-5,\binom{k+2}{2}-2,\ldots,1.$$

An ideal determining this Hilbert function has 2 generators in degree k with a relation in degree $k+1$, and no further generators until degree $\lceil \frac{n+1}{2} \rceil$. We assume $k < \lfloor \frac{n}{2} \rfloor$.

Proposition 4.11. *The projective dimension of*

$$Gor_{T(2,k,n)} = \binom{n+2}{2} - \binom{n-k+3}{2} + \frac{k^2+k+2}{2} = k(n+3) - (n+1).$$

Proof. Denote by g_1 and g_2 the two generators of J that occur in degree k. Since they have a linear relation in degree $k+1$, they must share a common degree $k-1$ factor; denote this by g. Then we can express the generators as $g_1 = g \cdot l_1$ and $g_2 = g \cdot l_2$, where l_1 and l_2 span a 2-dimensional subspace V of the vector space with basis $\langle x, y, z \rangle$. The number of parameters for g_1 and g_2 is counted by first choosing g in dim $Grass\left(1, \binom{k+1}{2}\right)$ ways, then choosing V in dim $Grass(1,3) = 2$ ways, giving a total of $\frac{k^2+k+2}{2}$ parameters for the generators.

If we let J stand for the ideal generated by g_1 and g_2, then the dimension of J_n as a vector space equals $\binom{n-k+3}{2} - 1$, so that the dual form in degree n

must be chosen from the $\binom{n+2}{2} - \binom{n-k+3}{2} + 1$ forms in $\mathbb{P}(J_n)^{\perp}$. Therefore the number of parameters for a dual form f up to nonzero multiple equals

$$\binom{n+2}{2} - \binom{n-k+3}{2}.$$

The projective dimension of the variety equals $\binom{n+2}{2} - \binom{n-k+3}{2} + \frac{k^2+k+2}{2} = k(n+3) - (n+1)$. □

Example 4.12. Consider the variety $Gor_{T(2,3,8)}$ of Gorenstein algebras having Hilbert function

$$T(2,3,8) = (1,3,6,8,10,8,6,3,1).$$

This can be determined by requiring $M_{3,5}(f)$ to have rank 8 and making no additional conditions on $M_{4,4}(f)$, allowing it to have the largest rank possible. The projective dimension of $I_9(M_{3,5}(f))$, the ideal of 9 by 9 minors of $M_{3,5}(f)$, is therefore $3 \cdot (8+3) - (8+1) = 24$.

In the same way we define $Gor_{T(3,k,n)}$ to be the variety of Gorenstein algebras where ideals determining $Gor_{T(3,k,n)}$ have 3 generators in degree k, 2 relations in degree $k+1$ and no further generators until degree $\lceil \frac{n+1}{2} \rceil$. We define $Gor_{T(1,k,n)}$ to be the variety of algebras whose ideals have one generator in degree k and no further generators until degree $\lceil \frac{n+1}{2} \rceil$.

Proposition 4.13. *The projective dimension of* $Gor_{T(3,k,n)}$ *is* $k(n+3) - (2n-2)$. *The projective dimension of* $Gor_{T(1,k,n)}$ *is* $k(n+3) - 1$.

Proof. We get these formulas by following the same arguments as in Proposition 4.11. □

The dimension of $Gor_{T(1,k,n)}$ is the same as the dimension of the generic and Catalecticant matrices of size $\binom{k+2}{2}$ by $\binom{n-k+2}{2}$ with corank 1.

The following is a table of dimensions for Hilbert functions with socle in degree 6 and order 2.

Table 4.14

T	*projective dimension*	*comments*
(1, 3, 5, 7, 5, 3, 1)	17	corank 1 condition on 6 by 15 catalecticant matrix
(1, 3, 5, 6, 5, 3, 1)	16	complete intersection
(1, 3, 5, 5, 5, 3, 1)	14	sum of 5 powers of linear forms
(1, 3, 4, 5, 4, 3, 1)	11	T(2, 2, 6)
(1, 3, 4, 4, 4, 3, 1)	11	sum of 4 powers of linear forms; complete intersection
(1, 3, 3, 3, 3, 3, 1)	8	sum of 3 powers of linear forms; T(3, 2, 6)

The above dimensions are for $Gor(T)$, not necessarily for the subset prametrizing ideals needing 5 generators. For example, the dimension of the 5-generator subsets $(1,3,4,4,4,3,1)$ and $(1,3,5,6,5,3,1)$ will be less than the dimension of $Gor(T)$. When fixing the rank of Catalecticant matrices does not uniquely specify T, then the determinantal variety defined by these ranks may be reducible.

Example 4.15. Let $n = 8$ and consider the determinantal variety V associated to a dual form f such that the rank of $M_{3,5}(f)$ equals 7. The valid Hilbert functions satisfying rank $M_{3,5}(f) = 7$ with socle in degree 8 are

$$T_1 = (1,3,6,7,8,7,6,3,1)$$
$$T_2 = (1,3,6,7,7,7,6,3,1)$$
$$T_3 = (1,3,5,7,7,7,5,3,1)$$
$$T_4 = (1,3,5,7,8,7,5,3,1)$$
$$T_5 = (1,3,5,7,9,7,5,3,1)$$

$V = \cup Gor_{T_i}$ and each Gor_{T_i} is an irreducible subvariety of V. We cannot specialize Gor_{T_1} to Gor_{T_2}, since the projective dimension of Gor_{T_2} is 20, while the dimension of Gor_{T_1} is 19. We also cannot specialize Gor_{T_2} to Gor_{T_1}, since Gor_{T_2} requires generators in degree 6 but Gor_{T_1} does not. Therefore $\overline{Gor_{T_1}} \not\subset \overline{GorT_2}$ and $\overline{Gor_{T_2}} \not\subset \overline{GorT_1}$, so V is not irreducible.

Remark. If we let n vary and parametrize V by a dual polynomial f where the rank of $M_{2,n-2}(f) = 3$, we do get an irreducible variety, since we have fixed the Hilbert function to be $(1, 3, 3, \ldots, 3, 3, 1)$. In general, whenever the parametrization of f is a rank condition on catalecticant which fixes all the other ranks, it fixes T. Since $I_t(M_{d,d}(f))$ fixes a Hilbert function, it is an irreducible ideal.

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ON THE COHOMOLOGY OF THE LIE ALGEBRA L_2

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We compute the 0-, 1-, and 2-dimensional homology of the vector field Lie algebra L_2 with coefficients in the modules $\mathcal{F}_{\lambda,\mu}$, and conjecture that the higher dimensional homology for any λ and μ is zero. We completely compute the 0- and 1-dimensional homology with coefficients in the more complicated modules $F_{\lambda,\mu}$. We also give a conjecture on this homology in any dimension for generic λ and μ.

Introduction.

Let us consider the infinite dimensional Lie algebra W_1^{pol} of polynomial vector fields $f(x)d/dx$ on $\mathbb{C}$. It is a dense subalgebra of W_1, the Lie algebra of formal vector fields on $\mathbb{C}$. We will compute the homology of the polynomial Lie algebra, and will use the notation $W_1^{\text{pol}} = W_1$. The Lie algebra W_1 has an additive algebraic basis consisting of the vector fields $e_k = x^{k+1}d/dx$, $k \geq -1$, in which the bracket is described by

$$[e_k, e_l] = (l-k)e_{k+l}.$$

Consider the subalgebras L_k, $k \geq 0$ of W_1, consisting of the fields such that they and their first k derivatives vanish at the origin. The Lie algebra L_k is generated by the basis elements $\{e_k, e_{k+1}, \dots\}$. The algebras W_1 and L_k are naturally graded by $\deg e_i = i$. Obviously the infinite dimensional subalgebras L_k of W_1 are nilpotent for $k \geq 1$.

The cohomology rings $H^*(L_k)$, $k \geq 0$ with trivial coefficients are known, there exist several different methods for the computation (see [**G, GFF, FF2, FR, V**]). The result is the following:

$$\dim H^q(L_k) = \binom{q+k-1}{k-1} + \binom{q+k-2}{k-2} \quad for \quad k \geq 1.$$

Not much is known about the cohomology with nontrivial coefficients for the Lie algebra L_k, $k > 1$. Among the known results, we mention the results on L_k, $k \geq 1$ on the cohomology $H^*(L_k; L_s)$ with any $s \geq 1$, see [**F**], and on L_k, $k \leq 3$ on the cohomology with coefficients in highest weight modules over the Virasoro algebra, see [**FF2**] and [**FF3**].

Let F_λ denote the W_1-module of the tensor fields of the form $f(z)dz^{-\lambda}$, where $f(z)$ is a polynomial in z and λ is a complex number; the action of W_1 on F_λ is given by the formula

$$(gd/dx)fdx^{-\lambda} = (gf' - \lambda fg')dx^{-\lambda}.$$

The module F_λ has an additive basis $\{f_j;\ j = 0, 1, \dots\}$ where $f_j = x^j dx^{-\lambda}$ and the action on the basis elements is

$$e_i f_j = (j - (i+1)\lambda) f_{i+j}.$$

Denote by $\mathcal{F}_\lambda$ the W_1-module which is defined in the same way, except that the index j runs over all integers. The W_1-modules F_λ with $\lambda \neq 0$ are irreducible, but as L_0-modules, they are reducible. For getting an L_0-submodule of F_λ, it is enough to take its subspace, generated by f_j, $j \geq \mu$, where μ is a positive integer. Denote the obtained L_0-module by $F_{\lambda,\mu}$.

More general, let us define the L_0-module $F_{\lambda,\mu}$ for arbitrary complex number μ, as the space, generated – like F_λ – by the elements f_j, $j = 0, 1, \dots$, on which L_0 acts by

$$e_i f_j = (j + \mu - (i+1)\lambda) f_{i+j}.$$

Finally define the modules $\mathcal{F}_{\lambda,\mu}$ over W_1 as $F_{\lambda,\mu}$ above, without requiring the positivity of j.

The homology of the Lie algebra L_1 with coefficients in $\mathcal{F}_{\lambda,\mu}$ and $F_{\lambda,\mu}$ are computed in [**FF1**]. We consider everywhere homology rather than cohomology, but the calculations are more or less equivalent. In the case of $\mathcal{F}_{\lambda,\mu}$ one can use the equality

$$(\mathcal{F}_{\lambda,\mu})' = \mathcal{F}_{-1-\lambda,-\mu}$$

which implies that

$$H^q\left(L_k; \mathcal{F}_{\lambda,\mu}\right)' = H_q\left(L_k; \mathcal{F}_{-1-\lambda,-\mu}\right).$$

In the case of $F_{\lambda,\mu}$ one can use the equality

$$\left(F_{\lambda,\mu}\right)' = \left(\mathcal{F}_{-1-\lambda,-\mu}\right)/F_{-1-\lambda,-\mu}$$

(see [**FF1**] for details).

Let us recall the results of [**FF1**]. Set $e(t) = (3t^2 + t)/2$ and define the k-th parabola ($k = 0, 1, 2, \dots$) as a curve on the complex plane with the parametric equation

$$\lambda = e(t) - 1$$

$$m - k = e(t) + e(t+k) - 1.$$

For $k_1, k_2 \in \mathbb{Z}$ we set

$$P(k_1, k_2) = (e(k_1) - 1,\ e(k_1) + e(k_1) - 1)$$

and let $\mathbf{P} = \{P(k_1, k_2) :\ k_1, k_2 \in \mathbb{Z}\}$. For a point P of $\mathbf{P}$ let us introduce

$$k(P) = |k_2 - k_1|$$

and

$$K(P) = |k_1| + |k_2|.$$

If $P \in \mathbf{P}$, then $K(P) \geq k(P), K(P) = k(P) \mod 2$ and P lies in the $k(P)$-th parabola. For $k \neq 0$ all the points of the k-th parabola with integer coefficients belong to $\mathbf{P}$. On the 0-th parabola there is one point from $\mathbf{P}$ with $K = 0$, and two points with $K = 2$, two points with $K = 4$, and in general, two points with every even number K. For $k \geq 0$ on the k-th parabola lie $2k+2$ points from $\mathbf{P}$ with $K = k$ and four points with $K = k+2$, four with $k+4$, and in general, four with $K = k + 2i$.

Theorem [**FF1**, Theorem 4.1].

$$\dim H_q^{(m)}(L_1; \mathcal{F}_{\lambda,\mu}) = \begin{cases} 2 & \textit{if} \quad (\lambda, \mu + m) \in \mathbf{P} \textit{ and } K(\lambda, \mu + m) < q \\ 1 & \textit{if} \quad (\lambda, \mu + m) \in \mathbf{P} \textit{ and } K(\lambda, \mu + m) = q \\ 0 & \textit{otherwise.} \end{cases}$$

Corollary. *If λ is not of the form $e(k) - 1$ with $k \in \mathbb{Z}$ and if $\mu \in \mathbb{Z}$, then*

$$H_*(L_1; \mathcal{F}_{\lambda,\mu}) = 0.$$

The homology $H_q(L_1; F_{\lambda,\mu})$ is also computed in [**FF1**]. We will not formulate the result in details, only some important for us facts.

Theorem (Modification of Theorem 4.2, [**FF1**]).

1) *If (λ, μ) is a generic point so that $(\lambda, \mu + m)$ does not lie on any of the parabolas for any integer m, then*

$$H_*(L_1; F_{\lambda,\mu}) = H_*(L_2).$$

2) *If $(\lambda, \mu + j)$ lies on the parabola for some j, then $H_q(L_1; F_{\lambda,\mu})$ is bigger than $H_1(L_2)$ at least for some q.*

3) *In all cases*

$$H_q(L_2) = 2q + 1 \leq \dim H_q(L_1; F_{\lambda,\mu}) \leq 4q + 1$$

and the boundaries are reached.

The next problem is to compute homology of L_2 with coefficients in the modules $\mathcal{F}_{\lambda,\mu}$ and $F_{\lambda,\mu}$. That is the aim of this paper. The results are the following.

Theorem 1.

$$H_0^{(m)}(L_2; \mathcal{F}_{\lambda,\mu}) = \begin{cases} \mathbb{C} & \textit{if} \quad \lambda = -1, m + \mu = -1 \\ 0 & \textit{otherwise.} \end{cases}$$

Theorem 2.

$$\dim H_1^{(m)}(L_2; \mathcal{F}_{\lambda,\mu}) = \begin{cases} 2 & \textit{if} \quad \lambda = m + \mu = -1 \\ 1 & \textit{if} \quad \lambda = -1, m + \mu = 1, 2, 3 \\ & \quad \textit{or}\ \lambda = 0\ \textit{and}\ m + \mu = 0 \\ & \quad \textit{or}\ \lambda = 1\ \textit{and}\ m + \mu = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

These results are analogous to the ones in [**FF1**] and one can expect that the picture will be similar for higher homology as well. With this in mind, the following result is a surprise.

Theorem 3.

$$\dim H_2^{(m)}(L_2; \mathcal{F}_{\lambda,\mu}) = \begin{cases} 1 & \textit{if} \quad \lambda = -1, m + \mu = -1, 1, 2, 3 \\ & \quad \textit{or}\ \lambda = 0\ \textit{and}\ m + \mu = 0 \\ & \quad \textit{or}\ \lambda = 1\ \textit{and}\ m + \mu = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

That means that the singular values of the parameters for the two-dimensional homology are the same, as the ones for the one-dimensional homology, which is not the case for the homology of L_1. Moreover, some partial computational results make the following conjecture plausible.

Conjecture 1. $H_q(L_2; \mathcal{F}_{\lambda,\mu}) = 0$ *for every* λ, μ *for* $q > 2$.

Let us try to explain the behavior of this homology. The main difference of the L_2 case from the L_1 case is that $H_q(L_1; \mathcal{F}_{\lambda,\mu}) = 0$ for *generic* λ and μ, while $H_q(L_2; \mathcal{F}_{\lambda,\mu}) = 0$ for *all* λ and μ (if $q > 2$). This might have the following explanation. By the Shapiro Lemma (see [**CE**, Ch. XIII/4, Prop. 4.2]),

$$H_q(L_2; \mathcal{F}_{\lambda,\mu}) = H_q\left(L_1; \operatorname{Ind}_{L_2}^{L_1} \mathcal{F}_{\lambda,\mu}\right)$$

and $\operatorname{Ind}_{L_2}^{L_1}\mathcal{F}_{\lambda,\mu}$ may be regarded as a limit case of the tensor product of modules of the type $F_{\lambda',\mu'} \otimes \mathcal{F}_{\lambda,\mu}$. Namely, $\operatorname{Ind}_{L_2}^{L_1}\mathcal{F}_{\lambda,\mu} = F \otimes \mathcal{F}_{\lambda,\mu}$ where F is the L_1-module spanned by $g_j, j \geq 0$, with the L_1-action $e_1 g_j = g_{j+1}$, $e_i g_j = 0$ for $i > 1$; the isomorphism is defined by the formula

$$e_1^k f_j \to \sum_{m=0}^{k} \binom{k}{m} g_m \otimes e_1^{k-m} f_j$$

$\Big($on the left hand side $e_1^k f_j$ means the action of e_1 in $\operatorname{Ind}_{L_2}^{L_1}\mathcal{F}_{\lambda,\mu}$, on the right hand side $e_1^{k-m} f_j$ means the action of e_1 in $\mathcal{F}_{\lambda,\mu}\Big)$. On the other hand, $F = \lim_{\lambda\to\infty} F_{\lambda,a\lambda}$ for any $a \neq 2$: put

$$g_j(\lambda) = (a-2)\lambda((a-2)\lambda+1)\dots((a-2)\lambda+j-1)f_j \in F_{\lambda,a\lambda};$$

then

$$e_i g_j(\lambda) = \frac{((a-i-1)\lambda+j)g_{i+j}(\lambda)}{((a-2)\lambda+j)\dots((a-2)\lambda+j+i-1)}$$

which tends to the action of L_1 in F when $\lambda \to \infty$.

Perhaps the homology

$$H_q(L_1; F_{\lambda',\mu'} \otimes \mathcal{F}_{\lambda,\mu})$$

depending not on two but on four parameters, has singular values for some $\lambda, \mu, \lambda', \mu'$ for each q. The problem of computing the cohomology $H_q(L_2; \mathcal{F}_{\lambda,\mu})$ is the two-parameter limit version of the previous problem, and it is not surprising that the singular solutions of the first problem have effect on the second problem only for small q values.

Our calculation yields also some results for $H_*(L_2; F_{\lambda,\mu})$. We will formulate them in Section 3, Theorem 4 and 5.

From Theorem 4 it follows that for generic λ, μ,

$$\dim H_0(L_2; F_{\lambda,\mu}) = 2,$$

and for singular values of λ, μ, $\dim H_0(L_2; F_{\lambda,\mu}) > 2$.

From Theorem 5 it follows that for generic λ, μ,

$$\dim H_1(L_2; F_{\lambda,\mu}) = 8,$$

and for singular values of λ, μ, $\dim H_1(L_2; F_{\lambda,\mu}) > 8$.

Conjecture 2. *For generic* λ, μ,

$$\dim H_q(L_2; F_{\lambda,\mu}) = 2(q+1)^2$$

or in more details,

$$H_q^{(m)}(L_2; F_{\lambda,\mu}) \simeq H_q^{(m)}(L_3) \oplus H_q^{(m-1)}(L_3).$$

This conjecture is motivated by the following observation. By the Shapiro Lemma,

$$H_q^{(m)}(L_3) = H_q^{(m)}\left(L_2; \operatorname{Ind}_{L_3}^{L_2}\mathbb{C}\right).$$

The module $\operatorname{Ind}_{L_3}^{L_2}\mathbb{C}$ is spanned by h_j $(j \geq 0)$ with L_2-action $e_2 h_j = h_{j+1}$, $e_i h_j = 0$ for $i > 2$; the grading in this module is $\deg h_j = 2j$. Hence

$$H_q^{(m)}(L_3) = H_q^{(m)}\left(L_2; \operatorname{Ind}_{L_3}^{L_2}\mathbb{C} + \Sigma \operatorname{Ind}_{L_3}^{L_2}\mathbb{C}\right)$$

where Σ stands for the shift of grading by one. On other words,

$$H_q^{(m)}(L_3) \oplus H_q^{(m-1)}(L_3) = H_q^{(m)}(L_2; F)$$

where F is spanned by g_j, $j \geq 0$, with the L_2-action $e_2 g_j = g_{j+2}$, $e_i g_j = 0$ for $i > 2$. As above, $F = \lim_{\lambda\to\infty} F_{\lambda,a\lambda}$ (now $a \neq 3$), which suggests that

$$H_q^{(m)}(L_2; F) = H_q^{(m)}(L_2; F_{\lambda,\mu})$$

for generic λ, μ.

Similarly one can expect that for generic λ, μ

$$H_q^{(m)}(L_k; F_{\lambda,\mu}) = H_q^{(m)}(L_{k+1}) \oplus H_q^{(m-1)}(L_{k+1}) \oplus \cdots \oplus H_q^{(m-k+1)}(L_{k+1}).$$

Remark, that if it is true that generically $H_q(L_2; \mathcal{F}_{\lambda,\mu}) = 0$ then generically

$$H^q(L_2; \mathcal{F}_{\lambda,\mu}) = H_{q-1}(L_2; F_{-1-\lambda,-\mu})$$

$\Big(H^q(L_2; \mathcal{F}_{\lambda,\mu}) = H_q(L_2; F'_{\lambda,\mu}) = H_q(L_2; \mathcal{F}_{-1-\lambda,-\mu}/F_{-1-\lambda,-\mu})$, and the homology exact sequence associated with the short coefficient exact sequence

$$0 \to F_{-1-\lambda,-\mu} \to \mathcal{F}_{-1-\lambda,-\mu} \to \mathcal{F}_{-1-\lambda,-\mu}/F_{-1-\lambda,-\mu} \to 0$$

provides the above isomorphism$\Big)$. In particular, if the L_2-module $L_2' = F_{-2,-3}$ is "generic", then Conjecture 2 implies

$$\dim H^2(L_2; L_2) = \dim H_1(L_2; F_{-2,-3}) = 8.$$

Similarly for L_k we have the hypothetical result

$$H^2(L_k; L_k) = k(k+2).$$

The paper by Yu. Kochetkov and G. Post [**KP**] contains the announcement of the equality

$$\dim H^2(L_2; L_2) = 8,$$

as well as some further computations, including explicit formulas for 8 generating cocycles, which imply the description of infinitesimal deformations of the Lie algebra L_2.

I. Spectral sequence.

Let us compute the homology $H_q^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$. Define a spectral sequence with respect to the filtration in the cochain complex $C_*^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$. The space $C_q^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$ is generated by the chains

$$e_{i_1} \wedge \ldots \wedge e_{i_q} \otimes f_j$$

where $2 \leq i_1 < \ldots < i_q$, $j \in \mathbb{Z}$ and $i_1 + \ldots i_q + j = m$. Define the filtration by $i_1 + \ldots + i_q = p$. Denote by $F_p C_q^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$ the subspace of $C_q^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$, generated by monomials of the above form with $i_1 + \ldots + i_q \leq p$. Obviously, $\left\{F_p C_q^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})\right\}_p$ is an increasing filtration in the chain complex. The differential acts by the rule

$$\begin{aligned} &d\left(e_{i_1} \wedge \ldots \wedge e_{i_q} \otimes f_j\right) \\ &\quad = d\left(e_{i_1} \wedge \ldots \wedge e_{i_q}\right) \otimes f_j - \sum_{s=1}^{q} (-1)^s e_{i_1} \wedge \ldots \hat{e}_{i_s} \wedge \ldots \wedge \ldots e_{i_q} \otimes e_{i_s} f_j. \end{aligned}$$

As m is fixed, the filtration in bounded.

Denote the spectral sequence, corresponding to this filtration by $E(\lambda, \mu, m)$. Then we have

$$E_0^p = C_*^{(p)}(L_2; \mathbb{C})$$

and d_0^p is the differential $\delta_p : C_*^{(p)}(L_2; \mathbb{C}) \to C_{*-1}(L_2; \mathbb{C})$. The first term of the spectral sequence is

$$E_1^p = H_*^{(p)}(L_2; \mathbb{C}).$$

The homology of L_2 with trivial coefficients is known (see [**G**]):

$$H_q^{(p)}(L_2) = \begin{cases} \mathbb{C} & \text{if } \frac{3q^2+q}{2} \leq p \leq \frac{3(q+1)^2-(q+1)}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Hence the E_1 term of our spectral sequence looks as follows:

$H_0^{(0)}$		$H_1^{(2)}$	$H_1^{(3)}$	$H_1^{(4)}$			$H_2^{(7)}$	$H_2^{(8)}$	$H_2^{(9)}$	$H_2^{(10)}$	$H_2^{(11)}$				$H_3^{(15)}$	$H_3^{(16)}$	...

where all the spaces $H_q^{(p)}$ shown in this diagram are one dimensional.

The spaces E_1^p do not depend on λ and μ, but the differentials of the spectral sequence do. Let us introduce the notation

$$e_q^{\pm} = \frac{3q^2 \pm q}{2}.$$

The differentials

$$d_{p-r}^p : E_{p-r}^p \to E_{p-r}^r \quad \left(e_q^+ \leq p < e_{q+1}^-,\ e_{q-1}^+ \leq r < e_q^-\right)$$

form a partial multi-valued mapping $\tilde{\delta}_q : H_q(L_2) \to H_{q-1}(L_2)$. We shall define a usual linear operator $\delta_q : H_q(L_2) \to H_{q-1}(L_2)$ such that (1) if $\tilde{\delta}_q(\alpha)$ is defined for some $\alpha \in H_q(L_2)$ then $\delta_q(\alpha) \in \tilde{\delta}_q(\alpha)$; (2) $\delta_{q-1} \circ \delta_q = 0$. (Certainly, the mapping δ_q will depend on λ, μ, m.) Then the limit term of the spectral sequence $E(\lambda, \mu, m)$, that is $H_*^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$ will coincide with the homology of the complex

$$H_0(L_2) \xleftarrow{\delta_1} H_1(L_2) \xleftarrow{\delta_2} H_2(L_2) \xleftarrow{\delta_3} \dots \quad .$$

To define $\delta_1, \delta_2, \dots$ we fix for any q and any p, $E_q^+ \leq p < e_{q+1}^-$, a cycle $c_q^p \in C_q^{(p)}(L_2)$ which represents the generator of $H_q^{(p)}(L_2)$.

It is evident that for each c_q^p there exist chains

$$\begin{aligned} b_q^{p-u} &\in C_q^{(p-u)}(L_2), \quad && u \geq 1 \\ g_{q-1}^v &\in C_{q-1}^{(v)}(L_2), \quad && v < e_{q-1}^+ \end{aligned}$$

such that

$$d\left(c_q^p \otimes f_{m-p} - \sum_{u\geq 1} b_q^{p-u} \otimes f_{m-p+u}\right) = \sum_{r=e_{q-1}^+}^{e_q^- - 1} \alpha_{p,r} c_{q-1}^r \otimes f_{m-r} + \sum_{v<e_{q-1}^+} g_{q-1}^v \otimes f_{m-v}$$

where $\alpha_{p,r}$ are complex numbers depending on λ, μ, m. These numbers compose the matrix of some linear mapping $H_q(L_2) \to H_{q-1}(L_2)$, and this mapping is our δ_q.

The chains $b_q^{p,u}$ and g_{q-1}^v may be chosen in the following way. Since $dc_q^p = 0$, the differential $d\left(c_q^p \otimes f_{m-p}\right)$ has the form $\sum_{w<p} h_{q-1}^w \otimes f_{m-w}$ with $h_{q-1}^w \in C_{q-1}^{(w)}(L_2)$. Here the leading term h_{q-1}^{p-1} is a cycle, $dh_{q-1}^{p-1} = 0$. Since $H_{q-1}^{p-1}(L_2) = 0$, we have $h_{q-1}^{p-1} = db_q^{p-1}$ with $b_q^{p-1} \in C_q^{(p-1)}(L_2)$. Now, the leading term of $d\left(c_q^p \otimes f_{m-p} - b_q^{p-1} \otimes f_{m-p+1}\right)$. belongs to $C_{q-1}^{(p-1)}(L_2)$ and it is again a cycle. We apply to it the same procedure and do it until the leading term of $d\left(c_q^p \otimes f_{m-p} - \sum b_q^{p-i} \otimes f_{m-p+i}\right)$ belongs to $C_{q-1}^{(e_q^- -1)}(L_2)$. This is still a cycle, but it is not necessarily a boundary, for $H_{q-1}^{e_q^- -1}(L_2) \neq 0$. Now we choose $b_q^{e_q^- -1} \in C_q^{\left(e_q^- -1\right)}(L_2)$ such that $db_q^{e_q^- -1}$ is our leading term up to some multiple of $c_{q-1}^{e_q^- -1}$. Then we do the same for $C_{q-1}^{\left(e_q^- -2\right)}(L_2)$, and so on until we reach $C_{q-1}^{e_{q-1}^+ -1}(L_2)$.

The matrix $|\alpha_{p,r}|$ depends on the choice of the cycles c_q^p. It depends also on the particular choice of the chains b_q^{p-u}, but only up to a triangular transformation. In particular, the kernels and the images of the mappings δ_q, and hence the homology $\operatorname{Ker}\delta_q/\operatorname{Im}\delta_{q+1}$, are determined by the cycles c_q^p.

Remark that $\dim H_q(L_2) = 2q+1$ and hence the matrix of δ_q is a $(2q-1) \times (2q+1)$-matrix depending on λ, μ, m. We get

$$(*) \qquad \dim H_q^{(m)}(L_2; \mathcal{F}_{\lambda,\mu}) = 2q + 1 - \operatorname{rank}\delta_q - \operatorname{rank}\delta_{q-1}.$$

II. Computations of $H_q^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$.

1. The space $H_0^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$.

As the action of W_1 on $\mathcal{F}_{\lambda,\mu}$ is

$$e_i \otimes f_j \to [j + \mu - \lambda(i+1)]f_{i+j}$$

and the nontrivial cycles of $H_1(L_2)$ are $c_1^2 = e_2$, $c_1^3 = e_3$, $c_1^4 = e_4$, the differentials are the following:

$$\begin{aligned} e_2 \otimes f_{m-2} &\to (m - 2 + \mu - 3\lambda)f_m, \\ e_3 \otimes f_{m-3} &\to (m - 3 + \mu - 4\lambda)f_m, \\ e_4 \otimes f_{m-4} &\to (m - 4 + \mu - 5\lambda)f_m. \end{aligned}$$

The coefficients in the right hand sides depend on λ and $m+\mu$, which is natural, because the whole complex $C_*^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$ depends only on λ and $m+\mu$. On the other hand, there is an isomorphism $\mathcal{F}_{\lambda,\mu} = \mathcal{F}_{\lambda,\mu+1}$, $f_j \to f_{j+1}$ with the shift of grading by 1. Therefore we may put $m = 0$ and the differential matrix $\delta_1 : H_1(L_2) \to H_0(L_2)$ has the form

$$(\mu - 2 - 3\lambda \mid \mu - 3 - 4\lambda \mid \mu - 4 - 5\lambda).$$

The rank of the matrix is 0 if $\lambda = m = -1$ and 1 in all the other cases. From this it follows

Theorem 1.

$$\dim H_0^{(m)}(L_2;\mathcal{F}_{\lambda,\mu}) = \begin{cases} 1 & \textit{if } \lambda=-1, m+\mu=-1 \\ 0 & \textit{otherwise.} \end{cases}$$

2. The space $H_1^{(m)}(L_2;\mathcal{F}_{\lambda,\mu})$.

The nontrivial cycles of $C_2(L_2;\mathbb{C})$ are

$$\begin{aligned} c_2^7 &= e_2\wedge e_5 - 3e_3\wedge e_4 \\ c_2^8 &= e_2\wedge e_6 - 2e_3\wedge e_5 \\ c_2^9 &= 3e_2\wedge e_7 - 5e_3\wedge e_6 \\ c_2^{10} &= e_2\wedge e_8 - 3e_4\wedge e_6 \\ c_2^{11} &= 5e_2\wedge e_9 - 7e_3\wedge e_8 \end{aligned}$$

of weight $7,8,9,10,11$.

Let us put $\mu - k\lambda - 1 = A(k,1)$. Direct calculation shows that

$$\begin{aligned} d\,((e_2\wedge e_5 &- 3e_3\wedge e_4)\otimes f_{-7} - A(3,7)e_2\wedge e_3\otimes f_{-5}) \\ &= -3A(4,7)e_4\otimes f_{-4} \\ &\quad + [3A(5,7) - A(3,7)A(3,5)]e_3\otimes f_{-3} \\ &\quad + [-A(6,7) + A(3,7)A(4,5)]e_2\otimes f_{-2}, \end{aligned}$$

hence

$$\begin{aligned} \delta_2\,(c_2^7) &= [-A(6,7) + A(3,7)A(4,5)]c_1 1^2 \\ &\quad + [3A(5,7) - A(3,7)A(3,5)]c_1^3 - 3A(4,7)c_1^4. \end{aligned}$$

Thus we have

$$\begin{aligned} \alpha_{7,2} &= -A(6,7) + A(3,7)A(4,5) \\ \alpha_{7,3} &= 3A(5,7) - A(3,7)A(3,5) \\ \alpha_{7,4} &= -3A(4,7). \end{aligned}$$

In the same way we calculate $\alpha_{p,r}$ for $p = 8,9,10,11$ and $r = 2,3,4$. We get the following 5×3-matrix:

$A(3,7)A(4,5)$ $-A(6,7)$	$-A(3,7)A(3,5)$ $+3A(5,7)$	$-3A(4,7)$
$1/2A(3,8)A(5,6)$ $-2A(4,8)A(4,5)$ $-A(7,8)$	$2A(4,8)A(3,5)$ $+2A(6,8)$	$-1/2A(3,8)A(3,6)$
$-5/2A(4,9)A(5,6)$ $-3A(8,9)$	$3A(3,9)A(5,7)$ $+5A(7,9)$	$-3A(3,9)A(4,7)$ $+5/2A(4,9)A(3,6)$
$-1/2A(3,10)A(4,8)A(4,5)$ $-3/2A(5,10)A(5,6)$ $-A(9,10)$	$1/2A(3,10)A(4,8)A(3,5)$ $+1/2A(3,10)A(6,8)$	$3/2A(5,10)A(3,6)$ $+3A(7,10)$
$7/2A(4,11)A(4,8)A(4,5)$ $+A(3,11)A(8,9)$ $-5A(10,11)$	$-A(3,11)A(3,9)A(5,7)$ $-7/2A(4,11)A(4,8)A(3,5)$ $-7/2A(4,11)A(6,8)$ $+7A(9,11)$	$A(3,11)A(3,9)A(4,7)$

We have to compute the rank of the matrix (δ_2). It is clear that the rank can not be bigger than 2. Direct computation shows that $\mathrm{rk}(\delta_2) = 1$ if and only if $\lambda = -1$, $\mu = -1, 1, 2, 3$; $\lambda = \mu = 0$; $\lambda = \mu = 1$. From this, using formula $(*)$, it follows

Theorem 2.

$$\dim H_1^{(m)}(L_2; \mathcal{F}_{\lambda,\mu}) = \begin{cases} 2 & \textit{if} \quad \lambda = m + \mu = -1 \\ 1 & \textit{if} \quad \lambda = -1, m + \mu = 1, 2, 3 \\ & \quad \textit{or } \lambda = 0 \textit{ and } m + \mu = 0 \\ & \quad \textit{or } \lambda = 1 \textit{ and } m + \mu = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

3. The spaces $H_q^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$ for $q \geq 2$.

The next differential δ_3 is a 5×7-matrix. Its rank can not be bigger than 3 for any λ and μ. On the other hand, computation shows that $\mathrm{rk}(\delta_3) = 3$ for every λ, μ; namely, the first three rows of the matrix are linearly independent for every λ, μ. From this it follows that the dimension of the space $H_2^{(m)}(L_2; F_{\lambda,\mu})$ drops only if the rank of the previous matrix (δ_2) does. This proves

Theorem 3.

$$\dim H_2^{(m)}\left(L_2; \mathcal{F}_{\lambda,\mu}\right) = \begin{cases} 1 & \textit{if} \quad \lambda = -1, m+\mu = -1, 1, 2, 3 \\ & \quad \textit{or } \lambda = 0 \textit{ and } m+\mu = 0 \\ & \quad \textit{or } \lambda = 1 \textit{ and } m+\mu = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

By this theorem, for generic λ, μ, $\dim H_2^{(m)}\left(L_2; \mathcal{F}_{\lambda,\mu}\right) = 0$.

It seems very likely that the next differential matrices (δ_k), $k \geq 4$, have the same rank for every λ and μ ($\mathrm{rk}(\delta_k) = q$) which would imply our

Conjecture 1. *$H_q\left(L_2; \mathcal{F}_{\lambda,\mu}\right) = 0$ for every λ, μ for $q > 2$.*

III. Computations of $H_q^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$.

Recall that the L_0-modules $F_{\lambda,\mu}$ differ from the W_1-modules $\mathcal{F}_{\lambda,\mu}$ only in requiring the non-negativity of j for the generators f_j. Consequently the spectral sequence is basically the same, only it is truncated as follows:

$$E_r^p(\lambda, \mu, m) = 0 \quad \text{if} \quad m - p < 0.$$

The space $C_q^{(m)}(L_2; F_{\lambda,\mu})$ is generated by the chains

$$e_{i_1} \wedge \ldots \wedge e_{i_q} \otimes f_j$$

with $2 \leq i_1 \leq \ldots \leq i_q$, $j \geq 0$ and $i_1 + \ldots + i_q = m$. This way, for computing homology, we have to compute the rank of truncated matrices, consisting of some of the upper rows of the previous matrices.

Let us compute the space $H_0(L_2; F_{\lambda,\mu})$. Obviously,

$$H_0^{(0)}(L_2; F_{\lambda,\mu}) = H_0^{(1)}(L_2; F_{\lambda,\mu}) = \mathbb{C}.$$

For $m = 2$ the differential is the following:

$$e_2 \otimes f_0 \to (\mu - 3\lambda) f_2$$

which shows that if $\mu = 3\lambda$, then $\dim H_0^{(2)} = 1$, otherwise $H_0^{(2)}(L_2; F_{\lambda,\mu}) = 0$.

For $m > 2$

$$\dim H_0^{(m)}(L_2; F_{\lambda,\mu}) = \begin{cases} 1 & \text{if } \lambda = -1 \text{ and } m + \mu = -1 \\ 0 & \text{otherwise.} \end{cases}$$

So we get

Theorem 4.

$$H_0^{(m)}(L_2; F_{\lambda,\mu}) = \begin{cases} \mathbb{C} & \text{if } m = 0, 1 \\ & \quad \text{or } m = 2 \text{ and } \mu = 3\lambda \\ & \quad \text{or } \lambda = -1 \text{ and } m + \mu = -1 \\ 0 & \text{otherwise.} \end{cases}$$

Corollary. *For generic* $\lambda, \mu \quad H_0(L_2; F_{\lambda,\mu}) = 2.$

Direct computation proves the result for the space $H_1^{(m)}(L_2; F_{\lambda,\mu})$.

Theorem 5.

$$\dim H_1^{(2)}(L_2; F_{\lambda,\mu}) = \begin{cases} 1 & \text{if } \mu = 3\lambda \\ 0 & \text{otherwise,} \end{cases}$$

$$\dim H_1^{(3)}(L_2; F_{\lambda,\mu}) = \begin{cases} 2 & \text{for } \lambda = -1, \mu = -4 \\ 1 & \text{otherwise,} \end{cases}$$

$$\dim H_1^{(4)}(L_2; F_{\lambda,\mu}) = \dim H_1^{(5)}(L_2; F_{\lambda,\mu}) = \dim H_1^{(6)}(L_2; F_{\lambda,\mu})$$
$$= \begin{cases} 3 & \text{for } \mu = -4, \lambda = -1 \\ 2 & \text{otherwise,} \end{cases}$$

$$\dim H_1^{(7)}(L_2; F_{\lambda,\mu}) = \begin{cases} 2 & \text{if } \mu = -8, \lambda = -1 \text{ or } \mu = 0, \lambda = 0 \\ 1 & \text{otherwise,} \end{cases}$$

$$\dim H_1^{(8)}(L_2; F_{\lambda,\mu}) = \begin{cases} 2 & \text{if } \mu = -9, \lambda = -1 \\ 1 & \text{for } \lambda \text{ and } \mu \text{ lying on the curve} \\ & -36\lambda + 147\lambda^2 - 27\lambda^3 + 8\mu - 72\lambda\mu + 27\lambda^2\mu \\ & +9\mu^2 - 9\lambda\mu^2 + \mu^3 = 0 \\ 0 & \text{otherwise;} \end{cases}$$

for $m > 8, \quad \dim H_1^{(m)}(L_2; F_{\lambda,\mu}) = \dim H_1^{(m)}(L_2; \mathcal{F}_{\lambda,\mu})$ *(see Theorem* 2*).*

Corollary. *For generic* λ, μ, $\dim H_1(L_2; F_{\lambda,\mu}) = 8.$

Conjecture 2. *For generic* λ, μ,

$$\dim H_q(L_2; F_{\lambda,\mu}) = 2(q+1)^2,$$

or, in more details,

$$H_q^{(m)}(L_2; F_{\lambda,\mu}) \simeq H_q^{(m)}(L_3; \mathbb{C}) \otimes H_q^{(m-1)}(L_3; \mathbb{C}).$$

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GENERIC DIFFERENTIABILITY OF CONVEX FUNCTIONS ON THE DUAL OF A BANACH SPACE

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We study a class of Banach spaces which have the property that every continuous convex function on an open convex subset of the dual possessing a weak * continuous subgradient at points of a dense G_δ subset of its domain, is Fréchet differentiable on a dense G_δ subset of its domain. A smaller more amenable class consists of Banach spaces where every minimal weak * cusco from a complete metric space into subsets of the second dual which intersect the embedding from a residual subset of the domain is single-valued and norm upper semi-continuous at the points of a residual subset of the domain. It is known that all Banach spaces with the Radon-Nikodym property belong to these classes as do all with equivalent locally uniformly rotund norm. We show that all with an equivalent weakly locally uniformly rotund norm belong to these classes. The condition closest to a characterisation is that the Banach space have its weak topology fragmentable by a metric whose topology on bounded sets is stronger than the weak topology. We show that the space $\ell_\infty(\Gamma)$, where Γ is uncountable, does not belong to our special classes.

We say that a Banach space is a *dual differentiability space* (*DD* space) if every continuous convex function on an open convex subset of the dual possessing a weak * continuous subgradient at points of a dense G_δ subset of its domain, is Fréchet differentiable on a dense G_δ subset of its domain. Spaces of this class include those with the Radon-Nikodym property, and all those which can be equivalently renormed to be locally uniformly rotund. In the paper [**K-G**, p. 472] it was shown that spaces which can be equivalently renormed to have every point of the unit sphere a denting point of the closed unit ball are spaces of this class, and in the paper [**G-M1**, p. 264] it was shown that spaces which can be equivalently renormed to have every point of the unit sphere an α denting point of the closed unit ball, (α is Kuratowski's index of non-compactness), are spaces of this class; Troyanski [**T1**, p. 306] and [**T2**, p. 179] has shown that spaces with either of these properties can be equivalently renormed to be locally uniformly rotund. In paper [**G-M2**, p. 111], the denting point property was weakened using an index of non-WCG.

Information about the class of DD spaces is more easily obtained through the study of a subclass defined by certain set-valued mappings having special continuity properties. A set-valued mapping Φ from a topological space A into subsets of a topological space X is *upper semi-continuous* at $t \in A$ if given an open subset W where $\Phi(t) \subseteq W$ there exists an open neighbourhood U of t such that $\Phi(U) \subseteq W$. If X is a linear topological space and $\Phi(t)$ is non-empty compact and convex for each $t \in A$ and Φ is upper semi-continuous on A we call Φ a *cusco* on A. A cusco Φ on A is said to be a *minimal cusco* if its graph does not contain the graph of any other cusco on A.

We say that a Banach space X is a *generic continuity space* (GC space) if every minimal weak * cusco Φ from a complete metric space A into subsets of the second dual X^{**} for which the set $\left\{t \in A : \Phi(t) \cap \widehat{X} \neq \emptyset\right\}$ is residual in A, is single-valued and norm upper semi-continuous at the points of a residual subset of A.

An open subset of a complete metric space is itself completely metrisable and a continuous convex function ϕ on an open convex subset of a Banach space generates a subdifferential mapping $x \mapsto \partial\phi(x)$ which is a minimal weak * cusco. The subdifferential mapping being single-valued and norm upper semi-continuous at a point is equivalent to the convex function being Fréchet differentiable at the point . So the class of GC spaces is contained in the class of DD spaces.

In Section 1 we show that for any Banach space X, minimal weak * cuscos from a complete metric space A into subsets of the second dual X^{**} which satisfy a certain generic property are always single-valued and norm upper semi-continuous at the points of a residual subset of A. We use this general result to show that Banach spaces which satisfy certain geometrical properties are GC spaces. In particular, we show that those Banach spaces which have an equivalent weakly locally uniformly rotund norm are GC spaces. In Section 2 we show that a Banach space is a GC space if its weak topology is fragmentable by a metric whose topology on bounded sets is stronger than the weak topology. We conclude in Section 3 by showing that the Banach space $\ell_\infty(\Gamma)$, where Γ is an uncountable set, is not a GC space.

1. A general property implying geometrical conditions for membership of the class of GC spaces.

For our general result we need the following characterisations of a minimal cusco.

Lemma 1.1. [G-M1, Lemma 2.5]. *Consider a cusco Φ from a topological space A into subsets of a separated locally convex space X. The following are* *equivalent*

(i) Φ *is a minimal cusco on* A,

(ii) *given any open set* U *in* A *and closed convex set* K *in* X *where* $\Phi(U) \not\subseteq K$ *there exists a non-empty open set* $V \subseteq U$ *such that* $\Phi(V) \cap K = \emptyset$,

(iii) *given any open set* U *in* A *and open half-space* W *in* X *where* $\Phi(U) \cap W \neq \emptyset$ *there exists a non-empty open set* $V \subseteq U$ *such that* $\Phi(V) \subseteq W$.

We also use a continuity condition defined in terms of *Kuratowski's index of non-compactness.* Given a bounded set E in a metric space X such an index is

$$\alpha(E) \equiv \inf\{r : E \text{ is covered by a finite family of sets of diameter less than } r\}.$$

Given a set-valued mapping Φ from a topological space A into subsets of a metric space X we say that Φ is α *upper semi-continuous* at $t \in A$ if given $\epsilon > 0$ there exits an open neighbourhood U of t such that $\alpha(\Phi(U)) < \epsilon$. Such α upper semi-continuous mappings have single-valued properties.

Lemma 1.2. [**G-M1**, p. 253]. *Consider a minimal weak * cusco* Φ *from a Baire space* A *into subsets of the second dual* X^{**} *of a Banach space* X. *If* Φ *is* α *upper semi-continuous on a dense subset of* A *then* Φ *is single-valued and norm upper semi-continuous at the points of a residual subset of* A.

The proof of our general theorem follows a similar method of proof as was used to prove Lemma 1.2 which is similar to a theorem of Christensen, [**Chr**, p. 651].

Theorem 1.3. *A minimal weak * cusco* Φ *from a complete metric space* A *into subsets of the second dual* X^{**} *of a Banach space* X *where the set*

$$E \equiv \left\{t \in A : \Phi(t) \subseteq \overline{\Phi(t) \cap \widehat{X}}^{w^*}\right\}$$

is residual in A, *is single-valued and norm upper semi-continuous at the points of a residual subset of* A.

Proof. Given $\epsilon > 0$ consider the open set $O_\epsilon \equiv \bigcup\{\text{open sets } U \text{ in } A : \alpha(\Phi(U)) \leq 2\epsilon\}$. Suppose that O_ϵ is not dense in A. Then there exists a non-empty open set V_0 in A such that $V_0 \cap O_\epsilon = \emptyset$. Consider a dense G_δ subset D of A contained in E. Now D is completely metrisable and we consider it with such a metric d.

We proceed by induction. Consider $t_1 \in V_0 \cap D$ and $\widehat{x}_1 \in \Phi(t_1) \cap \widehat{X}$. Now $\Phi(V_0) \not\subseteq \widehat{x}_1 + \epsilon B(X^{**})$ for otherwise $V_0 \cap O_\epsilon \neq \emptyset$. Since Φ is a minimal weak * cusco, by Lemma 1.1, there exists a non-empty open set V_1 such that $\overline{V}_1 \subseteq V_0$ and $\Phi(V_1) \cap (\widehat{x}_1 + \epsilon B(X^{**})) = \emptyset$. We may assume that the d-diam$(V_1 \cap D) < 1$.

Suppose that the first n iterations of this procedure have been completed. Then we have a non-empty open set V_n such that $\overline{V}_n \subseteq V_{n-1}$ and $\Phi(V_n) \cap (\text{co}\{\widehat{x}_1, \widehat{x}_2, \dots, \widehat{x}_n\} + \epsilon B(X^{**})) = \emptyset$ where $\widehat{x}_i \in \Phi(t_i) \cap \widehat{X}$ and $t_i \in V_{i-1} \cap D$ for $i \in \{1, 2, \dots, n\}$. Now consider $t_{n+1} \in V_n \cap D$ and $\widehat{x}_{n+1} \in \Phi(t_{n+1}) \cap \widehat{X}$. Again $\Phi(V_n) \not\subseteq \text{co}\{\widehat{x}_1, \widehat{x}_2, \dots, \widehat{x}_{n+1}\} + \epsilon B(X^{**})$ for otherwise $V_0 \cap O_\epsilon \supseteq V_n \neq \emptyset$. Since Φ is a minimal weak * cusco, by Lemma 1.1 there exists a non-empty open set V_{n+1} with d-diam$(V_{n+1} \cap D) < \frac{1}{2^n}$ such that $\overline{V}_{n+1} \subseteq V_n$ and $\Phi(V_{n+1}) \cap (\text{co}\{\widehat{x}_1, \widehat{x}_2, \dots, \widehat{x}_{n+1}\} + \epsilon B(X^{**})) = \emptyset$. Continuing in this way we form a Cauchy sequence $\{t_n\}$ in D which converges to some $t_\infty \in \bigcap_{n\in\mathbb{N}} \overline{V}_n = \bigcap_{n\in\mathbb{N}} V_n \subseteq D$.

Then for each $n \in \mathbb{N}$, $\Phi(t_\infty) \cap (\text{co}\{\widehat{x}_1, \widehat{x}_2, \dots, \widehat{x}_n\} + \epsilon B(X^{**})) = \emptyset$ and so

$$\Phi(t_\infty) \cap \left(\text{co} \bigcup_{n\in\mathbb{N}} \{\widehat{x}_n\} + \epsilon B(X^{**})\right) = \Phi(t_\infty) \cap \left(\bigcup_{n\in\mathbb{N}} \text{co}\{\widehat{x}_1, \widehat{x}_2, \dots, \widehat{x}_n\} + \epsilon B(X^{**})\right) = \emptyset.$$

So there exists an $f \in X^*$, which strongly separates $\Phi(t_\infty) \cap \widehat{X}$ and $\overline{\text{co}} \bigcup_{n\in\mathbb{N}} \{\widehat{x}_n\}$ and so there is a weak * open half space W generated by f containing $\overline{\Phi(t_\infty) \cap \widehat{X}}^{w^*}$ and disjoint from $\overline{\text{co}} \bigcup_{n\in\mathbb{N}} \{\widehat{x}_n\}$. Since $t_\infty \in E$, we have $\Phi(t_\infty) \subseteq W$. Since Φ is weak * upper semi-continuous at t_∞ there exists an open neighbourhood U of t_∞ such that $\Phi(U) \subseteq W$. However, for $n \in \mathbb{N}$ sufficiently large, $t_n \in U$ and then $\widehat{x}_n \in \Phi(t_n) \cap \widehat{X} \subseteq W$ contradicting the separation by f. We conclude that O_ϵ is dense in A and that Φ is α upper semi-continuous at the points of $\bigcap_{n\in\mathbb{N}} O_{\frac{1}{n}}$ a dense G_δ subset of A. Our result now follows from Lemma 1.2. □

We can now make the following deductions from Theorem 1.3.

Corollary 1.4. *A minimal weak * cusco Φ from a complete metric space A into subsets of the second dual X^{**} of a Banach space X where the set $\left\{t \in A : \Phi(t) \subseteq \widehat{X}\right\}$ is residual in A, is single-valued and norm upper semi-continuous at the points of a residual subset of A.*

A special case of a theorem of Namioka [**N**, p. 525] can be deduced from Theorem 1.3.

Corollary 1.5. *A weakly continuous single-valued mapping from a complete metric space A into a Banach space X is norm continuous at the points of a residual subset of A.*

A Banach space X is *weak Asplund* if every continuous convex function on an open convex subset A of X is Gâteaux differentiable on a residual subset of A. A Banach space X belongs to Stegall's class $\mathcal{S}$ if and only if every minimal weak * cusco Φ from a Baire space A into subsets of X^* is single-valued on a residual subset of A. It has been shown [**K-O**, Corol. 4.5] that a Banach space X belongs to Stegall's class $\mathcal{S}$ if and only if every minimal weak * cusco Φ from a complete metric space A into subsets of X^* is single-valued on a residual subset of A.

Corollary 1.6. *A Banach space X is*

(i) *a DD space if its dual X^* is weak Asplund,*

(ii) *a GC space if its dual X^* belongs to Stegall's class $\mathcal{S}$.*

Proof. We consider only the proof of (ii). A minimal weak * cusco Φ from a complete metric space A into subsets of X^{**} has the set $\{t \in A :$ $\Phi(t)$ is singleton$\}$ residual in A. So if the set $\left\{t \in A : \Phi(t) \cap \widehat{X} \neq \emptyset\right\}$ is residual in A then the set $\left\{t \in A : \Phi(t) \subseteq \widehat{X}\right\}$ is residual in A and we deduce from Corollary 1.4 that X is a GC space. □

We should note the Banach space ℓ_1 has dual ℓ_∞ which is not weak Asplund, [**P**, p. 13]. However ℓ_1 has the Radon-Nikodym property and so the property given in Corollary 1.6 is a sufficient but not necessary condition for a Banach space to be a DD space or a GC space.

It has recently been proved, that a Banach space belongs to Stegall's class $\mathcal{S}$ if it has an equivalent norm Gâteaux differentiable away from the origin, [**P-P-N**].

Corollary 1.7. *A Banach space X is a GC space if the dual X^* has an equivalent norm Gâteaux differentiable away from the origin.*

We note that the equivalent norm on X^* need not be a dual norm.

Corollary 1.8. *A Banach space X is a GC space if it can be mapped into a GC space Y, by a continuous linear mapping T whose conjugate T' has a dense range.*

Proof. Consider a minimal weak * cusco Φ from a complete metric space A into subsets of X^{**} where the set $\left\{t \in A : \Phi(t) \cap \widehat{X} \neq \emptyset\right\}$ is residual in A. As a conjugate, T'' is continuous when X^{**} and Y^{**} have their weak * topologies so $T'' \circ \Phi$ is a minimal weak * cusco from A into subsets of Y^{**}. Since Y is a GC space and the set $\left\{t \in A : T'' \circ \Phi(t) \cap \widehat{Y} \neq \emptyset\right\}$ is residual in A, so $T'' \circ \Phi$ is single-valued on a residual subset A. Since T' has dense range then T'' is one-to-one, so Φ is single-valued on a residual subset of A and we have by Theorem 1.3 that Φ is single-valued and norm upper semi-continuous at the points of a residual subset of A. □

It is well known that a closed linear subspace of a Banach space with the Radon-Nikodym property has the Radon-Nikodym property. The following is an extension of this result.

Theorem 1.9. *If a Banach space X is a GC space then every closed linear subspace Y of X is a GC space.*

Proof. The conjugate of the inclusion mapping maps X^* onto Y^* and so the result follows from Corollary 1.8. □

This subspace property holds for the larger class of DD spaces, but the proof uses a different technique.

Theorem 1.10. *If a Banach space X is a DD space then every closed linear subspace Y of X is a DD space.*

Proof. Consider ϕ a continuous convex function on an open convex subset B of Y^* where the set $\left\{g \in B : \partial\phi(g) \cap \widehat{Y} \neq \emptyset\right\} \supseteq E$ a dense G_δ subset of B. Consider T the inclusion mapping of Y into X. The conjugate T' maps X^* onto Y^*. Further, $\phi \circ T'$ is a continuous convex function on the open convex set $A \equiv (T')^{-1}(B)$ in X^*. Since T' is onto it is an open mapping and therefore $D \equiv (T')^{-1}(E)$ is a dense G_δ subset of A. But further, if $f_0 \in D$ then exists a $y_0 \in Y$ such that $\widehat{y}_0 \in \partial\phi(T' f_0)$. Then

$$\widehat{y}_0(T'f) - \widehat{y}_0(T'f_0) \leq \phi(T'f) - \phi(T'f_0) \text{ for all } f \in A$$

so

$$\widehat{y}_0(f) - \widehat{y}_0(f_0) \leq (\phi \circ T')(f) - (\phi \circ T')(f_0) \text{ for all } f \in A;$$

that is, $\widehat{y}_0 \in \partial(\phi \circ T')(f_0)$.

Then $\left\{f \in A : \partial(\phi \circ T')(f) \cap \widehat{X} \neq \emptyset\right\} \supseteq D$ a dense G_δ subset of A. Since X is a DD space there exists a dense G_δ subset G of A where $\phi \circ T'$ is Fréchet differentiable. That is, for $f \in G$,

$$\lim_{\lambda \to 0} \frac{(\phi \circ T')(f + \lambda g) - (\phi \circ T')(f)}{\lambda}$$

exists and is approached uniformly for all $g \in X^*$, $||g|| = 1$. Using the fact the T' is the restriction of each element of X^* to Y and that each restriction has a norm preserving extension on X then

$$\lim_{\lambda \to 0} \frac{\phi(T'(f + \lambda g)) - \phi(T'f)}{\lambda}$$

exists and is approached uniformly for all $T'g \in Y^*$, $||T'g|| = 1$. So ϕ is Fréchet differentiable on $T'(G)$ which is a dense subset of B. Since the set of points where a continuous convex function is Fréchet differentiable is always a G_δ subset, [**P**, p. 15], ϕ is Fréchet differentiable on a dense G_δ subset of B. We conclude that Y is a DD space. □

A Banach space X is said to be *weakly locally uniformly rotund* if for each $x_0 \in X$, $||x_0|| = 1$, given $\epsilon > 0$ and $f \in X^*$, $||f|| = 1$ there exists a $\delta(\epsilon, x_0, f) > 0$ such that $|f(x - x_0)| < \epsilon$ for all $x \in X$, $||x|| \leq 1$ when $||x + x_0|| > 2 - \delta$. A weakly locally uniformly rotund space is rotund but not necessarily locally uniformly rotund. However, such a geometrical property on a Banach space does have rotundity implications for the second dual space.

Lemma 1.11. *Consider a weakly locally uniformly rotund Banach space X. Given $x_0 \in X$, $||x_0|| = 1$, for every $F \in X^{**}$, $||F|| = 1$, $F \neq \widehat{x}_0$, we have $||F + \widehat{x}_0|| < 2$.*

Proof. Suppose that there exists an $F \in X^{**}$, $||F|| = 1$, $F \neq \widehat{x}_0$, such that $||F + \widehat{x}_0|| = 2$. Since $F \neq \widehat{x}_0$ there exists an $f_0 \in X^*$, $||f_0|| = 1$ and an $r > 0$ such that $|(F - \widehat{x}_0)(f_0)| > r$. Since X is weakly locally uniformly rotund, given $0 < \epsilon < \frac{r}{2}$ there exists a $\delta(\epsilon, x_0, f_0) > 0$ such that $|f_0(x - x_0)| < \epsilon$ for all $x \in X$, $||x|| \leq 1$ when $||x + x_0|| > 2 - \delta$. Since the norm on X^{**} is weak * lower semi-continuous the set $\{G \in X^{**} : ||G + \widehat{x}_0|| > 2 - \delta\}$ is weak *open in X^{**} and contains F. By Goldstine's Theorem $B\left(\widehat{X}\right)$ is weak * dense in $B(X^{**})$ so there exists some $\widehat{x} \in B\left(\widehat{X}\right)$ such that $||\widehat{x} + \widehat{x}_0|| > 2 - \delta$ and $|(F - \widehat{x})(f_0)| < \epsilon$. Then for such an $\widehat{x} \in B\left(\widehat{X}\right)$ we have $|f_0(x - x_0)| < \epsilon$ and therefore

$$|(F - \widehat{x}_0)(f_0)| \leq |(F - \widehat{x})(f_0)| + |f_0(x - x_0)| < 2\epsilon < r$$

which contradicts the initial separation property. □

We need the following property of minimimal weak * cuscos.

Lemma 1.12. [**K-G**, p. 471]. *Given a minimal weak * cusco Φ from a Baire space A into subsets of the dual X^* of a Banach space X, there exists* *a residual subset of A at each point t of which, $\Phi(t)$ lies in the face of a sphere of X^*.*

Theorem 1.13. *A Banach space X is a GC space if it can be equivalently renormed to be weakly locally uniformly rotund.*

Proof. Consider X so renormed. Then since Φ is a minimal weak * cusco on A we have by Lemma 1.12 that there exists a residual subset D of A at each point t of which, $\Phi(t)$ lies in the face of a sphere of X^{**}. So if the set $G \equiv \left\{t \in A : \Phi(t) \cap \widehat{X} \neq \emptyset\right\}$ is residual in A then $G \cap D$ is residual in A. But by Lemma 1.11, Φ is single-valued on $G \cap D$ and so $\Phi(G \cap D) \subseteq \widehat{X}$ and we deduce from Theorem 1.3 that X is a GC space. □

We do not need so strong a geometrical condition as weak local uniform rotundity. To be a GC space it would be sufficient for the space X to have an equivalent norm such that given $x_0 \in X$, $||x_0|| = 1$, for every $F \in X^{**} \setminus \widehat{X}$, $||F|| = 1$ we have $||F + \widehat{x}_0|| < 2$. Such an equivalent norm is not necessarily rotund. However, it is difficult to find a characterisation of this property on X.

2. Fragmentability conditions for membership of the class of *GC* spaces.

We aim to find fragmentability conditions which imply that a Banach space is a GC space.

Consider a bounded subset E in a Banach space X. Given $f \in X^*$, $||f|| = 1$ and $\delta > 0$, a *slice* of E defined by f and δ is the subset

$$S(E, f, \delta) \equiv \{x \in E : f(x) > \sup f(E) - \delta\}.$$

A slice of a bounded set E in the dual X^* defined by a weak * continuous linear functional on X^* is called a *weak * slice* of E.

We need the following local boundedness property of minimal weak * cuscos.

Lemma 2.1. *A minimal weak * cusco Φ from a Baire space A into subsets of the dual X^* of a Banach space X is locally bounded on a dense open subset of A.*

Proof. It is sufficient to show that there exists an open subset of A on which Φ is bounded. For each $n \in \mathbb{N}$, consider the set

$$E_n \equiv \{t \in A : \Phi(t) \subseteq nB(X^*)\}.$$

Clearly, $\bigcup_{n\in\mathbb{N}} E_n = A$. Since A is Baire there exists an $n_0 \in \mathbb{N}$ such that $\text{int}\overline{E_{n_0}} \neq \emptyset$. Consider an open set $U \subseteq \overline{E}_{n_0}$. Suppose for some $t_0 \in U \setminus E_{n_0}$ there exists an $f_0 \in \Phi(t_0) \setminus n_0 B(X^*)$. Then f_0 can be strongly separated from $n_0 B(X^*)$ by a weak * continuous linear functional on X^* which generates a weak * open half space W containing f_0 and $n_0 B(X^*) \subseteq C(W)$. Then since Φ is a minimal weak * cusco, by Lemma 1.1 there exists a non-empty open set $V \subseteq U$ such that $\Phi(V) \subseteq W$. But this contradicts the fact that there are points of E_{n_0} in V which map into $n_0 B(X^*)$. □

The following characterisation of the class of GC spaces simplifies our computation.

Theorem 2.2. *A Banach space X is a GC space if and only if every minimal weak * cusco Φ from a complete metric space A into subsets of X^{**} where $\Phi(t) \cap \widehat{X} \neq \emptyset$ for all $t \in A$ is single-valued and norm upper semi-continuous at the points of a residual subset of A.*

Proof. Consider a minimal weak * cusco Φ from a complete metric space A into subsets of X^{**} where $\left\{t \in A : \Phi(t) \cap \widehat{X} \neq \emptyset\right\} \supseteq A_1$ a dense G_δ subset of A. Then A_1 is completely metrisable, [**K-N**, p. 96]. Consider the set-valued mapping Φ_1 the restriction of Φ to A_1. Now Φ_1 is also a minimal weak * cusco on A_1 and $\Phi_1(t) \cap \widehat{X} \neq \emptyset$ for all $t \in A_1$. So Φ_1 is single-valued and norm upper semi-continuous at the points of a dense G_δ subset D of A_1 which is also a dense G_δ subset of A.

Consider $t_0 \in D$. Since Φ_1 is norm upper semi-continuous at t_0 there exists an open neighbourhood U of t_0 such that $\Phi_1(U \cap A_1) \subseteq B[\Phi(t_0); \epsilon]$. We will show that $\Phi(U) \subseteq B[\Phi(t_0); \epsilon]$. Suppose not, then since Φ is a minimal weak * cusco, by Lemma 1.1 there exists a non-empty open set $V \subseteq U$ such that $\Phi(V) \cap B[\Phi(t_0); \epsilon] = \emptyset$. But this contradicts the fact that A_1 is dense in A and Φ_1 is norm upper semi-continuous at t_0.

The converse is obvious. □

The following norm fragmenting theorem generalises a characterisation of Banach spaces with the Radon-Nikodym property.

Theorem 2.3. *A Banach space X is a GC space if there exists a weak * lower semi-continuous norm $|||\cdot|||$ on X^{**} and every non-empty bounded subset of X has slices of arbitrarily small $|||\cdot|||$-diameter.*

Proof. Consider a minimal weak * cusco Φ from a complete metric space A into subsets of X^{**} where $\Phi(t) \cap \widehat{X} \neq \emptyset$ for all $t \in A$. Consider the mapping $\widetilde{\Phi}$ from A into subsets of $\widehat{X}$ defined by

$$\widetilde{\Phi}(t) = \Phi(t) \cap \widehat{X}.$$

Given $\epsilon > 0$, consider the set

$$O_\epsilon \equiv \bigcup \left\{ \text{open sets } V \text{ such that } |||\cdot||| - \text{diam } \widetilde{\Phi}(V) < \epsilon \right\}.$$

Now O_ϵ is open; we show that it is dense in A. By Lemma 2.1 we may assume that $\widetilde{\Phi}$ is locally bounded. Consider any non-empty open set U in A where $\widetilde{\Phi}(U)$ is bounded. Then there is a weak * slice of $\widetilde{\Phi}(U)$ with $|||\cdot|||$-diameter less than ϵ. Since Φ is a minimal weak * cusco, by Lemma 1.1 there exists a non-empty open set $V \subseteq U$ such that $\widetilde{\Phi}(V)$ lies inside this slice and so $|||\cdot|||$-diam $\widetilde{\Phi}(V) < \epsilon$. So O_ϵ is dense in A. Then $D \equiv \bigcap_{n\in N} O_{\frac{1}{n}}$ is a dense G_δ of A and $\widetilde{\Phi}$ is single-valued and $|||\cdot|||$-upper semi-continuous at the points of D.

Consider $t_0 \in D$. Suppose that there exists an $F_0 \in \Phi(t_0) \setminus \widehat{X}$. For $r \equiv \frac{1}{2}|||F_0 - \widehat{x}_0|||$, consider $B_{|||\cdot|||}[\widehat{x}_0; r]$. Since $|||\cdot||$ is weak * lower semi-continuous, $B_{|||\cdot|||}[\widehat{x}_0; r]$ is weak * closed. So F_0 and $B_{|||\cdot|||}[\widehat{x}_0; r]$ can be strongly separated by a weak * continuous linear functional which generates a weak * open half-space W containing F_0 and $B_{|||\cdot|||}[\widehat{x}_0; r] \subseteq C(W)$. Since $\widetilde{\Phi}$ is $|||\cdot|||$-upper semi-continuous at t_0, there exists an open neighbourhood U of t_0, such that $\widetilde{\Phi}(U \cap D) \subseteq B_{|||\cdot|||}[\widehat{x}_0; r]$. Now $\Phi(U) \cap W \neq \emptyset$ and since Φ is a minimal weak * cusco, by Lemma 1.1 there exists a non-empty open set $V \subseteq U$ such that $\Phi(V) \subseteq W$. But this contradicts the fact that $\Phi(t) \cap C(W) \neq \emptyset$ for each $t \in V \cap D$. So we conclude that Φ is single-valued on D and maps into $\widehat{X}$. It follows from Theorem 1.3 that Φ is single-valued and norm upper semi-continuous at the points of a residual subset of A. □

We note that the weak * lower semi-continuous norm $||\cdot|||$ on X^{**} need not be an equivalent norm for X^{**}.

A Banach space has the Radon-Nikodym property if and only if every non-empty bounded subset has slices of arbitrarily small diameter, [**P**, p. 72]. So we could deduce the following known result from Theorem 2.3.

Corollary 2.4. *A Banach space with the Radon-Nikodym property is a GC space.*

It is possible to give a characterisation for *GC* spaces in terms of the behavior of set-valued mappings from a complete metric space into subsets of the original space. To do this we generalise the idea of minimality for set-valued mappings from the characterisation of minimal cuscos given in Lemma 1.1.

We say that a set-valued mapping Φ from a topological space A into subsets of a separated locally convex space X is *minimal* if for any open half-space W in X and open subset U in A where $\Phi(U) \cap W \neq \emptyset$ there exists a non-empty open set $V \subseteq U$ such that $\Phi(V) \subseteq W$.

We use the following selection property of minimal set-valued mappings.

Lemma 2.5. *Consider a Banach space X with a separated locally convex topology τ where the norm closed balls are also τ-closed and a τ-minimal set-valued mapping Φ from a topological space A into subsets of X. If there exists a selection $\tilde{\Phi}$ on a dense set D in A which is norm continuous on D then Φ is single-valued and norm upper semi-continuous at the points of D.*

Proof. Suppose that at $t_0 \in D$, Φ is not single-valued and norm upper semi-continuous. Then there exists an $r > 0$ and in every neighbourhood U of t_0 there exists a $t_1 \in U$ such that $\Phi(t_1) \not\subseteq B\left(\tilde{\Phi}(t_0); r\right)$. Now $x_1 \in \Phi(t_1) \setminus B\left(\tilde{\Phi}(t_0); r\right)$ can be strongly separated from $B\left[\tilde{\Phi}(t_0); \frac{r}{2}\right]$ by a τ-continuous linear functional which generates a τ-open half-space W containing x_1 and $B\left[\tilde{\Phi}(t_0); \frac{r}{2}\right] \subseteq C(W)$. Since $\tilde{\Phi}$ is norm continuous at t_0 there exists an open neighbourhood U of t_0, such that $\tilde{\Phi}(U \cap D) \subseteq B\left(\tilde{\Phi}(t_0); \frac{r}{2}\right)$. But Φ is τ-minimal and $\Phi(U) \cap W \neq \emptyset$. So there exists a non-empty open set $V \subseteq U$ such that $\Phi(V) \subseteq W$. But this contradicts $\tilde{\Phi}(V \cap D) \subseteq C(W)$. So we conclude that Φ is single-valued and norm upper semi-continuous at the points of D. □

The following theorem characterises a GC space X by the behavior of weakly minimal mappings into X.

Theorem 2.6. *For a Banach space X the following are equivalent*

(i) *X is a GC space,*

(ii) *every weakly minimal locally bounded set-valued mapping Φ from a complete metric space A into subsets of X is single-valued and norm upper semi-continuous at the points of a residual subset of A,*

(iii) *every weakly minimal locally bounded single-valued mapping ϕ from a complete metric space A into X is norm continuous at the points of a residual subset of A.*

Proof. (i) $\Rightarrow$ (ii). Consider a weakly minimal locally bounded set-valued mapping Φ from A into subsets of X, and weak * cusco $\overline{\Phi}$ from A into subsets of X^{**} generated by Φ where

$$\overline{\Phi}(t) = \bigcap \left\{ \overline{co}^{w^*} \widehat{\Phi(U)} \text{ where } U \text{ is a neighbourhood of } t \right\}, \text{ [B-F-K, p. 472]}.$$

Since Φ is weakly minimal then from Lemma 1.1 we see that $\overline{\Phi}$ is minimal weak * cusco. But also $\overline{\Phi}(t) \cap \widehat{X} \neq \emptyset$ for all $t \in A$. Since X is a GC space we deduce that $\overline{\Phi}$ is single-valued and norm upper semi-continuous at the points of a residual subset of A, and then so is Φ also.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (i) Consider a minimal weak * cusco Φ from a complete metric space A into subsets of X^{**} where $\Phi(t) \cap \widehat{X} \neq \emptyset$ for all $t \in A$. By Lemma 2.1, we may suppose that Φ is locally bounded on A. Consider a selection $\widetilde{\Phi}$ from A into X. Now $\widetilde{\Phi}$ is a weakly minimal, locally bounded single-valued mapping from A into X so is norm-continuous at the points of a residual subset D of A. It follows from Lemma 2.5 that Φ is single-valued and norm upper semi-continuous at the points of D. □

Although this characterisation enables our computation, it is somewhat unsatisfactory in that it does not give us significant information about the specific properties which identity GC spaces. When looking for a characterisation of GC spaces, it is logical to look for a condition which includes the sufficiency conditions which we have already given. A unifying condition can be found in the concept of fragmentability and its generalisation, [**R1**, p. 247].

Given a topological space X we say that a function $\lambda : X \times X \to \mathbb{R}$ is a *premetric* on X if

(i) $\lambda(x, y) \geq 0$ for all $x, y \in X$ and

(ii) $\lambda(x, y) = 0$ if and only if $x = y$, [**Sc**, p. 225].

We define what we will call the λ-*topology* on X as follows. A subset U of X is said to be λ-open if for every $x_0 \in U$ there exists an $r > 0$ such that $\{x \in X : \lambda(x, x_0) < r\} \subseteq U$. Given $x_0 \in X$ and $\epsilon > 0$, a subset of the form $\{x \in X : \lambda(x, x_0) < \epsilon\}$ is fundamental in defining the λ-topology but it is not necessarily λ-open. We say that λ *fragments* X if, given $\epsilon > 0$, for every non-empty subset E of X there exists a relatively open subset U of E such that

$$\lambda - \operatorname{diam}(U) \equiv \sup\{\lambda(x, y) : x, y \in U\} < \epsilon.$$

We note that the λ-topology on a subset E of X is *stronger* than the relative topology on E if for every $x_0 \in E$ and open set W containing x_0 there exists a $\delta > 0$ such that $\{x \in E : \lambda(x, x_0) < \delta\} \subseteq W$.

If a topological space X has a fragmenting premetric then there exists a fragmenting metric on X, [**R1**, p. 246]. A Banach space which has an equivalent rotund norm has a fragmenting metric for its weak topology, [**R2**]. We recall that $\ell_\infty(\mathbb{N})$ can be equivalently renormed to be rotund but $\ell_\infty(\Gamma)$, where Γ is uncountable, cannot, [**D**, p. 120; 123].

Theorem 2.7. *A Banach space X is a GC space if it possesses a premetric λ where every non-empty bounded set has slices of arbitrarily small λ-diameter, and where the λ-topology on bounded sets is stronger than the* *weak topology.*

Proof. Consider a weakly minimal locally bounded set-valued mapping Φ from a complete metric space A into subsets of X. Given $\epsilon > 0$, consider the set $O_\epsilon \equiv \cup\{$open sets V in A such that λ-diam $\Phi(V) < \epsilon\}$. Now O_ϵ is open in A; we show that it is dense in A. Consider any non-empty open set U in A where $\Phi(U)$ is bounded. Then there is a slice of $\Phi(U)$ with λ-diameter less than ϵ. Since Φ is weakly minimal, there exists a non-empty open set $V \subseteq U$ such that $\Phi(V)$ lies inside this slice and so λ-diam $\Phi(V) < \epsilon$. So O_ϵ is dense in A. Then $D \equiv \bigcap_{n\in\mathbb{N}} O_{\frac{1}{n}}$ is a dense G_δ subset of A where Φ is single-valued. Since the λ-topology is stronger than the weak topology on bounded set, Φ is single-valued and weakly continuous at the points of D. Now D is a dense G_δ subset of the complete metric space A so D is completely metrisable, [**K-N**, p. 96]. Then by Corollary 1.5 there exists a dense G_δ subset E of D and so of A where $\Phi|_D$ is norm continuous . We conclude from Lemma 2.5 that Φ is single-valued and norm upper semi-continuous at the points of E. Our result now follows from Theorem 2.6. □

We show that Theorem 2.7 includes Theorem 1.13. We do this using the following premetric. Given a rotund normed linear space X and using the notation $[x,y] \equiv \{\alpha x + (1-\alpha)y : 0 \le \alpha \le 1\}$, we define the function $\lambda : X \times X \to \mathbb{R}$ by

$$\lambda(x,y) = \max\{||[x,y]||\} - \min\{||[x,y]||\}, [\mathbf{Sc}, \text{ p. } 226].$$

Clearly, $\lambda(x,x) = 0$. If $x \ne y$ then by rotundity $\lambda(x,y) \ge \max\{||[,y]||\} - \frac{1}{2}||x+y|| > 0$. So λ is a premetric on X.

We need the following properties of this premetric. Given $x_0 \in X$ and $r > 0$ we use the notation

$$B_\lambda(x_0;r) \equiv \{x \in X : \lambda(x,x_0) < r\}.$$

Lemma 2.8. *Given a rotund normed linear space X,*

(i) $\lambda(Kx,y) \le \lambda(x,y) + 2|1-K|\,||x||$ *for all* $K \ne 0$ *and* $x,y \in X$,

(ii) $B_\lambda(x;r) \subseteq (||x||+r)B(X)$ *for all* $x \in X$,

(iii) *given* $x \in X$, *for* $K > 1$ *and* $0 < r < (K-1)||x||$,

$$B_\lambda(x;r) \subseteq B_\lambda(Kx; r + 2|1-K|\,||x||) \cap K||x||B(X).$$

Proof. (i) For $0 \le \alpha \le 1$, $||\alpha Kx + (1-\alpha)y|| \le ||\alpha x + (1-\alpha)y|| + \alpha|1-K|\,||x||$, so $\max\{||[Kx,y]||\} \le \max\{||[x,y]||\} + |1-K|\,||x||$. But also, $||\alpha x + (1-\alpha)y|| \le ||\alpha Kx + (1-\alpha)y|| + \alpha|1-K|\,||x||$, so $\min\{||[x,y]||\} \le \min\{||[Kx,y]||\} + |1-$

$K|\,||x||$. Therefore, $\max\{||[Kx,y]||\} - \min\{||[Kx,y]||\} \le \max\{||[x,y]||\} - \min\{||[x,y]||\} + 2|1-K|\,||x||$.

(ii) and (iii) come directly from the definition of λ and (i). □

We notice that if X is a weakly locally uniformly rotund normed linear space then given $x_0 \in X$, $x_0 \neq 0$ and $\epsilon > 0$ and $f \in X^*$, $||f|| = 1$, there exists $\delta(\epsilon, x_0, f) > 0$ such that $|f(x_0 - x)| < ||x_0||\epsilon$ when $x \in ||x_0||B(X)$ and $||x + x_0|| > ||x_0||(2-\delta)$. So if $\lambda(x, x_0) < ||x_0||\frac{\delta}{2}$ and $x \in ||x_0||B(X)$ then

$$\begin{aligned}\frac{1}{2}||x+x_0|| &> \min\{||[x,x_0]||\} > \max\{||[x,x_0]||\} - ||x_0||\frac{\delta}{2}\\ &> ||x_0||\left(1 - \frac{\delta}{2}\right)\end{aligned}$$

so $||x + x_0|| > ||x_0||(2-\delta)$ and it follows that $|f(x_0 - x)| < ||x_0||\epsilon$.

Proposition 2.9. *A Banach space X which has an equivalent weakly locally uniformly rotund norm has a premetric λ where every non-empty bounded subset of X has slices of arbitrarily small λ-diameter and where the λ-topology is stronger than the weak topology.*

Proof. Consider X so renormed and the premetric λ defined above. Consider a non-empty bounded subset A of X and write $s \equiv \sup\{||x|| : x \in A\}$. If $s = 0$ then it is trivially true. If $s \neq 0$ then given $\epsilon > 0$ there exists an $f \in X^*$, $||f|| = 1$ such that the set $E \equiv A \cap S(sB(X), f, \epsilon) \neq \emptyset$. For $x, y \in E$ and writing $r \equiv \max\{||x||, ||y||\} \le s$ we note that $x, y \in S(rB(X), f, \epsilon + r - s)$ and so $\lambda\text{-diam}\, E < \epsilon$.

To show that the λ-topology is stronger than the weak topology it is sufficient to show that each subbasic weak open set is λ-open. At 0 the norm and λ-topologies agree so we consider neighbourhoods of $x_0 \in X$, $x_0 \neq 0$. Given $\epsilon > 0$ consider the weak open subbasic set

$$W \equiv \{x \in X : |f(x) - f(x_0)| < 3\epsilon||x_0||\} \text{ where } f \in X^*,\ ||f|| = 1.$$

Now we have that there exists a $\delta(\epsilon, x_0, f) > 0$ such that $|f(x_0 - x)| < ||x_0||\epsilon$ when $\lambda(x, x_0) < ||x_0||\frac{\delta}{2}$ and $x \in ||x_0||B(X)$. Choose $1 < K < 2$ such that $K - 1 < \min\left\{\frac{\delta}{8}, \frac{\epsilon||x_0||}{|f(x_0)|+1}\right\}$ and then choose $0 < r < \min\left\{||x_0||\frac{\delta}{4}, (K-1)||x_0||\right\}$. From Lemma 2.8(iii) we have that

$$\begin{aligned}B_\lambda(x_0; r) &\subseteq B_\lambda(Kx_0; r + 2(K-1)||x_0||) \cap K||x_0||B(X)\\ &\subseteq B_\lambda\left(Kx_0; ||x_0||\frac{\delta}{2}\right) \cap K||x_0||B(X)\end{aligned}$$

by the choice of K and r.

So $B_\lambda(x_0;r) \subseteq B_\lambda(Kx_0; K||x_0||\frac{\delta}{2}) \cap K||x_0||B(X)$. Therefore $|f(Kx_0 - x)| < K||x_0||\epsilon$ when $x \in B_\lambda(x_0;r)$. But then

$$\begin{aligned} |f(x_0) - f(x)| &\leq |f(x_0) - Kf(x_0)| + |f(Kx_0) - f(x)| \\ &< (K-1)|f(x_0)| + K||x_0||\epsilon \\ &< \frac{\epsilon||x_0||\,|f(x_0)|}{|f(x_0)|+1} + K||x_0||\epsilon \\ &< 3\epsilon||x_0||. \end{aligned}$$

So $B_\lambda(x_0;r) \subseteq W$ and we conclude that the λ-topology is stronger than the weak topology on X. □

It is straight forward to show that Theorem 2.7 includes Theorem 2.3. This follows directly from the following lemma.

Lemma 2.10. *A Banach space X where there exists a weak * lower semi-continuous norm $|||\cdot|||$ on X^{**} has the $|||\cdot|||$-topology stronger than the weak topology on bounded subsets of X.*

Proof. Consider a bounded subset A of X, $x_0 \in A$ and a subbasic weak open neighbourhood of x_0 in A, $W \equiv \{x \in A : |f(x) - f(x_0)| < \epsilon\}$ for $\epsilon > 0$ and $f \in X^*$, $||f|| = 1$. Given $r > 0$ the closed ball $B^{**}_{|||\cdot|||}[\widehat{x}_0;r]$ is weak * closed so $B^{**}_{|||\cdot|||}[\widehat{x}_0;r] \cap (A \setminus W)$ is weak * compact. If $B^{**}_{|||\cdot|||}[\widehat{x}_0;\frac{1}{n}] \cap (A \setminus W) \neq \emptyset$ for all $n \in \mathbb{N}$ then there exists an $F \in \bigcap_{n\in\mathbb{N}} B^{**}_{|||\cdot|||}[\widehat{x}_0;\frac{1}{n}] \cap (A \setminus W)$. But this would contradict the fact that $F \neq \widehat{x}_0$. So there exists an $r > 0$ such that $B^{**}_{|||\cdot|||}(x_0;r) \subseteq W$ and we conclude that the $|||\cdot|||$-topology is stronger than the weak topology on A. □

3. A Banach space which is not a *GC* space.

The Banach space $\ell_\infty(\Gamma)$, where Γ is uncountable, is not a *GC* space. To show this we exhibit a complete metric space P and a weakly minimal, locally bounded set-valued mapping Φ from P into subsets of $\ell_\infty(\Gamma)$ where for each $p \in P$, $\Phi(p)$ is not singleton. Our argument is completed by an appeal to the characterisation given in Theorem 2.6. The construction is based on an example of Talagrand [**Ta**].

We denote by X the set of characteristic functions of countable subsets of Γ with the topology of uniform convergence on countable subsets of Γ. A base of neighbourhoods for $x_0 \in X$ is given by sets of the form $U(x_0, J) \equiv \{x \in X : x|_J = x_0|_J\}$ where J is a countable subset of Γ.

We use the technique of the Banach-Mazur game played on the topological space X, [**C**, p. 115]. This is a game between two players α and β where each player chooses alternately a non-empty open set contained in the other's previously chosen set. Player β begins by choosing U_1. When β chooses U_n then α chooses V_n where $U_n \supseteq V_n$; when α chooses V_n then β chooses U_{n+1} where $V_n \supseteq U_{n+1}$. The sequence of open sets

$$U_1 \supseteq V_1 \supseteq U_2 \supseteq V_2 \supseteq \cdots \supseteq U_n \supseteq V_n \supseteq \cdots$$

is called a *play*. The player α *wins* this play if $\bigcap_{n\in\mathbb{N}} V_n \neq \emptyset$. The game is said to be *α-favourable* if there exists a *winning tactic* by which α chooses V_n dependent only on how β chooses U_n so that α always wins.

Although the following lemma was proved in [**Ta**, p. 160], we will subsequently need to refer to the α-winning tactic used in our proof.

Lemma 3.1. *The topological space X is α-favourable.*

Proof. We define an α-tactic as follows:

For each open set U in X choose a point $x \in U$ and a basic neighbourhood

$$V \equiv U(x, J) \subseteq U.$$

Each play, $U_1 \supseteq V_1 \supseteq U_2 \supseteq V_2 \supseteq \cdots \supseteq U_n \supseteq V_n \supseteq \cdots$ generates a decreasing sequence of basic neighbourhoods

$$V_1 \equiv U(x_1, J_1) \supseteq V_2 \equiv U(x_2, J_2) \supseteq \cdots V_n \equiv U(x_n, J_n) \supseteq \cdots .$$

Clearly, $J_n \subseteq J_{n+1}$ for each $n \in \mathbb{N}$ and each x_{n+1} is an extension of $x_n|_{J_n}$ to J_{n+1}. So we can define a function x_* on Γ as an extension of $x_n|_{J_n}$ for each $n \in \mathbb{N}$ on $J \equiv \bigcup_{n\in\mathbb{N}} J_n$ and zero on $\Gamma \setminus J$. Since J is countable, $x_* \in X$. But also $x_* \in \bigcap_{n\in\mathbb{N}} U(x_n, J_n)$ so we have an α-winning tactic. $\square$

We note that $U(x_*, J) \subseteq \bigcap_{n\in\mathbb{N}} U(x_n, J_n)$ and $U(x_*, J)$ has infinitely many elements.

In Lemma 3.1 we produced an α-winning tactic. We now consider the set $\mathcal{P}$ of all plays

$$p \equiv (U_n, V_n) \equiv U_1 \supseteq V_1 \supseteq U_2 \supseteq V_2 \supseteq \cdots \supseteq U_n \supseteq V_n \supseteq \cdots$$

which follow such an α-winning tactic, with metric ρ defined by

$$\begin{aligned} &\rho(p,p) = 0 \text{ for each } p \in \mathcal{P} \text{ and} \\ &\quad \text{if } p' \equiv (U_n', V_n') \neq (U_n'', V_n'') \equiv p'' \text{ then} \\ &\rho(p',p'') = \frac{1}{n} \text{ where } n \text{ is the first integer where } U_n' \neq U_n''. \end{aligned}$$

If for some $n \in \mathbb{N}$, $U_n' = U_n''$ then from the definition of the play for such an α-winning tactic, $V_n' = V_n''$.

Lemma 3.2. *The metric space $\mathcal{P}$ is complete.*

Proof. Consider a Cauchy sequence $\{p^k \equiv (U_n^k, V_n^k)\}$ in $\mathcal{P}$. Then for every $n \in \mathbb{N}$ there exists some $k_n \geq n$ such that $U_i^{k_n} = U_i^k, V_i^{k_n} = V_i^k$ whenever $1 \leq i \leq n$ and $k \geq k_n$. So we can define a new play $p^* \in \mathcal{P}$ by

$$p^* \equiv (U_n^{k_n}, V_n^{k_n}) \text{ and } \rho(p^k, p^*) \to 0 \text{ as } k \to \infty.$$

□

A similar metric space was studied in [**K-O**, Prop. 2.1].

We now consider the natural embedding π of the topological space X into the Banach space $\ell_\infty(\Gamma)$. For $x_1, x_2 \in X$, $x_1 \neq x_2$ we have that $||\pi(x_1) - \pi(x_2)||_\infty = 1$ and so it is clear that this embedding is nowhere norm continuous on X. However, the natural embedding π of X into $\ell_\infty(\Gamma)$ with its weak topology is continuous at every point of X. We will establish this through two preliminary lemmas.

Given $x \in X$, we denote by $s(x)$ the *support* of x; that is, $s(x) = \{t \in \Gamma : x(t) = 1\}$. Our first result follows from Zorn's lemma.

Lemma 3.3. *Given $f \in \ell_\infty^*(\Gamma)$ which is not identically zero on $\pi(X)$ there exists a non-empty subset A of X which is maximal with respect to the properties*

(i) *$\{s(x) : x \in A\}$ is disjoint family in Γ; that is, for $x_1, x_2 \in A$, $x_1 \neq x_2$ we have $s(x_1) \cap s(x_2) = \emptyset$, and*

(ii) *$f(\pi(x)) \neq 0$ for each $x \in A$.*

Lemma 3.4. *The set A is countable.*

Proof. Given $\epsilon > 0$, consider the set $A_\epsilon \equiv \{x \in A : |f(\pi(x))| \geq \epsilon\}$. Now $A = \bigcup_{n \in \mathbb{N}} A_{\frac{1}{n}}$ so it is sufficient to prove that for every $\epsilon > 0$, A_ϵ is finite. Suppose that for some $r > 0$, A_r is infinite. Then one of the sets $A_r^+ \equiv \{x \in A : f(\pi(x)) > r\}$ or $A_r^- \equiv \{x \in A : f(\pi(x)) < -r\}$ will be infinite. We may suppose that A_r^+ is infinite. For any finite subset A' of A_r^+ we have from property (i) of Lemma 3.3 that $\sum_{x \in A'} \pi(x)$ belongs to the closed unit ball $B(\ell_\infty(\Gamma))$. But $f\left(\sum_{x \in A'} \pi(x)\right) = \sum_{x \in A'} f(\pi(x)) > |A'|r$ where $|A'|$ denotes the number of elements in the finite set A'. But this implies that f is not bounded on $B(\ell_\infty(\Gamma))$ which contradicts the continuity of f. □

We are now in a position to establish our continuity property.

Lemma 3.5. *The natural embedding π of the topological space X into $\ell_\infty(\Gamma)$ with its weak topology is continuous at every point of X.*

Proof. Consider $f \in \ell_\infty^*(\Gamma)$. If f is identically zero on $\pi(X)$ then the result is obvious. Suppose f is not identically zero on $\pi(X)$. Then from Lemma 3.4,

$$J^* \equiv \bigcup\{s(x) : x \in A\} \text{ is a countable subset of } \Gamma.$$

Denote by x^* the characteristic function of J^* on Γ. For every $x \in X$ we have $x = x.x^* + x.(1 - x^*)$, so $f(\pi(x)) = f(\pi(x.x^*)) + f(\pi(x.(1 - x^*)))$. But $s(x.(1 - x^*)) \subseteq s(x) \cap (\Gamma \setminus J^*)$ so $x.(1 - x^*) \in X \setminus A$. Since A is maximal with respect to properties (i) and (ii) of Lemma 3.3, we deduce that $f(\pi(x.(1 - x^*))) = 0$. Therefore, $f(\pi(x)) = f(\pi(x.x^*))$ for all $x \in X$. Now consider $x_0 \in X$ and a basic neighbourhood $U(x_0, J^*)$. For any $x \in U(x_0, J^*)$ we have $x|_{J^*} = x_0|_{J^*}$ and so $x.x^* = x_0.x^*$. Then $f(\pi(x)) = f(\pi(x.x^*)) = f(\pi(x_0.x^*)) = f(\pi(x_0))$. This implies the required continuity of the natural embedding π. □

We now consider the set-valued mapping Φ from $\mathcal{P}$ into subsets of $\ell_\infty(\Gamma)$ defined for the play $p \equiv (U_n, V_n) \in \mathcal{P}$ by

$$\Phi(p) = \bigcap_{n\in\mathbb{N}} \pi(U_n) = \bigcap_{n\in\mathbb{N}} \pi(V_n).$$

It is this set-valued mapping which establishes that $\ell_\infty(\Gamma)$ is not a GC space.

Theorem 3.6. *The set-valued mapping Φ from $\mathcal{P}$ into subsets of $\ell_\infty(\Gamma)$ is weakly minimal, locally bounded and for each $p \in \mathcal{P}$, $\Phi(p)$ is not singleton.*

Proof. Clearly, for each $p \in \mathcal{P}$, $\Phi(p) \subseteq B(\ell_\infty(\Gamma))$. For each play $p \equiv (U_n, V_n)$ we note from Lemma 3.1 that the set $E_p \equiv \bigcap_{n\in\mathbb{N}} U_n = \bigcap_{n\in\mathbb{N}} V_n$ is a subset of X which contains more than one point. So for each $p \in \mathcal{P}$, $\Phi(p) = \bigcap_{n\in\mathbb{N}} \pi(U_n)$ is not singleton.

Consider $f \in \ell_\infty^*(\Gamma)$ generating a weak open half-space W in $\ell_\infty(\Gamma)$ and play $p^o \equiv (U_n^o, V_n^o) \in \mathcal{P}$ such that $x^o \in \Phi(p^o) \cap W$. Now by Lemma 3.5, the natural embedding π of X into $\ell_\infty(\Gamma)$ is weakly continuous so $\pi^{-1}(W)$ is a non-empty open subset of X. Given $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $n_0 > \frac{1}{\delta}$ consider any play $p' \equiv (U_n', V_n') \in \mathcal{P}$ such that $U_i' = U_n^o$, $V_i' = V_i^o$ for all $1 \le i \le n_0$ and $U'_{n_0+1} = U^o_{n_0+1} \cap \pi^{-1}(W)$. Now $\rho(p', p^o) < \frac{1}{n_0} < \delta$. But since $\pi(U'_{n_0+1}) \subseteq W$ we have $\Phi(p') \subseteq W$. So Φ is weakly minimal. □

Note added in proof

Professor Isaac Namioka has recently given an example to show that $\ell_\infty(\mathbb{N})$ is not a GC space.

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MOON HYPERSURFACES AND SOME RELATED EXISTENCE RESULTS OF CAPILLARY HYPERSURFACES WITHOUT GRAVITY AND OF ROTATIONAL SYMMETRY

FEI-TSEN LIANG

Let $\Omega_*(R)$ be a domain in $\mathbb{R}^n$ bounded by two spherical caps Σ_1 and Σ_2 of respective radii $\frac{n-1}{n}$ and R, with $\frac{n-1}{n} < R < 1$. (cf. Figure 1 for $n = 3$). We consider the vertical cylinder Z over $\partial\Omega_*(R)$ and seek a hypersurface $u_R(x_1, \dots, x_n)$ over $\Omega_*(R)$ of constant mean curvature $H \equiv 1$ which meets Z in the angle π (vertically downward) over $\Sigma_1(R)$ and the angle 0 (vertically upward) over $\Sigma_2(R)$; intuitively and essentially, this amounts to seeking a solution to the problem

$$(0.1)\qquad \begin{cases} \operatorname{div} Tu_R = & n \\ \nu \cdot Tu_R = & \begin{cases} -1 & \text{on } \Sigma_1(R) \\ 1 & \text{on } \Sigma_2(R), \end{cases} \end{cases}$$

ν being outward unit normal.

0. Introduction.

In view of the shape of the base domain $\Omega_*(R)$, we shall, as in [**FG**] for $n = 2$, refer to $\Omega_*(R)$ as *n-dimensional moon domains* and as in [**F2**], refer to the solution of (0.1) as *moon (hyper)-surfaces*. Such a moon surface ($n = 2$) is chosen to majorize the gradient of solution $u(x)$ of

$$(0.2)\qquad \operatorname{div} Tu = 2$$

in $B_R, R_0^{(2)} < R < 1$, with $R_0^{(2)} = 0.565406\dots$ being the unique value of R for which $\Sigma_1(R)$ passes through the center of the circle including $\Sigma_2(R)$. This enables us to show the existence of apriori gradient bounds for solution of the equation (0.2) in $B_R, R_0^{(2)} < R < 1$, in [**FG**].

0.1. We note that, an integration of (0.1) over the section $\Omega_*(R)$ yields

$$(0.3)\qquad |\Sigma_2(R)| - |\Sigma_1(R)| = n|\Omega_*(R)|.$$

Thus, the condition (0.3) is necessary for existence of the moon hypersurfaces u_R.

In §3 and §5.1 of this paper, the existence of *n-dimensional moon domains* $\Omega_*(R)$, $1 > R > \frac{n-1}{n}$, characterized by *the condition* (0.3), will be verified, for $n = 3$ and $n > 3$, respectively. The existence of *moon hypersurfaces*, for $n = 3$ and $n > 3$, will be proved in §1 and §5.2, respectively. These results may help us to extend the above-mentioned apriori gradient estimates to higher dimensions.

0.2. As in [**F2**] and §3 of [**L1**] for $n = 2$, we shall, in §2 and §5.3, for $n = 3$ and $n > 3$, respectively, in a suitable sense indicated there, *construct the moon (hyper)-surface as a limit of solutions* u_ϵ *to* (1.2) *defined throughout the sphere* B_R *including* $\Sigma_2(R)$. This result will also be applied in [**L2**] to show that absolute gradient estimates cannot hold for solutions of

$$\operatorname{div} Tu = n \tag{0.4}$$

in B_R, $R < R_0^{(n)}$, $R_0^{(n)}$ being the unique value of R for which $\Sigma_1(R)$ passes through the center of the sphere including $\Sigma_2(R)$. As calculated in the ending of §4, we have

$$R_0^{(3)} = \frac{-2 + 2\sqrt{19}}{9} = 0.746421987\cdots \text{ (cf. (4.11))}.$$

For $n > 3$, $R_0^{(n)}$ is determined as in §5.1.1.

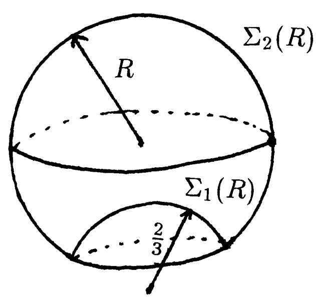

Figure 1. (n=3)

0.3. The proof of the existence of the moon hypersurfaces u_R and the existence of that sequence of solutions converging to it are reduced to the general existence results in Finn [**F1**]. That is, in §1, we shall verify, for $n = 3$,

(0.5.1)
$$\phi[\Omega^o] \equiv |\partial\Omega^o \cap \Omega_*| + |\partial\Omega^o \cap \Sigma_1| - |\partial\Omega^o \cap \Sigma_2| + n|\Omega^o| > 0$$

(0.5.2)
$$\psi[\Omega^o] \equiv |\partial\Omega^o \cap \Omega_*| - |\partial\Omega^o \cap \Sigma_1| + |\partial\Omega \cap \Sigma_2| - n|\Omega| > 0$$

for every Caccioppoli set $\Omega^0 \subseteq \Omega_*$, $\Omega^0 \neq \phi, \Omega_*$; in §2, we shall verify, for $n = 3$, for ϵ sufficiently small

(0.6.1)
$$\phi[\Omega^0] \equiv |\partial\Omega^0 \cap \Omega_*| - (1-\epsilon)|\partial\Omega^0 \cap \Sigma_2| - \widehat{\beta}_\epsilon \left|\partial\Omega^0 \cap \widehat{\Sigma}\right| + n|\Omega^0| > 0,$$

(0.6.2)
$$\psi[\Omega^0] \equiv |\partial\Omega^0 \cap \Omega_*| + (1-\epsilon)|\partial\Omega^0 \cap \Sigma_2| + \widehat{\beta}_\epsilon \left|\partial\Omega^0 \cap \widehat{\Sigma}\right| - n|\Omega^0| > 0,$$

for every Caccioppoli set $\Omega^0 \subseteq B_R$, $1 > R > 2/3$, $\Omega^0 \neq \phi$, B_R where $\widehat{\Sigma} = \partial B_R - \Sigma_2$ and $\widehat{\beta}_\epsilon(R)$ is a constant depending on R, and defined by the equation (2.1); $-1 < \widehat{\beta}_\epsilon < 1$ for $1 > R > 2/3$ and $-1 < \beta_\epsilon < 0$ for $1 > R > R_0^{(3)}$. The verification of (0.5.1), (0.5.2), (0.6.1) and (0.6.2), however, is not a straightforward generalization of that of the two dimensional case, *due to the fact that the hypersurfaces of constant mean curvature are in general not spherical.* A new approach is inexcusably required. We will draw on the technique of the rearrangement of level curves. The rotational symmetry of both the boundary surface ∂B_R and the boundary data will therefore play a crucial role in our investigation. Also, in this connection, we find that, in both cases of §1 and §2, it is more easy and natural to discuss $\psi[\Omega^0]$ than $\phi[\Omega^0]$; thus because of the respective equivalence of (0.5.1), (0.6.1) and (0.5.2), (0.6.2), we will restrict our attention to (0.5.2) and (0.6.2). In either case, a minimizing body for $\psi[\Omega^0]$ exists and, using our new technique, the only possible non-empty minimizing body for $\psi[\Omega^0]$ is shown to have a spherical cap of radius 2/3 and passing through $\partial\Sigma_1$ as its boundary in the sphere B_R (obtained by completing Σ_2). This only possible non-empty minimizing body includes or is included in a hemisphere in the case of §1 or §2, respectively, and has $\psi > 0$ in either case, thereby proving that the empty set is the one and only minimizing body for $\psi[\Omega^0]$. (0.5.2) and (0.6.2) are immediate consequences of this.

The main tool used in this case of §1 is, what is known as the classical isoperimetric inequality. We, however, find difficulties in applying this technique to the case of §2, mainly due to the boundary data $1-\epsilon$ being unequal to 1. *Steiner symmetrization* is suitably modified to prove that the minimizing body for $\psi[\Omega^0]$ in (0.6.2) is a surface of revolution, with the extremely useful help of the analyticity of the boundary surface in B_R of a minimizing body for $\psi[\Omega]$ and $n = 3$, (which is provided by Massari [**Ma**]).

0.4. For simplicity of writing and convenience of visualization, we deal exclusively with the case of three dimensional domains in §1, §2, §3 and §4. In the chapter §5, we will extend the results in these chapters to domains of dimension higher than three. We note that, for $n > 7$, Massari's Theorem [**Ma**] does not yield the analyticity of the boundary surface in B_R of a minimizing body for $\psi[\Omega]$. This difficulty of extension, however, as we shall observe in §5.3, is insubstantial. Reviewing the argument used in §2 and §5.3, incidentally, will enable us to formulate in §6 some existence results of capillary hypersurfaces whose domain of definition and boundary data are of rotational symmetry about the same axis.

1. Existence of the Moon Hypersurfaces for $n = 3$.

In this section, we shall prove.

Theorem 1.1. *Let $\Omega_* \subset \mathbb{R}^3$ be a "moon domain", bounded by two spherical caps Σ_1 and Σ_2 with the respective radii $\frac{2}{3}$ and R, $1 > R > \frac{2}{3}$, which satisfies the condition*

$$|\Sigma_1| - |\Sigma_2| = 3|\Omega_*|. \tag{1.1}$$

Then the problem

$$(1.2)\ \operatorname{div} Tu = 3 \quad in \quad \Omega_*,$$

$$\int_{\Omega_*} (w_{p_i}\zeta_i + 3\eta)\, dx + \int_{\Sigma_1} \eta\, d\sigma - \int_{\Sigma_2} \eta\, d\sigma = 0 \quad \text{for all } \eta \in H^{1,1}(\Omega_x)$$

$$\text{where} \quad \zeta_i = \eta_{x_i}, \quad w = \sqrt{1 + |p|^2}, \quad p = (p_1, p_2, p_3), \quad p_i = v_{x_i}$$

has a solution $u(x)$, unique up to an additive constant.

1.1. Background information. As in §2 of [**L1**], we reduce the proof of Theorem 1.1 to the general existence results in Finn [**F1**], which, although have been formulated for two dimensional domains, can be easily extended to higher dimensions by the same argument.

As in [**F1**], the capillary problem in the absence of gravity can be reduced to the variational problem for a functional

$$\xi[u] = \int_\Omega \sqrt{1 + |Du|^2} + nH \int_\Omega u\, dx - \int_{\partial\Omega} \beta(s) u\, d\sigma,$$

with $\beta(s)$, $-1 \le \beta(s) \le 1$, being piecewise Lipschitz continuous on the boundary $\partial\Omega$ of a bounded domain $\Omega \subseteq \mathbb{R}^n$, and H being a constant. As in §2 of [**L1**], for future reference we formulate

Proposition 1. *If, for a piecewise Lipschitz domain* Ω, *both the conditions*

$$\phi\left[\Omega^0\right] = \left|\partial\Omega^0 \cap \Omega\right| - \int_{\partial\Omega^0\cap\partial\Omega} \beta\, ds + nH\left|\Omega^0\right| > 0 \tag{1.3}$$

and

$$\psi\left[\Omega^0\right] = \left|\partial\Omega^0 \cap \Omega\right| + \int_{\partial\Omega^0\cap\partial\Omega} \beta\, ds - nH\left|\Omega^0\right| > 0 \tag{1.4}$$

hold for every Caccioppoli set $\Omega^0 \neq \phi,\ \Omega\ (\Omega^0 \subseteq \Omega)$. *Then there is a minimizing function* $u(x) \in BV_{\mathrm{loc}}(\Omega)$ *for* $\xi[u]$. *Furthermore, the minimizing function is unique up to an additive constant, is regular and locally bounded in* Ω, *satisfies in* Ω *the Eq.* (0.1) *and the variational condition*

$$\int_\Omega \left(w_{p_i}\zeta_i + nH\eta\right) dx - \int_{\partial\Omega} \beta\eta = 0 \tag{1.5}$$

for any $\eta \in H^{1,1}(\Omega)$, $\zeta_i = \eta_{x_i}$; *here*

$$W = \sqrt{1+|p|^2}, \quad p = (p_1, p_2, \ldots, p_n), \quad p_i = u_{x_i}.$$

1.2. The Proof of Theorem 1.1. In view of Proposition 1, it suffices to show (0.5.1) and (0.5.2) for every Caccioppoli set $\Omega^0 \subseteq \Omega_*$, $\Omega^0 \neq \phi, \Omega_*$.

To show this, we first observe that if $\overline{\Omega}^0 \subset \Omega_*$, then

$$\begin{aligned} \phi\left[\Omega^0\right] &\equiv \left|\partial\Omega^0 \cap \Omega\right| + 3\left[\Omega^0\right| > 0 \\ \psi\left[\Omega^0\right] &\equiv \left|\partial\Omega^0 \cap \Omega\right| - 3\left[\Omega^0\right| > 0 \end{aligned} \tag{1.6}$$

where the last inequality is an immediate consequence of the following Proposition. (Henceforth, we denote the characteristic function of a Caccioppoli set E as $\varphi_{E'}$, and the integral $\int_{B_R} |D\varphi_E|$, denoted as the perimeter of E in B_R, is defined by $\int_{B_R} |D\varphi_E| = \sup \int_{B_R} \varphi_E \operatorname{div} g$ among all vector functions $g \in C_0^1(B_R)$, $|g| \leq 1$. This integral equals the surface area of ∂E in B_R whenever this boundary is smooth.)

Proposition 2. *If* A *is a Caccioppoli set with* $A \subseteq B_R$, $0 < R < 1$, *then*

$$\int_{B_R} |D\varphi_A| - 3\int_{B_R} \varphi_A\, dx > 0.$$

Proof. Let $v(x)$, defined on B_1, describe the lower unit hemisphere, then

$$\operatorname{div} Tv = 3 \quad \text{in } B_1. \tag{1.7}$$

If $A \subseteq B_R$, $0 < R < 1$, we can integrate the Eq. (1.7) in A, obtaining

$$3\int_{B_R} \varphi_A \, dx = \int_A \operatorname{div} Tv \, dx = -\langle D\varphi_A, Tv\rangle,$$

and hence, as $|Tv| < 1$ in $A, 3\int_{B_R} \varphi_A \, dx < \int_{B_R} |D\varphi_A|$. □

(We note that this result is alternatively obtained in Giusti [**G1**], pages 114 and 115.)

Thus it now suffices to consider all those sets intersecting $\partial\Omega_*$ with a set of positive area. We shall show that (0.5.2) *holds for all those* $\Omega^0 \subsetneqq \Omega_*$ *which have either or both of* $|\partial\Omega^0 \cap \Sigma_1|$ *and* $|\partial\Omega^0 \cap \Sigma_2| > 0$. Once we show this, since, for all the Caccioppoli sets $\Omega^0 \subseteq \Omega_*$,

$$\begin{aligned}(1.8)\quad \phi\left[\Omega^0\right] &\equiv \left|\partial\Omega^0 \cap \Omega\right| + \left|\partial\Omega^0 \cap \Sigma_1\right| - \left|\partial\Omega^0 \cap \Sigma_2\right| + 3\left|\Omega^0\right| \\ &= \left|\partial\Omega^0 \cap \Omega\right| - \left|\Sigma_1 \cap \partial\left(\Omega_* - \Omega^0\right)\right| \\ &+ \left|\Sigma_2 \cap \partial\left(\Omega_* - \Omega^0\right)\right| - 3\left|\Omega - *\Omega^0\right|, \qquad \text{(by (1.1))} \\ &= \psi\left[\Omega_* - \Omega^0\right],\end{aligned}$$

(0.5.2) implies that there also holds (0.5.1) for all the Caccioppoli sets $\Omega^0 \subseteq \Omega_*$, $\Omega^0 \neq \phi$, Ω_*. The proof of Theorem 1.1 can thus be completed.

To show this, we first observe that if Ω^0 has $|\partial\Omega^0 \cap \Sigma_1| = 0$, then $\psi[\Omega^0] = |\partial\Omega^0 \cap \Omega_*| + |\partial\Omega^0 \cap \Sigma_2| - 3|\Omega_0| > 0$, again due to *Proposition* 2.

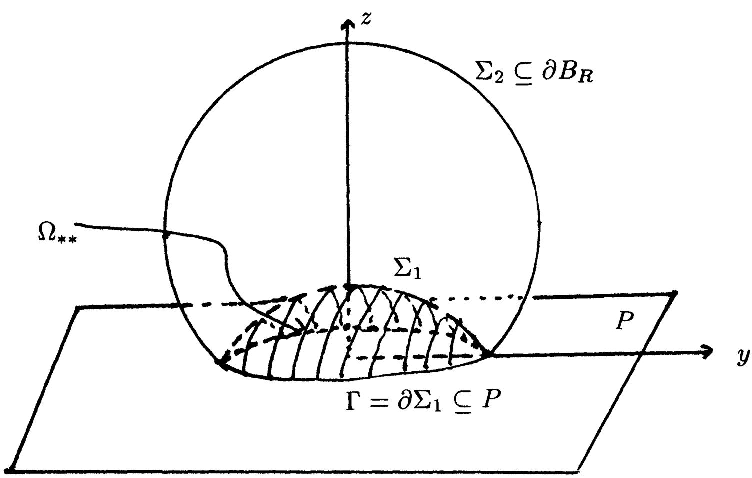

Figure 2.

Thus, it suffices to consider all those Caccioppoli sets Ω^0 *with* $|\partial\Omega^0 \cap \Sigma_1| >$ 0 *and* $|\partial\Omega^0 \cap (\Omega_* \cup \Sigma_2)|$ *being connected. We observe also that, for all such sets we can always assume that* $\partial\Omega^0 \cap \Sigma_1 = \Sigma_1$, for otherwise we could add to Ω^0 and ϵ-neighbourhood of Σ_1 and then pass to limit as $\epsilon \to 0^+$. (Here we note that the *boundary data* β_ϵ *being* $\equiv -1$ *on* Σ_1 enables us to do so.) *We call the collection of these sets as* S. For sets in S, we have

$$\psi\left[\Omega^0\right] \equiv \left|\partial\Omega^0 \cap \Omega_*\right| - \left|\Sigma_1\right| + \left|\partial\Omega^0 \cap \Sigma_2\right| - 3\left|\Omega^0\right|; \tag{1.9}$$

to minimize this expression (1.9) among all these sets in S, however, is equivalent to minimizing

$$\psi^*\left[\Omega^0\right] = \left|\partial\Omega^0 \cap \Omega_*\right| + \left|\partial\Omega^0 \cap \Sigma_2\right| - 3\left|\Omega^0 \cup \Omega_{**}\right| \tag{1.10}$$

in the same collection of sets, where Ω_{**} is that part of $B_R - \Omega_*$ lying above the unique plane P passing through the circle $\Gamma = \partial\Sigma_1$ (see Figure 2). Here and in the following, we assume B_R to be the sphere that is obtained by completing Σ_2, P to be the x, y plane and that side of P containing the center of B_R to be "above" P.

As in §2 of [**L1**], we consider a minimizing sequence $\left\{\Omega_j^0\right\}$ for the functional $\psi^*[\Omega^0]$ in (1.10), and use the same argument to conclude from Theorem 1.19 in Giusti [**G2**] that there is a subsequence of $\{\varphi_{\Omega_j^0}\}$ that converges in $L^1(\Omega)$ to $\varphi_{\overline{\Omega}'}$ and that setting $\tilde{\Sigma} = \partial\tilde{\Omega} \cap \Omega_*$

$$\left|\tilde{\Sigma}\right| = \int_{\Omega_*} \left|D\varphi_{\tilde{\Omega}}\right| \le \inf \int_{\Omega_*} \left|D\varphi_{\Omega_j^0}\right|.$$

Further, we have

$$\psi^*\left[\tilde{\Omega}\right] \le \inf \psi^*\left[\Omega_j^0\right],$$

by a reasoning similar to that used for the proof of Lemma 6.3 in Finn [**F1**].

We proceed to characterize the geometry of $\tilde{\Sigma}$.

Proposition 3. *If* $\tilde{\Sigma} \neq \phi$, *then* $\tilde{\Sigma}$ *must be a spherical cap passing through* $\partial\Sigma_1$.

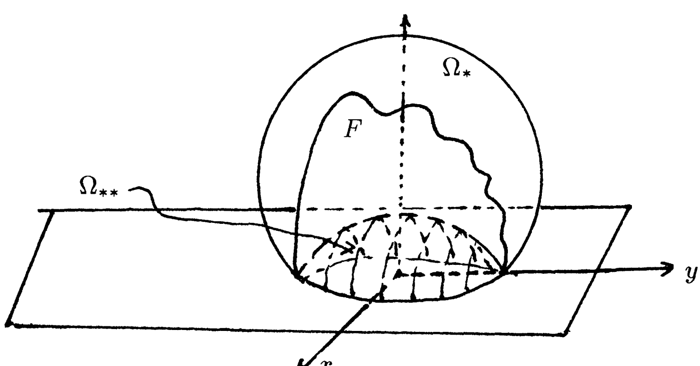

Figure 3.

Proof of Proposition 3. We consider an arbitrary body F in $\Omega_* \cup \Omega_{**} \cup \Sigma_2$ (cf. Figure 3), passing through $\Gamma = \partial\Sigma_1$, and bounded below by the disk $P \cap B_R$. From the discussion below Figure 2 and above (1.9), we may, without loss of generality, assume that $F \setminus \Omega_{**}$ is in the collection S. Now that $\psi^*[F \cap \Omega_*] = |\partial F \cap (\Omega_* \cup \Sigma_2)| - 3|F|$, we shall prove Proposition 3 by constructing a body $\widehat{F}$ such that $\widehat{F} \setminus \Omega_{**}$ is in the collection S, and that

$$\left|\widehat{F}\right| = |F|,$$
$$|\partial F \cap (\Omega_* \cup \Sigma_2)| \geq \left|\partial \widehat{F} \cap (\Omega_* \cup \Sigma_2)\right|,$$

where the last equality holds only when $\partial F \cap (\Omega_* \cup \Sigma_2)$ is a spherical cap passing through Γ.

We observe first that, for each value V with

$$|\Omega_{**}| < V < \left|\overline{\Omega}_* \cup \Omega_{**}\right|,$$

a spherical cap passing through Γ and situating above P exists, the volume enclosed by which and disk $P \cap B_R$ is equal to V. (Cf. Figure 4).

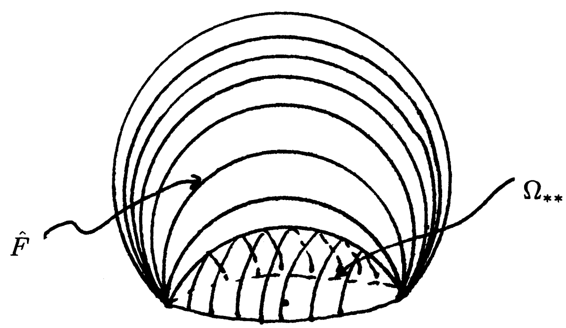

Figure 4.

Now that

$$|\Omega_{**}| < |F| < \left|\overline{\Omega}_* \cup \Omega_{**}\right|,$$

a body with

$$\left|\widehat{F}\right| = |F|$$

exists which has a spherical cap $\widehat{\Sigma}$ as its boundary in Ω_*. Obviously, $\widehat{F} \setminus \Omega_{**}$ is in the collection S. Furthermore, we may extend the spherical cap $\widehat{\Sigma}$ to a full sphere Σ which is the boundary of a ball B. Then

$$|F| + |B - F| = \left|\widehat{F}\right| + \left|B - \widehat{F}\right| = |B|,$$

and the isoperimetric inequality for three dimensions (Cf. [**MM**], p. 92) asserts that

$$|\partial F \cap (\Omega_* \cup \Sigma_2)| + \left|\Sigma \setminus \widehat{\Sigma}\right| \geq |\Sigma|,$$

that is,

$$|\partial F \cap (\Omega_* \cup \Sigma_2)| \geq \left|\widehat{\Sigma}\right| = \left|\partial \widehat{F} \cap (\Omega_* \cup \Sigma_2)\right|,$$

and equality holds only when $F = \widehat{F}$. □

Also, by the analyticity of $\overline{\Sigma}$ (see [**Ma**]), we may use an argument similar to that one used to prove Lemma 6.4 in page 148 of [**F1**] to conclude.

Lemma 1. *If $\overline{\Sigma} \neq \phi$ then $\overline{\Sigma}$ must consist of surfaces of constant mean curvature 3/2 and $\overline{\Omega}$ lies on the side of $\overline{\Sigma}$ into which the curvature vector points.*

Putting Proposition 3 and Lemma 1 together, we see that a non-empty $\tilde{\Sigma}$ must be a spherical cap of radius 2/3, which can possibly occur only when Σ_1 is a subset of a hemisphere of radius 2/3 and $\tilde{\Sigma}$ strictly includes a hemisphere of radius 2/3. In case that Σ_1 is included in a hemisphere, denting Σ_0 as the spherical cap of radius 2/3, included in Ω_* and Ω_0 as the body enclosed by Σ_0 and Σ_1, we shall show

$$\psi^* \left[\overline{\Omega}_0 \cup \Omega_{**}\right] - \psi^* \left[\Omega_{**}\right] > 0, \quad \text{where } \psi^* \left[\Omega_{**}\right] \equiv |\Sigma_1| - 3\left|\Omega_{**}\right|$$

and hence

$$\begin{aligned}\psi\left[\Omega_0\right] &= \psi^* \left[\Omega_0 \cup \Omega_{**}\right] - |\Sigma_1| + 3\left|\Omega_{**}\right| \\ &> \psi^* \left[\Omega_{**}\right] - |\Sigma_1| + 3\left|\Omega_{**}\right| \\ &= 0,\end{aligned}$$

thereby proving (0.5.2), as minimizing ψ and ψ^* are one and the same matter.

In fact, adopting spherical coordinates with origin at the center 0 of $B_{2/3}$ including Σ_0, we choose $\theta_1 < \pi/2$ so that the equation $r = \dfrac{2\sin(\pi - \theta_1)}{3}$ is that for the circle $\Gamma\,(\equiv \partial\Sigma_1)$. Thus, (cf. Figure 5) as calculated in (3.7) and (3.8) for $R = \frac{2}{3}$ (cf. (3.1), (3.3))

$$\begin{aligned}\psi^* \left[\overline{\Omega}_0 \cup \Omega_{**}\right] - \psi^* \left[\Omega_{**}\right] &= \left(\frac{8}{27}\pi + \frac{8}{27}\pi\cos^3\theta_1\right) - \left(\frac{8}{27}\pi - \frac{8}{27}\pi\cos^3\theta_1\right) \\ &= \frac{16}{27}\pi\cos^3\theta_1 \\ &> 0,\end{aligned}$$

as desired.

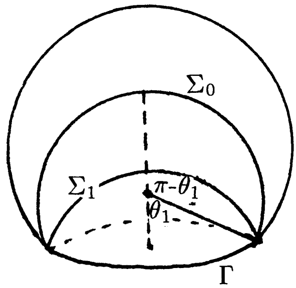

Figure 5.

2. Moon Hypersurfaces constructed as a generalized solution over B_R in the sense of Miranda for $n = 3$.

As in Sec. 7.11 of Finn [**F1**], II of Finn [**F2**], or §3 of [L1], let us extend the spherical cap Σ_2 to a full sphere ∂B_R, and write $\widehat{\Sigma} = \partial B_R - \Sigma_2$ (cf. Figure 6). Then if ϵ is small enough, it will be verified in §5 that there is unique $\widehat{\beta}_\epsilon(R)$, $-1 < \widehat{\beta}_\epsilon < 1$ for $1 > R \geq 2/3$ and $-1 < \widehat{\beta}_\epsilon < 0$ for $1 > R \geq R_0^{(3)}$, such that data

$$\beta_\epsilon = \begin{cases} 1 - \epsilon & \text{on } \Sigma_2 \\ \widehat{\beta}_\epsilon & \text{on } \widehat{\Sigma}, \end{cases}$$

satisfies the necessary condition

$$(1 - \epsilon)\,|\Sigma_2| + \widehat{\beta}_\epsilon \left|\widehat{\Sigma}\right| = 3\,|B_R| \tag{2.1}$$

for the existence of a minimizing function $u_\epsilon(x) \in BV_{\text{loc}}(B_R)$, which minimizing the functional

$$\xi^\epsilon[u] = \int_{B_R} \sqrt{1 + |\nabla u|^2} + 3\int_{B_R} u\,dx - \int_{\partial B_R} \beta_\epsilon(s) u\,ds,$$

and thus (cf. Proposition 1) satisfies

$$\operatorname{div} T u_\epsilon = 3$$

in B_R; here (2.1) is necessary because substituting $\eta(x) = 1$ (in $\overline{B}_R$) into the variational condition (1.5) for this particular function $\xi^\epsilon[u]$ yields (2.1).

We shall show that (a) *this minimizing function* $u_\epsilon(x)$ *indeed exists if* ϵ *is small enough, and* (b) *as* $\epsilon \to 0$, $|\nabla u_\epsilon|$ *cannot be bounded in* ϵ *for any subset of* $|\Sigma_1|$ *of positive area.*

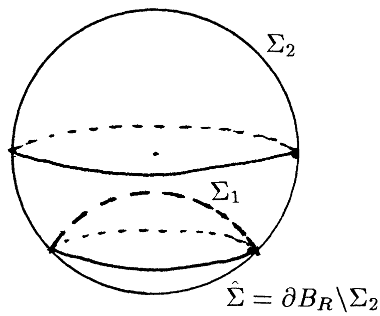

Figure 6.

2.1. *To prove* (a), in view of Proposition 1, it suffices to show that, for sufficiently small ϵ, (0.6.1) and (0.6.2) hold for every Caccioppoli set $\Omega^0 \subseteq B_R$, $\Omega^0 \neq \phi$, B_R. To show this, as in §1, we first observe that, if $\overline{\Omega^0} \subseteq B_R$, then

$$\phi\left[\Omega^0\right] \equiv \left|\partial\Omega^0\right| + 3\left|\Omega^0\right| > 0,$$
$$\psi\left[\Omega^0\right] \equiv \left|\partial\Omega^0\right| - 3\left|\Omega^0\right| > 0,$$

where the last inequality readily follows from *Proposition* 2 in §1. Thus, it suffices to consider all those sets whose intersection with ∂B_R is a set of positive area. We shall show that (0.6.2) for $n = 3$ holds for all those $\Omega^0 \subsetneqq \Omega_*$ which have $\left|\partial\Omega^0 \cap \Sigma_2\right| > 0$ or $\left|\partial\Omega^0 \cap \hat{\Sigma}\right| > 0$. As in §1, we note that *proof of* (a) *will be completed once we verify the truth of* (0.6.2), *because there holds by virtue of* (2.1),

$$\phi\left[\Omega^0\right] = \psi\left[B_R - \Omega^0\right],$$

for each Caccioppoli set $\Omega^0 \subseteq B_R$.

To show (0.6.2) for $n = 3$, we first observe that *if* ϵ *is small enough,*

$$\psi\left[\Omega^0\right] \equiv \left[\partial\Omega^0 \cap B_R\right] + (1-\epsilon)\left|\partial\Omega^0 \cap \Sigma_2\right| - 3\left|\Omega^0\right| > 0$$

for all the Caccioppoli sets Ω_0 *with* $\left|\partial\Omega^0 \cap \hat{\Sigma}\right| = 0$. *This follows from Proposition* 2 *and* Giusti [**G1**], Lemma 1.

Thus it suffices to consider all those Caccioppoli sets Ω^0 *with* $\left|\partial\Omega^0 \cap \hat{\Sigma}\right| > 0$ *and* $\partial\Omega^0 \cap (B_R \cup \Sigma_2)$ *being connected.*

As in §1, we may try to minimize $\psi[\Omega^0]$ among all the Caccioppoli sets $\overline{\Omega}$ and consider a minimizing sequence $\left\{\Omega_j^0\right\}$ for $\psi[\Omega^0]$; the same reasoning concludes that there exists a subsequence of the $\left\{\varphi_{\Omega_j^0}\right\}$ converging in $L^1(\Omega)$ to $\varphi_{\tilde{\Omega}}$ such that

$$\psi\left[\tilde{\Omega}\right] \leq \inf \psi\left[\Omega_j^0\right].$$

Set $\tilde{\Sigma} = \tilde{\Omega} \cap B_R$. If $\tilde{\Sigma} \neq \phi$, we have shown that $\left|\partial\tilde{\Omega} \cap \widehat{\Sigma}\right| > 0$.

Due to the very fact that $\beta_\epsilon \neq 1$, it seems infeasible to proceed further as in §1. We may, however, take a different approach and arrive at the same conclusion. The main idea of the following discussion is provided by Steiner's solution to the two dimensional isoperimetric problem.

Our main aim is to show

Proposition 4. *The only non-empty candidate for $\tilde{\Sigma}$ is the spherical cap Σ_1. In other words, the only non-empty candidate for $\tilde{\Omega}$ is $B_R - \Omega_*$.*

We again let P to be the unique plane passing through the circle $\Gamma \equiv \partial\Sigma_1 \cap \partial B_R$ and designate P as the x, y plane so that the center of B_R has the z-coordinate $z > 0$.

To prove Proposition 4, we shall proceed to verify

Proposition 3*. *If $\tilde{\Sigma} \neq \phi$, then $\tilde{\Sigma}$ is made up of surfaces of revolution about the Z-axis.*

We will reduce the proof of Proposition 3* to that of the following

Proposition 3.** *If $\tilde{\Sigma} \neq \phi$, then at each point of $\tilde{\Sigma}$, the tangent of the horizontal cross-section of $\tilde{\Sigma}$ through this point is the normal of the unique vertical plane $ax + by = 0, a, b$: constants, passing through this point (and the origin).*

The equivalence of Proposition 3* and Proposition 3** is obvious; in fact, at each height z_0, Proposition 3** yields that

$$x\dot{x} + y\dot{y} = 0$$

for each connected subarc $(x(t), y(t), z_0)$ of the horizontal cross-section of $\tilde{\Sigma}$, which holds if and only if

$$x^2 + y^2 = \text{ constant},$$

i.e., $(x(t), y(t), z_0)$ describes a circle with the center on the z-axis. This amounts to Proposition 3*.

We thus proceed to give a

Proof of Proposition 3**. Consider a vertical plane $\widehat{P}: ax+by=0$, a, b: constants, which divides $\overline{\Omega}$ into two non-empty parts Ω_1 and Ω_2 (and of course passes through a great circle of ∂B_R). We can assume $\psi[\Omega_1] \leq \psi[\Omega_2]$. Reflecting the body Ω_1 in the plane $\widehat{P}$, we obtain a body Ω_1' on the opposite side of the plane $\widehat{P}$ such that

$$\Omega_1 \cup \Omega_1' \subseteq \overline{B}_R.$$

Then

$$\begin{aligned}\psi[\Omega_1 \cup \Omega_1'] &= \psi[\Omega_1] + \psi[\Omega_1'] - 2\left|\partial\Omega_2 \cap \widehat{P}\right| \\ &\leq \psi[\Omega_1] + \psi[\Omega_2] - \left|\partial\Omega_1 \cap \widehat{P}\right| - \left|\partial\Omega_2 \cap \widehat{P}\right| \\ &= \psi\left[\tilde{\Omega}\right]\end{aligned}$$

since $\psi[\Omega_1'] = \psi[\Omega_1] \leq \psi[\Omega_2]$ (cf. Remark 1 below) and $\left|\partial\Omega_1 \cap \widehat{P}\right| = \left|\partial\Omega_2 \cap \widehat{P}\right|$, by construction. The minimizing property of $\overline{\Omega}$ yields $\psi[\Omega_1 \cup \Omega_1'] = \psi\left[\overline{\Omega}\right]$ (and hence $\psi[\Omega_1] = \psi[\Omega_2]$). The body $\Omega_1 \cup \Omega_1'$ is therefore another minimizing body for $\psi[\Omega]$ and the theorem of Massari [**Ma**] thus yields the *analyticity* of the boundary surface of $\Omega_1 \cup \Omega_1'$ in B_R. In other words, $\widehat{\Sigma}_1 \cup \widehat{\Sigma}_1'$ is an analytic surface in B_R, where $\widehat{\Sigma}_1 = \partial\Omega_1 \cap B_R$ and $\widehat{\Sigma}_1'$ is the reflection of $\widehat{\Sigma}_1$ in the plane $\widehat{P}$. In particular, each horizontal cross-section of $\widehat{\Sigma}_1 \cup \widehat{\Sigma}_1'$ must consist of smooth arcs, which is possible only if Proposition 3** holds, (for otherwise a cusp would have appeared at a certain horizontal cross-section of $\widehat{\Sigma}_1 \cup \widehat{\Sigma}_1'$). □

Remark 1. We note that $\psi[\Omega_1'] = \psi[\Omega_1]$ because of the rotational symmetry of both the boundary surface ∂B_R and the boundary data β_ϵ.

In Proposition 3*, *we know that* $\partial\tilde{\Sigma} \cap \partial B_R \subseteq \widehat{\Sigma} \cup \partial\Sigma_1$ by the fact that ϵ can be arbitrarily small and the reasoning used in the proof of Proposition 3 in §1. Thus, (0.6.2) yields that $\partial\tilde{\Sigma} \cap \partial B_R$ must be a connected subset of $\widehat{\Sigma}$, for otherwise replacing a part of $\tilde{\Sigma}$ below $\partial\Sigma_1$ by that part of $\widehat{\Sigma}$ surrounding it yields a smaller value for ψ. Thus, the reasoning used in the proof of Proposition 3 yields that $\tilde{\Sigma}$ must be spherical. Also, the reasoning following the proof of Proposition 3 excludes *that spherical cap situated above* Σ_1 *and passing through* $\partial\Sigma_1$.

Furthermore, in Proposition 3*, were $\tilde{\Sigma}$ situating below Σ_1, then a rigid motion of it would result in a body meeting $\widehat{\Sigma}$ with the same surface area and therefore yielding the same value for ψ (cf. Figure 7), which, however, would by no means be symmetric with respect to the z-axis, violating Proposition 3*. We thus precluded the occurence of $\tilde{\Sigma}$ being a spherical cap other than Σ_1. *Proposition* 4 *is finally proved.*

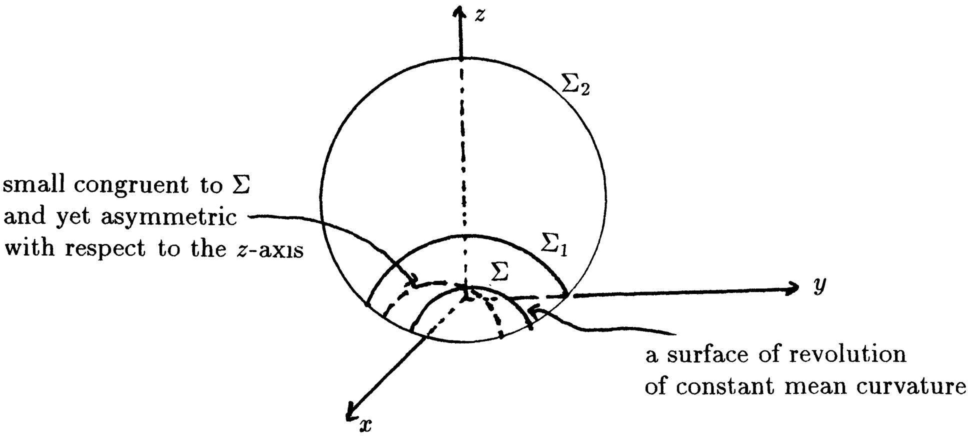

Figure 7.

Now that Proposition 4 has been proved, our proof of (a) is complete by observing that

$$\begin{aligned}\psi[\Omega_*] &= \phi[B_R - \Omega_*] \\ &= (3|\Omega_*| + |\Sigma_1| - |\Sigma_2|) + \epsilon|\Sigma_2| \\ &= \epsilon|\Sigma_2|, \qquad \text{by (0.3)} \\ &> 0.\end{aligned}$$

2.2. *Next, to prove* (b), we note that *a proof for* (b) *given in* §3.2 *of* [**L1**] *for the two dimensional domains extends in an obvious may to arbitrary dimensional domains* and we do not repeat it here.

We, however, recall that, in the course of our proof, we have incidentally proved

Proposition 5.

$$\int_{\Sigma_1} \nu \cdot Tu_\epsilon \, ds \longrightarrow -|\Sigma_1|, \qquad \text{as } \epsilon \to 0$$

and $Tu_\epsilon(x) \to \nu(x_0)$, *as* $\epsilon \to 0$, *uniformly for* $x_0 \in \Sigma_2$.

We therefore gain the rough impression that *the solution of* (0.6) *in* Ω_* *has been constructed as a limit of solutions* u_ϵ *defined throughout* B_R, as stated in §0.0.2. We may proceed to gain a rigid and precise understanding on this.

As in §3.3.3 of [**L1**], according to a theorem of Miranda [**M**], we know that a subsequence of $\{u_\epsilon\}$ can be found which converges in B_R to a generalized solution $u(x)$ of the equation (0.4), $n = 3$, in $L^1_{\text{loc}}(B_R)$. Set P and N and normalize the solutions u_ϵ in essentially the same way as we have done in §3.3.3 of [**L1**]. We again have

Proposition 6. *Both the sets N and $B_R - P$ minimize the functional*

$$\psi[\Omega^0] \equiv \left|\partial\Omega^0 \cap B_R\right| + \widehat{\beta}_0 \left|\partial\Omega^0 \cap \widehat{\Sigma}\right| + \left|\partial\Omega^0 \cap \Sigma_2\right| - 3|\Omega^0|, \tag{2.3}$$

among all the Caccioppoli sets $\Omega^0 \subseteq B_R$, $\Omega^0 \neq \phi$ or B_R, where

$$0 < \widehat{\beta}_0 = \frac{3|B_R| - |\Sigma_2|}{\left|\widehat{\Sigma}\right|} < 1.$$

Repeating our reasoning for proving *Proposition* 4, we again know that the minimizing body for (2.3) must be either empty or else $B_R - \Omega_*$. In consideration of our normalization, the results in (b) and the reasoning used in §3.3.4 of [**L1**] therefore again yield that $P = \phi$ and $N = B_R - \Omega_*$. We thus prove that the regularity domain of u coincides with Ω_*. Also, the reasoning used in the ending of §3.3.3 of [**L1**] or Theorem 7.8 in [**F3**] again yields the identity of the function u and the solution to (1.2) (or (0.1)) in Ω_*. We therefore arrive at an accurate interpretation of what we asserted.

3. The Existence of Three Dimensional Moon Domains $\Omega_*(R)$ for $1 > R > 2/3$.

Consider the function

$$f(r;\theta) = \sigma_\theta(r) - 3v_\theta(r) \tag{3.1}$$

where $\sigma_\theta(x)$ is the area of the spherical cap $D_{\rho(\theta,r)}$ whose boundary $\partial D_{\rho(\theta,r)}$ is a circle of radius $\rho = r\sin\theta$ on ∂B_r and $V_\theta(r)$ is the volume enclosed by the spherical cap $D_{\rho(\theta,r)}$ and the plane passing through the circle $\partial D_{\rho(\theta,r)}$ (cf. Figure 8). We readily see that, if $\Omega_*(R)$, $1 > R > 2/3$, exists, the equation of the circle $\Gamma \equiv \partial\Sigma_1(R)$ is $\rho = \frac{2}{3}\sin\theta_1(R)$ where $\theta_1(R)$ is the root of the equation

$$f(R;\pi - \psi(\theta)) - f(2/3;\theta) = 0 \qquad \text{(cf. (1.1) or (0.3))}, \tag{3.2}$$

with

$$\psi(\theta) = \sin^{-1}\frac{2\sin\theta}{3R}. \tag{3.3}$$

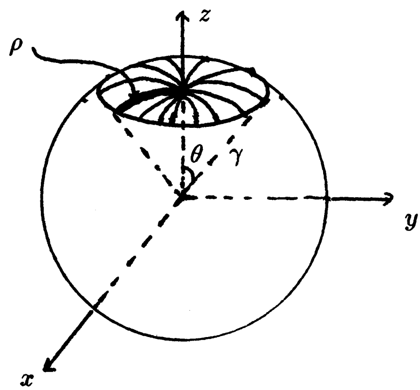

Figure 8.

We shall justify, for $1 > R > 2/3$, the existence of $\Omega_*(R)$ by showing the existence of a root $\theta = \theta_1(R)$ for the equation

$$g(R;\theta) = 0,$$

with

$$g(R;\theta) = f(R;\pi - \psi(\theta)) - f(2/3;\theta). \tag{3.4}$$

We have

$$\sigma_\theta(x) = 2\pi r^2 \int_0^\theta \sin\theta\, d\theta = 2\pi r^2(1 - \cos\theta), \tag{3.5}$$

and

$$V_\theta(r) = \pi \int_{r\cos\theta}^{r} (r^2 - z^2)\, dz = \pi r^3(2/3 - \cos\theta + 1/3\cos^3\theta). \tag{3.6}$$

Hence, by (3.1),

$$\begin{aligned} f\left(\frac{2}{3};\theta\right) &= \frac{8\pi}{9}(1 - \cos\theta) - \frac{8}{27}\pi(2 - 3\cos\theta + \cos^3\theta) \\ &= \frac{8}{27}\pi - \frac{8}{27}\pi\cos^3\theta, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &f(R, \pi - \psi(\theta)) \\ &\quad = 2\pi R^2(1 + \cos\psi(\theta)) - \pi R^3(2 + 3\cos\psi(\theta) - \cos^3\psi(\theta)), \end{aligned} \tag{3.8}$$

with

$$\cos\psi(\theta) = \sqrt{1 - \left[\frac{2\sin\theta}{3R}\right]^2} \qquad \text{(cf. (3.3))}. \tag{3.9}$$

By (3.4), (3.7), (3.8) and (3.9), we have

$$g(R,0) = 4\pi R^2 - 4\pi R^3 \begin{cases} > 0, & \text{if } R < 1, \\ = 0, & \text{if } R = 1, \end{cases} \tag{3.10}$$

and

$$g(R,\pi) = 4\pi R^2 - 4\pi R^3 - \frac{16}{27}\pi \begin{cases} = 0, & \text{if } R = 2/3, \\ < 0, & \text{if } 1 \geq R > 2/3. \end{cases} \tag{3.11}$$

The existence of a root $\theta = \theta_1(R)$, $0 \leq \theta_1(R) < \pi$, for the equation $g(R;\theta) = 0$ readily follows from (3.10) and (3.11).

4. In (4.1), if ϵ is sufficiently small, $-1 < \widehat{\beta}_\epsilon(R) < 1$ for $1 > R > 2/3$ and $-1 < \widehat{\beta}_\epsilon(R) < 0$ for $1 > R > R_0^{(3)}$.

In (2.1), we have to set

$$\widehat{\beta}_\epsilon = \frac{3\,|B_R| - (1-\epsilon)|\Sigma_2|}{\left|\widehat{\Sigma}\right|}.$$

It follows at once that $\widehat{\beta}_\epsilon(R) < 1$ for ϵ sufficiently small, since $3|B_R| = 4\pi R^3 < 4\pi R^2 = |\Sigma_2| + \left|\widehat{\Sigma}\right|$, for $1 > R \geq 2/3$. On the other hand, using (0.3),

$$\widehat{\beta}_\epsilon(R) = \frac{3\,|B_R - \Omega_*| - |\Sigma_1| + \epsilon|\Sigma_2|}{\left|\widehat{\Sigma}\right|}. \tag{4.1}$$

To show that $-1 < \widehat{\beta}_\epsilon(R)$ for sufficiently small ϵ we only need verify

$$|\Sigma_1| - 3\,|B_R - \Omega_*| < \left|\widehat{\Sigma}\right|. \tag{4.2}$$

To do so, we, as in §1, denote P as the plane passing through the circle $\partial\Sigma_1$ and denote Ω_{**} as the body enclosed by $P \cap B_R$ and Σ_1 (cf. Figure 2). Then, we have

$$\left|\widehat{\Sigma}\right| > |\text{ the planar disk } P \cap B_R|, \tag{4.3}$$

and

$$|\Sigma_1| - 3\,|B_R - \Omega_*| < |\Sigma_1| - 3|\Omega_{**}|. \tag{4.4}$$

However, the inequality

$$|P \cap B_R| > |\Sigma_1| - 3|\Omega_{**}| \tag{4.5}$$

follows immediately from the fact proved in §1 that Ω_{**} strictly minimizes $\psi^*[\Omega^0]$ (cf. (1.10) and (1.12)) among all the Caccioppoli sets passing through the circle $\partial\Sigma_1$ and situating entirely at one side of the plane P (including P). The inequality (4.2) is thus proved.

We note that, alternatively, (4.5) can be proved by a direct calculation. Namely, using the notations in §3,

$$|P \cap B_R| = \frac{4}{9}\pi \sin^2\theta_1,$$

and

$$|\Sigma_1| - 3|\Omega_{**}| = f\left(\frac{2}{3};\theta_1\right) = \frac{8}{27}\pi - \frac{8}{27}\pi\cos^3\theta_1 \qquad \text{(cf. (3.7))}.$$

Hence

$$\begin{aligned}
&|P \cap B_R| - (|\Sigma_1| - 3\,|B_R - \Omega_*|)\\
&= \frac{4}{27}\pi(3\sin^2\theta - 2 + 2\cos^3\theta_1)\\
&= \frac{4}{27}\pi[1 - \cos^2\theta(3 - 2\cos\theta)]\\
&\geq 0,
\end{aligned}$$

for all θ.

We now proceed to prove $\widehat{\beta}_\epsilon(R) < 0$ for $1 > R > R_0^{(3)}$. We have, as $|B_R - \Omega_*| < 2|\Omega_{**}|$, that

$$\begin{aligned}
&|\Sigma_1| - 3\,|B_R - \Omega_*|\\
&> |\Sigma_1| - 6|\Omega_{**}|\\
&= \frac{8}{27}\pi(-1 + 3\cos\theta_1 - 2\cos^3\theta_1)\\
&= \frac{8}{27}\pi(1 - \cos\theta_1)(2\cos^2\theta_1 + 2\cos\theta_1 - 1)\\
&= \frac{16}{27}\pi(1 - \cos\theta_1)\left[\cos\theta_1 + \frac{\sqrt{3}+1}{2}\right]\left[\cos\theta_1 - \frac{\sqrt{3}-1}{2}\right].
\end{aligned}$$

In view of (2.1), we therefore only need verify

$$\theta_1(R) < \cos^{-1}\left[\frac{\sqrt{3}-1}{2}\right], \qquad \text{for } 1 > R \geq R_0^{(3)}. \tag{4.6}$$

To do so, we may observe that, there holds the following

Proposition 7. $\theta_1(R_1) \gneqq \theta_1(R_2)$, *if* $R_1 \lneqq R_2$.

Proof. This is an immediate consequence of (0.5.2). *In fact*, if $R_1 \lneqq R_2$ and $\theta_1(R_1) \leq B_1(R_2)$, then after a rigid motion, $\Sigma_1(R_1) \subseteq \Sigma_1(R_2)$ and $\Omega_*(R_1) \subseteq \Omega_*(R_2)$ with

$$\psi[\Omega_*(R_1)] = |\Sigma_2(R_1)| - |\Sigma_1(R_1)| - 3|\Omega_*(R_1)| > 0,$$

in accordance with (0.5.2) and yet contradicting our original definition (0.3)of $\Omega_*(R_1)$. □

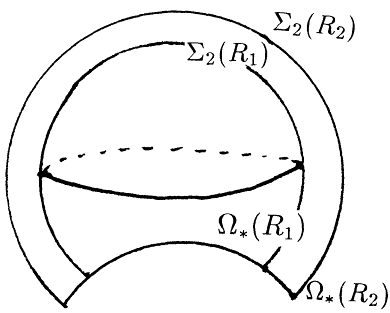

Figure 9.

Thus, to verify (4.6), it suffices to show that

$$\theta_1\left(R_0^{(1)}\right) < \cos^{-1}\left[\frac{\sqrt{3}-1}{2}\right]. \tag{4.7}$$

To do so, we may observe that, as $\Sigma_1\left(R_0^{(3)}\right)$ passes though the center of $B_{R_0^{(3)}}$, we have, using the notation as in §3,

$$\psi\left(\theta_1\left(R_0^{(3)}\right)\right) = \frac{\pi}{2} - \frac{\theta_1\left(R_0^{(3)}\right)}{2}, \qquad \text{(cf. Figure 10),} \tag{4.8}$$

that is,

$$R_0^{(3)} \cos\frac{\theta_1\left(R_0^{(3)}\right)}{2} = \frac{2}{3}\sin\theta_1\left(R_0^{(3)}\right)$$

$$= \frac{4}{3} \sin \frac{\theta_1 \left(R_0^{(3)} \right)}{2} \cos \frac{\theta_1 \left(R_0^{(3)} \right)}{2}$$

and hence

$$\sin \frac{\theta_1 \left(R_0^{(3)} \right)}{2} = \frac{3}{4} R_0^{(3)}, \tag{4.9}$$

which yields

$$\cos \theta_1 \left(R_0^{(3)} \right) = 1 - 2 \sin^2 \frac{\theta_1 \left(R_0^{(3)} \right)}{2} = 1 - \frac{9}{8} \left(R_0^{(3)} \right)^2 . \tag{4.10}$$

Substituting (4.8) into (3.2) or (3.4), we shall obtain $R_0^{(3)}$ as the root of the equation

$$f \left(R_0^{(3)}; \frac{\pi}{2} + \frac{\theta_1 \left(R_0^{(3)} \right)}{2} \right) - f \left(\frac{2}{3}; \theta_1 \left(R_0^{(3)} \right) \right) = 0,$$

or, using (3.1), (3.7) and (3.8),

$$\begin{aligned} &2 \left(R_0^{(3)} \right)^2 \left(1 + \sin \frac{\theta_1 \left(R_0^{(3)} \right)}{2} \right) \\ &\qquad - \left(R_0^{(3)} \right)^3 \left(2 + 3 \sin \frac{\theta_1 \left(R_0^{(3)} \right)}{2} - \sin^3 \frac{\theta_1 \left(R_0^{(3)} \right)}{2} \right) \\ &= \frac{8}{9} \left(1 - \cos \theta_1 \left(R_0^{(3)} \right) \right) - \frac{8}{27} \left(2 - 3 \cos \theta_1 \left(R_0^{(3)} \right) + \cos^3 \theta_1 \left(R_0^{(3)} \right) \right), \end{aligned}$$

or, using (4.9) and (4.10)

$$\begin{aligned} &2 \left(R_0^{(3)} \right)^2 \left(1 + \frac{3}{4} R_0^{(3)} \right) - \left(R_0^{(3)} \right)^2 \left(2 + \frac{9}{4} R_0^{(3)} - \frac{27}{64} R_0^{(3)} \right)^3 \\ &= \left(R_0^{(3)} \right)^2 - \frac{9}{8} \left(R_0^{(3)} \right)^4 + \frac{27}{64} \left(R_0^{(3)} \right)^6, \end{aligned}$$

that is,

$$\frac{1}{8} \left[9 \left(R_0^{(3)} \right)^4 + 4 \left(R_0^{(3)} \right)^3 - 8 \left(R_0^{(3)} \right)^2 \right] = 0,$$

or,

$$9 \left(R_0^{(3)} \right)^2 + 4 \left(R_0^{(3)} \right)^2 - 8 = 0.$$

Hence,

$$R_0^{(3)} = \frac{-2+\sqrt{76}}{9} = \frac{-2+2\sqrt{19}}{9} = 0.746421987\ldots \tag{4.11}$$

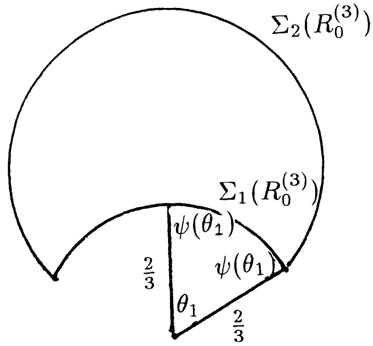

Figure 10.

Thus, using (4.10) and (4.11)

$$\theta_1\left(R_0^{(3)}\right) = \cos^{-1}\left[1 - \frac{9}{8}\left(R_0^{(3)}\right)^2\right] = \cos^{-1}\left[\frac{-1+\sqrt{19}}{9}\right].$$

As $\left[\frac{-1+\sqrt{19}}{9}\right] > \frac{\sqrt{3}-1}{2}$, (4.7) (and hence (4.6)) follows.

5. On Still Higher Dimensional Cases.

5.1. We first verify the *existence of the n-dimensional moon domain* $\Omega_*(R)$, $1 > R > \dfrac{n-1}{n}$, characterized by the equation

$$|\Sigma_2(R)| - |\Sigma_1(R)| = n|\Omega_*(R)| \tag{5.1}$$

where $\partial\Omega_* = \Sigma_2 \cup \Sigma_1$, Σ_2 and Σ_1 being spherical caps in $\mathbb{R}^n$ of the respective radii $R_1 = \dfrac{n-1}{n}$ and R. As in §2, we set

$$f(r;\theta) = \sigma_\theta(r) - nv_\theta(r), \tag{5.2}$$

where the definition for $\sigma_\theta(r)$ and $v_\theta(r)$ in the beginning of §3 extends to the present setting in an obvious way. If $\Omega_*(R)$ exists, the equation of the $(n-1)$-dimensional sphere $\Gamma \equiv \partial\Sigma_1(R)$ is $\rho = \dfrac{n-1}{n}\sin\theta_1$ where θ_1 is the root of the equation

$$f(R;\pi-\psi(\theta)) - f\left(\frac{n-1}{n};\theta\right) = 0,$$

with

$$\psi(\theta) = \sin^{-1} \frac{(n-1)\sin\theta}{nR}.$$

Denoting ω_N as the volume of the N-dimensional sphere and setting, again,

$$g(R;\theta) = f(R;\pi - \psi(\theta)) - f\left(\frac{n-1}{n};\theta\right), \tag{5.3}$$

we have

$$\begin{aligned} g(R;0) &= f(R;\pi) \\ &= 2(n-1)\omega_{n-1}R^{n-1} - 2(n-1)\omega_{n-1}R^n \\ &> 0, \qquad \text{if } R < 1, \end{aligned} \tag{5.4}$$

and

(5.5)

$$\begin{aligned} g(R;\pi) &= f(R;\pi) - f\left(\frac{n-1}{n};\pi\right) \\ &= 2(n-1)\omega_{n-1}\left\{(R^{n-1} - R^n) - \left[\left(\frac{n-1}{n}\right)^{n-1} - \left(\frac{n-1}{n}\right)^n\right]\right\} \\ &\begin{cases} = 0, & \text{if } R = \dfrac{n-1}{n} \\ < 0, & \text{if } 1 > R > \dfrac{n-1}{n}, \end{cases} \end{aligned}$$

since $r = \dfrac{n-1}{n}$ is the zero of the derivative of the concave function $h(r) = r^{n-1} - r^n$. From (5.4) and (5.5) follows the existence of a root $\theta = \theta_1(R)$, $0 \leq \theta_1(R) \leq \pi$, for the equation $g(R;\pi) = 0$, of which the existence of $\Omega_*(R)$ is an immediate consequence.

5.1.1. Using the above notation, we may here describe a procedure for determining the value $R_0^{(n)}$, $n > 3$. Indeed, since $R_0^{(n)}$ is the unique value of R for which $\Sigma\left(R_0^{(n)}\right)$ passes through the center of $B_{R_0^{(n)}}$, we may, as in §4, obtain $R_0^{(n)}$ as the root of the equation

$$f\left(R;\frac{\pi}{2} + \frac{\theta_1}{2}\right) - f\left(\frac{2}{3};\theta_1\right) = 0,$$

with

$$\sin\frac{\theta_1}{2} = \frac{n}{2(n-1)}R;$$

and

$$\cos\theta_1 = 1 - 2\sin^2\frac{\theta_1}{2} = 1 - \frac{n^2}{2(n-1)^2}R^2;$$

here $f(r;\theta)$ is defined by (5.2).

The number $R_0^{(n)}$, as mentioned of in the end of 0.2 is of significance once we place it into perspective in the context of the results in [**L2**].

5.2. Having verified the eistence of $\Omega_*(R)$ for $n > 3$, we proceed to prove *the existence of comparison hypersurfaces in* $\Omega_*(R)$, $1 > R > \frac{n-1}{n}$, which is the solution to the problem (0.4) and (1.5), setting $H = 1$ and $\beta = -1, +1$ on Σ_1, Σ_2, respectively. We again, using Proposition 2, reduce this to the proof of the ineq.

$$(5.6) \quad \psi^*[\Omega^0] \equiv |\partial\Omega^0 \cap \Omega| + |\partial\Omega^0 \cap \Sigma_2| - n|\Omega^0 \cup \Omega_{**}| > \psi^*[\Omega_{**}] \equiv |\Sigma_1| - n|\Omega_{**}|$$

for all Ω^0 passing through $\Gamma \equiv \partial\Sigma_1$, with the whole Σ_1 as a part of its boundary and situating entirely in one of the two half spaces provided by the hyperplane passing through Γ; here Ω_{**} is the region bounded by Σ_1 and this hyperplane. Repeating the variational procedure indicated in §1, we again justify the existence of a minimizing body for $\psi^*[\Omega^0]$. Set $\tilde{\Sigma} = \partial\tilde{\Omega} \cap \Omega_*$. We readily see that Proposition 3 holds here; that is, nonempty $\overline{\Sigma}$ must be a spherical cap passing through Γ, which as Lemma 1 can also be extended, must be a spherical cap of radius $\frac{n-1}{n}$ strictly including a hemisphere and can possibly occur only when Σ_1 is included in a hemisphere. However, if Σ_1 is included in a hemisphere, denoting Ω_0 as the body enclosed by Σ_1 and that spherical cap of radius $\frac{n-1}{n}$ included in Ω_*, we have, adopting the notation in §5.1,

$$\begin{aligned}
&\psi^*[\Omega_0 \cup \Omega_{**}] - \psi^*[\Omega_{**}] \\
&= f(2/3; \pi - \theta_1(2/3)) - f(2/3; \theta_1(2/3)) \\
&= (\sigma_{\pi-\theta_1(2/3)}(2/3) - \sigma_{\theta_1(3/2)}(2/3)) - (v_{\pi-\theta_1(2/3)}(2/3) - v_{\theta_1(2/3)}(2/3)) \\
&= (n-1)\omega_{n-1}(2/3)^{n-1}\int_{\theta_1}^{\pi-\theta_1}\sin\theta\,d\theta - \omega_{n-1}\int_{2/3\cos(\pi-\theta_1)}^{2/3\cos\theta_1}\left(\frac{4}{9} - Z_1^2\right)^{\frac{n-1}{2}} dz_1 \\
&= (n-1)\omega_{n-1}\left(\frac{2}{3}\right)^{n-1}\int_{\theta_1}^{\pi-\theta_1}\sin\theta\,d\theta - \omega_{n-1}\left(\frac{2}{3}\right)^{n}\int_{\theta_1}^{\pi-\theta_1}\sin^n\theta\,d\theta \\
&> 0, \qquad \text{obviously.}
\end{aligned}$$

We therefore prove (5.6) and the existence of the comparison hypersurfaces for $n > 3$.

5.3. A careful examination of the work of §5 tells us that; in order to *construct such a sequence of solutions to* (0.4) *in* $B_R \subseteq \mathbb{R}^n$, $n > 3$, *whose limit, in the sense indicated in* 2.2 *is our moon hypersurface in* $\Omega_*(R) \subseteq \mathbb{R}^n$, we only need to (1) *verify that* $-1 < \widehat{\beta}_\epsilon(R) < 1$, *for* $1 > R > \dfrac{n-1}{n}$ *and* ϵ *sufficiently small*, where

$$\widehat{\beta}_\epsilon(R) = \frac{n|B_R| - (1-\epsilon)|\Sigma_2(R)|}{\left|\widehat{\Sigma}\right|}$$

$\widehat{\Sigma} = \partial B_R - \Sigma_2$, the full sphere ∂B_R being obtained by extending $\Sigma_2(R)$.

(2) Prove the statement of Proposition 3** and Proposition 4 *in spite of the difficulty arised by the possible existence of singular subsets of* $\tilde{\Sigma} \cap B_R$ *in the case of* $n > 7$. The fact that $\widehat{\beta}_\epsilon < 1$ readily follows from the inequality

$$n|B_R| = n\omega_n R^n < n\omega_n R^{n-1} = |\partial B_R| = |\Sigma_2| + \left|\widehat{\Sigma}\right|.$$

The fact $\widehat{\beta}_\epsilon > -1$, by (5.1), amounts to the fact that

$$\left|\widehat{\Sigma}\right| > |\Sigma_1| - n|B_r - \Omega_*|$$

which is a consequence of the inequality

$$\left|D_{\rho\left(\theta_1, \frac{n-1}{n}\right)}\right| > |\Sigma_1| - n|\Omega_{**}|,$$

(see the beginning of §3 for notation) obtained immediately from the fact that Ω_{**} minimizing $\psi^*[\Omega]$ (cf. (5.6)) among all sets indicated below (5.6).

As of (2), we may, first of all, put Proposition 4, 3* and 3** in a precise form in the higher dimensional setting. In fact, to extend the existence results in §2 to the case where $n > 3$, it suffices to verify that (0.6.2) holds for every Caccioppoli set $\Omega^0 \subseteq B_R$, $\Omega^0 \neq \phi, B_R$, $1 > R > \dfrac{n-1}{n}$. To do so, as in §2, we may observe that it suffices to consider those Caccioppoli sets with $\left|\partial\Omega^0 \cap \widehat{\Sigma}\right| > 0$ and $\partial\Omega^0 \cap (B_R \cup \Sigma_2(R))$ being connected. Thus, as in §2, we may try to minimize $\psi[\Omega^0]$ in (0.6.2) among all the Caccioppoli sets $\Omega \subseteq B_R$ and the same reasoning concludes that a subsequence of minimizing sequence $\{\Omega_j^0\}$ for $\psi\{\Omega^0\}$, $\Omega_j^0 \subseteq B_R$, exists such that $\left\{\varphi_{\Omega_j^0}\right\}$ converges in $L^1(\Omega)$ to $\varphi_{\tilde{\Omega}}$ such that

$$\psi\left[\tilde{\Omega}\right] \leq \inf \psi[\Omega_j^0].$$

Set $\tilde{\Sigma} = \partial\tilde{\Omega} \cap B_R$. If $\overline{\Sigma} \neq \phi$, we have observed that $\left[\tilde{\Sigma} \cap \widehat{\Sigma}\right] > 0$, and we may assume $\partial\tilde{\Omega}^0 \cap (B_R \cup \Sigma_2(R))$ to be connected.

For our present purpose, we only have to show, as in §2.1.

Proposition $\bar{4}$. *The only non-empty candidate for $\tilde{\Sigma}$ is the spherical cap Σ_1. In other words, the only non-empty candidate for $\tilde{\Omega}$ is $B_R - \Omega_*$.*

We again let P to be the unique plane passing through the $(n-2)$-dimensional sphere $\Gamma \equiv \partial\Sigma_1 \cap \partial B_R$ and designate P as the $x_1, x_2, \dots, x_{n-1}$ plane so that the center of B_R has the x_n-coordinate $x_n > 0$.

To prove Proposition 4, we shall also proceed to verify

Proposition $\tilde{3}^*$. *If $\tilde{\Sigma} \neq \phi$, then $\tilde{\Sigma}$ is of rotational symmetry about the x_n-axis.*

In §2.1, Proposition 3* is proved with the aid of a theorem of Massari [**Ma**], which, as mentioned above, does not exclude the possibility of existence of singular points of a minimizing body in the case that $n > 7$; however, it gives an estimate for the dimension of singular parts, which has been improved by Federer. Their results yields

Theorem Of Massari And Federer. *If $\overline{\Sigma} \neq \phi$, then the reduced boundary $\partial^*\overline{\Omega}$ of $\overline{\Omega}$ is an analytic manifold of dimension $n-1$ and*

$$H_s\left[\left(\overline{\Sigma} \setminus \partial^*\overline{\Omega}\right) \cap B_R\right] = 0, \quad \forall\, s > n-7,\ s \in \mathbb{R},$$

where H_s denotes the Hausdorff s-measure.

To prove Proposition $\widetilde{3}^*$, as in §2.1, we consider a vertical plane $\widehat{P}:\ a_1x^1 + a_2x^2 + \cdots + a_{n-1}x^{n-1} = 0,\ a_1, \dots, a_{n-1}$: constants, which divides $\overline{\Omega}$ into two non-empty parts Ω_1 and Ω_2. We may assume, without loss of generality, that $\psi[\Omega_1] \leq \psi[\Omega_2]$. Reflecting the body Ω_1 in the opposite side of the plane $\widehat{P}$, then

$$\Omega_1 \cup \Omega_1' \subseteq B_R,$$

and, as in §2.1, we have $\psi[\Omega_1 \cup \Omega_1'] \leq \psi\left[\tilde{\Omega}\right]$ and hence $\psi[\Omega_1 \cup \Omega_1'] = \psi\left[\tilde{\Omega}\right]$, in view of the minimizing property of $\widetilde{\Omega}$. Thus, we have

Proposition $\widetilde{3}^{**}$. *If $\widetilde{\Sigma} \neq \phi$, then at each regular point of $\widetilde{\Sigma}$, the normal of the horizontal cross-section of $\widetilde{\Sigma}$ through this point is orthogonal to the normal of the unique vertical plane $a_1x^1 + a_2x^2 + \cdots + a_{n-1}x^{n-1} = 0,\ a_1, a_2, \dots, a_{n-1}$: constants, passing though this point (and the origin).*

At height x_0^n, if the horizontal cross-section includes regular points of $\widetilde{\Sigma}$, we may choose a regular point $(x_0^1, \dots, x_0^n)$ of $\widetilde{\Sigma}$, then, for each connected curve $(x^1(t), \dots, x^{n-1}(t), x_0^n)$ through this point and included in a regular part of the horizontal cross-section of $\widetilde{\Sigma}$ at the height x_0^n, Proposition 3** yields that

$$\dot{x}^1 x^1 + \dot{x}^2 x^2 + \cdots + \dot{x}^{n-1} x^{n-1} = 0,$$

which holds if and only if

$$(x^1)^2 + (x^2)^2 + \cdots + (x^{n-1})^2 = \text{ constant},$$

i.e. $(x^1(t), \ldots, x^{n-1}(t), x_0^n)$ lies on a sphere with its center at $(0, \ldots, 0, x_0^n)$. Thus, each regular point of this horizontal cross-section of $\widetilde{\Sigma}$ must be included in a region on an $(n-2)$-dimensional sphere with its center at $(0, 0, \ldots, x_0^n)$ and, furthermore, denoting C as the component of this horizontal cross-section including this spherical cap, we note that C must be a whole *closed* sphere; indeed, were C bounding a region in the hyperplane $x^n = x_0^n$ and C includes only a *portion* of and *not* the whole sphere, then C would have to include at least two disjoint spherical regions and *the dimension of singular parts of this cross-section would be* $n-2$, contradicting above-mentioned regularity result of Massari and Federer; however, were C bounding no region then a portion of $\widetilde{\Sigma}$ with positive $(n-1)$-dimensional Hausdorff measure would *not* be a portion of the boundary of any component of $\widetilde{\Omega}$ (with positive n-dimensional Hausdorff measure) and removing this portion of $\widetilde{\Sigma}$ would result in a smaller value of ψ, contradicting the minimality of $\widetilde{\Omega}$ and $\widetilde{\Sigma}$. Thus, the proof of Proposition $\widetilde{3}^*$ is complete. The argument following the proof of Proposition 3** in §2.1 again applies in our present setting and enables us to prove Proposition 1, from which, as indicated above, follows (0.6.2) and the existence of that sequence of solutions to (0.4) in $B_R \subseteq \mathbb{R}^n$, described in §0.0.2 and beginning of this section.

6. Some existence Results of Capillary Hypersurfaces without Gravity and of Rotational Symmetry.

As in Finn [**F1**] and quoted in Proposition §1 of this paper, we may reduce the capillary problem in the absence of gratitivity to the variational problem

$$\xi[u] = \int_\Omega \sqrt{1 + |\nabla u|^2} + nH \int_\Omega u\, dx - \int_{\partial\Omega} \beta(s) u\, ds, \tag{6.1}$$

with $\beta(s)$, $-1 \leq \beta(s) \leq 1$, being piecewise Lipschitz on the boundary of a piecewise Lipschitz domain $\Omega \subseteq \mathbb{R}^n$, and H being a constant. As quoted in Proposition 1, a necessary and sufficient condition for the existence of a minimizing function $u(x) \in BV_{\text{loc}}(\Omega)$ for the functional (6.1) is that both the conditions (1.3) and (1.4) hold for every Caccioppoli set $\Omega^0 \neq \phi$, $\Omega\,(\Omega^0 \subseteq \Omega)$. Furthermore, since H is constant, the conditions (1.3) and (1.4) are equivalent. Thus, in §1, §2, §5.1 and §5.3 of this work, we have restricted our attention to verifying (1.4); the argument used in §2 and §5.3 yields the existence of a minimizing body $\tilde{\Omega}$ for $\psi[\Omega^0]$ and setting $\tilde{\Sigma} = \partial\tilde{\Omega} \cap \Omega$, the argument used to verify Proposition 3* yields.

Proposition 3***. *Suppose Ω and $\beta(s)$ are rotational symmetry of the same axis. If $\tilde{\Sigma} \neq \phi$ then $\tilde{\Sigma}$ is of rotational symmetry about this axis.*

We may, without loss of generality assume that this axis of symmetry is the x_n-axis. Suppose, in addition, that $\beta(s)$ is piecewise constant; i.e., there exist relatively open subsets Σ^i of $\partial\Omega$, such that, if $i < j$, Σ^i is "below" Σ^j (in the sense that, for two arbitrarily chosen points $x^i \in \Sigma^i$ and $x^j \in \Sigma^j$, then x_n component of x^i is less than that of x^j), and,

(6.2) $$\beta(s)|_{\Sigma^i} = \text{ constant } c_i, \quad \cup\Sigma^i = \partial\Omega.$$

Then, the argument used in §2.1 to exclude those $\tilde{\Sigma}$ situating below Σ_1 can be applied to yield

Corollary 1. *Suppose, in addition to the hypothesis of Proposition* 3***, *$\beta(s)$ is piecewise constant, as indicated in* (6.2). *Then, if $\tilde{\Omega} \neq \phi$ or Ω, there occurs at least one of the following possibilities:*

Possibility 1. $\partial\tilde{\Sigma} \cap \Sigma^1 = \phi$ or Σ^1.

Possibility 2. There exists at least one i, $i \geq 1$, such that

$$\partial\Sigma^i \setminus \partial\Sigma^{i+1} \subseteq \partial\tilde{\Omega}.$$

Possibility 3. In (6.2), $\bigcup_{i=1}^{k} \Sigma^i = \partial\Omega$ for some integer $k < \infty$ and

$$\partial\tilde{\Omega} \cap \Sigma^k = \phi \text{ or } \Sigma^k.$$

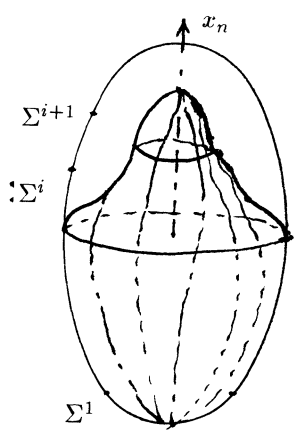

Figure 11.

Indeed, Σ^i being open, were Corollary 1 false, a rigid motion of $\tilde{\Omega}$ would result in a body meeting Σ^j, for each j, with the same area as $\tilde{\Omega}$ and is therefore another minimizing body for the functional ψ, which, however, would *not* be of rotational symmetry of the axis indicated in Proposition 3^{***}.

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STABLE RELATIONS II: CORONA SEMIPROJECTIVITY AND DIMENSION-DROP C^*-ALGEBRAS

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We prove that the relations in any presentation of the dimension-drop interval are stable, meaning there is a perturbation of all approximate representations into exact representations. The dimension-drop interval is the algebra of all M_n-valued continuous function on the interval that are zero at one end-point and scalar at the other. This has applications to mod-p K-theory, lifting problems and classification problems in C^*-algebras. For many applications, the perturbation must respect precise functorial conditions. To make this possible, we develop a matricial version of Kasparov's technical theorem.

1. Introduction.

Suppose $\mathcal{R}$ is a finite set of relations on a finite set G of generators so that $C^*\langle G|\mathcal{R}\rangle$ is isomorphic to the dimension-drop interval

$$\tilde{\mathbb{I}}_n = \{f \in C[0,1] \mid f(0), f(1) \in \mathbb{C}I\}.$$

For simplicity, we assume the relations are of the form $p(g_1, \dots, g_n) = 0$ for some $*$-polynomial p. *Weak stability* means that an approximate representation $(x_1, \dots, x_n)$, meaning an n-tuple of elements in a C^*-algebra A such that each $p(x_1, \dots, x_n)$ is close zero, can be perturbed slightly within A to an actual representation $(\tilde{x}_1, \dots, \tilde{x}_n)$. That this (and a little more) can be done was shown in [**8**], but only for one specific set of relations. The relations $\mathcal{R}$ are *stable* if the pertubation can be done so that whenever there is a $*$-homomorphism $\varphi : A \to B$ which sends $(x_1, \dots, x_n)$ to an exact representation, then $\varphi(\tilde{x}_j) = \varphi(x_j)$.

There are several advantages to stability over weak stability. It is far more useful when dealing with extensions of C^*-algebras and it depends only on the universal C^*-algebra, not the choice of relations for that C^*-algebra. The reason for our focus on the dimension-drop interval is primarily that this is the most complicated building block used in the inductive limits, called AD algebras, that appeared in Elliott's first classification paper [**7**].

See [**5**] for an application of stable relations to the extension problem for AD algebras. See [**4**] for a discussion of the role of the dimension-drop interval in mod-p K-theory. Our results will be stated in the more general context of dimension-drop graphs, but certainly the dimension-drop interval is the most important case.

In §2 we give a characterization, in terms of lifting properties, of the universal C^*-algebras for stable relations. Since this property, called semiprojectivity, depends only on the C^*-algebra, this frees us from having to specify generators and relations in many cases. We have a third, equivalent property involving corona algebras. This characterization formalizes some of the ideas used by Olsen and Pedersen [**11**] to show that nilpotents always lift.

For any C^*-algebra A we let $M(A)$ denote the multiplier algebra of A and $C(A)$ denote the corona algebra $M(A)/A$.

By a dimension-drop graph, we mean a C^*-algebra of the form

$$\{f \in C(X, M_n) \mid f(v) \in \mathbb{C}I \text{ for all vertices } v\}$$

where X is the underlying topological space for a graph and n is a positive integer. We call this a dimension-drop interval in the special case where X is the unit interval with 0 and 1 as vertices.

To handle these algebras we need several generalizations of Kasparov's Technical Theorem. The purpose of these results is to show that, inside of a corona algebra, one can find good substitutes for elements that would exist if only the corona algebra were a von Neumann algebra. For example, there is an acceptable substitute for the logarithm of a unitary with full spectrum. Also, if $M_n(A)$ sits inside the corona algebra, there are elements that function just like matrix units in the way they multiply against $M_n(A)$, even if A is not unital but only σ-unital.

These technical lemmas are very similar to the second splitting lemma in BDF [**3**, Lemma 7.3]. The basic form of these results is to show that every $\varphi : A \to C(E)$ factors through some injection $A \to A_1$. In the BDF case, A and A_1 are commutative and $C(E)$ is the Calkin algebra.

Once we have shown that a dimension-drop graph is universal for a stable set of relations, a host of perturbation, lifting and homotopy results follow regarding homomorphisms (and asymptotic morphisms) out of dimension-drop C^*-algebras. For most of these we refer the reader to [**8**] but we will mention one of these, [**8**, Theorem 3.8]. If a separable C^*-algebra A has the property that any finite set of its elements can be approximated by elements of a C^*-subalgebra isomorphic to a quotient of a dimension-drop graph, then A is the inductive limit of dimension-drop graphs.

A C^*-algebra that will figure prominently in all this the cone $\mathrm{C}M_n = M_n(C_0(0,1])$. By [**8**, Theorem 4.9] we know that $\mathrm{C}M_n$ is projective. This is a very useful fact as there are many copies of $\mathrm{C}\, M_n$ inside of a dimension-drop graph.

The author is grateful to Gert Pedersen for discussions which lead to much simplified proofs in Section four.

2. A characterization of stability.

We begin with a characterization of projectivity in terms of corona algebras that is suggested by [**11**]. This then generalizes to give a characterization of semiprojectivity and of stability for relations. One consequence is that two finite sets of relations that determine isomorphic universal C^*-algebras are either both stable, or both not.

All our definitions are with respect to the full category of not-necessarily-unital C^*-algebras and $*$-homomorphisms.

Definition 2.1. A C^*-algebra A is *projective* if, for every surjection $\pi : B \to C$ and every $*$-homomorphism $\varphi : A \to C$, there exists a $*$-homomorphism $\bar{\varphi} : A \to B$ such that $\pi \circ \bar{\varphi} = \varphi$. We call A *corona projective* if this holds only in the special case where $C = C(E)$ for some σ-unital C^*-algebra E.

Theorem 2.2. *Let A be a separable C^*-algebra. Then A is projective if and only if A is corona projective.*

Proof. The forward implication is trivial. Suppose that A is corona projective and that $\varphi : A \to C$ and a surjection $\pi : B \to C$ are given. Replacing B, if necessary, by the closed span of a lift of a dense sequence in $\varphi(A)$ reduces the problem to the case where B is separable.

Let $I = \ker(\pi)$ and let $I^\perp$ denote the annihilator of I in B. As $I \cap I^\perp = 0$ and $I + I^\perp$ is an essential ideal in B, we have the following commutative diagram with the left square a pull-back.

$$\begin{array}{ccccccc}
 & & B & \longrightarrow & B/I^\perp & \xrightarrow{\iota_1} & M(I+I^\perp)/I^\perp \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\
A & \xrightarrow{\varphi} & B/I & \longrightarrow & B/(I+I^\perp) & \xrightarrow{\iota_2} & M(I+I^\perp)/(I+I^\perp)
\end{array}$$

By the corona projectivity of A, we have

$$\psi : A \to M(I + I^\perp)/I^\perp$$

which is a lift of the composition of the bottom row:

We now claim that $\pi_2^{-1}(\mathrm{im}(\iota_2)) \subseteq \mathrm{im}(\iota_1)$. Suppose $b \in \pi_2^{-1}(\mathrm{im}(\iota_2))$. Thus $\pi_2(b) = \iota_2(c)$ for some c. But $c = \pi_1(a)$ for some a, so

$$\pi_2(\iota_1(a)) = \iota_2(\pi_1(a)) = \iota_2(c) = \pi_2(b).$$

This implies

$$\iota_1(a) - b \in \ker(\pi_2) = (I + I^{\perp})/I^{\perp} \subseteq B/I^{\perp} = \operatorname{im}(\iota_1)$$

and hence $b \in \operatorname{im}(\iota_1)$.

By the claim, we may regard ψ as a map into $B/I^{\perp}$. The pull-back property now shows that φ and ψ together determine the desired lifting to B. □

Following Blackadar [**1**] we define semiprojectivity as a lifting property. This turns out to have better closure properties than the version of semiprojectivity due to Effros and Kaminker [**6**], which is better suited to some homotopy calculations.

Definition 2.3. A C^*-algebra A is called *semiprojective* if, for every $*$-homomorphism $\varphi : A \to B/\overline{\bigcup I_n}$, where the I_n are increasing ideals in B, and with $\pi_m : B/I_m \to B/\overline{\bigcup I_n}$ the natural quotient map, there exists, for some m, a $*$-homomorphism $\bar{\varphi} : A \to B/I_m$ such that $\pi_m \circ \bar{\varphi} = \varphi$. We call A *corona semiprojective* if this holds only in the special case where $B/\overline{\bigcup I_n} \cong C(E)$ for some σ-unital C^*-algebra E. □

Theorem 2.4. *Let A be a separable C^*-algebra. Then A is semiprojective if and only if A is corona semiprojective.*

Proof. The proof is similar to that of Theorem 2.2 except that one uses the following diagram, with $I = \overline{\bigcup I_n}$.

$$\begin{array}{ccccccc} & & B/I_n & \longrightarrow & B/(I_n + I^{\perp}) & \xrightarrow{\iota_1} & M(I + I^{\perp})/I^{\perp} \\ & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_3} \\ A & \xrightarrow{\varphi} & B/I & \longrightarrow & B/(I + I^{\perp}) & \xrightarrow{\iota_2} & M(I + I^{\perp})/(I + I^{\perp}) \end{array}$$

Notice that $\overline{\bigcup I_n + I^{\perp}} = I + I^{\perp}$, so corona semiprojectivity applies, and the left square is still a pull-back since $I \cap (I_n + I^{\perp}) = I_n$. □

If A is unital, then it is easy to see that one need only check the corona semiprojectivity condition in the special case $\varphi(1) = 1$.

We now recall the definition of stability from [**8**]. We shall assume that $G = \{g_1, \dots, g_l\}$ is a finite set of generators and $\mathcal{R} = \{p_1, \dots, p_k\}$ is a finite set of $*$-polynomials with zero constant terms. By $C^*\langle G|\mathcal{R}\rangle$, we denote the universal (not-necessarily-unital) C^*-algebra generated by $g_1, \dots, g_l$ subject to

$$\|g_j\| \leq 1 \quad \text{and} \quad p_i(g_1, \dots, g_l) = 0.$$

By $C^*_\epsilon\langle G|\mathcal{R}\rangle$, we denote the universal unital C^*-algebra generated by $g_1, \dots, g_l$ subject to

$$\|g_j\| \le 1+\epsilon \quad \text{and} \quad \|p_i(g_1, \dots, g_l)\| \le \epsilon.$$

Sometimes, to be more explicit, we will denote the generators of $C^*_\epsilon\langle G|\mathcal{R}\rangle$ by $g_1^\epsilon, \dots, g_l^\epsilon$. We let P_ϵ denote the surjection

$$P_\epsilon : C^*_\epsilon\langle G|\mathcal{R}\rangle \to C^*\langle G|R\rangle$$

which sends g_j^ϵ to g_j.

If, for every $\eta > 0$, there exists $\epsilon > 0$ and a $*$-homomorphism

$$\sigma_\epsilon : C^*\langle G|\mathcal{R}\rangle \to C^*_\epsilon\langle G|\mathcal{R}\rangle$$

such that

$$\|\sigma_\epsilon(g_j) - g_j^\epsilon\| \le \eta, \quad j = 1, \dots, l$$

and $P_\epsilon \circ \sigma_\epsilon = \mathrm{id}$, then R is *stable.*

Theorem 2.5. *For a finitely presented C^*-algebra $C^*\langle G|\mathcal{R}\rangle$, the following conditions are equivalent:*

(1) *$\mathcal{R}$ is stable.*

(2) *$C^*\langle G|R\rangle$ is semiprojective.*

(3) *$C^*\langle G|R\rangle$ is corona semiprojective.*

Proof. The implication (1) $\Rightarrow$ (2) follows from [**8**, Theorem 3.2] while (2) $\Leftrightarrow$ (3) is a special case of Theorem 2.4. For (2) $\Rightarrow$ (1), applying semiprojectivity to the identity map immediately gives a map $\bar{\sigma}_{\bar{\epsilon}} : C^*\langle G|\mathcal{R}\rangle \to C^*_{\bar{\epsilon}}\langle G|\mathcal{R}\rangle$ with $P_{\bar{\epsilon}} \circ \bar{\sigma}_{\bar{\epsilon}} = \mathrm{id}$. Let σ_ϵ equal the composition of $\bar{\sigma}_{\bar{\epsilon}}$ with the natural surjection of $C^*_{\bar{\epsilon}}\langle G|\mathcal{R}\rangle$ onto $C^*_\epsilon\langle G|\mathcal{R}\rangle$ for ϵ sufficiently small, $0 < \epsilon < \bar{\epsilon}$. □

3. Generalizations of Kasparov's Technical Theorem.

Using the techniques of [**8**] and [**11**] we derive several generalizations of Kasparov's Technical Theorem (KTT). Our goal is to find the closest possible thing to matrix units inside a corona algebra for C^*-subalgebras of the form $A \otimes F$ where A is σ-unital and F is finite-dimensional.

All our theorems involve a subset D with which these ersatz matrix units are to commute. Easier proofs exist if one ignores D and sticks with the separable case. Indeed, one may use the projectivity of $\mathbf{C}M_n$, or $\bigoplus C_0(0,1]$, and [**12**, Proposition 3.12.1] along the lines of an observation of Cuntz described in [**2**, §12.4]. We will discuss this further in recent joint work with Gert Pedersen [**10**].

In this section, E will always denote a σ-unital C^*-algebra and $C(E)$ its corona algebra.

Theorem 3.1. *Suppose $A_1, \ldots, A_n$ are σ-unital C^*-subalgebras of $C(E)$. Let D be a separable, unital C^*-subalgebra of $C(E)$ such that*

$$A_j D A_k = 0, \quad j \neq k.$$

There exist $g_1, \ldots, g_n$ in $C(E) \cap D'$ such that

$$0 \leq g_j \leq 1, \quad j = 1, \ldots, n,$$

$$g_j g_k = 0, \quad j \neq k,$$

$$g_j a = a g_j = a, \quad \forall a \in A.$$

Proof. For $n = 2$ this is equivalent to KTT. Indeed, it is very close to the equivalent result [**11**, Theorem 3.7]. An induction argument gives the general case. □

Notice that $A_1 A_2 = 0$ implies that the C^*-algebra generated by $A_1 \cup A_2$ is isomorphic to $A_1 \oplus A_2$. Therefore, Kasparov's Technical Theorem implicitly involves a $*$-homomorphism $A_1 \oplus A_2 \to C(E)$. A natural setting for generalization is $M_n(A) \to C(E)$.

Theorem 3.2. *Suppose A is a σ-unital C^*-algebra, φ is a $*$-homomorphism*

$$\varphi : M_n(A) \to C(E)$$

and $\operatorname{im}(\varphi)$ *commutes with a separable subset D of $C(E)$. There exists a $*$-homomorphism*

$$\psi : \mathrm{C} M_n \to C(E) \cap D'$$

such that, setting $q_{ij} = \psi(t \otimes e_{ij})$,

$$q_{ij} \varphi(a \otimes e_{kl}) = \delta_{jk} \varphi(a \otimes e_{il}), \quad \forall a \in A.$$

Proof. Without loss of generality, D may be assumed to be a unital C^*-algebra. Applying Theorem 3.1 to

$$D, \varphi(A \otimes e_{11}), \ldots, \varphi(A \otimes e_{nn})$$

we obtain $g_1, \ldots, g_n$ in $C(E) \cap D'$ such that

$$0 \leq g_i \leq 1, \quad g_i g_j = 0 \ (i \neq j),$$

$$g_j\varphi(a \otimes e_{jj}) = \varphi(a \otimes e_{jj}).$$

Let h be a completely positive element of A. Since, for any a in A,

$$\begin{aligned} g_i\varphi(hah \otimes e_{jk}) &= g_i g_j \varphi(h \otimes e_{jj})\varphi(ah \otimes e_{jk}) \\ &= \delta_{ij}\varphi(hah \otimes e_{jk}) \end{aligned}$$

we conclude

(1) $$g_i\varphi(a \otimes e_{jk}) = \delta_{ij}\varphi(a \otimes e_{jk})$$

for all i, j, k and all $a \in A$.

Let $x = \varphi(h \otimes w)$ where

$$w = \begin{bmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{bmatrix}.$$

Since x is normal and both x and $|x| = \varphi(h \otimes I)$ commute with D, we may apply [**11**, Theorem 4.4]. Thus, there exists u in $C(E) \cap D'$, with $\|u\| \leq 1$, such that $x = u|x|$ and $x^* = u^*|x|$.

Multiplying $x = u|x|$ by $\varphi(ah \otimes e_{ij})$ yields

$$u\varphi(hah \otimes e_{ij}) = \varphi(hah \otimes e_{i+1,j}).$$

(Addition taken mod n.) Therefore, by this and a similar calculation based on $x^* = u^*|x|$,

(2) $$u\varphi(a \otimes e_{ij}) = \varphi(a \otimes e_{i+1,j}) \quad \text{and} \quad u^*\varphi(a \otimes e_{ij}) = \varphi(a \otimes e_{i-1,j}),$$

for all j, k and all $a \in A$.

We now make a first approximation on what shall be the images, under ψ, of the generators $t \otimes e_{j1}$ of $\mathrm{C}\, M_n$. Let

$$a_n = g_n u^{n-1} g_1,$$

and then for $j = n-1, \ldots, 2$,

$$a_{j-1} = g_{j-1} u^{j-2} |a_j|.$$

Clearly $a_j \in D'$ and

(3) $$|a_2| \leq |a_3| \leq \cdots \leq |a_n| \leq 1.$$

By induction, $a_j \in \overline{g_j C(E) g_1}$. This forces some of the relations determining $\mathrm{C}\,M_n$ (as in [**8**, Proposition 2.7]) to hold, namely

$$a_j a_k = 0, \quad j, k = 2, \dots, n,$$

$$a_j^* a_k = 0, \quad j \neq k. \tag{4}$$

We claim that, for all $b \in A$ and all i, j, k,

$$a_i \varphi(b \otimes e_{jk}) = \delta_{1j} \varphi(b \otimes e_{ik}) \text{ and } a_i^* \varphi(b \otimes e_{jk}) = \delta_{ij} \varphi(b \otimes e_{1k}). \tag{5}$$

For $i = n$ this follows directly from (1) and (2). But then

$$|a_n| \varphi(b \otimes e_{jk}) = \delta_{1j} \varphi(b \otimes e_{jk})$$

so one may handle the case $i = n - 1$, et cetera.

As done in the proof of [**8**, Lemma 4.8], for $j = 2, \dots, n$ we define

$$\tilde{a}_j = \lim_{m \to \infty} a_j((1/m) + a_j^* a_j)^{-1/2} (a_2^* a_2)^{1/2}.$$

By the calculations done in the proof of [**8**, Lemma 4.8] we conclude that setting $\psi(t \otimes e_{i1}) = \tilde{a}_i$ defines a homomorphism

$$\psi : \mathrm{C}\,M_n \to C(E) \cap D'.$$

For every $b \in A$, (5) implies

$$\tilde{a}_i \varphi(b \otimes e_{jk}) = \delta_{1j} \varphi(b \otimes e_{ik}) \text{ and } \tilde{a}_i^* \varphi(b \otimes e_{jk}) = \delta_{ij} \varphi(b \otimes e_{1k}) \tag{6}$$

whence

$$\psi(t \otimes e_{ij}) \varphi(b \otimes e_{kl}) = \delta_{jk} \varphi(b \otimes e_{il}).$$

□

4. Interval stretching in corona algebras.

We continue in this section to assume $C(E)$ is the corona algebra of some σ-unital C^*-algebra.

Let us consider a simple case of Kasparov's Technical Theorem. Given h_1, h_2 in $C(E)$ such that

$$0 \leq h_i \leq 1 \ (i = 1, 2) \quad \text{and} \quad h_1 h_2 = 0, \tag{7}$$

the conclusion is there exists an additional element so that now

$$0 \leq z \leq 1,\ 0 \leq h_i \leq 1\ (i = 1, 2),$$

$$h_1 z = 0,\ h_2 z = h_2 \text{ and } \quad h_1 h_2 = 0. \tag{8}$$

The universal C^*-algebra for these relations are as follows:

$$C^*\langle h_1, h_2 \mid (7) \text{ holds } \rangle \cong C_0([-1, 0) \cup (0, 1])$$

and

$$C^*\langle h_1, h_2, z \mid (8) \text{ holds } \rangle \cong C_0([-1, 0) \cup (0, 2]).$$

For this reason, we think of Kasparov's Technical Theorem as a device for stretching an interval algebra at a point.

We introduce some notation to be used for the rest of this section.

Let $X \subseteq \mathbb{C}$ denote the union of the unit circle and the interval $[-2, -1]$. Let

$$A_n = \{f \in C(X, M_n) \mid f(-2) \text{ is scalar}\}$$

and let $\alpha : M_n(C_0(0, 1))^\sim \to A_n$ denote the inclusion of the subalgebra of functions in $C(X, M_n)$ that are constant and scalar on $[-2, -1]$.

Lemma 4.1. *Let B denote any separable, unital C^*-algebra. Given a $*$-homomorphism*

$$\varphi : M_n(C_0(0, 1))^\sim \otimes B \to C(E)$$

whose image commutes with a separable subset $D \subseteq C(E)$, there exists $*$-homomorphism

$$\tilde{\varphi} : A_n \otimes B \to C(E)$$

such that $\tilde{\varphi} \circ (\alpha \otimes \mathrm{id}_B) = \varphi$ and whose image commutes with D.

Proof. Since A_n and $M_n(C_0(0, 1))^\sim$ are nuclear there is no ambiguity in the tensor product. As the tensor products involve unital C^*-algebras they are characterized as the universal C^*-algebras containing commuting copies of the two factors. By altering the subset D one easily shows that it suffices to prove this result only when $B = \mathbb{C}$.

Proposition 2.8 of [**8**] shows that $M_n(C_0(0, 1))^\sim$ is the universal unital C^*-algebra generated by $x, a_2, a_3, \ldots, a_n$ subject to the relations

$$\|a_j\| \leq 1, \quad j = 2, \ldots, n,$$

$$a_j a_k = 0, \quad 2 \leq j, k \leq n,$$

$$a_j^* a_k = 0, \quad j \neq k,$$

$$a_j^* a_j = x^* x,$$
$$x^* x = x x^* = -x - x^*.$$

Similarly, one may show that A_n is the universal unital C^*-algebra generated by $x, b_2, b_3, \ldots, b_n$ subject to the relations

$$\|b_j\| \leq 1, \quad j = 2, \ldots, n,$$
$$b_j b_k = 0, \quad 2 \leq j, k \leq n,$$
$$b_j^* b_k = 0, \quad j \neq k,$$
$$b_j^* b_j = b_k^* b_k, \quad 2 \leq j, k \leq n,$$
$$(b_j^* b_j - 1)(x x^* + x^* x) = 0,$$
$$x x^* = x^* x = -x - x^*,$$

and the inclusion α corresponds to the $*$-homomorphism determined by the assignment $x \mapsto x, a_j \mapsto b_j|x|$. Working with the same relations, but in nonunital category, one sees that this is a special case of Theorem 3.2.

□

Lemma 4.2. *Suppose J is an ideal in A and A is a sub-C^*-algebra of B. Let J_B denote the ideal of B generated by J. There is an isomorphism*

$$\Phi : B/J_B \to B *_A (A/J)$$

defined by $\Phi(b + J_B) = b$.

We will need to prove technical results regarding maps from general dimension-drop graphs into corona algebras. For clarity we will concentrate on the most important case, that of the dimension-drop intervals, $\tilde{\mathbb{I}}_n$. Recall

$$\tilde{\mathbb{I}}_n = \{f \in C[0,1] \mid f(0), f(1) \in \mathbb{C}I\},$$

this being the unital version of the dimension-drop interval.

Although isomorphic to $\tilde{\mathbb{I}}_n$ we also consider

$$\mathbb{J}_n = \{f \in C[-1,2] \mid f(-1) \text{ and } f(2) \text{ are scalar}\}.$$

Let $\iota : \tilde{\mathbb{I}}_n \to \mathbb{J}_n$ denote the inclusion that extends a function to be constant on $[-1,0]$ and on $[1,2]$.

Theorem 4.3. *Suppose $\varphi : \tilde{\mathbb{I}}_n \to C(E)$ is a $*$-homomorphism whose image commutes with a separable subset D. Then there exists a $*$-homomorphism $\bar{\varphi} : \mathbb{J}_n \to C(E) \cap D'$ such that $\bar{\varphi} \circ \iota = \varphi$.*

Proof. Consider $M_n(C_0(0,1))^\sim \otimes C[0,1]$ which we identify with

$$C_n = \{f \in C([0,1]^2, M_n) \mid f(0,t) = f(1,t) \in \mathbb{C}I,\ \forall t\}.$$

Restriction to the diagonal gives us a surjection

$$\rho : M_n(C_0(0,1))^\sim \otimes C[0,1] \to \tilde{\mathbb{I}}_n.$$

One can check that by the last lemma we have the commutative diagram

$$\begin{array}{ccc} (A_n \otimes C[0,1]) *_{C_n} \tilde{\mathbb{I}}_n & \xrightarrow{\cong} & \mathbb{J}_n \\ \uparrow{\scriptstyle (\alpha\otimes \mathrm{id})*\mathrm{id}} & & \uparrow{\scriptstyle \iota} \\ C_n *_{C_n} \tilde{\mathbb{I}}_n & \xrightarrow{\cong} & \tilde{\mathbb{I}}_n \end{array}$$

and so this result thus follows from Lemma 4.1. □

Remark. The generalization of Theorem 4.3 to the case of extending maps of dimension-drop graphs into corona algebras follows by the same methods, but the notation is significantly worse.

5. Stability for dimension-drop graphs.

Suppose X is a graph. We denote the associated dimension-drop C^*-algebra by

$$C_{\text{vert}}(X, M_n) = \{f \in C(X, M_n) \mid f(v) \in \mathbb{C}I \text{ for all vertices } v\}.$$

Theorem 5.1. *For every graph X, and every positive integer n, the C^*-algebra $C_{\text{vert}}(X, M_n)$ is universal for a stable set of relations.*

Proof. We may reduce to the case of X connected using Proposition 3.10 and [**8**, Theorem 5.1]. For connected graphs, the proof is by induction on the number of vertices. If there is but one vertex then

$$C_{\text{vert}}(X, M_n) \cong \left(\bigoplus_{j=1}^{J} M_n(C_0(0,1)) \right)^\sim$$

where J is the number of edges. This has stable relations by [**8**, Theorem 5.1].

Now suppose X has at least two vertices, v_0 and v_1. We will need an auxiliary space, $\tilde{X}$, which is obtained from X by stretching all edges attached to v_0 or v_1. Topologically, $\tilde{X}$ will be a copy of X. We shall use v_0 and v_1 to denote the appropriate vertices in $\tilde{X}$.

Choose a function

$$h_0 : \tilde{X} \to [-1, 2]$$

such that $h_0^{-1}([-1,0])$ consists of the union of half-closed subintervals, containing v_0, of each edge adjacent to v_0. We may assume a similar statement holds for $h_0^{-1}([1,2])$ and v_1.

We will identify X with the quotient of $\tilde{X}$ obtained by collapsing $h_0^{-1}([-1,0])$ to a point and $h_0^{-1}([1,2])$ to a different point. We will also consider two copies of the graph obtained from X by collapsing the two designated vertices together. We let $\tilde{Y}$ denote the quotient of $\tilde{X}$ obtained by identifying v_0 with v_1 and Y denote the quotient of $\tilde{X}$ obtained by collapsing $h_0^{-1}([-1,0]) \cup h_0^{-1}([1,2])$ to a point.

Accordingly, we will be making identifications of the various dimension-drop algebras with subalgebras of $C(\tilde{X}, M_n)$. Of course, $C_{\text{vert}}(\tilde{X}, M_n)$ is defined as such a subalgebra. The remaining identifications are:

$$\begin{aligned} C_{\text{vert}}(X, M_n) &= \{f \mid f(x) = f(v_0) \text{ if } h_0(x) \leq 0 \\ &\qquad\qquad \text{and } f(x) = f(v_1) \text{ if } h_0(x) \geq 1\}, \\ C_{\text{vert}}(Y, M_n) &= \{f \mid f(x) = f(v_0) \text{ if } h_0(x) \leq 0 \text{ or } h_0(x) \geq 1\} \\ C_{\text{vert}}(\tilde{Y}, M_n) &= \{f \mid f(v_0) = f(v_1)\}. \end{aligned}$$

Our strategy is based on the observation that $C_{\text{vert}}(X, M_n)$ is generated by the subalgebra $C_{\text{vert}}(Y, M_n)$ and the element

$$h = h_1 \otimes I \quad \text{where} \quad h_1(x) = \max(\min(h_0(x), 1), 0).$$

A way to express the relation between h and $C_{\text{vert}}(Y, M_n)$ is that

$$e^{2\pi i h} = e^{2\pi i h_1} \otimes I.$$

By Theorem 2.6, our task is reduced to proving corona semiprojectivity for $C_{\text{vert}}(X, M_n)$ while assuming it for $C_{\text{vert}}(\tilde{Y}, M_n)$. So suppose that we are given a unital $*$-homomorphism

$$\varphi : C_{\text{vert}}(X, M_n) \to C(E) \cong B/\overline{\bigcup I_m}.$$

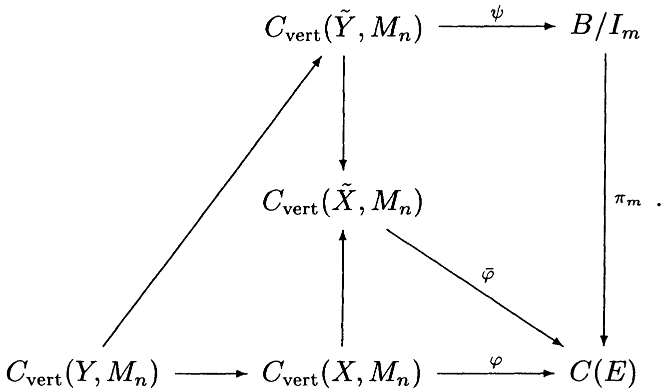
By Theorem 4.3 and the remark following, there is an extension of φ to

$$\bar{\varphi} : C_{\text{vert}}(\tilde{X}, M_n) \to C(E).$$

By the induction hypothesis, the restriction of $\bar{\varphi}$ to $C_{\text{vert}}(\tilde{Y}, M_n)$ can be lifted to

$$\psi : C_{\text{vert}}(\tilde{Y}, M_n) \to B/I_m$$

for some m. This leads to the following commutative diagram:



Let H be any lift of $\varphi(h)$ to B/I_m such that $0 \leq H \leq 1$. Now define

$$\tilde{H} = \psi(l(h_0) \otimes I) + \psi(m(h_0)^{1/2} \otimes I)H\psi(m(h_0)^{1/2} \otimes I)$$

where l and m are the functions

$$l(t) = \begin{cases} 0, & t \leq 0, \\ t, & 0 \leq t \leq 1, \\ 2-t, & 1 \leq t \leq 2, \end{cases} \quad m(t) = \begin{cases} -t, & t \leq 0, \\ 0, & 0 \leq t \leq 1, \\ t-1, & 1 \leq t \leq 2. \end{cases}$$

These are defined so that $l + mh_2 = h_2$ where h_2 is the function

$$h_2(t) = \begin{cases} 0, & t \leq 0, \\ t, & 0 \leq t \leq 1, \\ 1, & 1 \leq t \leq 2. \end{cases}$$

Notice also that $h_2(h_0) = h_1$.

Clearly $\tilde{H}$ is selfadjoint. In fact, it is also a lift of $\varphi(h)$ since

$$\begin{aligned} \pi_m(\tilde{H}) &= \bar{\varphi}(l(h_0) \otimes I) + \bar{\varphi}(m(h_0) \otimes I)\bar{\varphi}(h_2(h_0) \otimes I) \\ &= \bar{\varphi}((l + mh_2)(h_0) \otimes I) = \varphi(h). \end{aligned}$$

For any $f \otimes T \in C_{\text{vert}}(Y, M_n)$

$$(f \otimes T)(m(h_0)^{1/2} \otimes I) = 0 \quad \Rightarrow \quad \psi(f \otimes T)\tilde{H} = \tilde{H}\psi(f \otimes T).$$

By replacing $\tilde{H}$ by $h_2(\tilde{H})$, we have found a lift of $\varphi(h)$, with $0 \leq \tilde{H} \leq 1$, and a lift of $\varphi|_{C_{\text{vert}}(Y,M_n)}$ that commute.

Expressing this conclusion differently, we have shown that given a unital map

$$C_{\text{vert}}(X, M_n) \to C(E)$$

we can find an m and a map making the diagram commute where D is the universal unital C^*-algebra generated by a copy of $C_{\text{vert}}(Y, M_n)$ and a central element h such that $0 \le h \le 1$. I.e.,

$$D \cong C_{\text{vert}}(Y, M_n) \otimes C[0,1].$$

We have no further need for $\tilde{X}$ so v_0 and v_1 again denote the specified vertices in X. We regard Y as the quotient of X, with quotient map $\eta : X \to Y$ which collapses v_0 and v_1 to a single vertex we call w_0.

Let us identify D with

$$\{g \in C(Y \times [0,1], M_n) \mid g(v,t) \in \mathbb{C}I \text{ for all vertices}\}.$$

The copy of $C_{\text{vert}}(Y, M_n)$ and the extra element h appear as functions in D constant in one variable or the other. There is a sort of diagonal map

$$\Delta : X \to Y \times [0,1], \quad \Delta(x) = (\eta(x), h_1(x))$$

which induces a surjection $\beta : D \to C_{\text{vert}}(X, M_n)$.

We need also a quotient of D where the relation (9) holds approximately. Consider

$$Z_\delta = \{(\eta(x), t) \in Y \times [0,1] \mid |e^{2\pi i h_1(x)} - e^{2\pi i t}| \le \delta\},$$

where δ is a small number to be named later, and let

$$D_\delta = \{g \in C(Z, M_n) \mid g(v,t) \in \mathbb{C}I \text{ for all vertices}\}.$$

Since Δ maps into Z it induces

$$\beta_0 : D_\delta \to C_{\text{vert}}(X, M_n).$$

By increasing m we may assume that the map $D \to B/I_m$ factors through D_δ. Therefore, we are done if we exhibit a right-inverse to β_0. This exists because there is a retraction of Z_δ onto $\operatorname{im}(\Delta)$ which sends (v,t) to (v,t') for every vertex v. To be able to describe this retraction we break up Z_δ as $Z_\delta = Z_1 \cup Z_2 \cup Z_3$ where

$$\begin{aligned} Z_1 &= \{(\eta(x), t) \mid |h_1(x) - t| \le 1/4, 0 < t < 1\}, \\ Z_2 &= \{(\eta(x), t) \mid |h_1(x) + 1 - t| \le 1/4\}, \\ Z_3 &= \{(\eta(x), t) \mid |h_1(x) - 1 - t| \le 1/4\}. \end{aligned}$$

The retraction sends Z_2 to $(w_0, 1)$ and Z_3 to $(w_0, 0)$. Each point $(\eta(x), t)$ in Z_1 is sent to $(\eta(x), s)$ where s is the unique number in $(0,1)$ such that

$e^{2\pi is} = e^{2\pi ih_1(x)}$. By choosing δ sufficiently small, we ensure that $(v,t) \notin Z_2 \cup Z_3$ for any vertex v except for $v = w_0$. Therefore this is the desired retraction. □

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SINGULAR MODULI SPACES OF STABLE VECTOR BUNDLES ON $\mathbf{P}^3$

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The goal of this paper is to give an example of singular moduli space of rank 3 stable vector bundles on $\mathbf{P}^3$.

Introduction.

In 1977/78, M. Maruyama proved the existence of a moduli scheme $M_{\mathbf{P}^n}(r; c_1, ..., c_{\min(r,n)})$ parametrizing isomorphic classes of rank r stable vector bundles on $\mathbf{P}^n$ with given Chern classes $c_1,..., c_{\min(n,r)}$ (cf. [**M1, M2**]). The goal of this note is to give, to the best of my knowledge, the first example of singular moduli space of stable vector bundles on $\mathbf{P}^3$. It has been motivated by a recent work of Ancona and Ottaviani where they show that the moduli space $MI_{\mathbf{P}^5}(k)$ of stable instanton bundles on $\mathbf{P}^5$ with quantum number k=3 or 4 is singular. Moreover they claim that $MI_{\mathbf{P}^5}(3)$ and $MI_{\mathbf{P}^5}(4)$ are the first examples of singular moduli spaces of stable vector bundles on projective spaces (cf. [**AO**]). Ancona-Ottaviani's result together with the well known fact that $M_{\mathbf{P}^2}(r; c_1, c_2)$ is a smooth quasi-projective variety of dimension $2rc_2 - (r-1)c_1^2 + 1 - r^2$ gives rise the following question:

Is there any example of singular moduli space of stable vector bundles on $\mathbf{P}^3$?

As I pointed out before my aim is to give an affirmative answer to this question (cf. Theorem 2.10).

1. Preliminaries.

In this section we recall some well known results needed later on.

1.1. Let H(18, 39) be the open subscheme of $Hilb\mathbf{P}^3_k$ parametrizing smooth connected curves $C \subset \mathbf{P}^3$ of degree 18 and genus 39. (See [**EF**] for a precise description of H(18, 39).) Let $H_1 \subset H(18, 39)$ be the 72-dimensional irreducible, generically smooth component whose general point parametrizes an arithmetically Cohen-Macaulay curve $X \subset \mathbf{P}^3$ having a locally free resolution of the following type:

(1) $$0 \to \mathcal{O}(-7)^4 \to \mathcal{O}(-6)^4 \oplus \mathcal{O}(-4) \to I_X \to 0.$$

Let $H_2 \subset H(18,39)$ be the 72-dimensional irreducible, generically smooth component whose general point parametrizes an arithmetically Cohen-Macaulay curve $Y \subset \mathbf{P}^3$ having a locally free resolution of the following type:

$$(2) \qquad 0 \to \mathcal{O}(-6)^2 \oplus \mathcal{O}(-8) \to \mathcal{O}(-5)^4 \to I_Y \to 0.$$

It is well known that there exits an irreducible subset $H = H_1 \cap H_2 \subset H(18,39)$ of dimension 71 whose general point parametrizes an arithmetically Buchsbaum curve $C \subset \mathbf{P}^3$ having a locally free resolution of the following type:

$$(3) \qquad 0 \to \mathcal{O}(-8) \to \mathcal{O}(-7)^4 \oplus \mathcal{O}(-8) \to \mathcal{O}(-6)^4 \oplus \mathcal{O}(-4) \to I_C \to 0.$$

1.2. Remark. For all curve $Z \in H_1 \cup H_2$, $\omega_Z(2)$ is globally generated. From now on, for all curve $Z \in H_1 \cup H_2$, we set $\alpha := \dim H^0(\omega_Z(2))$ (=74; by Riemann-Roch's Theorem).

1.3. Fact. Let $0 \to E \to F \to G \to 0$ be an exact sequence of vector bundles. Then, we have the following exact sequence involving alternating and symmetric powers:

$$0 \to S^q E \to S^{q-1}E \otimes F \to .... \to E \otimes \Lambda^{q-1}F \to \Lambda^q F \to \Lambda^q G \to 0.$$

1.4. Hoppe's criterion for the stability of a vector bundle. Let X be a projective manifold with $Pic(X) \cong \mathbf{Z}$ and let E be a vector bundle on X. If $H^0(X, (\Lambda^q E)_{\text{norm}}) = 0$ for $1 \leq q \leq rk(E) - 1$, then E is stable. As usual, given a vector bundle E on X, we denote by E_{norm} the twist of E whose first Chern class c_1 verifies $-rk(E) + 1 \leq c_1 \leq 0$.

2. Main results.

2.1. Let us call $\mathcal{L}_1$ the irreducible family of sheaves E on $\mathbf{P}^3$ constructed as an extension:

$$\sigma = (\sigma_1, ..., \sigma_\alpha): \qquad 0 \to \mathcal{O}^\alpha \to E(1) \to I_X(2) \to 0$$

where $X \in H_1$ and $\sigma_1, ..., \sigma_\alpha \in H^0(\omega_X(2)) \cong Ext^1(I_X(2), \mathcal{O})$ are general global sections which generate the sheaf $\omega_Z(2)$ everywhere.

It is easy to see that E is a vector bundle on $\mathbf{P}^3$ of rank $\alpha + 1$.

2.2. Let us call $\mathcal{L}_2$ the irreducible family of sheaves F on $\mathbf{P}^3$ constructed as an extension:

$$\lambda = (\lambda_1, ..., \lambda_\alpha): \qquad 0 \to \mathcal{O}^\alpha \to F(1) \to I_Y(2) \to 0$$

where $Y \in H_2$ and $\lambda_1, ..., \lambda_\alpha \in H^0(\omega_Y(2)) \cong Ext^1(I_Y(2), \mathcal{O})$ are general global sections which generate the sheaf $\omega_Z(2)$ everywhere.

Again it is easy to see that F is a vector bundle on $\mathbf{P}^3$ of rank $\alpha + 1$.

2.3. And let $\mathcal{L} \subset \mathcal{L}_1 \cap \mathcal{L}_2$ be the irreducible family of sheaves G on $\mathbf{P}^3$ constructed as an extension:

$$\mu = (\mu_1, ..., \mu_\alpha) : \qquad 0 \to \mathcal{O}^\alpha \to G(1) \to I_C(2) \to 0$$

where $C \in H \subset H_1 \cap H_2$ and $\mu_1, ..., \mu_\alpha \in H^0(\omega_C(2)) \cong Ext^1(I_C(2), \mathcal{O})$ are general global sections which generate the sheaf $\omega_Z(2)$ everywhere.

Again it is easy to see that G is a vector bundle on $\mathbf{P}^3$ of rank $\alpha + 1$.

Proposition 2.4.

(1) *A general vector bundle $E \in \mathcal{L}_1$ has a locally free resolution of the following type:*

$$0 \to \mathcal{O}(-5)^4 \to \mathcal{O}(-4)^4 \oplus \mathcal{O}(-2) \oplus \mathcal{O}^\alpha \to E(1) \to 0.$$

(2) *A general vector bundle $F \in \mathcal{L}_2$ has a locally free resolution of the following type:*

$$0 \to \mathcal{O}(-6) \oplus \mathcal{O}(-4)^2 \to \mathcal{O}(-3)^4 \oplus \mathcal{O}^\alpha \to F(1) \to 0.$$

(3) *A general vector bundle $G \in \mathcal{L}$ has a locally free resolution of the following type:*

$$0 \to \mathcal{O}(-6) \to \mathcal{O}(-6) \oplus \mathcal{O}(-5)^4 \to \mathcal{O}(-4)^4 \oplus \mathcal{O}(-2) \oplus \mathcal{O}^\alpha \to G(1) \to 0.$$

Proof. (1) From the exact sequence:

$$0 \to \mathcal{O}^\alpha \to E(1) \to I_X(2) \to 0$$

and the locally free resolution of $I_X(2)$ (See 1.1):

$$0 \to \mathcal{O}(-5)^4 \to \mathcal{O}(-4)^4 \oplus \mathcal{O}(-2) \to I_X(2) \to 0$$

we get the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}(-5)^4 & = & \mathcal{O}(-5)^4 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}^\alpha & \longrightarrow & \mathcal{O}^\alpha \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-4)^4 & \longrightarrow & \mathcal{O}(-2) \oplus \mathcal{O}(-4)^4 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}^\alpha & \longrightarrow & E(1) & \longrightarrow & I_X(2) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

which gives what we want.

(2) and (3) Analogous. □

Corollary 2.5. *Given a vector bundle $E \in \mathcal{L}_1 \cup \mathcal{L}_2$, $E(t)$ is globally generated for all $t \geq 5$.*

2.6. Let $\mathcal{F}_1$ be the irreducible family of rank 3 vector bundles P on $\mathbf{P}^3$ defined as the cokernel:

$$0 \to \mathcal{O}(-5)^{\alpha-2} \xrightarrow[s_1,\dots,s_2]{} E \to P \to 0$$

where $E \in \mathcal{L}_1$ and $s_i \in H^0(E(5))$ are general global sections of $E(5)$.

2.7. Let $\mathcal{F}_2$ be the irreducible family of rank 3 vector bundles Q on $\mathbf{P}^3$ defined as the cokernel:

$$0 \to \mathcal{O}(-5)^{\alpha-2} \xrightarrow[f_1,\dots,f_2]{} F \to Q \to 0$$

where $F \in \mathcal{L}_2$ and $f_i \in H^0(F(5))$ are general global sections of $F(5)$.

2.8. Let $\mathcal{F} \subset \mathcal{L}_1 \cap \mathcal{L}_2$ be the irreducible family of rank 3 vector bundles R on $\mathbf{P}^3$ defined as the cokernel:

$$0 \to \mathcal{O}(-5)^{\alpha-2} \xrightarrow[g_1,\dots,g_2]{} G \to R \to 0$$

where $G \in \mathcal{L}$ and $g_i \in H^0(G(5))$ are general global sections of $G(5)$.

Proposition 2.9.

(1) *A general vector bundle P of $\mathcal{F}_1$ is a rank 3 stable vector bundle on $\mathbf{P}^3$ with Chern classes $(287, 42065, 4195775)$.*

(2) *A general vector bundle Q of $\mathcal{F}_2$ is a rank 3stable vector bundle on $\mathbf{P}^3$ with Chern classes $(287, 42065, 4195775)$.*

(3) *A general vector bundle R of $\mathcal{F}$ is a rank 3 stable vector bundle on $\mathbf{P}^3$ with Chern classes $(287, 42065, 4195775)$.*

Proof. (1) Using the exact sequence:

$$(*) \qquad 0 \to \mathcal{O}(-5)^{\alpha-2} \to E \to P \to 0$$

and the locally free resolution of E given in Proposition 2.4(1) we get:

$$c_t(P) = (1-3t)(1-t)^{74} / \left((1-6t)^4(1-5t)^{68}\right).$$

Hence $c_1(P) = 287$, $c_2(P) = 42065$ and $c_3(P) = 4195775$.

Let us see that P is stable. Using Hoppe's criterion we need to prove that $H^0(P)_{\mathrm{norm}} = H^0(\Lambda^2 P)_{\mathrm{norm}} = 0$. Since $c_1(P) > 0$ and $c_1(\Lambda^2 P) > 0$, we have $(P)_{\mathrm{norm}} = P(\lambda)$ and $(\Lambda^2 P)_{\mathrm{norm}} = (\Lambda^2 P)(\rho)$ for some $\rho, \lambda \leq -1$. So it siffices to prove that $H^0(P)(-1) = H^0(\Lambda^2 P)(-1) = 0$.

Using the exact sequence (*) and the locally free resolution of E given in Proposition 2.4(1) we easily get that $H^0 E(-1) = H^0 P(-1) = 0$. Again using the exact sequence (*) and taking wedge powers we get the exact sequence

$$0 \to S^2\mathcal{O}(-5)^{\alpha-2} \to \mathcal{O}(-5)^{\alpha-2} \otimes E \to \Lambda^2 E \to \Lambda^2 P \to 0$$

cutting in short exact sequences we get $H^0(\Lambda^2 P)(-1) = H^0(\Lambda^2 E)(-1) = 0$ where the last equality follows from the locally free resolution of E given in Proposition 2.4(1) taking wedge powers and cutting in short exact sequences.

(2) and (3) are analogous. □

Theorem 2.10. *The moduli space* $M_{\mathbf{P}^3}(3; -1, 14609, 1917791)$ *is singular.*

Proof. We have constructed two irreducible families $\mathcal{F}_1$ and $\mathcal{F}_2$ of rank 3 stable vector bundles on $\mathbf{P}^3$ with Chern classes (287,42065,4195775) which meets along an irreducible family $\mathcal{F}$. Hence in order to see that $M := M_{\mathbf{P}^3}(-1, 14609, 1917791) \cong M_{\mathbf{P}^3}(-287, 42065, 4195775)$ is singular it is enough to prove that $\mathcal{F}_1$ and $\mathcal{F}_2$ belongs to two different components of M. Using proposition 2.9 and 2.4 we get:

(1) If P is a general vector bundle of $\mathcal{F}_1$ then:

$$H^1_* P = H^3 P(3) = 0$$
$$h^0 P(3) = 1 + 10\alpha,\ h^2 P(3) = 0.$$

(2) If Q is a general vector bundle of $\mathcal{F}_2$ then:

$$H^1_* Q = H^3 Q(3) = 0$$
$$h^0 Q(3) = 10\alpha,\ h^2 Q(3) = 1.$$

Therefore, by semicontinuity $\mathcal{F}_1$ and $\mathcal{F}_2$ are contained in different components of M which gives what we want. □

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THE GODBILLON-VEY CYCLIC COCYCLE AND LONGITUDINAL DIRAC OPERATORS

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The goal of this paper is to prove the index theorem for the pairing of the Godbillon-Vey cyclic cocycle with the index class of the longitudinal Dirac operator for a codimension one foliation. Let $(X, \mathcal{F})$ be a foliated S^1-bundle over an arbitrary spin manifold M. The Dirac operator on M lifts to a longitudinal elliptic operator D, the longitudinal Dirac operator, on $(X, \mathcal{F})$. The index class of D is an element of the K_0-group of the foliation C^*-algebra $C^*(X, \mathcal{F})$. A densely defined cyclic even-cocycle on $C^*(X, \mathcal{F})$, the Godbillon-Vey cyclic cocycle, is constructed. The main result gives a topological formula for the pairing of the Godbillon-Vey cyclic cocycle with the index class of D. The proof of the main theorem uses a new technique, the pairing with the graph projections.

1. Introduction.

Over the past decade K-theory has come to play significant roles in the study of C^*-algebras. One such role is as a receptor of indices of pseudodifferential operators on foliated manifolds. If P is a longitudinal elliptic operator on a foliated manifold $(X, \mathcal{F})$, then the index of P is an element of the K_0-group of the foliation C^*-algebra $C^*(X, \mathcal{F})$ [**10**]. A transverse invariant measure ν for the foliation generates a trace on the C^*-algebra $C^*(X, \mathcal{F})$. This trace defines an additive map ϕ_ν from the K_0-group into the scalars. Evaluating ϕ_ν on the index of an operator, we obtain a numerical invariant (an analytic index), which depends on the transverse invariant measure ν. The index theorem of A. Connes [**6**] describes the analytic index in terms of the symbol of the operator and the foliation cycle corresponding to the transverse invariant measure.

For many interesting foliations, e.g. Anosov foliations, there does not exist a nontrivial transverse invariant measure. Thus, in order to obtain numerical invariants of operators on such foliations, we need an alternative. A natural candidate is the pairing between K-group and cyclic cohomology. In fact, a trace on a C^*-algebra may be regarded as a densely defined cyclic 0-cocycle. Our aim is to give an index formula for higher dimensional cyclic cocycles. In this direction several authors have obtained results for certain cocycles, see for example [**11**]. Connes and H. Moscovici [**9**] studied the pairing between cyclic cocycles associated with group cocycles and Dirac operators on a Golois covering. In order to compute the pairing they use idempotents constructed by A. Wasserman. Our arguments use graph projections associated with the operators; the advantage is that they provide a direct construction and result in a simple argument.

We focus on a particular cyclic cocycle for a special class of foliations. Let Γ be a discrete group acting freely on a manifold $\widetilde{M}$ so that $\widetilde{M}/\Gamma$ is a closed manifold. Suppose that a Γ-action on the circle S^1, by orientation preserving diffeomorphisms, is given. The S^1-bundle over $\widetilde{M}/\Gamma$ associated with the action is equipped with a foliation $\mathcal{F}$, whose leaves are transverse to the fiber of the bundle. The S^1-bundle X with $\mathcal{F}$ is called a foliated S^1-bundle. When the action satisfies a certain condition (Condition 2.2), the foliation C^*-algebra $C^*(X, \mathcal{F})$ is strongly Morita equivalent to the reduced crossed product $C(S^1) \rtimes \Gamma$. The foliation $\mathcal{F}$ is of codimension one, and transversely orientable. To such a foliation, is assigned a characteristic class, called the Godbillon-Vey class [**13**]. It is a 3-dimensional de Rham cohomology class of X. For foliated S^1-bundles, this characteristic class is interpreted as a group 2-cocycle with values in the space of 1-forms on S^1 [**5**]. Based on this picture, A. Connes studied an analytical interpretation of the Godbillon-Vey class [**8**]. He constructed a densely defined cyclic 2-cocycle τ on the C^*-algebra $C(S^1) \rtimes \Gamma$ and showed that the additive map, induced by τ, coincides with the map, which the Godbillon-Vey class induces on the geometric group $K^0(S^1, \Gamma)$, via the index map $K^0(S^1, \Gamma) \to K_0(C(S^1) \rtimes \Gamma)$.

If P is a longitudinal elliptic operator on a foliated S^1-bundle $(X, \mathcal{F})$, its index $\mathrm{ind}(P)$ is regarded as a class in $K_0(C(S^1) \rtimes \Gamma)$ via the strong Morita equaivalence. We will explicitly compute the value of the additive map mentioned above on the indices of longitudinal Dirac operators. More precisely, we will consider the case where an even-dimensional manifold $\widetilde{M}$ is endowed with a Γ-invariant metric and a Γ-invariant spin structure. We will study the index of the associated Dirac operator D. In order to carry out an explicit computation, the following points have to be taken care of. (1) Since $\mathrm{ind}(D)$ is defined to be a class in the K_0-group of the foliation C^*-algebra, we have to obtain a formula for a densely defined cyclic cocycle on $C^*(X, \mathcal{F})$ (Section 6). The strong Morita equivalence between $C^*(X, \mathcal{F})$ and $C(S^1) \rtimes \Gamma$ yields a homomorphism from $C(S^1) \rtimes \Gamma$ into $C^*(X, \mathcal{F})$. Thus, once we obtain a densely defined cyclic cocycle on $C^*(X, \mathcal{F})$, we can compare this cocycle with Connes's cocycle (Section 9). (2) The index $\mathrm{ind}(D)$ is described in terms of a parametrix of D [**10**], [**9**], and there is not a canonical choice of a parametrix. Thus it seems infeasible to compute the evaluation on such an element. Hence we need a projection "canonically" attached to the operator. The operator extends to a closed operator T; the graph of T is a closed subspace, and the associated orthogonal projection is called the graph projection of T. It will be shown that the graph projection represents $\mathrm{ind}(D)$. A disadvantage of using graph projections is that they lack the regularity which idempotents in [**9**], [**10**] can enjoy. Thus it has to be verified that the graph projection does indeed belong to the domain of the cyclic cocycle.

A use of graph projections in the index problem is a new idea. Once (1) and (2) above are done, the proof of the actual computation of the evaluation (Theorem 8.10) will be straightforward by employing Getzler's symbolic calculus method [**12**].

This work grew out of a study of the K_0-group of the C^*-algebras of Anosov foliations on the unit circle bundle $T_1\Sigma$ of a closed Riemann surface Σ of genus $g > 1$ furnished with a metric of constant negative curvature. Those C^*-algebras are strongly Morita equivalent to crossed product C^*-algebras $C(S^1) \rtimes \pi_1(\Sigma)$, where $\pi_1(\Sigma)$ acts on $C(S^1)$ through linear fractional transformations. Since Anosov foliations on $T_1\Sigma$ have nonzero Godbillon-Vey classes, there must be a class in K_0 on which the cyclic cocycle attains a nonzero value. Our motivation was to describe this class as clearly as possible. This matter will be discussed in Section 10.

2. Foliated Bundles and Its C^*-algebras.

In this section we study the properties of C^*-algebras associated with foliated bundles. On these C^*-algebras we will construct densely defined cyclic cocycles in Section 6.

Let M be a closed Riemannian manifold, and let $\widetilde{M} \to M$ be a Galois covering with deck transformation group Γ. Given a right Γ-action on a closed manifold V by diffeomorphisms, we can construct a fibre bundle $X \to M$ with fibre V. This is the associated bundle

$$p : X = \widetilde{M} \times_\Gamma V \to \widetilde{M}/\Gamma = M,$$

where the right Γ-action on $\widetilde{M} \times V$ is diagonal. The product foliation on $\widetilde{M} \times V$ with leaves $\widetilde{M} \times \{x\}, x \in V$, descends to a foliation $\mathcal{F}$ on X. The projection p restricted to any leaf of $\mathcal{F}$ is a covering map. We call the V-bundle $X \to M$ together with $\mathcal{F}$ a *foliated V-bundle*.

Condition 2.1. Through the paper we assume that a Γ-action on V satisfies the condition: for $g \in \Gamma$, if there exists an open set U in V such that $xg = x$ for all $x \in U$, then g is the identity element of Γ.

The Condition 2.1 guarantees that the holonomy groupoid G of $\mathcal{F}$ is a

Hausdorff space, and that

$$G \cong (\widetilde{M} \times \widetilde{M} \times V)/\Gamma,$$

where Γ acts by $(m,n,x)g = (mg,ng,xg)$, $(m,n,x) \in \widetilde{M} \times \widetilde{M} \times V$, $g \in \Gamma$. The groupoid structure of $(\widetilde{M} \times \widetilde{M} \times V)/\Gamma$ is described as follows. Denote by $[m,n,x]$ the class of $(m,n,x) \in \widetilde{M} \times \widetilde{M} \times V$. The source map s and the range map r are given by

$$r([m,n,x]) = [m,x],$$
$$s([m,n,x]) = [n,x].$$

Two elements $[m',n',x']$ and $[m,n,x]$ are composable if and only if there exists a $g \in \Gamma$ such that $n' = mg$, $x' = xg$. In this case,

$$[m',n',x'][m,n,x] = [m'g^{-1},n,x].$$

The lifting to $\widetilde{M}$ of the Riemannian metric on M induces a leafwise Riemannian metric. The latter gives rise to a left Haar system $\{\nu^x\}$ of the groupoid G [**18**].

We recall the definition of foliation C^*-algebras with coefficient [**11**]. Let E be a Hermitian vector bundle over X. Denote by $C_c^\infty(G,E)$ the space of all compactly supported smooth sections of the bundle $(s^*(E))^* \otimes r^*(E)$. So, if $f \in C_c^\infty(G,E)$, then

$$f(\gamma) \in \mathrm{Hom}(E_{s(\gamma)}, E_{r(\gamma)}), \ \gamma \in G.$$

The space $C_c^\infty(G,E)$ has a $*$-algebra stucture:

$$(f_1 * f_2)(\gamma) = \int_{G^{r(\gamma)}} f_1(\gamma')f_2(\gamma'^{-1}\gamma)\, d\nu^{r(\gamma)}(\gamma'),$$
$$f^*(\gamma) = (f(\gamma^{-1}))^*,$$

where $f_1(\gamma')f_2(\gamma'^{-1}\gamma)$ is the composition of maps, and

$$(f(\gamma^{-1}))^* \in \mathrm{Hom}(E_{s(\gamma)}, E_{r(\gamma)})$$

is the adjoint of $f(\gamma^{-1}) \in \mathrm{Hom}(E_{r(\gamma)}, E_{s(\gamma)})$.

Let $\tilde{r}, \tilde{s}$ be the lifting of r, s to $\widetilde{M} \times \widetilde{M} \times V \to \widetilde{M} \times V$, respectively. Thus

$$\tilde{r}(m,n,x) = (m,x) \quad \text{and} \quad \tilde{s}(m,n,x) = (n,x).$$

Denote by $\widetilde{E}$ the lifting to $\widetilde{M} \times V$ of E. It is easy to see that $C_c^\infty(G,E)$ is identified with the space $\mathcal{K}_c$ of those Γ-invariant smooth sections of $(\tilde{s}^*(E))^* \otimes$

$\tilde{r}^*(E)$ which have Γ-compact supports. Here we say that a subset of $\widetilde{M} \times \widetilde{M} \times V$ is Γ-compact, if its image in $(\widetilde{M} \times \widetilde{M} \times V)/\Gamma$ is compact (Definition 8.3 of [**3**]).

Let $\widetilde{M}_x = \widetilde{M} \times \{x\}$, $x \in V$, and let μ_x be the strictly positive smooth density on $\widetilde{M}_x$ corresponding to the Γ-invariant smooth density on $\widetilde{M}$ through the canonical identification of $\widetilde{M}_x$ and $\widetilde{M}$. Set

$$H_x = L^2\left(\widetilde{E}_x, \mu_x\right),$$

where $\widetilde{E}_x$ is the restriction of $\widetilde{E}$ to $\widetilde{M}_x$. Then the collection $\mathcal{H} = (H_x)_{x \in V}$ together with the space $C_c(\widetilde{E})$, of compactly supported continuous sections of the bundle $\widetilde{E}$ over $\widetilde{M} \times V$, defines a continuous field of Hilbert spaces over V. The Γ-action on $\widetilde{M} \times V$ and $\widetilde{E}$ gives rise to an action on $\mathcal{H}$. We denote this action by $\xi \to g\xi$, for $g \in \Gamma$, and a section ξ of $\mathcal{H}$. The space $\mathrm{End}_\Gamma(\mathcal{H})$ of Γ-equivariant bounded measurable fields of operators $T = (T_x)$, $T_x \in B(H_x)$, is a C^*-algebra, where the norm is given by

$$||T|| = \sup\{||T_x||; x \in V\}.$$

There is a faithful representation $\rho : \mathcal{K}_c \to \mathrm{End}_\Gamma(\mathcal{H})$. For $f \in \mathcal{K}_c$, the operator $\rho(f)$ is defined by

$$[\rho(f)_x \xi](m) = \int_{\widetilde{M}_x} f(m, n, x)\xi(n)\, d\mu_x(n), \tag{2.2}$$

for $\xi \in H_x$. The norm-closure of $\mathcal{K}_c$ with respect to the norm

$$||f|| = ||\rho(f)|| = \sup\{||\rho(f)_x||; x \in V\}, \; f \in \mathcal{K}_c,$$

is, by definition, the C^*-algebra $C^*(X, \mathcal{F}, E)$ of the foliated bundle $(X, \mathcal{F})$ with coefficient E.

Let $C(V) \rtimes \Gamma$ be the reduced crossed product C^*-algebra arising from the (left) Γ-action on $C(V)$ given by

$$(ga)(x) = a(xg), \quad g \in C(V).$$

The C^*-algebra $C(V) \rtimes \Gamma$ is exactly the reduced C^*-algebra associated with the following groupoid. As a topological space this is $V \times \Gamma$. The space of units is V, with $s(x, g) = xg$ and $r(x, g) = x$. Thus $C(V) \rtimes \Gamma$ contains the following dense $*$-subalgebra $C_c(V \times \Gamma)$:

$$(ab)(x, g) = \sum_{h \in \Gamma} a(x, h)b(xh, h^{-1}g),$$
$$a^*(x, g) = \overline{a(xg, g^{-1})}$$

for $a, b \in C_c(V \times \Gamma)$. For each $x \in V$, one has

$$s^{-1}(x) = \{(xg^{-1}, g) \in V \times \Gamma;\ g \in \Gamma\}.$$

Define a $*$-representation L_x of $C_c(V \times \Gamma)$ on $l^2(s^{-1}(x))$ by

$$[L_x(a)\xi](xg^{-1}, g) = \sum_{h \in \Gamma} a(xg^{-1}, h)\xi(xg^{-1}h, h^{-1}g),$$

where $a \in C_c(V \times \Gamma)$ and $\xi \in l^2(s^{-1}(x))$. Then

$$||a|| = \sup\{||L_x(a)||;\ x \in V\} < \infty,$$

and $C(V) \rtimes \Gamma$ is the completion of $C_c(V \times \Gamma)$ with respect to the norm$||\cdot||$.

If U_g denotes the characteristic function of $V \times \{g\}$, then U_g belongs to $C_c(V \times \Gamma)$, since V is compact. Any $a \in C_c(V \times \Gamma)$ can be expressed as a finite sum

$$a = \sum_{g \in \Gamma} a_g U_g, \quad a_g \in C(V).$$

The $*$-algebra $C_c(V \times \Gamma)$ is generated by $C(V)$ and $(U_g)_{g\in\Gamma}$, subject to relations: $U_g U_h = U_{gh}$, $g, h \in \Gamma$, $U_g^* = U_{g^{-1}}$, and $U_g a U_g^* = g(a)$, $a \in C(V)$.

Remark 2.3. The collection $\{l^2(s^{-1}(x))\}_{x\in V}$ forms a continuous field of Hilbert spaces, and the correspondence $x \to L_x(a)$ is a continuous field of bounded operators.

Proposition 2.4. *There exists a Hilbert $C(V) \rtimes \Gamma$-module ϵ such that $C^*(X, \mathcal{F}, E)$ is isomorphic to the C^*-algebra $\mathcal{K}(\epsilon)$ of compact operators of ϵ. In particular, $C^*(X, \mathcal{F}, E)$ is strongly Morita equivalent to $C(V) \rtimes \Gamma$.*

Proof. Choose a base point $* \in M$. The image T of $\{*\} \times V$ in X is a complete transversal of $\mathcal{F}$, where $G_T^T = s^{-1}(T) \cap r^{-1}(T)$ and $G_T = s^{-1}(T)$ are identified with $V \times \Gamma$ and $\widetilde{M} \times V$, respectively. Then Proposition 3 of [**14**] implies the assertion. □

We now describe the module ϵ, as we will need the description later. Let $\mathcal{S} = C_c(\widetilde{E})$. A right $C_c(V \times \Gamma)$-action on $\mathcal{S}$ is defined by

$$(\xi f)(m, x) = \sum_{g \in \Gamma} f(xg^{-1}, g)(g^{-1}\xi)(m, x), \quad \xi \in \mathcal{S}, \quad f \in C_c(V \times \Gamma).$$

A $C_c(V \times \Gamma)$-valued inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{S}$ is defined by

$$\langle\xi_1, \xi_2\rangle(x, g) = \int_{\widetilde{M}_x} (\xi_1(m, x), (g\xi_2)(m, x))_{\widetilde{E}}\, d\mu_x(m),$$

where $(\cdot,\cdot)_{\widetilde{E}}$ is the Hermitian product in $\widetilde{E}$. The module ϵ is the completion of $\mathcal{S}$ with respect to the norm $||\xi|| = ||\langle\xi,\xi\rangle||^{1/2}$.

The representation of $C^*(X,\mathcal{F},E)$ on ϵ given by Proposition 2.4 is:

$$(f*\xi)(\gamma) = \int_{G^{r(\gamma)}} f(\gamma')\xi(\gamma'^{-1}\gamma)\,d\nu^{r(\gamma)}(\gamma'),$$

for $f \in C_c(G,E)$, $\xi \in C_c(G_T, r^*E)$. Through the identification $G \cong (\widetilde{M}\times\widetilde{M}\times V)/\Gamma$, the left $C_c(G,E)$-action is described as

$$(f*\xi)(m,x) = \int_{\widetilde{M}_x} f(m,n,x)\xi(n,x)\,d\mu_x(n),$$

where $(m,x) \in \widetilde{M}\times V \cong G_T$, and f is regarded as a Γ-invariant family of integral kernels on $\widetilde{M}\times\widetilde{M}\times V$. Proposition 3 of [**14**] says that the left $C_c(G,E)$-action extends to a faithful representation of $C^*(X,\mathcal{F},E)$ on ϵ, and that the image of this representation is precisely the space $\mathcal{K}(\epsilon)$ of compact operators of the Hilbert C^*-module ϵ over $C(V)\rtimes\Gamma$.

Let $C_c^{\infty,0}(\widetilde{E})$ be the space of compactly supported sections of $\widetilde{E}$ over $\widetilde{M}\times V$ of class $C^{\infty,0}$ ([**6**]). In an obvious way $C_c^{\infty,0}(\widetilde{E})$ can be regarded as a subspace of sections of the field $\mathcal{H}$. Consider the $*$-algebra of intertwining operators of $\mathcal{H}$ which map $C_c^{\infty,0}(\widetilde{E})$ into itself. Its C^*-closure in $\mathrm{End}_\Gamma(\mathcal{H})$ is denoted by $\mathfrak{B}$.

Proposition 2.5. *There exists a $*$-monomorphism Φ from $\mathfrak{B}$ into the C^*-algebra $\mathcal{L}(\epsilon)$ of all bounded operators of the Hilbert C^*-module ϵ over $C(V)\rtimes\Gamma$.*

Proof. For $\xi \in C_c^{\infty,0}(\widetilde{E})$, denote by ξ_x the restriction of ξ onto $\widetilde{M}_x$. Then $\xi_x \in H_x = L^2(\widetilde{E}_x)$. For $f \in C_c(G_x^T)$, define $S_{\xi,x}(f) \in C_c^{\infty,0}(\widetilde{E})$ by

$$S_{\xi,x}(f)(m) = \sum_{g\in\Gamma}(g^{-1}\xi)(m,x)f(xg^{-1},g).$$

For $u \in C_c^{\infty,0}(\widetilde{E})$, define $T_{\xi,x}(u) \in C_c(G_x^T)$ by

$$T_{\xi,x}(u)(xg^{-1},g) = \langle (g^{-1}\xi)_x, u\rangle_x,$$

where $\langle,\rangle_x$ is the inner product of H_x.

We need the following lemma.

Lemma 2.6. *The linear maps $S_{\xi,x}$ and $T_{\xi,x}$ extend to bounded maps*

$$S_{\xi,x} : l^2(G_x^T) \to H_x,$$

and

$$T_{\xi,x} : H_x \to l^2(G_x^T).$$

Moreover,

(1) $S_{\xi,x}$ *is the adjoint of* $T_{\xi,x}$;

(2) *for* $\xi, \eta, \zeta \in C_c^{\infty,0}(\widetilde{E})$ *and* $f \in l^2(G_x^T)$, *one has*

$$S_{\xi,x}T_{\eta,x}(\zeta_x) = (\xi\langle\eta,\zeta\rangle)_x,$$
$$T_{\xi,x}S_{\eta,x}(f) = L_x(\langle\xi,\eta,\rangle)f;$$

(3) $||\xi|| = \sup\{||S_{\xi,x}||;\ x \in V\} = \sup\{||T_{\xi,x}||;\ x \in V\}$,
where $||\xi||$ *is the norm of* ξ *in* ϵ.

Proof. By a straightforward computation,

$$\langle S_{\xi,x}(f),\ u\rangle_x = \langle f,\ T_{\xi,x}(u)\rangle,$$

where $f \in C_c(G_x^T)$ and $u \in C_c(\widetilde{E})$, and the right-hand side is the inner product in $l^2(G_x^T)$. Let $a \in C_c(V \times \Gamma)$ and $\xi, \eta \in C_c^{\infty,0}(\widetilde{E})$. Then

$$S_{\xi,x}(a|_{G_x^T}) = (\xi a)_x,$$
$$T_{\xi,x}(\eta_x) = \langle\xi,\ \eta\rangle|G_x^T.$$

From this

$$S_{\xi,x}T_{\eta,x}(\zeta_x) = S_{\xi,x}(\langle\eta,\zeta\rangle|_{G_x^T})$$
$$= (\xi\langle\eta,\zeta\rangle)_x.$$

As for the second equality in the assertion (2), we have

$$\begin{aligned} T_{\xi,x}S_{\eta,x}(f) &= T_{\xi,x}((\eta a)_x) \\ &= \langle\xi,\eta a\rangle|G_x^T \\ &= (\langle\xi,\eta\rangle a)|G_x^T \\ &= L_x(\langle\xi,\eta\rangle)f, \end{aligned}$$

where a is an element of $C_c(V \times \Gamma)$ such that $a|_{G_x^T} = f$. Thus

$$||S_{\xi,x}(f)||^2 = \langle S_{\xi,x}(f), S_{\xi,x}(f)\rangle = \langle f, L_x(\langle\xi,\xi\rangle)f\rangle.$$

From this and facts that $C_c(G_x^T)$ is dense in $l^2(G_x^T)$, and that $L_x(\langle\xi,\xi\rangle)$ is positive, it follows that $S_{\xi,x}$ extends to a bounded linear map, and

$$||S_{\xi,x}|| = ||L_x(\langle\xi,\xi\rangle)||.$$

Consequently, $T_{\xi,x}$ also extends to a bounded linear map, and

$$||T_{\xi,x}|| = ||S_{\xi,x}||.$$

Finally,

$$\begin{aligned}||\xi||^2 = ||\langle\xi,\xi\rangle|| &= \sup ||L_x(\langle\xi,\xi\rangle)|| \\ &= \sup ||S_{\xi,x}||^2 \\ &= \sup ||T_{\xi,x}||^2.\end{aligned}$$

This completes the proof of the lemma. □

We return to the proof of Proposition 2.5.

Assume that $P = (P_x) \in \mathrm{End}_\Gamma(\mathcal{H})$ and its adjoint $P^* = (P_x^*)$ preserve the space $C_c^{\infty,0}(\widetilde{E})$. Since P is Γ-equivariant, it is readily seen that P defines a $C_c(V \times \Gamma)$-module homomorphism $\widehat{P}$ of $C_c^{\infty,0}(\widetilde{E})$. Furthermore, for $\xi \in C_c^{\infty,0}(\widetilde{E})$, one has

$$\begin{aligned}||\widehat{P}(\xi)|| &= \sup ||S_{P(\xi),x}|| \\ &= \sup ||P_x S_{\xi,x}|| \\ &\leq \sup ||P_x|| \sup ||S_{\xi,x}|| \\ &= ||P||\ ||\xi||.\end{aligned} \tag{2.7}$$

Thus $\widehat{P}$ is a bounded operator of ϵ. Similarly P^* defines a bounded operator $\widehat{P}^*$ with

$$\langle\widehat{P}\xi,\eta\rangle = \langle\xi,\widehat{P}^*\eta\rangle$$

for $\xi,\eta \in \epsilon$. Therefore $\widehat{P} \in \mathcal{L}(\epsilon)$.

We show that the correspondence $P \to \widehat{P}$ is injective. From the inequality (2.7),

$$||\widehat{P}||_{\mathcal{L}(\epsilon)} \leq ||P||.$$

Assume that $\widehat{P} = 0$. Let $P = \lim_{j\to\infty} P^{(j)}$ in norm in $\mathrm{End}_\Gamma(\mathcal{H})$ where we have that $P^{(j)}$ preserves $C_c^{\infty,0}(\widetilde{E})$. Then, for $\xi \in C_c^{\infty,0}(\widetilde{E})$, we have

$$\lim_{j\to\infty} ||\widehat{P}^{(j)}(\xi)|| = 0.$$

Notice that any $\xi \in C_c^{\infty,0}(\widetilde{E})$ is written as $\xi = \alpha\langle\beta,\gamma\rangle$ for some $\alpha,\beta,\gamma \in C_c^{\infty,0}(\widetilde{E})$. Then

$$P_x^{(j)}\xi_x = P_x^{(j)}(\alpha\langle\beta,\gamma\rangle)_x = P_x^{(j)} S_{\alpha,x} T_{\beta,x}(\gamma_x).$$

Therefore

$$\begin{aligned}\sup \|P_x^{(j)}\xi_x\| &\leq \sup \|P_x^{(j)} S_{\alpha,x}\| \sup \|T_{\beta,x}(\gamma_x)\| \\ &\leq C\|\widehat{P}^{(j)}(\alpha)\|\end{aligned}$$

for some $C > 0$. Thus $\sup \|P_x^{(j)}\xi_x\| \to 0$ as $j \to \infty$. Hence $P_x\xi_x = 0$ for all $x \in V$. This means that $P = 0$ in $\mathrm{End}_\Gamma(\mathcal{H})$. Thus $P \to \widehat{P}$ is an injective $*$-homomorphism, and in particular,

$$\|\widehat{P}\|_{\mathcal{L}(\epsilon)} = \|P\| = \sup \|P_x\|.$$

This ends the proof of Proposition 2.5. □

Remark 2.8. The foliation C^*-algebra $C^*(X, \mathcal{F}, E)$ is a subalgebra of $\mathfrak{B}$, and the restriction to $C^*(X, \mathcal{F}, E)$ of the embedding of $\mathfrak{B}$ into $\mathcal{L}(\epsilon)$ is exactly the isomorphism

$$C^*(X, \mathcal{F}, E) \to \mathcal{K}(\epsilon)$$

given in Proposition 2.4.

Remark 2.9. When the Γ-action on V does not satisfy the Condition 2.1, the structure of the holonomy groupoid is more complex, and $C^*(X, \mathcal{F}, E)$ is not strongly Morita equivalent to $C(V) \rtimes \Gamma$. Thus the arguments above do not apply to this case. However, if one uses the C^*-algebra of the fundamental groupoid, in place of the holonomy groupoid, then the results in this paper remain valid.

3. Algebra of Pseudodifferential Operators.

For a given foliated bundle $(X, \mathcal{F})$, the C^*-algebra $C^*(X, \mathcal{F}, E)$ defined in the preceding section contains pseudodifferential operators. In this section we will introduce a dense Banach subalgebra $\mathfrak{A}$ of $C^*(X, \mathcal{F}, E)$ and will show that $\mathfrak{A}$ is holomorphically closed.

Let $\widetilde{E}^0$ and $\widetilde{E}^1$ be Γ-equivariant Hermitian vector bundles over $\widetilde{M} \times V$. Let $P : C_c^{\infty,0}(\widetilde{E}^0) \to C^{\infty,0}(\widetilde{E}^1)$ be a continuous linear map. We say that P is a Γ-equivariant family of pseudodifferential operators of order r if

(1) P is Γ-equivariant,

(2) for each $x \in V$, the operator P restricts to $\widetilde{M}_x$ to give a pseudodifferential operator of order r

$$P_x : C_c^{\infty}(\widetilde{E}_x^0) \to C^{\infty,0}(\widetilde{E}_x^1),$$

(3) the distributional kernel of P has Γ-compact support.

Conditions (1) and (2) imply that the distributional kernel is regarded as a distribution on $\widetilde{M} \times \widetilde{M} \times V$ and is Γ-invariant.

Denote by $\Psi^r_\Gamma\left(\widetilde{E}^0, \widetilde{E}^1\right)$ the space of all Γ-equivariant families of pseudodifferential operators of order $\leq r$ from $\widetilde{E}^0$ to $\widetilde{E}^1$. When $\widetilde{E}^0 = \widetilde{E}^1 = \widetilde{E}$, we use $\Psi^r_\Gamma(\widetilde{E})$ instead of $\Psi^r_\Gamma\left(\widetilde{E}^0, \widetilde{E}^1\right)$. A basic fact is that if $P \in \Psi^r_\Gamma\left(\widetilde{E}^0, \widetilde{E}^1\right)$, $Q \in \Psi^s_\Gamma\left(\widetilde{E}^1, \widetilde{E}^2\right)$, then $QP \in \Psi^{r+s}_\Gamma\left(\widetilde{E}^0, \widetilde{E}^2\right)$. If $P \in \Psi^r_\Gamma\left(\widetilde{E}^0, \widetilde{E}^1\right)$, then its formal adjoint P^* belongs to $\Psi^r_\Gamma\left(\widetilde{E}^1, \widetilde{E}^0\right)$. So, in particular, $\Psi^0_\Gamma(\widetilde{E})$ is a $*$-algebra.

Recall [**11**] that by a tangential operator we mean a continuous linear operator $D : C^{\infty,0}_c(\widetilde{E}^0) \to C^{\infty,0}(\widetilde{E}^1)$ such that D is Γ-equivariant and that for each $x \in V, D$ restricts to $\widetilde{M}_x$ to give a continuous linear operator $D_x : C^\infty_c(\widetilde{E}^0_x) \to C^\infty(\widetilde{E}^1_x)$.

Let Δ_x be the Laplacian on $\widetilde{M}_x$ twisted by $\widetilde{E}_x$. Then Δ_x acts on the sections of $\widetilde{E}_x$. Denote by $W^s_x(\widetilde{E})$ the completion of $C^\infty_c(\widetilde{E}_x)$ with respect to the Sobolev s-norm:

$$\|f\|^2_{s,x} = \langle f, (I + \Delta_x)^s f\rangle_x,$$

where $\langle\cdot,\cdot\rangle_x$ is the inner product of $H_x = L^2(\widetilde{E}_x)$. We obtain a continuous field $W^s_\tau(\widetilde{E}) = \left(W^s_x(\widetilde{E})\right)_{x\in V}$ of Hilbert spaces over V, which we shall call a *tangential Sobolev field* [**15**, p. 78].

A tangential operator D is *smoothing* if D induces a bounded operator

$$W^s_\tau(\widetilde{E}) \to W^t_\tau(\widetilde{E})$$

for all $s, t \in \mathbb{R}$. A smoothing operator is *compactly smoothing* if its distributional kernel has Γ-compact support.

For a tangential operator P, and $s, t \in \mathbb{R}$, set

$$\|P_x\|_{s,t} = \sup\left\{(\|P_x\xi\|_{s,x})/\|\xi\|_{t,x}; \quad \xi \in C^\infty_c(\widetilde{E}_x)\right\},$$

and

$$\|P\|_{s,t} = \sup\{\|P_x\|_{s,t}; \quad x \in V\}.$$

Of course, $\|P_x\|_{s,t}$, $\|P\|_{s,t}$ might be infinite. However it is true that if $P \in \Psi^r_\Gamma(\widetilde{E})$, then

$$\|P\|_{s-r,s} < \infty,$$

for any s. In particular, P extends to an intertwining operator

$$W^s_\tau(\widetilde{E}) \to W^{s-r}_\tau(\widetilde{E}).$$

If P belongs to $\Psi_\Gamma^{-\infty}(\widetilde{E}) = \bigcap_r \Psi_\Gamma^r(\widetilde{E})$, then P is a compactly smoothing operator. Moreover, one can see that $\Psi_\Gamma^{-\infty}(\widetilde{E})$ is contained in $C^*(X, \mathcal{F}, E)$, here $\widetilde{E}$ is the lifting of E to $\widetilde{M} \times V$.

Let $S^*\mathcal{F}$ be the unit cosphere bundle of $\mathcal{F}$, and let π be the canonical projection $S^*\mathcal{F} \longrightarrow X$. Let E^0, E^1 be Hermitian bundles over X, and let $\widetilde{E}^0, \widetilde{E}^1$ be the liftings to $\widetilde{M} \times V$ of E^0, E^1, respectively. The principal symbol map is $\sigma_r : \Psi_\Gamma^r\left(\widetilde{E}^0, \widetilde{E}^1\right) \to C^{\infty,0}(S^*\mathcal{F}, \mathrm{Hom}(\pi^*E^0, \pi^*E^1))$. We say that $P \in \Psi_\Gamma^r\left(\widetilde{E}^0, \widetilde{E}^1\right)$ is elliptic if $\sigma_r(P)$ is invertible.

Proposition 3.1. ([**15**, Prop. 7.12], [**6**, p. 128]). *Let $P \in \Psi_\Gamma^r\left(\widetilde{E}^0, \widetilde{E}^1\right)$ be elliptic. Then there exists $Q \in \Psi_\Gamma^{-r}\left(\widetilde{E}^1, \widetilde{E}^0\right)$ such that $I - PQ$ and $I - QP$ are compactly smoothing.*

The operator Q given by Proposition 3.1 is called a *parametrix* of P.

Every $P \in \Psi_\Gamma^0(\widetilde{E})$ is regarded as an intertwining operator in $\mathrm{End}_\Gamma\left(W_\tau^0(\widetilde{E})\right)$. Thus $\Psi_\Gamma^0(\widetilde{E}) \subseteq \mathfrak{B}$. Let $\wp_0$ denote the C^*-closure of $\Psi_\Gamma^0(\widetilde{E})$ in $\mathrm{End}_\Gamma\left(W_\tau^0(\widetilde{E})\right)$. The principal symbol map σ_0 extends to a $*$-homomorphism

$$\sigma : \wp_0 \longrightarrow C(S^*\mathcal{F}, \mathrm{End}(\pi^*E)),$$

and the sequence

$$0 \to C^*(X, \mathcal{F}, E) \to \wp_0 \xrightarrow{\sigma} C(S^*\mathcal{F}, \mathrm{End}(\pi^*E)) \to 0$$

is exact.

Fix an $N > \dim M$. For $P \in \Psi_\Gamma^{-1} = \Psi_\Gamma^{-1}(\widetilde{E})$, set

$$|||P||| = \max\left(||P||_{1-N,-N}, ||p||_{N,N-1}\right).$$

Then by the interpolation method of Calderon, for all $-N \leq s \leq N-1$, one has

$$||P||_{s,s} \leq ||P||_{s+1,s} \leq |||P|||.$$

Certainly, $|||\cdot|||$ is a norm on Ψ_Γ^{-1}. A staightforward computation shows that

$$|||PQ||| \leq |||P|||\ |||Q|||$$

for $P, Q \in \Psi_\Gamma^{-1}$.

Let $\mathfrak{A}$ be the Banach algebra completion of Ψ_Γ^{-1} with respect to $|||\cdot|||$.

Lemma 3.2. *There exists an injective homomorphism $\alpha : \mathfrak{A} \to \wp_0$.*

Proof. Since $||P||_{0,0} \leq |||P|||$, there exists a homomorphism $\alpha : \mathfrak{A} \to \wp_0$. We prove the injectivity of α. Let $\{P_j\}$ be a Cauchy sequence in Ψ_Γ^{-1} with respect to $|||\cdot|||$. It suffices to show that if $\alpha(P_j) \to 0$ in $\wp_0$, then $P_j \to 0$ in $\mathfrak{A}$. Since $\{P_j\}$ is a Cauchy sequence in $\mathfrak{A}$, it is a Cauchy sequence also with respect to $||\cdot||_{s+1,s}$, $-N \leq s \leq N-1$. Therefore, there exist intertwining operators $P^{(s)}$, of fields of Hilbert spaces $W^s_\tau(\widetilde{E}) \to W^{s+1}_\tau(\widetilde{E})$ such that

$$||P_j - P^{(s)}||_{s+1,s} \to 0 \quad \text{as} \quad j \to \infty.$$

Recall that $C^{\infty,0}_c(\widetilde{E})$ is a total subspace of $W^s_\tau(\widetilde{E})$. For $\xi \in C^{\infty,0}_c(\widetilde{E})$ and for $s \geq -1$, we have

$$\begin{aligned} ||P^{(s)}\xi||_0 &\leq ||(P_j - P^{(s)})\xi||_0 + ||P_j\xi||_0 \\ &\leq ||(P_j - P^{(s)})\xi||_{s+1} + ||P_j||_{0,0}||\xi||_0 \\ &\leq ||(P_j - P^{(s)})||_{s+1,s}||\xi||_s + ||P_j||_{0,0}||\xi||_0 \to 0 \end{aligned}$$

as $j \to \infty$.

Hence $P^{(s)}\xi = 0$ for all $\xi \in C^{\infty,0}_c(\widetilde{E})$. Consequently $P^{(s)} = 0$.

Assume, now, that $s + 1 < 0$. Then

$$\begin{aligned} ||P^{(s)}\xi||_{s+1} &\leq ||(P_j - P^{(s)})\xi||_{s+1} + ||P_j\xi||_{s+1} \\ &\leq ||(P_j - P^{(s)})\xi||_{s+1} + ||P_j\xi||_0 \\ &\leq ||(P_j - P^{(s)})||_{s+1,s}||\xi||_s + ||P_j||_{0,0}||\xi||_0 \to 0 \end{aligned}$$

as $j \to \infty$.

Hence $P^{(s)}\xi = 0$ for all $\xi \in C^{\infty,0}_c(\widetilde{E})$. Thus $P_j \to 0$ in $\mathfrak{A}$. □

From now on, we regard $\mathfrak{A}$ as a subalgebra of $\wp_0$. In particular, an element $P \in \mathfrak{A}$ is interpreted as a collection of operators $P = (P_s)$ such that $P_s : W^s_\tau(\widetilde{E}) \to W^{s+1}_\tau(\widetilde{E})$ is bounded for $-N \leq s \leq N-1$, and such that

$$P_s | W^s_\tau(\widetilde{E}) = P_t \quad \text{if} \quad s < t.$$

Let $\mathfrak{A}^+$ be $\mathfrak{A}$ with unit adjointed. As an algebra, $\mathfrak{A}^+$ is identified with the algebra generated by $\mathfrak{A}$ and the identity I of $\wp_0$. Then a sequence $\{\lambda_i I + P_i\}$ in $\mathfrak{A}^+$ converges to $\lambda I + P$ in $\mathfrak{A}^+$ if and only if

$$\lambda_i \to \lambda \qquad \text{in} \quad \mathbb{C},$$

and

$$P_i \to P \qquad \text{in} \quad \mathfrak{A}.$$

Theorem 3.3. *The dense subalgebra $\mathfrak{A}^+$ of $C^*(X, \mathcal{F}, E)^+$ is holomorphically closed.*

In order to prove Theorem 3.3 we need the two lemmata below.

Lemma 3.4. *If $P+I$, $P \in \mathfrak{A}$, is invertible in $C^*(X,\mathcal{F},E)^+$, then $(I+P)_s : W_\tau^s(\widetilde{E}) \to W_\tau^{s+1}(\widetilde{E})$ is invertible for $|s| \leq N$.*

Proof. Let $0 \leq s \leq 1$. Obviously, $(I+P)_s : W_\tau^s(\widetilde{E}) \to W_\tau^s(\widetilde{E})$ is injective. By the Open Mapping Theorem, if $(I+P)_s$ is surjective, then it is invertible. Let $\eta \in W_\tau^0(\widetilde{E})$. Since $(I+P)_0 : W_\tau^0(\widetilde{E}) \to W_\tau^0(\widetilde{E})$ is invertible, there exists a $\xi \in W_\tau^0(\widetilde{E})$ such that $(I+P)\xi = \eta$. Then $\xi = \eta - P\xi \in W_\tau^1(\widetilde{E}) + W_\tau^s(\widetilde{E}) \subseteq W_\tau^s(\widetilde{E})$, since $P\xi \in W_\tau^1(\widetilde{E})$. Thus $(I+P)_s : W_\tau^s(\widetilde{E}) \to W_\tau^s(\widetilde{E})$ is injective.

By an induction, using the fact that P maps $W_\tau^{N-1}(\widetilde{E})$ into $W_\tau^N(\widetilde{E})$, we can show that $(I+P)_s$ is invertible for $0 \leq s \leq N$.

As for $-N \leq s \leq 0$, use the nondegenerate pairing

$$W_\tau^{-s}(\widetilde{E}) \times W_\tau^s(\widetilde{E}) \to \mathbb{C}$$

and the fact that $(\xi, (I+P)_s\eta) = ((I+P^*)_{-s}\xi, \eta)$ to deduce the conclusion. □

By Lemma 3.4, we know that when $I+P$ is invertible, it induces invertible operators at each level $W_\tau^s(\widetilde{E}) \to W_\tau^s(\widetilde{E})$.

Lemma 3.5. *Let $I+P$, $P \in \Psi_\Gamma^{-1}$, be invertible in $\wp_0$. Then $(I+P)^{-1} \in \mathfrak{A}^+$.*

Sublemma. *If $Q \in \Psi_\Gamma^0$ is invertible in $\wp_0$, then there exists a sequence $\{A_i\}$ in Ψ_Γ^0 such that $I - A_iQ$ is compactly smoothing, and that*

$$||I - A_iQ||_{s,t} \to 0 \quad \text{as} \quad i \to \infty \quad \text{for all} \quad s,t.$$

Proof of Sublemma. Since Q is invertible in $\wp_0$, its principal symbol $\sigma(P)$ is invertible, i.e. Q is elliptic. Then there exists $R \in \Psi_\Gamma^0$ such that $I - QR$, $I - RQ$ are compactly smoothing.

Since Q is invertible in $\wp_0$, there exists a sequence $\{B_i\}$ in Ψ_Γ^0 such that

$$||Q^{-1} - B_i||_{0,0} \to 0 \quad \text{as} \quad i \to \infty.$$

Put $A_i = 2R + B_i - RQB_i - B_iQR - RQR + RQB_iQR$. Then $A_i \in \Psi_\Gamma^0$. We have

$$I - A_iQ = (I - RQ)(I - B_iQ)(I - RQ).$$

Since $S = I - RQ$ is compactly smoothing,

$$\begin{aligned} ||I - A_iQ||_{s,t} &= ||S(Q^{-1} - B_i)QS||_{s,t} \\ &\leq ||S||_{s,0}||Q^{-1} - B_i||_{0,0}||Q||_{0,0}||S||_{0,t} \to 0 \quad \text{as} \quad i \to \infty, \end{aligned}$$

and the sublemma is proved. □

Proof of Lemma 3.5. By the sublemma, there exists a sequence $\{A_i\}$ of order zero ψDO's such that $I - A_i(I+P)$ is compactly smoothing, and such that

$$||I - A_i(I+P)||_{s,t} \to 0 \quad \text{as} \quad i \to 0 \quad \text{for all} \quad s,t.$$

Notice that $I - A_i = A_iP + (I - A_i(I+P))$ belongs to Ψ_Γ^{-1}. Thus $A_i = I + B_i$ with $B_i \in \Psi_\Gamma^{-1}$. Set

$$T_i = I - (I + B_i)(I+P) \in \Psi_\Gamma^{-\infty}.$$

We have $(I+P)^{-1} - I = B_i + T_i(I+P)^{-1}$. The operator $(I+P)^{-1} - I$ maps $W_\tau^s(\tilde{E})$ into $W_\tau^{s+1}(\tilde{E})$ for $-N \leq s \leq N-1$. Therefore

$$|| \left((I+P)^{-1} - I\right) - B_i||_{s+1,s} \quad \text{is finite,}$$

and

$$\begin{aligned} || \left((I+P)^{-1} - I\right) - B_i||_{s+1,s} &= ||T_i(I+P)^{-1}||_{s+1,s} \\ &\leq ||T_i||_{s+1,s}||(I+P)^{-1}||_{s,s} \to 0 \quad \text{as} \quad i \to \infty. \end{aligned}$$

This means that $(I+P)^{-1} = I+Q$, with $Q \in \mathfrak{A}$. Thus $(I+P)^{-1} \in \mathfrak{A}^+$. □

Proof of Theorem 3.3. The proof uses the well-known fact that an algebra is holomorphically closed if and only if the resolvents are contained in the algebra itself. Since $\mathfrak{A}$ is an ideal of $C^*(X,\mathcal{F},E)^+$, no elements of $\mathfrak{A}$ are invertible in $C^*(X,\mathcal{F},E)^+$. So it is sufficient to consider elements of the form $I+P$, $P \in \mathfrak{A}$. Since $P \in \mathfrak{A}$, there exists a sequence $\{P_i\}$ in Ψ_Γ^{-1} such that

$$|||P_i - P||| \to 0 \quad \text{as} \quad i \to \infty.$$

Then, in particular, $||(I+P) - (I+P_i)||_{0,0} \to 0$ as $i \to \infty$. As $I+P$ is invertible in $C^*(X,\mathcal{F},E)^+$, one may assume that $I+P_i$ is also invertible in $C^*(X,\mathcal{F},E)^+$ for all i. From

$$(I+P)^{-1} - I = (I+P)^{-1}(I - (I+P)) = -(I+P)^{-1}P,$$

it follows that $(I+P)^{-1} - I$ maps $W_\tau^s(\tilde{E})$ into $W_\tau^{s+1}(\tilde{E})$ for $-N \leq s \leq N-1$. As bounded operators on $W_\tau^s(\tilde{E})$, one has that

$$\begin{aligned} &||(I+P_i)^{-1} - (I+P)^{-1}||_{s,s} \\ &\qquad \leq \left(||(I+P) - (I+P_i)||_{s,s}||(I+P)^{-1}||_{s,s}^2\right) \\ &\qquad\qquad / \left(1 + ||(I+P_i) - (I+P)||_{s,s}||(I+P)^{-1}||_{s,s}^2\right) \end{aligned}$$

From this, it follows that $\sup\{\|(I+P_i)^{-1}\|_{s,s};\ i\} < \infty$. Moreover, one can see that $\sup\{\|(I+P_i)^{-1}\|_{s,s};\ i,\ |s| \le N\} < \infty$. Then we have

$$\begin{aligned}
&\|((I+P)^{-1}-I)-((I+P_i)^{-1}-I)\|_{s+1,s} \\
&\quad\le \|(I+P)^{-1}-(I+P_i)^{-1}\|_{s+1,s} \\
&\quad\le \|(I+P)^{-1}[(I+P_i)-(I+P)](I+P_i)^{-1}\|_{s+1,s} \\
&\quad\le \|(I+P)^{-1}\|_{s+1,s+1}\|P_i-P\|_{s+1,s}\|(I+P_i)^{-1}\|_{s,s}.
\end{aligned}$$

Since $\|(I+P_i)^{-1}\|_{s,s}$ is uniformly bounded, as $i \to \infty$ one has

$$\|((I+P)^{-1}-I)-((I+P_i)^{-1}-I)\|_{s+1,s} \to 0.$$

This means that $|||(I+P)^{-1}-(I+P_i)^{-1}||| \to 0$. By Lemma 3.5, $(I+P_i)^{-1} \in \mathfrak{A}^+$. Consequently $(I+P)^{-1} \in \mathfrak{A}^+$. □

Applying Theorem 3.3 to the bundle E^k, one obtain that $M_k(\mathfrak{A})^+$ is holomorphically closed in $M_k(C^*(X,\mathcal{F},E))^+$. From this we get the following (see [**3**]).

Proposition 3.6. *The canonical inclusion $\mathfrak{A} \subseteq C^*(X,\mathcal{F},E)$ induces an isomorphism*

$$K_0[\mathfrak{A}] \xrightarrow{\cong} K_0[C^*(X,\mathcal{F},E)].$$

4. Modular Automorphism Groups.

A volume form on the fibre of the foliated bundle $(X,\mathcal{F})$ gives rise to a weight on $C^*(X,\mathcal{F},E)$. We will show that the modular automorphism group, associated with the weight, leaves the Banach algebra $\mathfrak{A}$ invariant, and induces a one-parameter group of automorphisms.

Throughout the rest of the paper, assume that V is oriented, and Γ acts on V by orientation preserving diffeomorphisms. Let ω_V be a volume form on V. For $g \in \Gamma$, a positive real-valued function λ_g on V is determined by

$$\lambda_g \omega_V = g(\omega_V).$$

The correspondence $g \to \lambda_g$ satisfies the cocycle condition:

$$\lambda_{gh} = g(\lambda_h)\lambda_g, \qquad g,h \in \Gamma. \tag{4.1}$$

Let ϕ be the state on $C(V) \rtimes \Gamma$ associated with the volume form ω_V. Then

$$\phi(f) = \int_V f_e \omega_V$$

if

$$f = \sum f_g U_g \in C_c(V \times \Gamma).$$

The modular automorphism group (σ_t) of ϕ leaves $C(V) \rtimes \Gamma$ invariant. We have

$$\sigma_t(f) = \sum \lambda_g^{-it} f_g U_g$$

for $f = \sum f_g U_g$. Actually, σ_t is implemented by the following unitary Δ^{it} on $L^2(V) \otimes l^2(\Gamma)$ defined by

$$\left[\Delta^{it}\xi\right](x,g) = \lambda_g^{-it}(x)\xi(xg).$$

Let $\tilde{\omega}$ be a Γ-invariant volume form on $\widetilde{M}$. Choose an orientation on X so that for a Γ-invariant volume form ω on $\widetilde{M} \times V$, there exists a positive smooth function ψ on $\widetilde{M} \times V$ such that

$$\tilde{\omega} \wedge \omega_V = \psi\omega.$$

As above, let $\widetilde{E}$ be a Γ-equivariant Hermitian bundle over $\widetilde{M} \times V$. Recall $\mathcal{S} = C_c(\widetilde{E})$. Define a linear operator Δ^{it} $(t \in \mathbb{R})$ on $\mathcal{S}$ by

$$\Delta^{it}(\xi) = \psi^{-it}\xi, \qquad \xi \in \mathcal{S}. \tag{4.2}$$

Lemma 4.3. *The linear operator Δ^{it} extends to a bounded operator Δ^{it} : $\epsilon \to \epsilon$ which satisfies:*

(1) $\langle \Delta^{it}(\xi), \Delta^{it}(\eta)\rangle = \sigma_t(\langle \xi, \eta\rangle), \qquad \xi, \eta \in \epsilon,$

(2) $\Delta^{it}(\xi a) = (\Delta^{it}(\xi))\sigma_t(a), \qquad \xi \in \epsilon, \quad a \in C(V) \rtimes \Gamma,$

(3) $\Delta^{is}\Delta^{it}(\xi) = \Delta^{i(s+t)}(\xi), \qquad t, s \in \mathbb{R},\ \xi \in E.$

Proof. (1) By the definition of $C(V) \rtimes \Gamma$-valued inner product and (4.2), the equality holds for $\xi, \eta \in \mathcal{S}$, $t \in \mathbb{R}$. Then

$$\begin{aligned}\sup\{||\Delta^{it}(\xi)||/||\xi||;\ \xi \in \mathcal{S},\ \xi \neq 0\}\\ = \sup\left\{||\sigma_t(\langle\xi,\xi\rangle)||^{1/2}/||\langle\xi,\xi\rangle||^{1/2};\ \xi \in \mathcal{S},\ \xi \neq 0\right\} = 1.\end{aligned}$$

Hence Δ^{it} extends to a bounded operator on a Banach space ϵ, and the equality holds for all $\xi \in \epsilon$.

(2) A straightforward computation shows that the equality (2) is true for $\xi \in \epsilon$, $a \in C(V) \rtimes \Gamma$. By continuity, the equality holds for all $\xi \in \epsilon$ and $a \in C(V) \rtimes \Gamma$.

(3) From the definition of Δ^{it} and continuity, the conclusion follows.

□

Statement (2) of Lemma 4.3 means that Δ^{it} is not $C(V) \rtimes \Gamma$-linear.

Lemma 4.4. (1) *If* $P \in \mathcal{L}(\epsilon)$, *then* $\Delta^{it} P \Delta^{-it} \in \mathcal{L}(\epsilon)$, *and* $||\Delta^{it} P \Delta^{-it}|| = ||P||$.
(2) *We have* $\Delta^{it} \mathcal{K}(\epsilon) \Delta^{-it} \subseteq \mathcal{K}(\epsilon)$.

Proof. (1) Let $\xi \in \epsilon$, and let $a \in C(V) \rtimes \Gamma$. By Lemma 4.3,

$$\begin{aligned} \left(\Delta^{it} P \Delta^{-it}\right)(\xi a) &= \Delta^{it} P \left(\left(\Delta^{-it}(\xi)\right) \sigma_{-t}(a)\right) \\ &= \Delta^{it} \left(P\left(\Delta^{-it}(\xi)\right) \sigma_{-t}(a)\right) \\ &= \left(\Delta^{it} P \Delta^{-it}(\xi)\right) a. \end{aligned}$$

We have also that

$$\begin{aligned} \langle \left(\Delta^{it} P^* \Delta^{-it}\right)(\xi), \eta \rangle &= \sigma_t \langle P^* \Delta^{-it}(\xi), \Delta^{-it}(\eta) \rangle \\ &= \sigma_t \langle \Delta^{-it}(\xi), P \Delta^{-it}(\eta) \rangle \\ &= \langle \xi, \Delta^{it} P \Delta^{-it}(\eta) \rangle. \end{aligned}$$

This means that $(\Delta^{it} P \Delta^{-it})^* = \Delta^{it} P^* \Delta^{-it}$. Obviously, $\Delta^{it} P \Delta^{-it}$, $\Delta^{it} P^* \Delta^{-it}$ are bounded. Thus

$$\Delta^{it} P \Delta^{-it} \in \mathcal{L}(\epsilon).$$

Since $\Delta^{it} : \epsilon \to \epsilon$ is a surjective isometry,

$$||\Delta^{it} P \Delta^{-it}|| = ||P||.$$

(2) Let $\xi, \eta \in \epsilon$. By the definition of rank one operators $\theta_{\xi,\eta}$ and Lemma 4.4,

$$\Delta^{it} \theta_{\xi,\eta} \Delta^{-it} = \theta_{\Delta^{it}\xi, \Delta^{it}\eta}.$$

Therefore $\Delta^{it} \mathcal{K}(\epsilon) \Delta^{-it} \subseteq \mathcal{K}(\epsilon)$. □

Definition 4.5. For $P \in \mathcal{L}(\epsilon)$, set

$$\widehat{\sigma}_t(P) = \Delta^{it} P \Delta^{-it} \in \mathcal{L}(\epsilon).$$

Proposition 4.6. *The operator* $\{\widehat{\sigma}_t\}_{t \in \mathbb{R}}$ *on* $\mathcal{L}(\epsilon)$ *amounts to a one-parameter group of automorphisms of the* C^**-algebra* $\mathcal{L}(\epsilon)$. *Moreover,* $\{\widehat{\sigma}_t\}$ *preserves* $\mathcal{K}(\epsilon)$.

Proof. It is easy to see that $t \to \widehat{\sigma}_t$ is strongly continuous. The conclusion follows from Lemma 4.4. □

Notice that Δ^{it} preserves $C_c^{\infty,0}(\widetilde{E})$.

Lemma 4.7. *If $P \in \Psi_\Gamma^r(\widetilde{E})$, then $\Delta^{it}P\Delta^{-it} \in \Psi_\Gamma^r(\widetilde{E})$.*

Proof. We only have to show that $\Delta^{it}P\Delta^{-it}$ is Γ-equivariant. (Other properties of elements of $\Psi_\Gamma^r(\widetilde{E})$ are obvious.) For $g \in \Gamma$, $\xi \in C_c^{\infty,0}(\widetilde{E})$, we have

$$g(\Delta^{it}(\xi)) = g(\psi^{-it}\xi) = g(\psi)^{-it}g(\xi) = \lambda_g^{-it}\psi^{-it}g(\xi) = \lambda_g^{-it}\Delta^{it}(\xi).$$

Hence

$$\begin{aligned} g(\Delta^{it}P\Delta^{-it}) = \lambda_g^{-it}\Delta^{it}gP\Delta^{-it} &= \lambda_g^{-it}\Delta^{it}Pg\Delta^{-it} \\ &= \lambda_g^{-it}\Delta^{it}P\lambda_g^{it}\Delta^{-it} = (\Delta^{it}P\Delta^{-it})g, \end{aligned}$$

because the multiplication by $\lambda_g^{it} \in C^\infty(V)$ commutes with operators Δ^{it} and P. □

Lemma 4.8. *The linear operator Δ^{it} extends to a bounded operator on $W_\tau^s(\widetilde{E})$ for all s.*

Proof. Recall that the L^2-inner product induces a well-defined pairing

$$\langle \cdot, \cdot \rangle_x : W_x^s \times W_x^{-s} \to \mathbb{C}$$

such that $|\langle \xi, \eta \rangle_x| \leq ||\xi_x||_s||\eta_x||_{-s}$. Let $s \geq 0$. Set $Q = \psi^{it}\Lambda^{2s}\psi^{-it}$. Thanks to Lemma 4.7, $Q \in \Psi_\Gamma^{2s}(\widetilde{E})$. We have

$$\begin{aligned} ||\Delta^{it}(\xi)_x||_s^2 &= \langle \Delta^{it}(\xi)_x, \Lambda^{2s}\Delta^{it}(\xi)_x \rangle \\ &= \langle \xi_x, (Q\xi)_x \rangle \\ &\leq ||\xi_x||_s||Q_x\xi_x||_{-s} \\ &\leq ||\xi_x||_s||Q_x||_{-s,s}||\xi_x||_s. \end{aligned}$$

Therefore $\Delta^{it} : W_\tau^s(\widetilde{E}) \to W_\tau^s(\widetilde{E})$ is bounded for $s \geq 0$. Then by nondegeneracy of the pairing $W_x^s \times W_x^{-s} \to \mathbb{C}$, we see that $\Delta^{it} : W_\tau^s(\widetilde{E}) \to W_\tau^s(\widetilde{E})$ is bounded for all s. □

By Lemma 4.8, there exists a constant $C > 0$ such that

$$|||\Delta^{it}P\Delta^{-it}||| \leq C|||P||| \quad \text{for} \quad P \in \Psi_\Gamma^{-1}(\widetilde{E}).$$

By continuity, $\widehat{\sigma}_t(P) = \Delta^{it}P\Delta^{-it}$, $P \in \mathfrak{A}$, gives rise to an $\mathbb{R}$-action on the Banach algebra $\mathfrak{A}$. Denote by δ the generator of $(\widehat{\sigma}_t)$, i.e.

$$\delta(P) = \lim_{t\to 0} i(\widehat{\sigma}_t(P) - P)/t, \quad P \in \mathfrak{A}.$$

Then δ is a closed derivation of $\mathfrak{A}$, whose domain contains $\Psi_\Gamma^{-1}(\widetilde{E})$. Set $\varphi = \log \psi$. Then by a straightforward computation we obtain that

$$\delta(P) = [\varphi, P] = \varphi P - P\varphi, \quad P \in \Psi_\Gamma^{-1}(\widetilde{E}),$$

where φ is regarded as pointwise multiplication operator.

Proposition 4.9. *If* $P \in \Psi_\Gamma^{-1}(\widetilde{E})$, *then* $\delta(P) \in \Psi_\Gamma^{-2}(\widetilde{E})$.

Proof. Recall first the definition of ψ, i.e.

$$\widetilde{\omega} \wedge \omega_V = \psi\omega.$$

From this, $g(\widetilde{\omega}) \wedge g(\omega_V) = g(\psi)g(\omega)$, $g \in \Gamma$. Since $\widetilde{\omega}$ and ω are Γ-invariant, and $g(\omega_V) = \lambda_g \omega_V$, we have

$$\lambda_g \widetilde{\omega} \wedge \omega_V = g(\psi)\omega.$$

Therefore we have

$$\lambda_g \psi = g(\psi), \quad g \in \Gamma, \tag{4.10}$$

and

$$\log \lambda_g + \varphi = g(\varphi). \tag{4.11}$$

Since $\varphi \in C^\infty(\widetilde{M} \times V)$, both φP and $P\varphi$ are continuous linear operators $C_c^{\infty,0}(\widetilde{E}) \to C_c^{\infty,0}(\widetilde{E})$, and $(\varphi P)_x = \varphi_x P_x$, $(P\varphi)_x = P_x \varphi_x$ are ψDO's on $\widetilde{M}_x$ for every $x \in V$. By asymptotic expansion of the symbols, we can see that $\varphi_x P_x - P_x \varphi_x$ is a ψDO of order -2. Hence we only have to show that $[\varphi, P]$ is Γ-invariant. We have

$$\begin{aligned} g(\varphi P - P\varphi) &= g(\varphi)Pg - Pg(\varphi)g \\ &= (g(\varphi)P - Pg(\varphi))g \\ &= (\varphi P - P\varphi)g + (\log \lambda_g P - P \log \lambda_g) \qquad \text{by (4.11)} \\ &= (\varphi P - P\varphi)g, \end{aligned}$$

because $\log \lambda_g$ commutes with P.

5. Godbillon-Vey Classes.

Throughout this section, V denotes the circle S^1 with the canonical volume form dx. The foliation $\mathcal{F}$ on $X = \widetilde{M} \times_\Gamma V$ is transversely orientable and codimension one. To such a foliation, a characteristic class $gv(\mathcal{F})$, called the Godbillon-Vey class, is assigned. In this section we will give a description of $gv(\mathcal{F})$ in terms of function ψ introduced in the preceding section. We will use this description in Section 8.

Let θ be an arbitrary 1-form on X defining $\mathcal{F}$. By integrability, there exists a 1-form η such that $d\theta = \eta \wedge \theta$. The Godbillon-Vey class then given by $[\eta \wedge d\eta] \in H^3_{DR}(X)$ ([**13**]).

Let $\widetilde{\theta}, \widetilde{\eta}$ be the lifting of θ, η respectively to $\widetilde{M} \times V$. Let Ω be the pullback of $\widetilde{\omega}$ by $\widetilde{M} \times V \to \widetilde{M}$. Then $\omega = \Omega \wedge \widetilde{\theta}$ is a Γ-invariant volume form on $\widetilde{M} \times V$.

Since $\widetilde{\theta}$ and $\omega_V = dx$ define the same foliation on $\widetilde{M} \times V$, there exists a nowhere vanishing smooth function f on $\widetilde{M} \times V$ such that $\widetilde{\theta} = f\omega_V$. Then

$$\omega = \Omega \wedge \widetilde{\theta} = f\Omega \wedge \omega_V = f\psi\omega.$$

So $f = 1/\psi$. Consequently, $\widetilde{\theta} = (1/\psi)\omega_V$. From this

$$d\widetilde{\theta} = \widetilde{\eta} \wedge \widetilde{\theta} = (1/\psi)\widetilde{\eta} \wedge \omega_V.$$

On the other hand

$$d\widetilde{\theta} = d(1/\psi\omega_V) = d(1/\psi) \wedge \omega_V,$$

for ω_V is closed. From these,

$$(1/\psi)\widetilde{\eta} \wedge \omega_V = d(1/\psi) \wedge \omega_V. \tag{5.1}$$

Recall that $\varphi = \log \psi$ and $\widetilde{\omega} \wedge \omega_V = \psi\omega$. Thus $-d\varphi \wedge \omega_V = \eta \wedge \omega_V$.

The tangent bundle T of $\widetilde{M} \times V$ has a splitting

$$T = T' \oplus T'',$$

where T' (resp. T'') consists of vectors tangential to $\widetilde{M}_x$, $x \in V$ (resp. $\{a\} \times V$, $a \in \widetilde{M}$). Set

$$\Omega^{n,m} = C^\infty(\Lambda^n(T')^* \otimes \Lambda^m(T'')^*).$$

The exterior derivative d splits as

$$d = d' + (-1)^n d'' \quad \text{on} \quad \Omega^{n,m},$$

where d' and d'' are exterior derivatives in the direction of $\widetilde{M}$ and V, respectively.

The $(1,0)$-component and $(0,1)$-component of the 1-form $\widetilde{\eta}$ are denoted $\widetilde{\eta}'$ and $\widetilde{\eta}''$, respectively. Since the wedge product with ω_V induces an injection

$$\Omega^{1,0} \to \Omega^{1,1},$$

it follows from (5.1) that

$$-d'\varphi = -(d\varphi)' = \widetilde{\eta}'.$$

Then

$$\begin{aligned} \widetilde{\eta} \wedge d\widetilde{\eta} &= (\widetilde{\eta}' + \widetilde{\eta}'') \wedge d(\widetilde{\eta}' + \widetilde{\eta}'') \\ &= -\widetilde{\eta}' \wedge d''\widetilde{\eta}' + \widetilde{\eta}' \wedge d'\widetilde{\eta}'', \end{aligned} \tag{5.2}$$

because $d\widetilde{\eta}' = -d'd'\varphi = 0$ and $\Omega^{n,m} = 0$ for $m > 1$. We have

$$\begin{aligned} d(\widetilde{\eta}' \wedge \widetilde{\eta}'') &= (d'\widetilde{\eta}' - d''\widetilde{\eta}') \wedge \widetilde{\eta}'' - \widetilde{\eta}' \wedge (d'\widetilde{\eta}'' + d''\widetilde{\eta}'') \\ &= -\widetilde{\eta}' \wedge d'\widetilde{\eta}''. \end{aligned} \tag{5.3}$$

Notice that $\widetilde{\eta}' \wedge \widetilde{\eta}''$ is Γ-invariant, sine the Γ-action on $\widetilde{M} \times V$ preserves the decomposition $T = T' \oplus T''$.

Proposition 5.4. *The Godbillon-Vey class of $\mathcal{F}$ is given by the cohomology class*

$$[-d'\varphi \wedge d''d'\varphi] \in H^3_{DR}(X).$$

Proof. By (5.2) and (5.3),

$$\widetilde{\eta} \wedge d\widetilde{\eta} = -d'\varphi \wedge d''d'\varphi - d(\widetilde{\eta}' \wedge \widetilde{\eta}'').$$

Since $\widetilde{\eta} \wedge d\widetilde{\eta}$ and $\widetilde{\eta}' \wedge \widetilde{\eta}''$ are Γ-invariant, so is $d'\varphi \wedge d''d'\varphi$. Therefore $-d'\varphi \wedge d''d'\varphi$ defines a 3-form on X, and

$$[\eta \wedge d\eta] = [-d'\varphi \wedge d''d'\varphi] \in H^3_{DR}(X).$$

□

Remark 5.5. Equality (4.11) together with the fact that $\log \lambda_g$ on $\widetilde{M} \times V$ is constant in the direction of $\widetilde{M}$ implies that $d'\varphi$ is Γ-invariant.

6. Cyclic Cocycles.

In this section we will construct a densely defined cyclic cocycle on the algebra $\mathfrak{A}$. This cocycle can be interpreted as an analytical variant of $gv(\mathcal{F})$.

As in the preceding sections, let $\widetilde{E}$ be a given Γ-equivariant bundle over $\widetilde{M} \times S^1$. Define a new right Γ-action by

$$v \cdot g = \lambda_g(x)^{-1} vg, \qquad v \in \widetilde{E}_{(m,x)}, \quad g \in \Gamma, \tag{6.1}$$

where vg is the given Γ-action. Denote by $\widetilde{E}'$ the vector bundle $\widetilde{E}$ equipped with this new action, and denote by $g[\xi]$ the action of $g \in \Gamma$ on $\xi \in C_c^{\infty,0}(\widetilde{E}')$. Then

$$g[\xi] = \lambda_g g(\xi). \tag{6.2}$$

With respect to the new action (6.1), the Hermitian metric of $\widetilde{E}$ is no more Γ-invariant. However, we have the relation

$$(v \cdot g, w \cdot g) = \lambda_g(x)^{-2}(v, w), \qquad v, w \in \widetilde{E}_{(m,x)}.$$

This enables us to obtain continuous fields of tangential Sobolev spaces.

Let $P \in \Psi_\Gamma^r(\widetilde{E})$. Then

$$\begin{aligned} g[P(\xi)] &= \lambda_g g(P(\xi)) = g[\xi] = \lambda_g P(g(\xi)) \\ &= P(\lambda_g g(\xi)) \\ &= P(g[\xi]), \end{aligned}$$

here we used the fact that P commutes with the multiplication operator λ_g. Thus $P \in \Psi_\Gamma^r(\widetilde{E}')$.

Conversely, if $Q \in \Psi_\Gamma^r(\widetilde{E}')$, then

$$\begin{aligned} \lambda_g g(Q(\xi)) &= g[Q(\xi)] = Q(g[\xi]) \\ &= Q(\lambda_g g(\xi)) \\ &= \lambda_g Q(g(\xi)). \end{aligned}$$

Since $\lambda_g > 0$, we have $g(Q(\xi)) = Q(g(\xi))$, i.e. $Q \in \Psi_\Gamma^r(\widetilde{E})$.

Denote by $\partial_2\varphi$ the partial derivative of φ in the direction of S^1. Regard the pointwise multiplication by $\partial_2\varphi$ as an operator $C_c^{\infty,0}(\widetilde{E}) \to C_c^{\infty,0}(\widetilde{E}')$, and consider the commutator of operators

$$[\partial_2\varphi, P] = (\partial_2\varphi)P - P(\partial_2\varphi) \quad \text{for} \quad P \in \Psi_\Gamma^r(\widetilde{E}).$$

Proposition 6.3. *We have* $[\partial_2\varphi, P] \in \Psi_\Gamma^{r-1}(\widetilde{E}, \widetilde{E}')$.

Proof. The proof is similar to that of Proposition 4.9. We only have to show that $[\partial_2\varphi, P]$ is Γ-equivariant. We then have

$$\begin{aligned} g[(\partial_2\varphi)P - P(\partial_2\varphi)] &= \lambda_g g((\partial_2\varphi)P - P(\partial_2\varphi)) \\ &= \lambda_g(g(\partial_2\varphi)gP - Pg(\partial_2\varphi)g) \\ &= \lambda_g g(\partial_2\varphi)Pg - \lambda_g Pg(\partial_2\varphi)g \\ &= (\partial_2\varphi + \partial_2(\log\lambda_g))Pg - P(\partial_2\varphi + \partial_2(\log\lambda_g))g \\ &= ((\partial_2\varphi)P - P(\partial_2\varphi))g + \partial_2(\log\lambda_g)Pg - P\partial_2(\log\lambda_g)g. \end{aligned}$$

Since $\log\lambda_g$ is constant along $\widetilde{M}_x$, $x \in S^1$, so is $\partial_2(\log\lambda_g)$. Thus $\partial_2(\log\lambda_g)$ commutes with P. Hence

$$g[\partial_2\varphi, P] = [\partial_2\varphi, P]g,$$

i.e. $[\partial_2\varphi, P]$ is Γ-equivariant. □

Let N be as in Section 3, and let $N \geq r \geq 0$. As in Section 3, we can define a norm $|||\cdot|||$ on $\Psi_\Gamma^{-r}(\widetilde{E}, \widetilde{E}')$ by

$$|||P||| = \max\{||P||_{-N+r,-N}, ||P||_{N,N-r}\}.$$

Denote by $OP_\Gamma^{-r}(\widetilde{E}, \widetilde{E}')$ the completion of $\Psi_\Gamma^{-r}(\widetilde{E}, \widetilde{E}')$ with respect to $|||\cdot|||$. It is easy to see that if $P \in \Psi_\Gamma^p(\widetilde{E})$, $Q \in \Psi_\Gamma^q(\widetilde{E}, \widetilde{E}')$, then PQ, $QP \in \Psi_\Gamma^{p+q}(\widetilde{E}, \widetilde{E}')$.

Proposition 6.4. *The space* $OP_\Gamma^{-2}(\widetilde{E}, \widetilde{E}')$ *is a Banach* $\mathfrak{A}$*-module.*

Proof. Straightforward. □

Notice that the correspondence $P \to [\partial_2\varphi, P]$ is an unbounded derivation from $\mathfrak{A}$ into $OP_\Gamma^{-2}(\widetilde{E}, \widetilde{E}')$ with domain $\Psi_\Gamma^{-1}(\widetilde{E})$. Closability of the multiplication operator $\partial_2\varphi$ implies that the derivation $P \to [\partial_2\varphi, P]$ is closable. Denote by δ_1 its closure with domain $\mathrm{Dom}(\delta_1)$.

Consider the multiplication operator Δ^{it} on both $\widetilde{E}$ and $\widetilde{E}'$.

Proposition 6.5. *If* $Q \in \Psi_\Gamma^r(\widetilde{E}, \widetilde{E}')$, *then* $\Delta^{it}Q\Delta^{-it} \in \Psi_\Gamma^r(\widetilde{E}, \widetilde{E}')$.

Proof. It is sufficient to show that $\Delta^{it}Q\Delta^{-it}$ is Γ-equivariant. Let $\xi \in C_c^{\infty,0}(\widetilde{E})$. Then

$$\begin{aligned} g[(\Delta^{it}Q\Delta^{-it})\xi] &= \lambda_g g(\Delta^{it}Q\Delta^{-it}\xi) = \lambda_g g(\Delta^{it})g(Q\Delta^{-it}\xi) \\ &= g(\Delta^{it})\lambda_g g(Q\Delta^{-it}\xi) = g(\Delta^{it})g[Q\Delta^{-it}\xi] \\ &= g(\Delta^{it})Q(g(\Delta^{-it}))g(\xi). \end{aligned} \tag{6.6}$$

By (4.10), $g(\Delta^{it}) = \Delta^{it}$. Hence the equality (6.6) is equal to

$$\lambda_g^{-it}\Delta^{it}Q(\lambda_g^{it}\Delta^{-it}g(\xi)) = (\Delta^{it}Q\Delta^{-it})(\xi).$$

This is the end of the proof. □

For $Q \in \Psi_\Gamma^{-2}(\widetilde{E}, \widetilde{E}')$, set $\widehat{\sigma}_t'(Q) = \Delta^{it}Q\Delta^{-it}$, $t \in \mathbb{R}$.

Lemma 6.7. *The linear operator $\widehat{\sigma}_t'$ extends to an automorphism of $OP_\Gamma^{-2}(\widetilde{E}, \widetilde{E}')$.*

Proof. By Lemma 4.8, the operator Δ^{it} is bounded on Sobolev spaces. Therefore

$$||\Delta^{it}Q\Delta^{-it}||_{s,s-2} \leq C_s||Q||_{s,s-2}.$$

In particular, there exists $C > 0$ such that

$$||\Delta^{it}Q\Delta^{-it}||_{-N+2,-N} \leq C||Q||_{-N+2,-N},$$
$$||\Delta^{it}Q\Delta^{-it}||_{N,N-2} \leq C||Q||_{N,N-2}.$$

It follows that $|||\widehat{\sigma}_t'(Q)||| \leq C|||Q|||$. □

It is clear that $(\widehat{\sigma}_t')$ is a one-parameter group of automorphisms. Denote by δ_2' the generator of $(\widehat{\sigma}_t')$, and by $\mathrm{Dom}(\delta_2')$ its domain.

Proposition 6.8. *If $Q \in \Psi_\Gamma^{-2}(\widetilde{E}, \widetilde{E}')$, then $Q \in \mathrm{Dom}(\delta_2')$, and $\delta_2'(Q) = [\varphi, Q]$.*

Proof. Same as that for the derivation δ_2. □

Proposition 6.9. *If $P \in \Psi_\Gamma^{-1}(\widetilde{E})$, then $\delta_1(\delta_2(P)) = \delta_2'(\delta_1(P))$.*

Proof. From Proposition 6.8 and the definition of δ_1, the conclusion follows. □

Recall that the underlying Hermitian vector bundle structures of $\widetilde{E}$ and $\widetilde{E}'$ are the same. Therefore $L^2(\widetilde{E}) = L^2(\widetilde{E}')$. Then, if $Q \in \Psi_\Gamma^{-r}(\widetilde{E}, \widetilde{E}')$, $r \geq 0$, the operator P_x can be regarded as a bounded operator on $L^2(\widetilde{E}_x)$. Let σ be a compactly supported smooth function on $\widetilde{M} \times S^1$, and let σ_x be the restriction of σ to $\widetilde{M}_x$, $x \in S^1$.

Proposition 6.10. *Let $s > \dim M$. Then $\sigma_x\Lambda_x^{-s/2}$ and $\Lambda_x^{-s/2}\sigma_x$ are Hilbert-Schmidt class operators.*

Proof. Recall that $\Lambda = (I + \Delta)^{1/2}$. For the Laplacian Δ' on M, we have that

$$(I + \Delta')^{-1/2} \in \mathcal{L}^p \quad \text{for any} \quad p > \dim M.$$

From this, $((I+\Delta')^{1/2})^{-s/2}$ is a Hilbert-Schmidt class operator. If Q is a ψDO of order $-1/2$ on M, then Q is a Hilbert-Schmidt class operator, because

$$Q = Q\left((I+\Delta')^{1/2}\right)^{s/2}\left((I+\Delta')^{1/2}\right)^{-s/2}.$$

In particular, the Schwartz kernel of Q is measurable and square-integrable.

Let $P \in \Psi_\Gamma^{-s/2}(\widetilde{E})$. Then the Schwartz kernel of P is measurable [**6**]. The observation above, combined with Γ-compactness of the support of the Schwartz kernel, implies that $\sigma_x P_x$ and $P_x \sigma_x$ are Hilbert-Schmidt class operators.

Let $P \in \Psi_\Gamma^{-s/2}(\widetilde{E})$ be a parametrix of $\Lambda^{s/2}$, so that $T = P\Lambda^{s/2} - I$ is a compactly smoothing operator. We have

$$\Lambda_x^{-s/2} = P_x - T_x \Lambda_x^{-s/2},$$

as operators on $L^2(\widetilde{E}_x)$. From this, $\sigma_x \Lambda_x^{-s/2} = \sigma_x P_x - \sigma_x T_x \Lambda_x^{-s/2}$. Since both $\sigma_x P_x$ and $T_x \sigma_x$ are Hilbert-Schmidt class operators, so is $\sigma_x \Lambda_x^{-s/2}$. As $\Lambda_x^{-s/2}$ is self-adjoint, we see that $\Lambda_x^{-s/2}\sigma_x$ is also a Hilbert-Schmidt class operator. □

Corollary 6.11. *Let σ, σ' be compactly supported smooth functions on $\widetilde{M} \times S^1$. Then for every $P \in \Psi_\Gamma^{-s}(\widetilde{E}, \widetilde{E}')$ with $s > \dim M$, the operator $\sigma_x P_x \sigma'_x$ is a trace class operator on $L^2(\widetilde{E}_x)$, for any $x \in S^1$. Moreover, there exists a constant $C > 0$ such that*

$$|Tr(\sigma_x P_x \sigma'_x)| \leq C||P||_{s/2,-s/2}.$$

Proof. We have

$$\sigma_x P_x \sigma'_x = (\sigma_x \Lambda_x^{-s/2})(\Lambda_x^{s/2} P_x \Lambda_x^{s/2})(\Lambda_x^{-s/2}\sigma'_x).$$

Consequently, $\sigma_x P_x \sigma'_x$ is of trace class, and

$$\begin{aligned}|Tr(\sigma_x P_x \sigma'_x)| &\leq ||(\sigma_x \Lambda_x^{-s/2})(\Lambda_x^{s/2} P_x \Lambda_x^{s/2})(\Lambda_x^{-s/2}\sigma'_x)||_1 \\ &\leq ||\sigma_x \Lambda_x^{-s/2}||_2 ||\Lambda_x^{s/2} P_x \Lambda_x^{s/2}||_{0,0} ||\Lambda_x^{-s/2}\sigma'_x||_2 \\ &\leq ||\sigma_x \Lambda_x^{-s/2}||_2 ||\Lambda_x^{-s/2}\sigma'_x||_2 ||\Lambda_x^{s/2} P_x \Lambda_x^{s/2}||_{s/2,-s/2},\end{aligned}$$

where $||\cdot||_1$ (resp. $||\cdot||_2$) is the trace class norm (resp. Hilbert-Schmidt norm).

Continuity of the family $(\Lambda_x^{-s/2})_x$ implies the existence of $C > 0$ such that

$$||\sigma_x \Lambda_x^{-s/2}||_2,\ ||\Lambda_x^{-s/2}\sigma'_x||_2 < C.$$

Thus

$$|Tr(\sigma_x P_x \sigma'_x)| \le C||P||_{s/2,-s/2}.$$

□

Let σ be a compactly supported smooth function on $\widetilde{M} \times S^1$ such that

$$\sum_{g\in\Gamma} g(\sigma)^2 = 1;$$

i.e. $\{g(\sigma)^2\}_{g\in\Gamma}$ is a Γ-invariant partition of unity on $\widetilde{M} \times S^1$.

Definition 6.12. For $P \in \Psi_\Gamma^{-s}(\widetilde{E}, \widetilde{E}')$ with $s > \dim M$, set

$$\text{(6.13)} \qquad \text{trace}_\Gamma(P) = \int_{S^1} Tr(\sigma_x P_x \sigma_x)\, dx.$$

Notice that the integrand in (6.13) is continuous. A modification of the proof of Lemma 4.9 of [**1**] shows that $\text{trace}_\Gamma(P)$ is independent of the choice of σ.

Let $P \in \Psi_\Gamma^{-s}(\widetilde{E}, \widetilde{E}')$, $Q \in \Psi_\Gamma^{-r}(\widetilde{E}) = \Psi_\Gamma^{-r}(\widetilde{E}')$.

Then PQ, $QP \in \Psi_\Gamma^{-s+r}(\widetilde{E}, \widetilde{E}')$.

Proposition 6.14. *Let* $r + s > \dim M + 2$. *Assume that either* $0 \le r \le 2$, *or* $0 \le s \le 2$. *Then*

$$\text{trace}_\Gamma(PQ) = \text{trace}_\Gamma(QP).$$

Proof. Since P and Q have Γ-compact Schwartz kernels, there exists a finite subset S of Γ satifying:

$$\begin{aligned}
&\text{(i)} && S = S^{-1},\\
&\text{(ii)} && \operatorname{supp} g(\sigma)_x \cap \operatorname{supp}\sigma \ne \varnothing \Rightarrow g \in S,\\
&\text{(iii)} && \sigma_x P_x \Sigma g(\sigma)_x = \sigma_x P_x \Sigma' g(\sigma)_x, \qquad \text{and}\\
& && \sigma_x Q_x \Sigma g(\sigma)_x = \sigma_x Q_x \Sigma' g(\sigma)_x,
\end{aligned}$$

where the summation Σ (resp. Σ') is taken over all $g \in \Gamma$ (resp. $g \in S$). Then

$$\begin{aligned}
Tr(\sigma_x(P_x Q_x)\sigma_x) &= Tr(\sigma_x(P_x \Sigma g(\sigma)_x^2 Q_x)\sigma_x)\\
&= Tr(\sigma_x(P_x \Sigma' g(\sigma)_x^2 Q_x)\sigma_x)\\
&= \Sigma' Tr(\sigma_x P_x g(\sigma)_x g(\sigma)_x Q_x \sigma_x).
\end{aligned}$$

The last expression is equal to

$$\Sigma' Tr(g(\sigma)_x Q_x \sigma_x \sigma_x P_x g(\sigma)_x),$$

because either $\sigma_x P_x g(\sigma)_x$ or $g(\sigma)_x Q_x \sigma_x$ is a trace class operator, by Corollary 6.11 and our assumption on s, r.

Let $U(g)$ be the canonical unitary mapping $L^2(\widetilde{E}_{xg}) \to L^2(\widetilde{E}_x)$. It is easy to check that, as multiplication operator,

$$g(\sigma)_x = U(g)\sigma_{xg}U(g)^{-1}.$$

Then we have

$$\begin{aligned} &g(\sigma)_x Q_x \sigma_x \sigma_x P_x g(\sigma)_x \\ &\quad = U(g)\sigma_{xg}U(g)^{-1}Q_x U(g)U(g)^{-1}(\sigma_x)^2 U(g)U(g)^{-1}P_x U(g)\sigma_{xg}U(g)^{-1}. \end{aligned}$$

Since Q is Γ-equivariant, $U(g)^{-1}Q_x U(g) = Q_{xg}$. As for P, we have

$$U(g)^{-1}P_x U(g) = \lambda_{g^{-1}}(x)^{-1}P_{xg} = \lambda_g(x)P_{xg}.$$

Hence

$$\begin{aligned} \operatorname{trace}_\Gamma(PQ) &= \int_{S^1} \Sigma' Tr(g(\sigma)_x Q_x \sigma_x \sigma_x P_x g(\sigma)_x)\, dx \\ &= \int_{S^1} \Sigma' Tr(\sigma_{xg} Q_{xg} g^{-1}(\sigma)_{xg} g^{-1}(\sigma)_{xg} \lambda_g(x) P_{xg} \sigma_{xg})\, dx \\ &= \int_{S^1} \Sigma' Tr(\sigma_{xg} Q_{xg} g^{-1}(\sigma)_{xg} g^{-1}(\sigma)_{xg} P_{xg} \sigma_{xg})\, d(xg) \\ &= \int_{S^1} \Sigma' Tr(\sigma_x Q_x g^{-1}(\sigma)_x g^{-1}(\sigma)_x P_x \sigma_x)\, dx \\ &= \int_{S^1} \Sigma' Tr(\sigma_x Q_x g(\sigma)_x g(\sigma)_x P_x \sigma_x)\, dx \\ &= \int_{S^1} Tr(\sigma_x Q_x \Sigma' (g^{-1}(\sigma)_x)^2 P_x \sigma_x)\, dx \\ &= \int_{S^1} Tr(\sigma_x Q_x \Sigma (g^{-1}(\sigma)_x)^2 P_x \sigma_x)\, dx \\ &= \operatorname{trace}_\Gamma(QP). \end{aligned}$$

□

By Corollary 6.11, $\operatorname{trace}_\Gamma$ is continuous with respect to $\|\cdot\|_{s/2,-s/2}$, provided that $s > \dim M$. This implies that $\operatorname{trace}_\Gamma$ extends to a continuous linear functional on $OP_\Gamma^{-s}(\widetilde{E}, \widetilde{E}')$ with $s > \dim M$. (Caution: our $\operatorname{trace}_\Gamma$ is not the same as $\operatorname{trace}_\Gamma$ of [**1**]. Our $\operatorname{trace}_\Gamma$ is not an actual trace on any algebra, it is just a linear functional, while Atiyah's $\operatorname{trace}_\Gamma$ is an actual trace on an algebra.)

Lemma 6.15. (1) *$\operatorname{trace}_\Gamma([\partial_2\varphi, P]) = 0$ for all $P \in \Psi_\Gamma^{-s}(\widetilde{E})$ with $s >$ $\dim M$.*

(2) $\operatorname{trace}_\Gamma(\delta_2'(Q)) = 0$ *for all* $Q \in \Psi_\Gamma^{-s}(\widetilde{E}, \widetilde{E}')$ *with* $s > \dim M$.

Proof. (1) Notice that $P_x\sigma_x$ and $P_x(\partial_2\varphi)_x\sigma_x$ are trace class operators. Then

$$\begin{aligned} Tr(\sigma_x(\partial_2\varphi)_x P_x\sigma_x) &= Tr(P_x\sigma_x\sigma_x(\partial_2\varphi)_x) \\ &= Tr(P_x(\partial_2\varphi)_x\sigma_x\sigma_x) \\ &= Tr(\sigma_x P_x(\partial_2\varphi)_x\sigma_x). \end{aligned}$$

Thus $Tr(\sigma_x[\partial_2\varphi, P]_x\sigma_x) = 0$. Hence $\operatorname{trace}_\Gamma([\partial_2\varphi, P]) = 0$.

(2) The proof is the same as that of (1). □

Furnish $\mathfrak{E} = \operatorname{Dom}(\delta_1) \cap \operatorname{Dom}(\delta_2)$ with the locally convex topology given by the graph norms associated with δ_1 and δ_2.

We will construct a densely defined cyclic cocycle on $\mathfrak{A}$. Let us first consider the case where $\dim M = 2$. Set

$$(6.16) \quad \begin{aligned} \tau_2\left(P^0, P^1, P^2\right) &= \operatorname{trace}_\Gamma\left(P^0\delta_1\left(P^1\right)\delta_2\left(P^2\right)\right) \\ &\quad - \operatorname{trace}_\Gamma\left(P^0\delta_2\left(P^1\right)\delta_1\left(P^2\right)\right) \quad \text{for} \quad P^0, P^1, P^2 \in \mathfrak{E} \subseteq \mathfrak{A}. \end{aligned}$$

Proposition 6.17. *The trilinear functional* τ_2 *is a cyclic* 2*-cocycle.*

Proof. If $P^0, P^1, P^2 \in \mathfrak{E}$, then the products

$$P^0\delta_1\left(P^0\right)\delta_2\left(P^0\right) \quad \text{and} \quad P^0\delta_2\left(P^0\right)\delta_1\left(P^0\right)$$

belong to $OP_\Gamma^{-5}(\widetilde{E}, \widetilde{E}')$. Since δ_1 and δ_2 are derivations, τ_2 is a Hochschild cocycle. By Proposition 6.14 and Lemma 6.15, τ_2 is a cyclic cocycle on $\Psi_\Gamma^{-1}(\widetilde{E}) \subset \mathfrak{E}$. Then by continuity and the fact that $\Psi_\Gamma^{-1}(\widetilde{E})$ is dense in $\mathfrak{E}$, we can see that τ_2 is a cyclic cocycle on $\mathfrak{E}$. □

Proposition 6.18. *The densely defined cyclic cocycle* τ_2 *is a* 2*-trace on* $\mathfrak{A}$ *in the sense of* [**8**].

Proof. We have that

$$\widehat{\tau}_2(a^0dx^1a^1dx^2) = \operatorname{trace}_\Gamma(a^0\delta_1(x^1)a^1\delta_2(x^2)) - \operatorname{trace}_\Gamma(a^0\delta_2(x^1)a^1\delta_1(x^2)),$$

and

$$\begin{aligned} &|\operatorname{trace}_\Gamma(a^0\delta_1(x^1)a^1\delta_2(x^2))| \le C||a^0\delta_1(x^1)a^1\delta_2(x^2)||_{3/2,-3/2} \\ &\le C||a^0||_{3/2,1/2}||\delta_1(x^1)||_{1/2,1/2}||a^1||_{1/2,-1/2}||\delta_2(x^2)||_{-1/2,-3/2} \\ &\le C_{1,2}|||a^0|||\ |||a^1|||, \end{aligned}$$

for some constant $C_{1,2}$ depending only on x^1 and x^2.

Similarly

$$|\operatorname{trace}_\Gamma(a^0\delta_2(x^1)a^1\delta_1(x^2))| \le C'_{1,2}|||a^0|||\;|||a^1|||.$$

This completes the proof. □

Let us now consider higher dimensional cases. Let $\dim M = 2n$. The formula (6.16) defines a cyclic cocycle on $\Psi_\Gamma^{-\infty}(\widetilde{E})$, but not on $\mathfrak{C}$ when $n > 1$. Consider the cyclic $2n$-cocycle $S^{n-1}\tau_2$, instead. For $P^0, \ldots, P^{2n}$, we have

$$\begin{aligned}(6.19)\qquad &((n-1)!)^{-1}(2\pi i)^{1-n}S^{n-1}\tau_2(P^0,\ldots,P^{2n})\\ &= \sum_{1\le i\le j\le n}\{\operatorname{trace}_\Gamma(P^0P^1\cdots P^{2i-2}\delta_1(P^{2i-1})P^{2i}\\ &\qquad \cdots P^{2j-1}\delta_2(P^{2j})P^{2j+1}\cdots P^{2n})\\ &\qquad - \operatorname{trace}_\Gamma(P^0P^1\cdots P^{2i-2}\delta_2(P^{2i-1})P^{2i}\\ &\qquad \cdots P^{2j-1}\delta_1(P^{2j})P^{2j+1}\cdots P^{2n})\}.\end{aligned}$$

Denote by $\tau_{2n}(P^0,\ldots,P^{2n})$ the right-hand side of (6.19). Notice that $\tau_{2n}(P^0,\ldots,P^{2n})$ makes sense when $P^0,\ldots,P^{2n} \in \mathfrak{C}$.

The proof of Proposition 6.18 can be generalized to show that τ_{2n} is a $2n$-trace on $\mathfrak{A}$.

Definition 6.20. When $\dim M = 2n$, the Godbillon-Vey cyclic cocycle gv is the $2n$-trace

$$gv = (n-1)!\tau_{2n}.$$

By [**8**, Lemma 2.3; Corollary 2.4], gv extends to a cyclic $2n$-cocycle on a holomorphically closed dense subalgebra of $\mathfrak{A}$, consequently it induces an additive map from $K_0[\mathfrak{A}]$ into the scalars. By Proposition 3.6, the canoninical inclusion $\mathfrak{A} \subseteq C^*(X,\mathcal{F},E)$ induces an isomorphism of K_0-groups. Hence gv induces a map $K_0[C^*(X,\mathcal{F},E)] \to \mathbb{C}$. In Section 8 we will compute the value of this map on a specific class in $K_0[C^*(X,\mathcal{F},E)]$.

7. Dirac Operators and Graph Projections.

In this section we will show that the graph projection of a longitudinal Dirac operator belongs to the domain of the $2n$-trace gv on $\mathfrak{A}$.

Let $\widetilde{M}$ be as in the preceding sections. Assume further that $\widetilde{M}$ is even-dimensional and is furnished with a Γ-invariant spin structure. Denote by $\widetilde{D}$ the associated Dirac operator on $\widetilde{M}$ acting on the bundle $\widetilde{S}$ of (complex) spinors. Since $\widetilde{M}$ is even, the bundle $\widetilde{S}$ has a $\mathbb{Z}_2$-grading ε. Thus

$$(7.1)\qquad \widetilde{S} = \widetilde{S}^+ \oplus \widetilde{S}^-,$$

where $\widetilde{S}^{\pm}$ are ± 1 eigenspaces of ε, respectively. With respect to the decomposition (7.1), the operator $\widetilde{D}$ has the form

$$\widetilde{D} = \begin{pmatrix} 0 & \widetilde{D}^- \\ \widetilde{D}^+ & 0 \end{pmatrix},$$

where $\widetilde{D}^{\pm}$ are first-order, elliptic differential operators. Since the Γ-action on $\widetilde{S}$ preserves $\widetilde{S}^{\pm}$ respectively, $\widetilde{D}^{\pm}$ are Γ-equivariant operators. Moreover, $\widetilde{D}$ is essentially selfadjoint and has a closed extension. The closure $\widetilde{D}^{**}$ of $\widetilde{D}$ has the form

$$\widetilde{D}^{**} = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix},$$

where T is the closure of $\widetilde{D}^*$, and $\widetilde{D}^{**}$ is selfadjoint.

The graph $G(T)$ of T is, by the definition of T, a closed subspace of $L^2(\widetilde{S}^+) \oplus L^2(\widetilde{S}^-) = L^2(\widetilde{S})$. Denote the corresponding orthogonal projection by e, and set

$$X = \widetilde{D}^{**}\varepsilon = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}.$$

Lemma 7.2. *We have*

$$e = \begin{pmatrix} (I+T^*T)^{-1} & (I+T^*T)^{-1}T^* \\ T(I+T^*T)^{-1} & T(I+T^*T)^{-1}T^* \end{pmatrix} = (I+X) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (I+X)^{-1}.$$

Proof. Define $\imath : L^2(\widetilde{S}^+) \to L^2(\widetilde{S}^+) \oplus L^2(\widetilde{S}^-)$ by

$$\imath = \begin{pmatrix} I \\ T \end{pmatrix} (I+T^*T)^{-1/2} = \begin{pmatrix} (I+T^*T)^{-1/2} \\ T(I+T^*T)^{-1/2} \end{pmatrix}.$$

It is easy to see that $\imath^*\imath = 1$. Since $(I+T^*T)^{-1/2}$ is an isomorphism from $L^2(\widetilde{S}^+)$ onto the domain $\mathrm{Dom}(T)$ of T, the image of $\imath$ is precisely the graph $G(T)$. Thus the projection e is given by

$$e = \imath\imath^* = \begin{pmatrix} (I+T^*T)^{-1} & (I+T^*T)^{-1}T^* \\ T(I+T^*T)^{-1} & T(I+T^*T)^{-1}T^* \end{pmatrix}.$$

As for the second equality, from the equality

$$(I+X)^{-1} = (I-X^2)^{-1}(I-X) = \begin{pmatrix} (I+T^*T)^{-1} & 0 \\ 0 & (I+TT^*)^{-1} \end{pmatrix} \begin{pmatrix} 0 & T^* \\ -T & 0 \end{pmatrix},$$

it follows that

$$e = (I + X) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (I + X)^{-1}.$$

□

Set

$$p_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad p_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Set $u = (I + X)\varepsilon$. Then

$$u^2 = (I + X)\varepsilon(I + X)\varepsilon = (I + X)(I - X) = I - X^2,$$

because $X\varepsilon = -\varepsilon X$.

Let $\widehat{e} = e - p_-$. Then using the equality

$$\varepsilon = (I + X)p_+ - p_-(I + X),$$

we can see that

$$\begin{aligned} \widehat{e} &= e - p_- = (I + X)p_+(I + X)^{-1} - p_- \\ &= ((I + X)p_+ - p_-(I + X))(I + X)^{-1} \\ &= \varepsilon(I + X)^{-1} \\ &= u^{-1}. \end{aligned} \tag{7.3}$$

From this,

$$\widehat{e}^2 = u^{-2} = (I - X^2)^{-1}. \tag{7.4}$$

A straightforward computation shows that

$$\widehat{e} = \begin{pmatrix} (I + T^*T)^{-1} & (I + T^*T)^{-1}T^* \\ T(I + T^*T)^{-1} & -(I + TT^*)^{-1} \end{pmatrix}. \tag{7.5}$$

As in the preceding sections, suppose that Γ acts on S^1 by orientation preserving diffeomorphisms. For each $x \in S^1$, identify $\widetilde{M}_x = \widetilde{M} \times \{x\}$ with $\widetilde{M}$ in a natural way. Via this identification, we obtain a vector bundle $\widetilde{S}_x$ and a differential operator $\widetilde{D}_x$. By abuse of language, denote the family $(\widetilde{D}_x)$ by $\widetilde{D}$. It is clear that $\widetilde{D}$ is a Γ-equivariant family of elliptic operators, acting on a Γ-equivariant vector bundle $\widetilde{S} = (\widetilde{S}_x)$, i.e.

$$\widetilde{D} \in \Psi^1_\Gamma(\widetilde{S}).$$

The Γ-equivariant differential operator $\widetilde{D}$ on $\widetilde{M} \times S^1$ descends to a longitudinal elliptic operator D on $X = \widetilde{M} \times_\Gamma S^1$, which we call a *longitudinal Dirac operator*.

The operator $\widetilde{D}$ is of the form

$$\widetilde{D} = \begin{pmatrix} 0 & \widetilde{D}^- \\ \widetilde{D}^+ & 0 \end{pmatrix},$$

and $\widetilde{D}^+ = (\widetilde{D}^+_x) \in \Psi^1_\Gamma(\widetilde{S}^+, \widetilde{S}^-)$. Consequently, we can consider a continuous field $e = (e_x)$ of projections: each e_x is the orthogonal projection of $L^2(\widetilde{S}^+_x) \oplus L^2(\widetilde{S}^-_x)$ onto the graph of the closure of $\widetilde{D}^+_x$. The matrix p_- can be regarded as the orthogonal projection of $\left(L^2(\widetilde{S}_x)\right)_{x\in S^1}$ onto $\left(L^2(\widetilde{S}^-_x)\right)_{x\in S^1}$. Then, obviously $p_- \in \Psi^0_\Gamma(\widetilde{S})$.

We devote the rest of the section to show that $\widehat{e}$ belongs to the domain of the cyclic cocycle gv. For this purpose we employ the method of bounded propagation [**20**], [**21**], [**23**]. Since the Dirac operator $\widetilde{D}$ is the lifting of the Dirac operator on a closed manifold M, it has bounded propagation speed.

Recall that the space $S^0(\mathbb{R})$ of symbols of order zero is the collection of all C^∞-functions f on $\mathbb{R}$ such that for each $j = 0, 1, 2, \dots$, it holds that

$$\sup\{(1 + |x|)^j |f^{(j)}(x)| : x \in \mathbb{R}\} < \infty.$$

We need the following:

Proposition 7.6. ([**15**, Thm. 7.25], [**20**, Thm. 21]). *Let $P \in \Psi^1_\Gamma(\widetilde{E})$ be a longitudinal, tangentially essentially selfadjoint, first-order elliptic differential operator of bounded propagation speed. If the Fourier transform $\widehat{f}$ of $f \in S^0(\mathbb{R})$ is compactly supported, then*

$$f(P) \in \Psi^0_\Gamma(\widetilde{E}).$$

If the Fourier transform $\widehat{g}$ of a Schwartz function g is compactly supported, $g(P)$ is compactly smoothing.

Let $\rho_+ : \mathbb{R} \to [0, 1]$ be a C^∞-function such that

$$\rho_+ \equiv 1 \qquad \text{on} \quad t \leq 1 - \delta,$$

and

$$\rho_+ \equiv 0 \qquad \text{on} \quad t \geq 1 + \delta$$

for some sufficiently small $0 < \delta < 1$. Set $\rho_-(t) = \rho_+(-t)$. For $\lambda \geq 2$, set

$$\rho_\lambda(t) = \rho_+(t - (\lambda - 1))\rho_-(t + \lambda - 1)$$

to obtain a C^∞-function $\rho_\lambda(t) : \mathbb{R} \to [0,1]$ such that

$$\rho_\lambda(t) \equiv 1 \qquad \text{on} \quad |t| \le \lambda - 1 - \delta,$$

and

$$\rho_\lambda(t) \equiv 0 \qquad \text{on} \quad |t| \ge \lambda - 1 + \delta.$$

Lemma 7.7. *For any positive integer i, there exists a positive constant C_i such that*

$$|\rho_\lambda^{(i)}(t)| \le C_i \qquad \textit{for all} \quad \lambda, t.$$

Proof. By the construction of ρ_λ, it is straightforward. □

Set

$$\varphi_\lambda(x) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{ixt} \rho_\lambda(t) e^{-|t|}\, dt. \tag{7.8}$$

Lemma 7.9. (1) *The function φ_λ belongs to $S^0(\mathbb{R})$, and its Fourier transform is $\rho_\lambda(t)e^{-|t|}$.*

(2) *The function $\psi_\lambda(x) = (2\pi)^{-1/2}(1+x^2)\varphi_\lambda(x) - 1$ is a Schwartz function with compactly supported Fourier transform.*

(3) *As $\lambda \to \infty$, ψ_λ converges to zero in $C_0(\mathbb{R})$.*

Proof. (1) Using integration by parts twice, we get that

$$\begin{aligned}(2\pi)^{-1/2}\varphi_\lambda(x) = \frac{1}{1+x^2} &+ \int_0^\infty \rho_\lambda''(t) e^{(ix-1)t}(ix-1)^{-2}\, dt \\ &+ \int_{-\infty}^0 \rho_\lambda''(t) e^{(ix+1)}(ix+1)^{-2}\, dt.\end{aligned} \tag{7.10}$$

From this, it follows that $\sup\{(1+x^2)|\varphi_\lambda(x)|;\ x \in \mathbb{R}\} < \infty$. This, in turn, means that $\varphi_\lambda \in L^1(\mathbb{R})$, because φ_λ is continuous. Then by the Fourier inversion formula,

$$\widehat{\varphi}_\lambda(t) = \rho_\lambda(t) e^{-|t|}.$$

For a given nonnegative integer j, consider

$$h_\lambda(t) = (it)^j \rho_\lambda(t).$$

Notice that $h_\lambda^{(k)}(0) = 0$ for $k = 0, 1, \dots, j$. Then

$$\begin{aligned}(2\pi)^{-1/2}\varphi_\lambda^{(j)}(t) &= \int_{\mathbb{R}} (it)^{-j} e^{ixt} \rho_\lambda(t) e^{-|t|}\, dt \\ &= \int_0^\infty h_\lambda(t) e^{(ix-1)t}\, dt + \int_{-\infty}^0 h_\lambda(t) e^{(ix+1)t}\, dt \\ &= (-1)^j \int_0^\infty h_\lambda^{(j)}(t) e^{(ix-1)t} (ix-1)^{-j}\, dt \\ &\quad + \int_{-\infty}^0 h_\lambda^{(j)}(t) e^{(ix+1)t}(ix+1)^{-j}\, dt.\end{aligned}$$

It is easy to see that there exists a constant $C > 0$ such that

$$|\varphi_\lambda^{(j)}(x)| \le C(|ix-1|^{-j} + |ix+1|^{-j}) \quad \text{for all} \quad x.$$

Thus

$$\sup\{(1+|x|)^j |\varphi_\lambda^{(j)}(x)|; \quad x \in \mathbb{R}\} < \infty.$$

(2) The equality (7.10) implies that $\psi_\lambda \in C_0(\mathbb{R})$. We need the following Sublemma, which we will prove later.

Sublemma. *As distributions, we have the identity*

$$\left(1 - \frac{d^2}{dt^2}\right) e^{-|t|} = 2\delta_0,$$

where δ_0 is the delta function at $t = 0$.

We now have that

$$\begin{aligned}\widehat{\psi}_\lambda &= \left(1 - \frac{d^2}{dt^2}\right)\rho_\lambda e^{-|t|} - \delta_0 \qquad (7.11)\\ &= -\rho_\lambda'' e^{-|t|} + 2\rho_\lambda' e^{-|t|} sgn(t) \qquad \text{(as distributions)}.\end{aligned}$$

Since both sides of (7.11) are compactly supported C^∞-functions, they are actually equal as C^∞-functions. It is now clear that ψ_λ is a Schwartz function.

(3) The Fourier transform induces an isomorphism from $C_0(\mathbb{R})$ onto $C^*(\mathbb{R})$. So

$$||\psi_\lambda||_{C_0(\mathbb{R})} = ||\widehat{\psi}_\lambda||_{C^*(\mathbb{R})} \le ||\widehat{\psi}_\lambda||_{L^1(\mathbb{R})}.$$

By our construction, $\rho_\lambda'', \rho_\lambda'$ are bounded uniformly in λ. Therefore the equality (7.11) implies that

$$||\widehat{\psi}_\lambda||_{L^1(\mathbb{R})} \to 0 \qquad \text{as} \quad \lambda \to \infty.$$

This concludes the proof of Lemma 7.9. □

Proof of Sublemma. Let $f(t) = e^{-|t|}$. For $g \in C_c^\infty(\mathbb{R})$, applying integration by parts twice, we obtain that

$$\begin{aligned}\int_{\mathbb{R}} f(t)g''(t)\,dt &= \int_0^\infty f(t)g''(t)\,dt + \int_{-\infty}^0 f(t)g''(t)\,dt \\ &= -2g(0) + \int_{\mathbb{R}} f(t)g(t)\,dt.\end{aligned}$$

Therefore

$$\begin{aligned}\left\langle \frac{1}{2}\left(1 - \frac{d^2}{dt^2}\right) f, g\right\rangle &= \left\langle f, \frac{1}{2}\left(1 - \frac{d^2}{dt^2}\right) g\right\rangle \\ &= \frac{1}{2}\int_{\mathbb{R}} f(t)g(t)\,dt - \frac{1}{2}\int_{\mathbb{R}} f(t)g''(t)\,dt \\ &= g(0) \\ &= \langle \delta_0, g\rangle.\end{aligned}$$

□

For $P \in \Psi_\Gamma^r(\widetilde{E})$, by a straightforward computation we get that

$$||P||_{k,k+r} = ||(I+\Delta)^{k/2}P(I+\Delta)^{-(k+r)/2}||_{0,0}. \tag{7.12}$$

In the definition of tangential Sobolev spaces for the bundle $\widetilde{S}$, we can use $\widetilde{D}^2$ in place of the Laplacian, thanks to the standard elliptic estimate. Thus we may assume that the Sobolev s-norm is given by

$$||\xi||_s = \left\|\left(I + \widetilde{D}^2\right)^{s/2}\xi\right\|_0 \qquad \text{for} \quad \xi \in C_c^\infty.$$

Consider an (unbounded) intertwining operator $T = (T_x)$ of $W_\tau^0(\widetilde{S}) = (L^2(\widetilde{S}_x))_x$, where T_x is the closure of $\widetilde{D}_x^+$. As before, set

$$X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}.$$

Then

$$\widehat{e} = (I+X)\varepsilon\left(I+\widetilde{D}^2\right)^{-1}. \tag{7.13}$$

By Proposition 7.5 and Lemma 7.9,

$$\varphi_\lambda(\widetilde{D}) \in \Psi_\Gamma^0(\widetilde{S}),$$

and

$$\psi_\lambda(\widetilde{D}) = \sqrt{2\pi}\left(I + \widetilde{D}^2\right)\varphi_\lambda(\widetilde{D}) - I \in \Psi_\Gamma^{-\infty}(\widetilde{S}).$$

These imply that $\varphi_\lambda(\widetilde{D}) \in \Psi_\Gamma^{-2}(\widetilde{S})$. Hence

$$(I + X)\varepsilon\varphi_\lambda(\widetilde{D}) \in \Psi_\Gamma^{-1}(\widetilde{S}).$$

The equality (7.13) means, in particular that $\widehat{e}$ is an operator of order -1. Therefore we can consider the norm $\left\|\widehat{e} - \sqrt{2\pi}(I + X)\varepsilon\varphi_\lambda(\widetilde{D})\right\|_{k,k-1}$.

By (7.12)

$$\begin{aligned}
&\left\|\widehat{e} - \sqrt{2\pi}(I + X)\varepsilon\varphi_\lambda(\widetilde{D})\right\|_{k,k-1} \\
&= \left\|(I + X)\varepsilon\left(I + \widetilde{D}^2\right)^{-1} - \sqrt{2\pi}(I + X)\varepsilon\varphi_\lambda(\widetilde{D})\right\|_{k,k-1} \\
&= \left\|\left(I + \widetilde{D}^2\right)^{k/2}(I + X)\varepsilon\left(\left(I + \widetilde{D}^2\right)^{-1} - \sqrt{2\pi}\varphi_\lambda(\widetilde{D})\right)\left(I + \widetilde{D}^2\right)^{(1-k)/2}\right\|_{0,0} \\
&= \left\|(I + X)\varepsilon\left(\left(I + \widetilde{D}^2\right)^{-1} - \sqrt{2\pi}\varphi_\lambda(\widetilde{D})\right)\left(I + \widetilde{D}^2\right)^{1/2}\right\|_{0,0} \\
&= \left\|(I + X)\varepsilon\left(I + \widetilde{D}^2\right)^{-1}\left(I - \sqrt{2\pi}\left(I + \widetilde{D}^2\right)\varphi_\lambda(\widetilde{D})\right)\right\|_{0,0} \\
&\le \left\|(I + X)\varepsilon\left(I + \widetilde{D}^2\right)^{-1}\right\|_{0,0}\left\|I - \sqrt{2\pi}\left(I + \widetilde{D}^2\right)\varphi_\lambda(\widetilde{D})\right\|_{0,0}.
\end{aligned}$$

In this computation we have used the fact that $(I + \widetilde{D}^2)^{1/2}$ commutes with $(I + X)\varepsilon(I + \widetilde{D}^2)^{-1} - \sqrt{2\pi}\varphi_\lambda(\widetilde{D})$. Now by Lemma 7.9, (3),

$$\left\|\widehat{e} - \sqrt{2\pi}(I + X)\varepsilon\varphi_\lambda(\widetilde{D})\right\|_{k,k-1} \to 0 \quad \text{as} \quad \lambda \to \infty.$$

Thus $\widehat{e}$ is in the closure of $\Psi_\Gamma^{-1}(\widetilde{S})$ with respect to the norm $|||\cdot|||$. Therefore $\widehat{e} \in \mathfrak{A}$.

We show that $\widehat{e}$ belongs to the domain of δ_2. Recall that $\widehat{e} = u^{-1} = ((I+X)\varepsilon)^{-1} = (\widetilde{D}+\varepsilon)^{-1}$. If φ is bounded, then the commutator $[\varphi, (\widetilde{D}+\varepsilon)^{-1}]$ is a bounded operator, and

$$\lim_{\lambda\to\infty}\left[\varphi, \sqrt{2\pi}(I + X)\varepsilon\varphi_\lambda(\widetilde{D})\right] = \left[\varphi, (\widetilde{D} + \varepsilon)^{-1}\right].$$

Unfortunately, φ is unbounded in general (see (4.11)). Thus $\left[\varphi, (\widetilde{D} + \varepsilon)^{-1}\right]$ is defined only on a subspace which may not be dense. So, even if $\left[\varphi, (\widetilde{D} + \varepsilon)^{-1}\right]$

extends to a bounded operator, the extension may not be unique. However, "formally" we have the equality

$$\begin{aligned}\left[\varphi,(\widetilde{D}+\varepsilon)^{-1}\right]&=\varphi(\widetilde{D}+\varepsilon)^{-1}-(\widetilde{D}+\varepsilon)^{-1}\varphi\\&=(\widetilde{D}+\varepsilon)^{-1}[\widetilde{D}+\varepsilon,\varphi](\widetilde{D}+\varepsilon)^{-1},\end{aligned}$$

and $(\widetilde{D}+\varepsilon)^{-1}[\widetilde{D}+\varepsilon,\varphi](\widetilde{D}+\varepsilon)^{-1}$ is a bounded operator, because $[\widetilde{D}+\varepsilon,\varphi]=[\widetilde{D},\varphi]\in\Psi^0_\Gamma(\widetilde{S})$. Thus it is natural to expect that

$$\delta_2(\widehat{e})=(\widetilde{D}+\varepsilon)^{-1}[\widetilde{D}+\varepsilon,\varphi](\widetilde{D}+\varepsilon)^{-1}.$$

Notice that $[\varphi,(\widetilde{D}+\varepsilon)\sqrt{2\pi}\varphi_\lambda(\widetilde{D})]\in\Psi_\Gamma^{-2}(\widetilde{S})$, and that $(\widetilde{D}+\varepsilon)^{-1}[\widetilde{D}+\varepsilon,\varphi](\widetilde{D}+\varepsilon)^{-1}$ is an operator of order -2 (not a ψDO). We will show (Proposition 7.17) that

$$\left\|\left[\varphi,(\widetilde{D}+\varepsilon)\sqrt{2\pi}\varphi_\lambda(\widetilde{D})\right]-(\widetilde{D}+\varepsilon)^{-1}\left[\widetilde{D}+\varepsilon,\varphi\right](\widetilde{D}+\varepsilon)^{-1}\right\|_{s,s-2}\to 0$$

as $\lambda\to\infty$ for any s. It is enough to show that

$$\left\|(\widetilde{D}+\varepsilon)\left[\varphi,(\widetilde{D}+\varepsilon)\sqrt{2\pi}\varphi_\lambda(\widetilde{D})\right](\widetilde{D}+\varepsilon)-[\widetilde{D}+\varepsilon,\varphi]\right\|_{s,s}\to 0$$

as $\lambda\to\infty$. Recall that $\psi_\lambda(x)=1-(1+x^2)\varphi_\lambda(x)$.

Lemma 7.14. *We have*

$$\left\|\left[\varphi,\psi_\lambda(\widetilde{D})\left(I+\widetilde{D}^2\right)\right]\right\|_{0,0}\to 0\quad as\quad \lambda\to\infty.$$

Proof. For simplicity, set $\alpha_\lambda(x)=(1+x^2)\psi_\lambda(x)$. Then

$$\psi_\lambda(\widetilde{D})\left(I+\widetilde{D}^2\right)=\alpha_\lambda(\widetilde{D})=\int\widehat{\alpha}_\lambda(s)e^{is\widetilde{D}}\,ds.$$

Since $[\varphi,\widetilde{D}]$ extends to a bounded operator, by Duhamel's formula,

$$\left[\varphi,\psi_\lambda(\widetilde{D})\left(I+\widetilde{D}^2\right)\right]=\int_{\mathbb{R}}\int_0^1\widehat{\alpha}_\lambda(s)e^{is\widetilde{D}t}[\varphi,\mathrm{is}\,\widetilde{D}]e^{is\widetilde{D}(1-t)}\,dt\,ds.$$

From this

$$\left\|\left[\varphi,\psi_\lambda(\widetilde{D})\left(I+\widetilde{D}^2\right)\right]\right\|_{0,0}\le\|[\varphi,\widetilde{D}]\|_{0,0}\int_{\mathbb{R}}|\widehat{\alpha}_\lambda(s)|\,|s|\,ds.$$

By the definition of ψ_λ, when $\lambda\to\infty$, the integral $\int_{\mathbb{R}}|\widehat{\alpha}_\lambda(s)|\,|s|\,ds$ behaves like $\lambda e^{-\lambda}$; i.e. there exists a constant $C>0$ such that

$$\int_{\mathbb{R}}|\widehat{\alpha}_\lambda(s)|\,|s|\,ds\le C\lambda e^{-\lambda}.$$

Thus

$$\left\| \left[\varphi, \psi_\lambda(\widetilde{D}) \left(I + \widetilde{D}^2\right)\right] \right\|_{0,0} \leq C\lambda e^{-\lambda}.$$

□

Lemma 7.15. *We have*

$$\left\| \left[\varphi, \psi_\lambda(\widetilde{D}) \left(I + \widetilde{D}^2\right)^{1/2}\right] \right\|_{0,0} \to 0 \quad as \quad \lambda \to \infty.$$

Proof. We have

$$\begin{aligned}
&\left[\varphi, \psi_\lambda(\widetilde{D}) \left(I + \widetilde{D}^2\right)^{1/2}\right] \\
&= [\varphi, \psi_\lambda(\widetilde{D})] \left(I + \widetilde{D}^2\right)^{1/2} + \psi_\lambda(\widetilde{D}) \left[\left(I + \widetilde{D}^2\right)^{1/2}, \varphi\right] \\
&= \left[\varphi, \psi_\lambda(\widetilde{D}) \left(I + \widetilde{D}^2\right)\right] \left(I + \widetilde{D}^2\right)^{-1/2} \\
&\quad + \psi_\lambda(\widetilde{D}) \left(I + \widetilde{D}^2\right)^{1/2} \left[\left(I + \widetilde{D}^2\right)^{1/2}, \varphi\right] \left(I + \widetilde{D}^2\right)^{-1/2} \\
&\quad + \psi_\lambda(\widetilde{D}) \left[\left(I + \widetilde{D}^2\right)^{1/2}, \varphi\right].
\end{aligned}$$

Then

$$\begin{aligned}
&\left\| \left[\varphi, \psi_\lambda(\widetilde{D}) \left(I + \widetilde{D}^2\right)^{1/2}\right] \right\|_{0,0} \\
&\leq \left\| \left[\varphi, \psi_\lambda(\widetilde{D}) \left(I + \widetilde{D}^2\right)\right] \right\|_{0,0} \left\| \left(I + \widetilde{D}^2\right)^{-1/2} \right\|_{0,0} \\
&\quad + \|\psi_\lambda(\widetilde{D})\|_{0,0} \left\| \left(I + \widetilde{D}^2\right)^{1/2} \left[\left(I + \widetilde{D}^2\right)^{1/2}, \varphi\right] \left(I + \widetilde{D}^2\right)^{-1/2} \right\|_{0,0} \\
&\quad + \|\psi_\lambda(\widetilde{D})\|_{0,0} \left\| \left[\left(I + \widetilde{D}^2\right)^{1/2}, \varphi\right] \right\|_{0,0}
\end{aligned}$$

(notice that $[(I + \widetilde{D}^2)^{1/2}, \varphi]$ is an operator of order 0). By Lemma 7.14, we get the conclusion. □

Lemma 7.16. *We have*

$$\left\| \left[\varphi, \psi_\lambda(\widetilde{D})\right] \right\|_{s,s-1} \to 0 \quad as \quad \lambda \to \infty.$$

Proof. By (7.12),

$$\left\| \left[\varphi, \psi_\lambda(\widetilde{D})\right] \right\|_{s,s-1} = \left\| \left(I + \widetilde{D}^2\right)^{s/2} [\varphi, \psi_\lambda(\widetilde{D})] \left(I + \widetilde{D}^2\right)^{(1-s)/2} \right\|_{0,0}.$$

Case (i). $s \geq 0$. In this case

$$\begin{aligned}&\left(I+\widetilde{D}^2\right)^{s/2}\left[\varphi,\psi_\lambda(\widetilde{D})\right]\left(I+\widetilde{D}^2\right)^{(1-s)/2}\\&=\left[\left(I+\widetilde{D}^2\right)^{s/2},\varphi\right]\psi_\lambda(\widetilde{D})\left(I+\widetilde{D}^2\right)^{(1-s)/2}+\left[\varphi,\psi_\lambda(\widetilde{D})\right]\left(I+\widetilde{D}^2\right)^{(1-s)/2}\\&\quad-\psi_\lambda(\widetilde{D})\left[\left(I+\widetilde{D}^2\right)^{s/2},\varphi\right]\left(I+\widetilde{D}^2\right)^{(1-s)/2};\end{aligned}$$

here we have used the fact that $\left[\left(I+\widetilde{D}^2\right)^{s/2},\varphi\right]\in\Psi_\Gamma^{s-1}(\widetilde{S})$ provided that $s \geq 0$. We have

$$\begin{aligned}&\left\|\left[\left(I+\widetilde{D}^2\right)^{s/2},\varphi\right]\psi_\lambda(\widetilde{D})\left(I+\widetilde{D}^2\right)^{(1-s)/2}\right\|_{0,0}\\&\leq\left\|\left[\left(I+\widetilde{D}^2\right)^{s/2},\varphi\right]\right\|_{0,s-1}\|\psi_\lambda(\widetilde{D})\|_{s-1,s-1}\left\|\left(I+\widetilde{D}^2\right)^{(1-s)/2}\right\|_{s-1,0},\end{aligned}$$

which converges to zero as $\lambda\to\infty$.

Similarly,

$$\left\|\psi_\lambda(\widetilde{D})\left[\left(I+\widetilde{D}^2\right)^{s/2},\varphi\right]\left(I+\widetilde{D}^2\right)^{(1-s)/2}\right\|_{0,0}\to 0\quad\text{as}\quad\lambda\to\infty.$$

Then, by Lemma 7.15, we obtain the conclusion.

Case (ii). $s<0$. In this case $-s/2+1/2>0$ and $[\varphi,(I+\widetilde{D}^2)^{(1-s)/2}]$ is a ψDO. Making use of $[\varphi,(I+\widetilde{D}^2)^{(1-s)/2}]$ in the place of $[(I+\widetilde{D}^2)^{s/2},\varphi]$ in Case (i), we can deduce the conclusion. □

Proposition 7.17. *The element $\widehat{e}$ is in the domain of δ_2, and*

$$\begin{aligned}\delta_2(\widehat{e})&=(\widetilde{D}+\varepsilon)^{-1}[\widetilde{D}+\varepsilon,\varphi](\widetilde{D}+\varepsilon)^{-1}\\&=(\widetilde{D}+\varepsilon)^{-1}[\widetilde{D},\varphi](\widetilde{D}+\varepsilon)^{-1}.\end{aligned}$$

Proof. As mentioned above, it is sufficient to show that

$$(\widetilde{D}+\varepsilon)\left[\varphi,(\widetilde{D}+\varepsilon)\sqrt{2\pi}\varphi_\lambda(\widetilde{D})\right](\widetilde{D}+\varepsilon)$$

converges to $[\widetilde{D}+\varepsilon,\varphi]$ as $\lambda\to\infty$, as operator of order zero. By a straightforward computation,

$$\begin{aligned}&\left[\widetilde{D}+\varepsilon,\varphi\right]-(\widetilde{D}+\varepsilon)\left[\varphi,(\widetilde{D}+\varepsilon)\sqrt{2\pi}\varphi_\lambda(\widetilde{D})\right](\widetilde{D}+\varepsilon)\\&=[\widetilde{D}+\varepsilon,\varphi]\psi_\lambda(\widetilde{D})+[\varphi,\psi_\lambda(\widetilde{D})](\widetilde{D}+\varepsilon).\end{aligned}$$

We have that

$$||[\varphi, \psi_\lambda(\widetilde{D})](\widetilde{D}+\varepsilon)||_{s,s} \le ||[\varphi, \psi_\lambda(\widetilde{D})]||_{s,s-1}||\widetilde{D}+\varepsilon||_{s-1,s}.$$

Then by Lemma 7.16,

$$||[\varphi, \psi_\lambda(\widetilde{D})](\widetilde{D}+\varepsilon)||_{s,s} \to 0 \quad \text{as} \quad \lambda \to \infty.$$

By construction, $\psi_\lambda(\widetilde{D})$ commutes with $\left(I+\widetilde{D}^2\right)$. Hence

$$\begin{aligned} ||\psi_\lambda(\widetilde{D})||_{s,s} &= \left\| \left(I+\widetilde{D}^2\right)^{s/2} \psi_\lambda(\widetilde{D}) \left(I+\widetilde{D}^2\right)^{-s/2} \right\|_{0,0} \\ &= ||\psi_\lambda(\widetilde{D})||_{0,0} \to 0 \quad \text{as} \quad \lambda \to \infty. \end{aligned}$$

From these it follows that $||[\widetilde{D}+\varepsilon, \varphi]\psi_\lambda(\widetilde{D})||_{s,s} \to 0$ as $\lambda \to \infty$. Consequently,

$$\left[\varphi, (\widetilde{D}+\varepsilon)\sqrt{2\pi}\varphi_\lambda(\widetilde{D})\right] \to (\widetilde{D}+\varepsilon)^{-1}[\widetilde{D}+\varepsilon, \varphi](\widetilde{D}+\varepsilon)^{-1}.$$

Recall that $(\widetilde{D}+\varepsilon)\sqrt{2\pi}\varphi_\lambda(\widetilde{D}) \to \widehat{e}$ in $\mathfrak{A}$. Therefore, by closedness of δ_2 we obtain that

$$\delta_2(\widehat{e}) = (\widetilde{D}+\varepsilon)^{-1}[\widetilde{D}+\varepsilon, \varphi](\widetilde{D}+\varepsilon)^{-1}.$$

□

By the same argument, we can verify that $\widehat{e}$ is also in the domain of δ_1, and that

$$\delta_1(\widehat{e}) = (\widetilde{D}+\varepsilon)^{-1}[\widetilde{D}+\varepsilon, \partial_2\varphi](\widetilde{D}+\varepsilon)^{-1}. \tag{7.18}$$

8. Main Theorem.

In this section we will compute the pairing between the $2n$-trace gv and the class of the graph projection of the longitudinal Dirac operator. Throughout this section $\dim M = 2n$.

Let D be the longitudinal Dirac operator for the foliated S^1-bundle $(X, \mathcal{F})$. Denote by $C^*(X, \mathcal{F}, S)^\sim$ the C^*-algebra generated by $C^*(X, \mathcal{F}, S)$ and the projection p_- in $\wp_0$. We then have a split exact sequence:

$$0 \to C^*(X, \mathcal{F}, S) \to C^*(X, \mathcal{F}, S)^\sim \to \mathbb{C}p_- \to 0.$$

In Section 7, we showed that

$$\widehat{e} = e - p_- \in \mathfrak{A} \subseteq C^*(X, \mathcal{F}, S).$$

Set $\Theta = [e] - [p_-]$. Then $\Theta \in K_0[C^*(X, \mathcal{F}, S)]$.

Proposition 8.1. *The class* Θ *is equal to* $\operatorname{ind}(D^+)$.

Proof. Recall [**9**, Lemma 6.1] that

$$\operatorname{ind}(D^+) = \left[\begin{pmatrix} S_0^2 & S_0(I+S_0)\widetilde{Q} \\ S_1\widetilde{D}^+ & I - S_1^2 \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] \in K_0[C^*(X, \mathcal{F}, S)],$$

where $\widetilde{Q}$ is a parametrix of $\widetilde{D}^+$, and

$$S_0 = I - \widetilde{Q}\widetilde{D}^+ \in C^*(X, \mathcal{F}, S^+),$$
$$S_1 = I - \widetilde{D}^+\widetilde{Q} \in C^*(X, \mathcal{F}, S^-).$$

Set

$$u = \begin{pmatrix} S_0(I+\widetilde{D}^-\widetilde{D}^+)^{-1} & S_0(I+\widetilde{D}^-\widetilde{D}^+)^{-1}\widetilde{D}^- \\ \widetilde{D}^+(I+\widetilde{D}^-\widetilde{D}^+)^{-1} & \widetilde{D}^+(I+\widetilde{D}^-\widetilde{D}^+)^{-1}\widetilde{D}^- \end{pmatrix},$$

and

$$v = \begin{pmatrix} S_0 & (I+S_0)\widetilde{Q} \\ \widetilde{D}^+S_0 & \widetilde{D}^+(I+S_0)\widetilde{Q} \end{pmatrix}.$$

Then,

$$u, v \in C^*(X, \mathcal{F}, S)^\sim,$$

and

$$uv = \begin{pmatrix} S_0^2 & S_0(I+S_0)\widetilde{Q} \\ S_1\widetilde{D}^+ & I - S_1^2 \end{pmatrix},$$

and $vu = e$. Thus

$$\operatorname{ind}(D^+) = \Theta \text{ in } K_0[C^*(X, \mathcal{F}, S)].$$

□

Denote by $C^*(X, \mathcal{F}, S)^+$ the C^*-algebra $C^*(X, \mathcal{F}, S)$ with unit adjoined. Notice that $C^*(X, \mathcal{F}, S)^+$ is identified with the C^*-subalgebra of $\wp_0$ generated by $C^*(X, \mathcal{F}, S)$ and $I \in \wp_0$. The $2n$-cocycle gv, constructed in Section 6, extends to $C^*(X, \mathcal{F}, S)^+$ by setting

$$gv(a^0, a^1, \ldots, a^{2n}) = 0,$$

if one of $a^0, a^1, \dots, a^{2n}$ is a scalar multiple of I.

In terms of $C^*(X, \mathcal{F}, S)^+$, the class Θ is expressed as a difference

$$\Theta = [p] - [q],$$

where

$$p = \begin{pmatrix} (I + \widetilde{D}^-\widetilde{D}^+)^{-1} & 0\,0 & (I + \widetilde{D}^-\widetilde{D}^+)^{-1}\widetilde{D}^- \\ 0 & 0\,0 & 0 \\ 0 & 0\,1 & 0 \\ \widetilde{D}^+(I + \widetilde{D}^-\widetilde{D}^+)^{-1} & 0\,0 & \widetilde{D}^+(I + \widetilde{D}^-\widetilde{D}^+)^{-1}\widetilde{D}^- \end{pmatrix},$$

and

$$q = \begin{pmatrix} 0\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{pmatrix}.$$

Notice that $p, q \in M_2(C^*(X, \mathcal{F}, S)^+)$. Then it is easy to see that

$$\langle gv, [p] - [q] \rangle = (2\pi i)^{2n} n! gv(\widehat{e}, \dots, \widehat{e}).$$

The main focus of the section is to explicitly compute $gv(\widehat{e}, \dots, \widehat{e})$.

We have

$$gv(\widehat{e}, \dots, \widehat{e}) = (n-1)! \sum \{\mathrm{trace}_\Gamma(\widehat{e}^{2i+1}\delta_1(\widehat{e})\widehat{e}^{2j}\delta_2(\widehat{e})\widehat{e}^{2n-2i-2j-2}) - \mathrm{trace}_\Gamma(\widehat{e}^{2i+1}\delta_2(\widehat{e})\widehat{e}^{2j}\delta_1(\widehat{e})\widehat{e}^{2n-2i-2j-2})\},$$

where the summation is taken over all i and j such that $0 \leq i, j$ and $i + j \leq n - 1$.

Lemma 8.2. *We have*

$$(1) \quad \widehat{e}^{2i+1}\delta_1(\widehat{e})\widehat{e}^{2j}\delta_2(\widehat{e})\widehat{e}^{2n-2i-2j-2} = \left(I + \widetilde{D}^2\right)^{-(i+1)} [\widetilde{D}, \partial_2\varphi] \left(I + \widetilde{D}^2\right)^{-(j+1)} [\widetilde{D}, \varphi] \times \left(I + \widetilde{D}^2\right)^{-(n-i-j-1)} (\widetilde{D} + \varepsilon)^{-1},$$

and

$$(2) \quad \widehat{e}^{2i+1}\delta_2(\widehat{e})\widehat{e}^{2j}\delta_1(\widehat{e})\widehat{e}^{2n-2i-2j-2} = \left(I + \widetilde{D}^2\right)^{-(i+1)} [\widetilde{D}, \varphi] \left(I + \widetilde{D}^2\right)^{-(j+1)} [\widetilde{D}, \partial_2\varphi] \times \left(I + \widetilde{D}^2\right)^{-(n-i-j-1)} (\widetilde{D} + \varepsilon)^{-1}.$$

Proof. Recall that $u = (I + X)\varepsilon = \widetilde{D} + \varepsilon$. In Section 7 we showed that

$$\delta_1(\widehat{e}) = (\widetilde{D} + \varepsilon)^{-1}[\widetilde{D} + \varepsilon, \partial_2\varphi](\widetilde{D} + \varepsilon)^{-1} = u^{-1}[\widetilde{D}, \partial_2\varphi]u^{-1},$$

and

$$\delta_2(\widehat{e}) = u^{-1}[\widetilde{D}, \varphi]u^{-1}.$$

Therefore

$$\begin{aligned}
&\widehat{e}^{2i+1}\delta_1(\widehat{e})\widehat{e}^{2j}\delta_2(\widehat{e})\widehat{e}^{2n-2i-2j-2} \\
&= (u^{-1})^{2i+1}u^{-1}[\widetilde{D}, \partial_2\varphi]u^{-1}(u^{-1})^{2j}u^{-1}[\widetilde{D}, \varphi]u^{-1}(u^{-1})^{2n-2i-2j-2} \\
&= \left(I + \widetilde{D}^2\right)^{-(i+1)}[\widetilde{D}, \partial_2\varphi]\left(I + \widetilde{D}^2\right)^{-(j+1)}[\widetilde{D}, \varphi] \\
&\quad \times \left(I + \widetilde{D}^2\right)^{-(n-i-j-1)}(\widetilde{D} + \varepsilon)^{-1}.
\end{aligned}$$

Similarly we obtain the second equality. □

For i, j with $0 \le i, j$, and $i + j \le n - 1$, let

$$\begin{aligned}
A^{i,j} &= \widehat{e}^{2i+1}\delta_1(\widehat{e})\widehat{e}^{2j}\delta_2(\widehat{e})\widehat{e}^{2n-2i-2j-2}, \\
B^{i,j} &= \widehat{e}^{2i+1}\delta_2(\widehat{e})\widehat{e}^{2j}\delta_1(\widehat{e})\widehat{e}^{2n-2i-2j-2}.
\end{aligned}$$

Then

$$\begin{aligned}
gv(\widehat{e}, \dots, \widehat{e}) &= (n-1)!\sum\left(\mathrm{trace}_\Gamma\left(A^{i,j}\right) - \mathrm{trace}_\Gamma\left(B^{i,j}\right)\right), \\
\mathrm{trace}_\Gamma\left(A^{i,j}\right) &= \int_{S^1} tr\left(\sigma_x A^{i,j}_x \sigma_x\right) dx, \\
\mathrm{trace}_\Gamma\left(B^{i,j}\right) &= \int_{S^1} tr\left(\sigma_x B^{i,j}_x \sigma_x\right) dx,
\end{aligned}$$

where $A^{i,j}_x$ (resp. $B^{i,j}_x$) is the restriction of $A^{i,j}$ (resp. $B^{i,j}$) onto $\widetilde{M}_x = \widetilde{M} \times \{x\}$, $x \in S^1$. We must compute $tr(\sigma_x A^{i,j}_x \sigma_x)$ and $tr(\sigma_x B^{i,j}_x \sigma_x)$. In order to do so, we make use of Getzler's symbolic calculus method **[12]**. Fix an arbitrary $x \in S^1$. For a while we do analysis on the manifold $\widetilde{M}_x = \widetilde{M}$. In order to simplify the notation we supress the subindex, as long as it is clear on which manifold we are working on.

Consider a one-parameter family of operators on $\widetilde{M} = \widetilde{M}_x$,

$$\begin{aligned}
A^{i,j}(t) = &\left(I + t^2\widetilde{D}^2\right)^{-(i+1)}[t\widetilde{D}, \partial_2\varphi]\left(I + t^2\widetilde{D}^2\right)^{-(j+1)} \\
&\times [t\widetilde{D}, \varphi]\left(I + t^2\widetilde{D}^2\right)^{-(n-i-j-1)} \\
&\times \begin{pmatrix} (I + t^2\widetilde{D}^-\widetilde{D}^+)^{-1} & (I + t^2\widetilde{D}^-\widetilde{D}^+)^{-1}t\widetilde{D}^- \\ t\widetilde{D}^+(I + t^2\widetilde{D}^-\widetilde{D}^+)^{-1} & -(I + t^2\widetilde{D}^+\widetilde{D}^-)^{-1} \end{pmatrix}, \quad t > 0.
\end{aligned}$$

Similarly, define $B^{i,j}(t)$.

In the symbolic calculus method, a key notion is that of asymptotic order. Assign to the parameter t the order -1, and to a Clifford multiplication the order $+1$. The total order is called the *asymptotic order.* For instance, the following symbols have the asymptotic order 0 [**9**]:

(i) $\sigma\left(\left(\lambda + t^2\widetilde{D}^2\right)^{-1}\right)(m,\xi)$,

(ii) $\sigma([t\widetilde{D}, f])(m,\xi) = t\, df_m, \quad f \in C^\infty(\widetilde{M})$.

In (i) the operator $(\lambda + t\widetilde{D}^2)^{-1}$ is a ψDO. However, its distributional kernel does not have Γ-compact support. In (ii) df_m is a Clifford multiplication operator.

Although in [**12**] only compact manifolds are studied, the method developed there works for compactly supported ψDO's. In particular, the following "Fundamental Lemma" is valid for such ψDO's (we use the notation of [**12**] and omit the proof).

Lemma 8.3. ([**9**], [**12**]). (1) *If $A = A(t)$ has asymptotic order 0, then*

$$\sigma_{t^{-1}}(A(t)) = \sigma_0(A) + O(t),$$

where $\sigma_{t^{-1}}$ is the rescaled symbol, and $\sigma_0(A)$ is the asymptotic symbol of A.

(2) *If A, B are operators of asymptotic order 0, then*

$$\sigma_0(AB) = \sigma_0(A) * \sigma_0(B),$$

where $*$ is the Getzler multiplication of symbols.

(3) *If $\Pi(t) \in OpS^{-\infty}$, then*

$$Tr_s(\Pi(t)) = (2\pi)^{-\dim M} \int_{T^*\widetilde{M}} tr_s(\sigma_{t^{-1}}(\Pi(t)))(m,\xi)\, dm\, d\xi, \quad t > 0,$$

where $dmd\xi$ is the symplectic measure on $T^*\widetilde{M}$.

We return to the computation. It is easy to see that

$$tr\left(\sigma A^{i,j}(t)\sigma\right) = Tr_s\left(\Pi^A_{i,j}(t)\right),$$

and

$$tr\left(\sigma B^{i,j}(t)\sigma\right) = Tr_s\left(\Pi^B_{i,j}(t)\right),$$

where

$$\Pi^A_{i,j}(t) = \sigma\left(I + t^2\widetilde{D}^2\right)^{-(i+1)}[t\widetilde{D}, \partial_2\varphi]\left(I + t^2\widetilde{D}^2\right)^{-(j+1)} \times [t\widetilde{D}, \varphi]\left(I + t^2\widetilde{D}^2\right)^{-(n-i-j)}\sigma,$$

and

$$\Pi^B_{i,j}(t) = \sigma \left(I + t^2\widetilde{D}^2\right)^{-(i+1)} [t\widetilde{D}, \varphi] \left(I + t^2\widetilde{D}^2\right)^{-(j+1)} \times [t\widetilde{D}, \partial_2\varphi] \left(I + t^2\widetilde{D}^2\right)^{-(n-i-j)} \sigma.$$

Next, notice that the operators considered in [**9**] and [**12**] are the operator $\sqrt{-1}\widetilde{D}$. For simplicity, let $\not{D} = \sqrt{-1}\widetilde{D}$. Then $\not{D}^* = -\not{D}$ and $\not{D}^2 = -\widetilde{D}^2$. We have $[\widetilde{D}, \varphi] = -\sqrt{-1}[\not{D}, \varphi]$, and $[\widetilde{D}, \partial_2\varphi] = -\sqrt{-1}[\not{D}, \partial_2\varphi]$. From this it follows that

$$\Pi^A_{i,j}(t) = -\sigma \left(I - t^2 \not{D}^2\right)^{-(i+1)} [t\not{D}, \partial_2\varphi] \left(I - t^2 \not{D}^2\right)^{-(j+1)} \times [t\not{D}, \varphi] \left(I - t^2 \not{D}^2\right)^{-(n-i-j)} \sigma.$$

Similarly,

$$\Pi^B_{i,j}(t) = \sigma \left(I - t^2 \not{D}^2\right)^{-(i+1)} [t\not{D}, \varphi] \left(I - t^2 \not{D}^2\right)^{-(j+1)} \times [t\not{D}, \partial_2\varphi] \left(I - t^2 \not{D}^2\right)^{-(n-i-j)} \sigma.$$

The operators $\Pi^A_{i,j}$ and $\Pi^B_{i,j}$ satisfy the assumption of Lemma 8.3. Therefore

$$\begin{aligned} (8.4) \qquad tr\left(\sigma A^{i,j}(t)\sigma\right) &= Tr_s\left(\Pi^A_{i,j}(t)\right) \\ &= (2\pi)^{-2n} \int_{T^*\widetilde{M}} tr_s\left(\sigma_{t^{-1}}\left(\Pi^A_{i,j}\right)\right)(m,\xi)\, dm\, d\xi \\ &= (2\pi)^{-2n} \int_{T^*\widetilde{M}} tr_s\left(\sigma_0\left(\Pi^A_{i,j}\right)\right)(m,\xi)\, dm\, d\xi + O(t). \end{aligned}$$

Similarly

$$(8.5)\ tr\left(\sigma B^{i,j}(t)\sigma\right) = (2\pi)^{-2n} \int_{T^*\widetilde{M}} tr_s\left(\sigma_0\left(\Pi^B_{i,j}\right)\right)(m,\xi)\, dm\, d\xi + O(t).$$

We compute the asymptotic symbols $\sigma_0\left(\Pi^A_{i,j}\right)$ and $\sigma_0\left(\Pi^B_{i,j}\right)$. Symbols which are independent of ξ commute with those dependent on ξ, with respect to Getzler multiplication. By [**9**, Example (3.2)], $\sigma_0([t\not{D}, \varphi]) = d\varphi$ and $\sigma_0([t\not{D}, \partial_2\varphi]) = d(\partial_2\varphi)$. Hence

$$\sigma_0\left(\Pi^A_{i,j}\right) = -\sigma d(\partial_2\varphi) \wedge d\varphi\sigma\sigma_0\left(\left(I - t^2 \not{D}^2\right)^{-(n+2)}\right),$$

and

$$\begin{aligned}\sigma_0\left(\Pi^B_{i,j}\right) &= -\sigma d\varphi\wedge d(\partial_2\varphi)\sigma\sigma_0\left(\left(I-t^2\,\not{D}^{\,2}\right)^{-(n+2)}\right)\\ &= \sigma d(\partial_2\varphi)\wedge d\varphi\sigma\sigma_0\left(\left(I-t^2\,\not{D}^{\,2}\right)^{-(n+2)}\right)\\ &= -\sigma_0\left(\Pi^A_{i,j}\right).\end{aligned}$$

Using the formula:

$$\left(I-t^2\,\not{D}^{\,2}\right)^{-k-1} = \frac{1}{k!}\int_0^\infty s^k e^{-s}e^{st^2\not{D}^2}\,ds,$$

we obtain

$$\sigma_0\left(\Pi^A_{i,j}\right) = -\sigma d(\partial_2\varphi)\wedge d\varphi\sigma\frac{1}{(n+1)!}\int_0^\infty s^{n+1}e^{-s}\sigma_0(e^{-st^2\not{D}^2})\,ds.$$

By [**9**, p. 362],

$$\int_{T^*\widetilde{M}}\sigma_0(e^{st^2\,\not{D}^{\,2}})\,d\xi = \pi^n s^{-n}\det\left(\frac{sR/2}{\sinh sR/2}\right)^{1/2},$$

where R is the curvature tensor of the Γ-invariant metric on $\widetilde{M}$.

Applying the super trace, which amounts to multiplying $(2/i)^n$ and taking the top degree term, we get that (8.4) is equal to

$$\begin{aligned}&-(2\pi)^{-2n}\int_{T^*\widetilde{M}}\sigma d(\partial_2\varphi)\wedge d\varphi\sigma\frac{1}{(n+1)!}\\ &\qquad\times\int_0^\infty s^{n+1}e^{-s}\pi^n s^{-n}\det\left(\frac{sR/2}{\sinh sR/2}\right)^{1/2}ds\,dm\,d\xi + O(t)\\ &= -\left(\frac{2}{i}\right)^n(2\pi)^{-2n}\pi^n\int_{\widetilde{M}}\sigma^2 d(\partial_2\varphi)\wedge d\varphi\sigma\det\left(\frac{R/2}{\sinh R/2}\right)^{1/2}+O(t).\end{aligned}$$

Therefore

$$\begin{aligned}(8.6)\quad &\sum[tr\,(\sigma A^{i,j}(t)\sigma) - tr\,(\sigma B^{i,j}(t)\sigma)]\\ &= -\operatorname{Card}(\{(i,j);\ 0\le i,j,\ \text{ and }\ i+j\le n-1\})\cdot 2(2\pi)^{-2n}\pi^n\\ &\quad\times\left(\frac{2}{i}\right)^n\times\int_{\widetilde{M}}\sigma^2 d(\partial_2\varphi)\wedge d\varphi\sigma\det\left(\frac{R/2}{\sinh R/2}\right)^{1/2}+O(t)\\ &= -\left(\frac{2}{i}\right)^n\frac{n(n+1)}{2}\cdot 2(2\pi)^{-2n}\pi^n\\ &\quad\times\int_{\widetilde{M}}\sigma^2 d(\partial_2\varphi)\wedge d\varphi\sigma\det\left(\frac{R/2}{\sinh R/2}\right)^{1/2}+O(t).\end{aligned}$$

The piece of degree $(2n-2)$ of $\det\left(\frac{R/2}{\sinh R/2}\right)^{1/2}$ is homogeneous of degree $(n-1)$. Hence (8.6) is equal to the following (8.7)

$$-\left(\frac{2}{i}\right)^n (2\pi)^{-2n}\pi^n n(n+1)(-2\pi i)^{n-1} \times \int_{\widetilde{M}} \sigma^2 d(\partial_2\varphi)\wedge d\varphi\sigma \det\left(\frac{-(1/2\pi i)(R/2)}{\sinh(-(1/2\pi i)(R/2))}\right)^{1/2} + O(t).$$

Proposition 8.8. *As $t\to 0$, the term $\sum[tr(\sigma_x A^{i,j}(t)_x\sigma_x)-tr(\sigma_x B^{i,j}(t)_x\sigma_x)]$ converges to*

$$-\left(\frac{2}{i}\right)^n (2\pi)^{-2n}\pi^n n(n+1)(-2\pi i)^{n-1} \times \int_{\widetilde{M}} \sigma_x^2 d((\partial_2\varphi)_x)\wedge d(\varphi_x) \det\left(\frac{-(1/2\pi i)(R/2)}{\sinh(-(1/2\pi i)(R/2))}\right)^{1/2}.$$

Moreover, convergence is uniform in x.

Proof. Convergence follows from the equality (8.7).

Recall that we are dealing with a family of operators $\widetilde{D}=(\widetilde{D}_x)$ on $\widetilde{M}\times S^1$ such that $\widetilde{D}_x=\widetilde{D}_y$ via the canonical identification of $\widetilde{M}_x$ and $\widetilde{M}_y$, and $\varphi, \partial_2\varphi$ are smooth functions. It follows that, when one applies Lemma 8.3, (1), one obtains an estimate $O(t)$, which is uniform in x. Then the conclusion is immediate. □

Proposition 8.9. *We have that*

$$\begin{aligned}\sum\int_{S^1}&[tr(\sigma_x A_x^{i,j}\sigma_x)-tr(\sigma_x B_x^{i,j}\sigma_x)]\,dx,\\ &=\sum\int_{S^1}[tr(\sigma_x A^{i,j}(t)_x\sigma_x)-tr(\sigma_x B^{i,j}(t)_x\sigma_x)]\,dx \quad \text{for all} \quad t>0.\end{aligned}$$

Proof. The right-hand side of the identity above is precisely

$$((n-1)!)^{-1}gv(\widehat{e}_t,\dots,\widehat{e}_t),$$

where $\widehat{e}_t$ is the graph projection of the operator $t\widetilde{D}^+$, and $\widehat{e}_t = e_t - p_-$. Clearly, (e_t) is a continuous path of projections. Therefore

$$[e]-[p_-]=[e_t]-[p_-] \quad \text{in} \quad K_0.$$

Hence

$$gv(\widehat{e}_t,\dots,\widehat{e}_t)=gv(\widehat{e},\dots,\widehat{e}) \quad \text{for all} \quad t>0.$$

□

From Propositions 8.8 and 8.9, it follows that

$$
\begin{aligned}
gv(\widehat{e},\dots,\widehat{e}) &= gv(\widehat{e}_t,\dots,\widehat{e}_t)\\
&= \lim_{t\to 0} gv(\widehat{e}_t,\dots,\widehat{e}_t)\\
&= \lim_{t\to 0}(n-1)!\sum\int_{S^1}[tr(\sigma_x A^{i,j}(t)_x\sigma_x) - tr(\sigma_x B^{i,j}(t)_x\sigma_x)]\,dx\\
&= (n-1)!\sum\int_{S^1}\lim_{t\to 0}[tr(\sigma_x A^{i,j}(t)_x\sigma_x) - tr(\sigma_x B^{i,j}(t)_x\sigma_x)]\,dx\\
&= -(n+1)!(2\pi)^{-2n}\pi^n(-2\pi i)^{n-1}\left(\frac{2}{i}\right)^n\\
&\quad\times\int_{S^1}\int_{\widetilde{M}_x}\sigma_x^2 d((\partial_2\varphi)_x)\wedge d(\varphi_x)\det\left(\frac{-(1/2\pi i)(R/2)}{\sinh(-(1/2\pi i)(R/2))}\right)^{1/2}dx\\
&= -(n+1)!(2\pi)^{-2n}\pi^n(-2\pi i)^{n-1}\left(\frac{2}{i}\right)^n\\
&\quad\times\int_{S^1}\int_{\widetilde{M}}\sigma^2 d'd''\varphi\wedge d'\varphi\wedge\widehat{A}(R)\\
&= -(n+1)!(2\pi)^{-2n}\pi^n(-2\pi i)^{n-1}\left(\frac{2}{i}\right)^n\int_X d'd''\varphi\wedge d'\varphi\wedge\widehat{A}(R),
\end{aligned}
$$

where $X=\widetilde{M}\times_\Gamma S^1$, and $\widehat{A}(R)$ is the $\widehat{A}$-polynomial of $\widetilde{M}$ given in terms of the curvature R of the Γ-invariant Riemannian metric on $\widetilde{M}$. Since $d'd''\varphi\wedge d'\varphi$ is Γ-invariant, so is $d'd''\varphi\wedge d'\varphi\wedge\widehat{A}(R)$. Consequently the integration of $d'd''\varphi\wedge d'\varphi\wedge\widehat{A}(R)$ on X is well defined. By Proposition 5.4, the 3-form $-d'd''\varphi\wedge d'\varphi$ represents the Godbillon-Vey class $gv(\mathcal{F})$. On the manifold X, the cohomology class of $\widehat{A}(R)$ is exactly the pullback of $\widehat{A}$-class $\widehat{A}(M)$ of the spin manifold M. Thus

$$
\begin{aligned}
(8.10)\quad gv(\widehat{e},\dots,\widehat{e}) &= -(n+1)!(-1)^{n-1}(2\pi i)^{-1}\int_X d'd''\varphi\wedge d'\varphi\wedge\widehat{A}(R)\\
&= (n+1)!(-1)^{n-1}(2\pi i)^{-1}\left(gv(\mathcal{F})\cup\widehat{A}(M)\right)[X].
\end{aligned}
$$

Summarizing the arguments above, we have the main result:

Theorem 8.11. *Let X be a foliated S^1-bundle over a $2n$-dimensional closed spin manifold M, and let D be the longitudinal Dirac operator. Then*

$$\langle gv,\operatorname{ind}(D^+)\rangle = (-1)^{n-1}(n+1)(2\pi i)^{-n-1}\left(gv(\mathcal{F})\cup\widehat{A}(M)\right)[X].$$

Corollary 8.12. *If* $\left(gv(\mathcal{F}) \cup \hat{A}(M)\right)[X] \neq 0$, *then the class* $\Theta = \mathrm{ind}(D^+)$ *is nontrivial in* $K_0[C^*(X, \mathcal{F}, S)]$.

Example 8.13. Let $(T_1\Sigma, \mathcal{F}_A)$ be an Anosov foliation associated with the geodesic flow on the unit circle bundle $T_1\Sigma$ over a closed Riemann surface Σ of genus ≥ 2. Since $\dim \Sigma = 2$,

$$\langle gv, \mathrm{ind}(D^+)\rangle = -2(2\pi)^{-2} gv(\mathcal{F}_A)[T_1\Sigma].$$

It is known [**18**] that $gv(\mathcal{F}_A)[T_1\Sigma] \neq 0$. Therefore, $\Theta = \mathrm{ind}(D^+)$ is nontrivial in $K_0[C^*(T_1\Sigma, \mathcal{F}_A, S)]$. In the next section we will show that Θ together with other known elements generates the whole $K_0[C^*(T_1\Sigma, \mathcal{F}_A, S)]$.

Remark 8.14. In (8.10), the righthand side is always purely imaginary. This is due to the fact that the cyclic $2n$-cocycle gv is *purely imaginary*, i.e.

$$gv(a_{2n}^*, a_{2n-1}^*, \ldots, a_0^*) = -\overline{gv(a_0, a_1, \ldots, a_{2n})}$$

for $a_0, \ldots, a_{2n} \in \mathrm{Dom}(gv)$.

9. A relationship between the cocycle gv and Connes's cocycle.

In this section we will study the relationship between the cyclic cocycle gv and Connes's cocycle [**8**].

Let us recall his construction. Denote by τ_1 the transverse fundamental class for $C(S^1) \rtimes \Gamma$. That is

$$\tau_1(f^0, f^1) = \sum_{g_0 g_1 = 1} \int_{S^1} f^0_{g_0}(x g_0)\, df^1_{g_1}(x),$$

where $f^j = \sum f^j_g U_g \in C_c^\infty(S^1 \times \Gamma)$. Its derivative $\dot{\tau}_1$, defined by

$$\dot{\tau}_1(f^0, f^1) = \lim_{t \to 0} \frac{1}{it}(\tau_1(\sigma_t(f^0), \sigma_t(f^1)) - \tau_1(f^0, f^1)), \tag{9.1}$$

is (σ_t)-invariant. The cocycle which Connes studied is $i_{D\varphi}(\dot{\tau}_1)$. We will see that there exists a homomorphism Π from $C(S^1) \rtimes \Gamma$ into $C^*(X, \mathcal{F}, E)$ such that

$$\Pi^*(gv) = i_{D\varphi}(\dot{\tau}_1)$$

on

$$C_c^\infty(S^1 \times \Gamma) \subset C(S^1) \rtimes \Gamma.$$

There exists a compactly supported C^∞-function σ on $\widetilde{M}$ such that

$$\sum_{g \in \Gamma} g(\sigma) = 1;$$

i.e. $\{g(\sigma)\}_{g\in\Gamma}$ is a Γ-invariant partition of unity for $\widetilde{M}$. We can choose σ so that σ takes the value 1 on some open set U. We may further assume that the fundamental domain $\mathcal{D}$ is contained in $\operatorname{supp}\sigma$. Assume that $\widetilde{E}$ is a Γ-equivariant vector bundle on $\widetilde{M} \times S^1$, which is the pullback of a vector bundle E on M by the composition of two canonical maps

$$\widetilde{M} \times S^1 \to \widetilde{M} \xrightarrow{p} M.$$

The bundle $\widetilde{S}$ of spinors considered in the preceding two sections satisfies this assumption. Choose a section $\xi \in C_c^\infty(p^*E)$ such that $\operatorname{supp}\xi \subset U$, and

$$\int_{\widetilde{M}} \langle \xi, \xi \rangle \, d\mu(m) = 1.$$

In a natural way, ξ can be regarded as a compactly supported section of $\widetilde{E}$. By the choice of ξ, we have that

$$\operatorname{supp}\xi \cap \operatorname{supp} g(\xi) = \varnothing \qquad \text{unless} \quad g = 1. \tag{9.2}$$

Moreover

$$\int_{\widetilde{M}} \langle \xi, \xi \rangle_x \, d\mu_x(m) = 1 \tag{9.3}$$

for all $x \in S^1$. From this follows that

$$\langle \xi, \xi \rangle = 1 \in C(S^1) \rtimes \Gamma,$$

where $\langle \cdot, \cdot \rangle$ is the $C(S^1) \rtimes \Gamma$-valued inner product on ϵ in Section 2.

In general, for a right Hilbert module over a unital C^*-algebra $\mathfrak{A}$, if there exists an $\eta \in \epsilon$ such that $\langle \eta, \eta \rangle_{\mathfrak{A}} = 1$, then the map Π defined by

$$\Pi(a) = \theta_{\eta\cdot a, \eta}, \qquad a \in \mathfrak{A}, \tag{9.4}$$

is a $*$-homomorphism from $\mathfrak{A}$ into $\mathcal{K}(\epsilon)$, which induces an isomorphism of K-groups. Apply this principle to ξ above to obtain a $*$-homomorphism Π from $C(S^1) \rtimes \Gamma$ into $\mathcal{K}(\epsilon) \cong C^*(X, \mathcal{F}, E)$.

Let dx and $\widetilde{\omega}$ be as in Section 5. Let ψ be a real-valued C^∞-function on $\widetilde{M} \times S^1$. It is easy to see that $\omega = \psi\widetilde{\omega} \wedge dx$ is a Γ-invariant volume form on $\widetilde{M} \times S^1$ if and only if ψ is never zero, and $\psi = g(\psi)\lambda_g$ for any $g \in \Gamma$. Set

$$\psi = \sum_{g\in\Gamma} \lambda_g g(\sigma).$$

Since $\{g(\sigma)\}$ is a partition of unity, and $\lambda_g > 0$, the function ψ is always positive. Moreover

$$\begin{aligned}\lambda_g g(\psi) &= \sum_{h\in\Gamma} \lambda_g g(\lambda_h)(gh)(\sigma)\\ &= \sum_{h\in\Gamma} \lambda_{gh}(gh)(\sigma) = \psi.\end{aligned}$$

Thus $\omega = \psi\tilde{\omega} \wedge dx$ is a Γ-invariant volume form. Using the definitions given in Section 5, obtain (Δ^{it}) and $(\widehat{\sigma}_t)$.

Lemma 9.5. *The section ξ (as a section of $\widetilde{E}$ over $\widetilde{M} \times S^1$) has the property that*

$$\Delta^{it}(\xi) = \xi, \qquad t \in \mathbb{R}.$$

Proof. Obvious from the fact that $\psi \equiv 1$ on $\operatorname{supp}\xi$. □

Lemma 9.6. *The $*$-homomorphism Π given by (9.2) is $\mathbb{R}$-equivariant; i.e.*

$$\widehat{\sigma}_t(\Pi(a)) = \Pi(\sigma_t(a)), \quad \textit{for all} \quad a \in C(S^1) \rtimes \Gamma \quad \textit{and} \quad t \in \mathbb{R}.$$

Proof. For each $a \in C(S^1) \rtimes \Gamma$ and $t \in \mathbb{R}$, by Lemma 4.3,

$$\begin{aligned}\widehat{\sigma}_t(\Pi(a)) &= \Delta^{it}\theta_{\xi\cdot a,\xi}\Delta^{-it}\\ &= \theta_{\Delta^{it}(\xi\cdot a),\Delta^{it}(\xi)}\\ &= \theta_{\Delta^{it}(\xi)\cdot\sigma_t(a),\Delta^{it}(\xi)}\\ &= \theta_{\xi\cdot\sigma_t(a),\xi}\\ &= \Pi(\sigma_t(a)).\end{aligned}$$

□

For $a \in C_c^\infty(S^1 \times \Gamma)$, the operator $\Pi(a)$ is a compactly smoothing operator. Therefore $\operatorname{trace}_\Gamma(\Pi(a))$ is well defined.

Proposition 9.7. *For $a^0, a^1 \in C_c^\infty(S^1 \times \Gamma)$, we have*

$$\operatorname{trace}_\Gamma\left(\Pi(a^0)\delta_1(\Pi(a^1))\right) = \dot{\tau}_1(a^0, a^1).$$

Proof. We have, using (9.2) and (9.3), that

$$\begin{aligned}
&\operatorname{trace}_\Gamma\,(\Pi(a^0)\delta_1(\Pi(a^1)))\\
&= \int_{S^1}\int_{\mathcal{D}}\int_{\widetilde{M}} \sum_{g,h,g',h'} a^0(xg, g^{-1}h)((\partial_2\varphi)(n,x) - (\partial_2\varphi)(m,x))\\
&\qquad \times a^1(xg', g'^{-1}h')(\xi(nh,xh), \xi(ng',xg'))\\
&\qquad \times (\xi(mh',xh'), \xi(mg,xg))\, d\mu_x(n)\, d\mu_x(m)\, dx\\
&= \int_{S^1}\int_{\mathcal{D}}\int_{\widetilde{M}} \sum_{g,h} a^0(xg, g^{-1}h)((\partial_2\varphi)(n,x) - (\partial_2\varphi)(m,x))a^1(xg, g^{-1}h)\\
&\qquad \times ||\xi(nh,xh)||^2\, ||\xi(mg,xg)||^2\, d\mu_x(n)\, d\mu_x(m)\, dx\\
&= \int_{S^1}\int_{\mathcal{D}}\int_{\widetilde{M}} \sum_{h} a^0(x,h)a^1(xh,h^{-1})((\partial_2\varphi)(n,x) - (\partial_2\varphi)(m,x))\\
&\qquad \times ||\xi(nh,xh)||^2\, ||\xi(mg,xg)||^2\, d\mu_x(n)\, d\mu_x(m)\, dx.
\end{aligned}$$

Since $\psi \equiv 1$ on $\operatorname{supp}\xi$, we have $(\partial_2\varphi)(m,x) = 0$ if $m \in \mathcal{D}$. Hence

$$\begin{aligned}
&\operatorname{trace}_\Gamma\,(\Pi(a^0)\delta_1(\Pi(a^1)))\\
&= \int_{S^1}\int_{\mathcal{D}}\int_{\widetilde{M}} \sum_{h} a^0(x,h)a^1(xh,h^{-1})(\partial_2\varphi)(n,x)||\xi(nh,xh)||^2\, ||\xi(mg,xg)||^2\\
&\qquad \times d\mu_x(n)\, d\mu_x(m)\, dx\\
&= \int_{S^1}\int_{\widetilde{M}} \sum_{h} a^0(x,h)a^1(xh,h^{-1})(\partial_2\varphi)(n,x)||\xi(nh,xh)||^2\, d\mu_x(n)\, dx.
\end{aligned}$$

If $nh \notin \mathcal{D}$, then $||\xi(nh,xh)||^2 = 0$. By the choice of ψ, if $||\xi(nh,xh)||^2 \neq 0$, then

$$\psi(n,x) = \lambda_{h^{-1}}(x)h(\sigma)(n),$$

and

$$\varphi(n,x) = l(h^{-1})(x) + \log(h(\sigma)(n)).$$

Therefore $(d''\varphi)_{(n,x)} = dl(h^{-1})_x$. Consequently

$$\begin{aligned}
\operatorname{trace}_\Gamma\,(\Pi(a^0)\delta_1(\Pi(a^1))) &= \sum_h \int_{S^1} a^0(x,h)a^1(xh,h^{-1})dl(h^{-1})\\
&= \dot{\tau}_1(a^0,a^1).
\end{aligned}$$

□

Finally we can relate the two cocycles:

Proposition 9.8. *For a^0, a^1, a^2 in $C_c^\infty(S^1 \times \Gamma)$, we have*

$$i_{D_\varphi}(\dot{\tau}_1)(a^0, a^1, a^2) = (\Pi^* gv)(a^0, a^1, a^2).$$

Proof. This is immediate from Lemma 9.6, Proposition 9.7 and [**8**, Lemma 6]. □

Remark 9.9. Suppose that E is the trivial line bundle. Then the formula

$$\int_{S^1}\int_D\int_{\widetilde{M}} k^0(m,n,x) d'' k^1(n,m,x)\, dn\, dm\, dx \tag{9.10}$$

defines the transverse fundamental class on $\mathcal{K}_c$. The cocycle $(k^0, k^1) \to \mathrm{trace}_\Gamma(k^0\delta_1(k^1)) = \mathrm{trace}_\Gamma(k^0[d''\varphi, k^1])$ is the derivative, in the sense of (9.1), of the cocycle (9.10) with respect to the modular automorphism group $(\widehat{\sigma}_t)$.

10. The K_0-groups of the C^*-algebras of Foliated S^1-bundles.

In this section we will determine the generators of the group $K_0[C^*(X, \mathcal{F})]$ for an arbitrary foliated S^1-bundle over a closed Riemann surface.

Let Σ be a closed Riemann surface of genus $g \geq 2$, and let $\Gamma = \pi_1(\Sigma)$. To any (right)action of Γ on the circle S^1 by orientation preserving diffeomorphisms, a fibre bundle with fibre S^1 is associated (Section 2). By evaluating the Euler class of this bundle on the fundamental class of Σ, we get an integer χ, which is called the Euler characteristic.

This group Γ is an amalgamated free product $\Gamma = F_2 *_{\mathbb{Z}} F_{2g-2}$. By [**17**] we have an exact sequence, a part of which looks like

$$K_0(A_1) \oplus K_0(A_2) \to K_0(A) \to K_1(A_0) \to K_1(A_1) \oplus K_1(A_2),$$

where $A_0 = C(S^1) \rtimes \mathbb{Z}$, $A_1 = C(S^1) \rtimes F_2$, $A_2 = C(S^1) \rtimes F_{2g-2}$, and $A = C(S^1) \rtimes \Gamma$. The computations done in [**16**] enable us to obtain

$$K_0[A] \cong \mathbb{Z}^{2g} \oplus \mathbb{Z} \oplus \mathbb{Z}/\chi\mathbb{Z}. \tag{10.1}$$

The subgroup $\mathbb{Z}^{2g}$ in (10.1) is generated by Rieffel projections. It is straightforward to see that those $2g$ generators lie in the kernel of the map $K_0(A) \to \mathbb{C}$ induced by the pairing with the cyclic 2-cocycle $i_{D\varphi}(\dot{\tau}_1)$ described in the preceding section. The torsion subgroup $\mathbb{Z}/\chi\mathbb{Z}$ is generated by the class of the unit. As for the remaining generator, we know only of its existence, by applying an exact sequence to compute the K-groups. We will show that this missing generator is given by the class Θ associated with the Dirac operator.

Recall that the upper half plane $\mathbb{H}_+$ is the universal covering of Σ. The Γ-equivariant Hermitian vector bundles $\widetilde{S} = \widetilde{S}^+ \oplus \widetilde{S}^-$, associated with the Γ-invariant spin structure on $\mathbb{H}_+$, give rise to a Hilbert C^*-module ϵ_1 over $C^*\Gamma$ in the fashion used to create ϵ in Section 2. Let ξ be as in Section 9. Then ξ yields $*$-homomorphisms $\Pi : C(S^1) \rtimes \Gamma \to \mathcal{K}(\epsilon)$ and $\Pi_1 : C^*\Gamma \to \mathcal{K}(\epsilon_1)$, which induce isomorphisms of K-groups.

Proposition 10.2. *There exists a $*$-homomorphism from $\mathcal{K}(\epsilon_1)$ into $\mathcal{K}(\epsilon)$ such that the diagram*

$$\begin{array}{ccc} C^*\Gamma & \xrightarrow{\Pi_1} & \mathcal{K}(\epsilon_1) \\ \downarrow & & \downarrow \\ C(S^1) \rtimes \Gamma & \xrightarrow{\Pi} & \mathcal{K}(\epsilon) \end{array}$$

is commutative, where $C^*\Gamma \to C(S^1) \rtimes \Gamma$ is the canonical inclusion.

Proof. Recall that $\mathcal{K}(\epsilon)$ is generated by operators with Γ-compactly supported, Γ-invariant C^∞-kernels. Let $P \in \mathcal{K}(\epsilon_1)$ have the kernel k. Then the Γ-invariant C^∞-kernel $\widetilde{k}$ defined by

$$\widetilde{k}(m,n,x) = k(m,n), \qquad (m,n,x) \in \mathbb{H}_+ \times \mathbb{H}_+ \times S^1,$$

determines an operator $\widetilde{P} \in \mathcal{K}(\epsilon)$. Using the definition of norm, it is not hard to check that the correspondence $P \to \widetilde{P}$ extends to a $*$-homomorphism $j : \mathcal{K}(\epsilon_1) \to \mathcal{K}(\epsilon)$.

Commutativity of the diagram is also easy. □

The Dirac operator D^+ on Σ lifts to a Γ-equivariant differential operator

$$\widetilde{D}^+ : C_c^\infty(\mathbb{H}_+, \widetilde{S}^+) \to C_c^\infty(\mathbb{H}_+, \widetilde{S}^-).$$

The graph projection $\widetilde{e}^+$ associated with D^+ is a bounded operator on $L^2(\mathbb{H}_+, \widetilde{S}^+ \oplus \widetilde{S}^-)$ and determines a class

$$\Theta_0 = [\widetilde{e}^+] - [p_-] \in K_0[\mathcal{K}(\epsilon_1)].$$

Proposition 10.3. *The class Θ_0 and the class of unit $1 \in C^*\Gamma$ generate*

$$K_0[\mathcal{K}(\epsilon_1)] \cong K_0[C^*\Gamma] \cong \mathbb{Z}^2.$$

Proof. By the fact that the index map from the K-homology of Σ into $K_*[C^*\Gamma]$ is an isomorphism [**2**, Thm. 3], we can see that $K_0[C^*\Gamma]$ is isomorphic to $\mathbb{Z}^2$ and is generated by the class of the unit and the index $\mathrm{ind}_\Gamma(\widetilde{D}^+)$. As in Section 8, it is not hard to see that Θ_0 coincides with $\mathrm{ind}_\Gamma(\widetilde{D}^+)$. □

By the construction of j we can see that

$$j(\widetilde{e}^{+} - p_{-}) = e - p_{-}.$$

From this we see $j_*(\Theta_0) = \Theta$, where $j_* : K_0[\mathcal{K}(\epsilon_1)] \to K_0[\mathcal{K}(\epsilon)]$ is the induced map.

Theorem 10.4. *The class Θ is the missing generator of $K_0[C(S^1) \rtimes \Gamma] \cong K_0[\mathcal{K}(\epsilon_1)]$.*

Proof. We claim that Θ, together with the known generators, spans the K_0-group. Let A_0, A_1, A_2, and A be as above. We have a commutative diagram:

$$\begin{array}{ccccccc}
K_0[A_1] \oplus K_0[A_2] & \longrightarrow & K_0[A] & \xrightarrow{\delta} & K_1[A_0] & \longrightarrow & K_1[A_1] \oplus K_1[A_2] \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
K_0[C^*F_2] \oplus K_0[C^*F_{2g-2}] & \longrightarrow & K_0[C^*\Gamma] & \xrightarrow{\delta} & K_1[C^*\mathbb{Z}] & \longrightarrow & K_1[C^*F_2] \oplus K_1[C^*F_{2g-2}],
\end{array}$$

where horizontal rows are exact, and all the vertical arrows are induced from the canonical inclusions of C^*-algebras.

The map $K_1[C^*\mathbb{Z}] \to K_1[C^*F_2] \oplus K_1[C^*F_{2g-2}]$ is a zero map, and the kernel of $K_1[A_0] \to K_1[A_1] \oplus K_1[A_2]$ is an infinite cyclic group generated by the class of the unitary of $C(S^1) \rtimes \mathbb{Z}$ corresponding to the generator of $\mathbb{Z}$.

Since the class of the unit and the class Θ_0 generate $K_0[C^*\Gamma]$, we see that $\delta(\Theta_0)$ must be the generator of $K_1[C^*\mathbb{Z}]$. From this and the observation above, $\delta(\Theta)$ is the generator of the kernel of $K_1[A_0] \to K_1[A_1] \oplus K_1[A_2]$. Therefore the class Θ and the image of the map $K_0[A_1] \oplus K_0[A_2] \to K_0[A]$ generate $K_0[A]$. □

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NEVANLINNA'S COEFFICIENTS AND DOUGLAS ALGEBRAS

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Some relations between Douglas algebras and coefficients appearing in Nevanlinna's matrix parametrization of the solutions of the Nevanlinna Pick interpolation problem are studied.

1. Introduction.

Let U denote the analytic functions bounded by one in $\mathbb{D} = \{z : |z| < 1\}$. Given a sequence $\{z_n\} \subset \mathbb{D}$, we consider the classical Nevanlinna Pick interpolation problem

$$\text{(NP)} \qquad f(z_n) = w_n, \qquad n = 1, 2, \dots, \quad f \in U.$$

If this problem has more than one solution, R. Nevanlinna [**4**] found analytic functions P, Q, R and S such that the set of all solutions is given by

$$E = \left\{ \frac{P - Qw}{R - Sw}, \quad w \in U \right\}. \tag{1.1}$$

The functions P, Q, R and S are unique subject to the normalization $S(0) = 0$ and $PS - RQ = \pi$, where

$$\pi(z) = \prod_n \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z}_n z}$$

is the Blaschke product corresponding to $\{z_n\}$.

While the funcions P, Q, R and S arose from classical function theory, it turns out that they are also connected with more recent developments. It is part of Nevanlinna's theory that the functions $P/R, Q/R, S/R$ and $1/R$ belong to U and are linked with π in many ways. (See Lemma 1.)

Suppose (NP) has a solution f_0 satisfying $\sup\{|f_0(z)|, \ z \in D\} < 1$. Our main result is that then $P/R, Q/R, S/R$ and $1/R$ all belong to a certain subalgebra of H^∞ depending only on π which we shall denote by CDA_π. This algebra is part of the theory of Douglas algebras through the work of S.Y. Chang and D.E. Marshall ([**1**], [**2?**]). Our results in particular answer a problem raised by V. Tolokonnikov in [**11**] where other relations between Douglas algebras and the Nevanlinna Pick problem are studied.

Our methods are based on Nevanlinna's ideas in [**4**] and last but not least on the more recent treatment of the Nevanlinna Pick problem given by J. Garnett in [**2**], where dual extremal methods are used. We also give a new proof of a recent result of Tolokonnikov concerning questions whether (NP) has a unique solution.

Next we introduce some notations and well known results.

Let m denote normalized Lebesgue measure on the unit circle $\mathbb{T} = \{z : |z| = 1\}$. If $1 \leq p \leq \infty$, H^p denote the Hardy space consisting of all $f \in L^p(m)$ whose harmonic extension to D is analytic there. If $p = \infty$, the norm $||f||_p$ in $L^p(m)$ can also be given by

$$||f||_\infty = \sup\{|f(z)| : z \in D\} \qquad f \in H^\infty.$$

For basic properties of H^p, we refer to Garnett's book [**2**].

We recall that $I \in H^\infty$ is called an inner function if $|I(e^{i\alpha})| = 1$ almost everywhere with respect to m. Any Blaschke product is inner, but there are many others ([**2**, p. 75]).

Considering H^∞ as a subalgebra of $L^\infty(m)$, let $D_\pi = [H^\infty, \overline{\pi}]$ be the Douglas algebra generated by H^∞ and the restriction $\overline{\pi}|_\mathbb{T}$ of $\overline{\pi}$ to $\mathbb{T}$. Then let $QD_\pi = D_\pi \cap \overline{D_\pi}$ be the maximal C^*-subalgebra of D_π. Define also $QDA_\pi = QD_\pi \cap H^\infty$ and let CDA_π denote the subalgebra of H^∞ generated by all inner functions I invertible in D_π. It is evident that $CDA_\pi \subset QDA_\pi$. For more about these algebras, see [**1**], and [**2**] for example. Let I be an inner function. The property of I being invertible in D_π has a very concrete formulation: If $\{\zeta_n\} \subset D$ and $|\pi(\zeta_n)| \to 1$, then $|I(\zeta_n)| \to 1$.

The special solutions I_α to (NP) given by

$$I_\alpha = \frac{P - Qe^{i\alpha}}{R - Se^{i\alpha}}$$

play an important role in this theory. Nevanlinna showed that each I_α is inner [**4**], and in fact almost all I_α are Blaschke products [**9**]. A Nevanlinna Pick problem is called scaled if it has a solution f_0 satisfying $||f_0||_\infty < 1$.

For general properties of Douglas algebras and more on the Nevanlinna Pick problem, Garnett's book [**2**] is a good reference.

The letter C_i will be used for different absolute constants, while $C(t)$ indicates a constant depending on the parameter t.

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2. Main result.

If (NP) has more than one solution, R. Nevanlinna considered the "Wertevorrat" $\Delta(z) = \{f(z) : f \text{ is a solution of (NP)}\}, z \in \mathbb{D}$. Using (1.1), one can easily check that $\Delta(z)$ is a disc of center $c(z) = (-Q(z)\overline{S(z)} + P(z)\overline{R(z)})(|R(z)|^2 - |S(z)|^2)^{-1}$, and radius $\rho(z) = |\pi(z)|(|R(z)|^2 - |S(z)|^2)^{-1}$.

For later use, we collect some of the properties of Nevanlinna's coefficients.

Lemma 1. *Assume* (NP) *has more than one solution and consider the Nevanlinna's coefficients P, Q, R, S appearing in* (1.1). *Then*

(i) *P, Q, R, S have radial limit almost everywhere and $Q = -\pi\overline{R}$, $P = -\pi\overline{S}$, $|R|^2 - |S|^2 = 1$, $Q\overline{S} - P\overline{R} = 0$, almost everywhere on the unit circle.*

(ii) $|R(z)|^2 - |S(z)|^2 \geq 1$, $|R(z)|^2 - |P(z)|^2 \geq 1$, $\quad z \in \mathbb{D}$.

(iii) *For any $e^{i\alpha} \in \partial\mathbb{D}$, $(R - Se^{i\alpha})^{-2}$ is an exposed point of H^1.*

(iv) *If $u \in U$ and $f = (P - Qu)(R - Su)^{-1}$, one has*

$$||f||_\infty = \left\| \frac{\overline{S}/\overline{R} - u}{1 - uS/R} \right\|_{L^\infty(\partial\mathbb{D})}.$$

(v) *If* (NP) *is scaled, one has $\rho(z) \to 1$ as $|\pi(z)| \to 1$.*

(vi) *If* (NP) *is scaled and $\gamma = \inf\{||f_0||_\infty : f \text{ is a solution of (NP)}\}$, then $R \in H^p$ for all $p < \pi(\arcsin(\gamma))^{-1}$.*

Proof. (i), (ii), (iii) are well known (see [**8**] and the references there given to [**2**]). Using the relations in (i)

$$\left| \frac{P - Qu}{R - Su}(e^{i\theta}) \right| = \left| \frac{Q}{R}(e^{i\theta}) \right| \left| \frac{P/Q - u}{1 - uS/R}(e^{i\theta}) \right|$$
$$= \left| \frac{\overline{S}/\overline{R} - u}{1 - uS/R}(e^{i\theta}) \right|, \quad \text{a.e. } e^{i\theta} \in \partial\mathbb{D},$$

and this is (iv). A proof of (v) can be found in [**10**]. Now, let us prove (vi). Consider $I_\alpha = (P - Qe^{i\alpha})(R - Se^{i\alpha})^{-1}$, for fixed α, $0 \leq \alpha < 2\pi$. Using (i). one can easily check

$$I_\alpha \overline{\pi} = e^{i\alpha} \frac{(R - Se^{i\alpha})^{-2}}{|(R - Se^{i\alpha})^{-2}|}, \qquad \text{a.e. on } \partial\mathbb{D}.$$

Since $\gamma = \operatorname{dist}(I_\alpha \overline{\pi}, H^\infty) < 1$, there exists $g \in H^\infty$ satisfying

$$1 > \gamma = \left\| \frac{(R - Se^{i\alpha})^{-2}}{|(R - Se^{i\alpha})^{-2}|} - g \right\|_\infty .$$

Since $I_\alpha(0) \in \partial\Delta(0)$, one has $\operatorname{dist}(I_\alpha \overline{\pi}, H_0^\infty) = 1$, where $H_0^\infty = \{f \in H^\infty : f(0) = 0\}$. The proof of Lemma 4.3 in ([**2**, p. 386]) shows $|g(z)| \geq 1 - \gamma$, $z \in \mathbb{D}$. Let $\operatorname{Arg}(z)$ be the principal branch of the argument. One has,

$$\left|\operatorname{Arg}\left(g^{-1}(R - Se^{i\alpha})^{-2}\right)\right| \leq \arcsin(\gamma), \quad \text{a.e. on } \partial\mathbb{D}.$$

So, the same is true on $\mathbb{D}$ and using a result in ([**2**, p. 114]), one gets

$$\left(g^{-1}(R - Se^{i\alpha})^{-2}\right)^{-1} \in H^p, \quad p < \frac{\pi}{2\arcsin(\gamma)}.$$

Hence $(R - Se^{i\alpha})^2 \in H^p$, for $p < \pi(2\arcsin(\gamma))^{-1}$ and it follows $R \in H^p$, for $p < \pi(\arcsin(\gamma))^{-1}$. This finishes the proof of Lemma 1. □

Let (NP) be an scaled Nevanlinna problem, V. Tolokonnikov proved that the extremal solutions I_α are invertible in D_π [**11**]. Our next result is an extension of this.

Proposition . *Let* (NP) *be a scaled Nevanlinna Pick problem and* I_α *one of its extremal solutions,* $0 \leq \alpha < 2\pi$. *Then* $D_{I_\alpha} = D_\pi$.

Proof. As mentioned before, it is known that I_α is invertible in D_π. We present another proof of it. From (v) of Lemma 1, $\rho(z) \to 1$ whenever $|\pi(z)| \to 1$. Since $I_\alpha(z) \in \partial\Delta(z)$, one gets $|I_\alpha(z)| \to 1$. Hence, I_α is invertible in D_π and $D_{I_\alpha} \subset D_\pi$.

For the converse assume

$$|I_\alpha(z_n)| \to 1.$$

Since the Nevanlinna Pick problem (NP) is scaled, the "Wertevorrat" $\Delta(z_n)$ must meet a fixed disc inside the unit disc. Actually, $f_0(z_n), I_\alpha(z_n) \in \Delta(z_n)$, where f_0 is a solution to (NP) with $||f_0||_\infty < 1$. Hence, for large n,

$$|\pi(z_n)| \geq \rho(z_n) \geq \frac{1}{4}(1 - ||f_0||_\infty) > 0$$

and one deduces that π is invertible in D_{I_α}.

The Proposition can also be immediately deduced from the proof of Theorem 2.1 in [**1**]. □

Remark. The hypothesis on the scaling of the Nevanlinna Pick problem is essential. In fact, there exist non scaled Nevanlinna Pick problems and points $\beta_n \in \mathbb{D}$ such that

$$\sup\{|w| : w \in \Delta(\beta_n)\} \underset{n\to\infty}{\longrightarrow} 0, \qquad |\pi(\beta_n)| \underset{n\to\infty}{\longrightarrow} 1$$

see [**5**]. Then, $I_\alpha(\beta_n) \to 0$, $0 \le \alpha < 2\pi$, and no I_α is invertible in D_π.

The following result is known although we have not found it in the literature. We thank the referee for pointing out it to us.

Lemma 2. *Given* $u, |u| = 1$ *and* $z, |z| \le 1$, *one has that*

$$z = \int_0^{2\pi} \frac{z - ue^{i\alpha}}{1 - \overline{z}ue^{i\alpha}} \frac{d\alpha}{2\pi}$$

can be uniformly approximated by its Riemann sums.

Proof. Multiplying by $\overline{u}$ if necessary, one may assume $u = 1$. For $w = e^{2\pi i n^{-1}}$, one has

$$z - \frac{1}{n}\sum_{k=1}^{n} \frac{z - w^k}{1 - w^k\overline{z}} = \overline{z}^{n-1}\frac{1 - |z|^2}{1 - \overline{z}^n}, \qquad |z| < 1.$$

This can be shown expanding in a series and using

$$\sum_{k=1}^{n} w^{pk} = 0$$

unless $p \equiv 0 \bmod n$. By continuity the same holds if $\overline{z}^n \ne 1$. Now, the inequalities

$$\begin{aligned} \left| z - \frac{1}{n}\sum_{k=1}^{n} \frac{z - w^k}{1 - w^k\overline{z}} \right| &\le \frac{|z|^{n-1}(1 + |z|)(1 - |z|)}{1 - |z|^n} \\ &= \frac{1 + |z|}{1 + |z|^{-1} + \cdots + |z|^{-(n-1)}} \le \frac{2}{n} \end{aligned}$$

finish the proof. □

Assume (NP) is scaled. In [**11**] it is proved that the functions P/R, $\pi R^{-2}(S/R)^k$, $k \ge 0$, belong to CDA_π and it is asked if $R^{-1} \in CDA_\pi$. Next, we complete these results.

Theorem 1. *Let* (NP) *be a scaled Nevanlinna Pick problem, E the set of its solutions and*

$$E = \left\{ \frac{P - Qw}{R - Sw} : w \in U \right\}$$

its Nevanlinna's parametrization. Let D_π be the Douglas algebra generated by H^∞ and $\overline{\pi}|_{\mathbb{T}}$. Then, the functions $P/R, Q/R, S/R, 1/R$ belong to the algebra CDA_π.

Proof. Since $|S/R(e^{i\theta})| \leq 1$, Lemma 2 shows

$$\frac{1}{2\pi}\int_0^{2\pi} I_\alpha(e^{i\theta})\,d\alpha = P/R(e^{i\theta}), \quad \text{a.e. } e^{i\theta} \in \mathbb{T},$$

and the integral can be uniformly approximated by its Riemann sums. Since I_α are inner functions invertible in D_π, one gets $P/R \in CDA_\pi$.

Since Q/R is an inner function, one only has to show that Q/R is invertible in D_π. If $|\pi(z)| \to 1$, by (v) of Lemma 1, the disc $\Delta(z)$ tends to the unit disc, that is to say,

$$\rho(z) = \frac{|Q/R(z) - P/R(z)S/R(z)|}{1 - |S/R(z)|^2} \to 1$$
$$c(z) = \frac{P/R(z) - Q/R(z)\overline{S/R(z)}}{1 - |S/R(z)|^2} \to 0.$$

Hence,

$$\begin{aligned} 0 \leftarrow & \frac{P/R(z)S/R(z) - Q/R(z)}{1 - |S/R(z)|^2} + Q/R(z) \\ = & \frac{P/R(z)S/R(z) - Q/R(z)|S/R(z)|^2}{1 - |S/R(z)|^2} \end{aligned}$$

and one gets $|Q/R(z)| \to 1$. Therefore $Q/R \in CDA_\pi$.

Since by (i) of Lemma 1 $Q\overline{S} = P\overline{R}$ a.e. on the unit circle, one has $S/R = \overline{(P/R)}(Q/R) \in CD_\pi$ and since it is analytic, $S/R \in CDA_\pi$.

Using $R = \overline{Q}\pi$ a.e. on the unit circle, one gets $\overline{(1/R)}Q/R = \pi/R \in H^\infty$. Then, for $0 < \delta < 1$,

$$\delta\frac{1}{R} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\frac{Q}{R}e^{i\alpha} + \frac{1}{R}\delta}{1 + e^{i\alpha}\overline{(\delta/R)}Q/R}\,d\alpha$$

uniformly on the unit circle. Since Q/R is an inner function invertible in D_π, so is

$$\frac{Q/Re^{i\alpha} + \delta/R}{1 + e^{i\alpha}\overline{(\delta/R)}Q/R}, \qquad e^{i\alpha} \in \partial\mathbb{D},$$

and one gets $R^{-1} \in CDA_\pi$. □

3. An example.

The results of last section may suggest that if one takes $w \in CDA_\pi$, $w \in U$ in Nevanlinna's formula, the resulting function $f = (P - Qw)(R - Sw)^{-1}$ may also belong to CDA_π. This is of course the case if $||w||_\infty < 1$, because of the relation

$$f = (P/R - wQ/R) \sum_{n=0}^{\infty} (wS/R)^n.$$

It has been surprising to us that for general $w \in U \cap CDA_\pi$, the function f may not belong to CDA_π. In fact, f may not belong to the bigger algebra QA_π, which consists of the holomorphic functions in the unit disc which belong to $D_\pi \cap \overline{D_\pi}$. To show this, we need to construct a scaled Nevanlinna Pick problem such that the corresponding function R is not bounded. We will do the construction in the upper half plane.

Consider $z_n = iy_n$, where $y_{n+1} < cy_n$, for some fixed $0 < c < 1$ and $z_n^* = x_n + iy_n$, where $x_n > 0$ is a decreasing sequence, $\sup x_n y_n^{-1}$ is a small number to be chosen later, $x_n y_n^{-1} \to 0$ as $n \to \infty$, but

$$\sum_n (x_n y_n^{-1})^2 = +\infty. \tag{3.1}$$

Let B and B^* be the Blaschke products in the upper half plane with zeros $\{z_n\}$ and $\{z_n^*\}$ and B_1, B_1^* the Blaschke products with zeros $\{\varphi(z_n)\}$, $\{\varphi(z_n^*)\}$, where φ is a conformal map from the upper half plane to the unit disc.

Lemma 3. *With the notations above, the Nevanlinna Pick problem*

$$(*) \qquad f(\varphi(z_n)) = B_1^*(\varphi(z_n)), \qquad n = 1, 2, \ldots, \quad f \in U$$

is scaled. Moreover, if

$$\{f \in H^\infty : f \ \ solves \ \ (*)\} = \left\{ \frac{P - Qw}{R - Sw} : w \in U \right\}$$

is Nevanlinna's parametrization of the set of its solutions, one has

$$\lim_{\theta \to 0} |R(e^{i\theta})| = +\infty.$$

Proof. We will prove the Lemma in the upper half plane. Let $x \in \mathbb{R}$, as in ([**2**, p. 432]), one can compute

(3.2)
$$\operatorname{Arg} \frac{B^*(x)}{B(x)} = \sum_n \operatorname{Arg} \left(\frac{x - z_n}{x - \overline{z_n}} \right) - \operatorname{Arg} \left(\frac{x - z_n^*}{x - \overline{z_n^*}} \right) = 2 \int_0^{x_n} \frac{y_n}{(x - t)^2 + y_n^2} \, dt.$$

Now, if $F \in H^1$, one has

$$\left|\int_{\mathbb{R}} F(x) \operatorname{Arg} \frac{B^*(x)}{B(x)}\, dx\right| = 2\left|\sum_n \int_0^{x_n} \int_{\mathbb{R}} \frac{y_n}{(x-t)^2 + y_n^2} F(x)\, dx\, dt\right|$$
$$\leq 2 \sum_n \int_0^{x_n} |F(t + iy_n)|\, dt \leq 2K \sup(x_n y_n^{-1}) ||F||_1$$

because the linear measure σ on $\bigcup_n [iy_n, x_n + iy_n]$ is a Carleson measure, with $\sigma(Q) \leq \sup_n (x_n y_n^{-1}) l(Q)$ where Q is a square lying on the real line and $l(Q)$ is the length of its side. So, given $\varepsilon > 0$, if $\sup_n x_n y_n^{-1}$ is sufficiently small, one gets $|| \operatorname{Arg}(B^*/B)||_{BMO} < \varepsilon$, and hence

(3.3) $$\operatorname{Arg}(B^*/B) = u + \tilde{v}, \quad ||u||_\infty \leq C\varepsilon, \quad ||v||_\infty \leq C\varepsilon,$$

where $\tilde{v}$ is the conjugate function of v and C is an absolute constant ([**2**, p. 248]).

Now,

$$||B^*/B - e^{v + i\tilde{v}}||_\infty \leq 2C\varepsilon,$$

hence

(3.4) $$\operatorname{dist}(B^*/B, H^\infty) \leq 2C\varepsilon < 1$$

and $(*)$ is scaled.

On other hand,

$$||B/B^* - e^{-v - i\tilde{v}}||_\infty \leq 2C\varepsilon,$$

so

(3.5) $$\operatorname{dist}(B/B^*, H^\infty) \leq 2C\varepsilon < 1.$$

Now, (3.4) and (3.5) give that B^* is an extremal solution of $(*)$, that is to say, there exists $0 \leq \alpha < 2\pi$,

$$B^* = \frac{P - Qe^{i\alpha}}{R - Se^{i\alpha}}.$$

Thus, applying (3.3) and (i) of Lemma 1,

$$\exp(i(u + \tilde{v})) = B^*/B = \frac{(R - Se^{i\alpha})^{-2}}{|(R - Se^{i\alpha})^{-2}|}.$$

Consider $H = \exp(iu - \tilde{u} + v + i\tilde{v}) \in H^1$ and hence

$$\frac{H}{|H|} = \frac{(R - Se^{i\alpha})^{-2}}{|(R - Se^{i\alpha})^{-2}|}.$$

By (iii) of Lemma 1 $(R - Se^{i\alpha})^{-2}$ is an exposed point of H^1, so

$$H = M(R - Se^{i\alpha})^{-2}, \qquad M \in \mathbb{C},$$

and $|M(R - Se^{i\alpha})^{-2}(x)| = \exp(v(x) - \tilde{u}(x))$. Now, by (3.3),

$$\begin{aligned} v(x) - \tilde{u}(x) = -\widetilde{\operatorname{Arg}}(B^*/B)(x) &= \frac{-2}{\pi}\sum_n \int_0^{x_n} \frac{x-t}{(x-t)^2 + y_n^2}\,dt \\ &= \frac{1}{\pi}\sum_n \ln\left(\frac{(x-x_n)^2 + y_n^2}{x^2 + y_n^2}\right). \end{aligned}$$

Now, let $x > 0$. Using the inequality $\ln(t^{-1}) \le c(\delta)(1-t)$ for $\delta \le t \le 1$, one gets

$$\begin{aligned} \left|\sum_{x_n : |x_n - x| < x} \ln\left(\frac{(x-x_n)^2 + y_n^2}{x^2 + y_n^2}\right)\right| &= \sum_{x_n : |x_n - x| < x} \ln\left(\frac{(x-x_n)^2 + y_n^2}{x^2 + y_n^2}\right)^{-1} \\ &\le C \sum_{x_n : |x_n - x| < x} \frac{2x_n x - x_n^2}{x^2 + y_n^2} \\ &\le \frac{2C}{x} \sum_{x_n : |x_n - x| < x} x_n \le C_1. \end{aligned}$$

On the other hand, considering k with $x_k > 2x > x_{k+1}$ one has

$$\begin{aligned} \sum_{x_n : |x_n - x| \ge x} \ln\left(\frac{(x-x_n)^2 + y_n^2}{x^2 + y_n^2}\right) &= \sum_{n=1}^{k} \ln\left(\frac{(x-x_n)^2 + y_n^2}{x^2 + y_n^2}\right) \\ &\ge C\sum_{n=1}^{k} \frac{x_n(x_n - 2x)}{x^2 + y_n^2} \ge C_2 \sum_{n=1}^{k-1} x_n^2 y_n^{-2}. \end{aligned}$$

Also, if $x < 0$, $v(x) - \tilde{u}(x) \ge -C_3 + v(-x) - \tilde{u}(-x)$. So, (3.1) gives

$$\lim_{x \to 0} \left|(R - Se^{i\alpha})^{-2}(x)\right| = +\infty,$$

and thus $\lim_{x\to 0} |S/R(x)| = 1$. So, by (i) of Lemma 1, $\lim_{x \to 0} |R(x)| = +\infty$ and this finishes the proof of Lemma 3. $\square$

Now, consider the Nevanlinna Pick problem $(*)$ given by Lemma 3 and

$$\gamma = \inf\{||f||_\infty : f \text{ is solution of } (*)\}.$$

For $1 > t > \gamma$, Proposition of last section gives that there exists an inner function J, $tJ = (P - Qw_0)(R - Sw_0)^{-1} \in CDA_\pi$. Using Theorem 1 one can see that $w_0 \in CDA_\pi$. Now consider an interpolating sequence $\{\alpha_n\}$ approaching to 1, with $|\pi(\alpha_n)| \to 1$ as $n \to \infty$, where $\pi = B_1$, and let I be the Blaschke product with zeros $\{\alpha_n\}$. Then, by Lemma 3, $R^{-2}I$ is continuous up to the circle. Also (iv) of Lemma 1 gives

$$\left\| \frac{\overline{S}/\overline{R} - w_0}{1 - w_0 S/R} \right\|_{L^\infty(\partial\mathbb{D})} = t \tag{3.6}$$

and then $|w_0(e^{i\theta})| \le |S/R(e^{i\theta})| + c(1 - |S/R(e^{i\theta})|)$, $0 \le \theta < 2\pi$, for some fixed $c = c(t) < 1$. Therefore $w_1 = w_0 + (1-c)R^{-2}I \in U \cap CDA_\pi$.

Now, assume $f = (P - Qw_1)(R - Sw_1)^{-1} \in QA_\pi$. Thus,

$$f - tJ = \pi(w_1 - w_0)(R - Sw_0)^{-1}(R - Sw_1)^{-1} \in QA_\pi.$$

Let σ denote the pseudohyperbolic metric, $\sigma(z,w) = |z-w|\,|1-\overline{w}z|^{-1}$. Since $|\pi(\alpha_n)| \to 1$ as $n \to \infty$, writing $g = (w_1 - w_0)(R - Sw_0)^{-1}(R - Sw_1)^{-1}$, from the fact that $\pi g \in QA_\pi$ one can deduce

$$\max_{\sigma(z,\alpha_n)\le r} |g(z)| - \min_{\sigma(z,\alpha_n)\le r} |g(z)| \to 0, \qquad \text{as } n \to \infty,$$

for any $r < 1$, because otherwise, taking a subsequence of $\{\alpha_n\}$, for some fixed $r < 1$, there would exist $\delta > 0$ and $z_n, \sigma(\alpha_n, z_n) \le r$, such that

$$(1 - |z_n|)\,|g'(z_n)| \ge \delta.$$

Then, by subharmonicity, for $m < 1$, it would follow

$$\int_{D_n} |g'(w)|^2\, dm(w) \ge C_1(m)\delta$$

where D_n is the disc of center z_n and radius $m(1 - |z_n|)$. So,

$$\int_{D_n} |g'(w)|^2 (1 - |w|)\, dm(w) \ge C_2(m)\delta(1 - |z_n|)$$

and using a result in [**2**, p. 381], this would contradict the fact $\pi g \in QA_\pi$.

Since $g(\alpha_n) = 0$, one gets

$$\max_{\sigma(z,\alpha_n)\le r} |g(z)| \to 0, \qquad \text{as } n \to \infty. \tag{3.7}$$

But, (3.6) and (v) of Lemma 1 give

$$|1 - S/R(z)w_i(z)| \le C_1(t)\,(1 - |S/R(z)|^2) \le C_1(t)|R(z)|^{-2}, \quad i = 0, 1, \; z \in \mathbb{D},$$

so,

$$\max_{\sigma(z,\alpha_n)\le r} |g(z)| \ge \frac{1-c}{1-C_1} \max_{\sigma(z,\alpha_n)\le r} |I(z)|.$$

Since $\{\alpha_n\}$ is an interpolating sequence, this contradicts (3.7). Therefore $f \notin QA_\pi$.

4. A question about uniqueness.

The question whether (NP) has a unique solution is in general delicate. A necessary condition for uniqueness is of course that $||f||_\infty = 1$ for any solution f to (NP). If there is $f_0 \in H^\infty$ with $||f_0||_\infty < 1$ solving the reduced problem $f(z_n) = w_n$, $n \geq N$ for some $N \geq 2$, we shall call (NP) semiscaled. In [**11**], Tolokonnikov obtained the following nice result

Theorem 2. (Tolokonnikov). *If a Nevanlinna Pick problem is semiscaled, but not scaled, then any solution is inner and hence must be unique.*

It should be observed that previous results due to T. Nakazi [**3**] an K.O. Oyma [**7**] easily follow from Theorem 2.

Proof. Let us use the notation from the introduction and assume that the Nevanlinna Pick problem (NP) is scaled. One can assume $N = 1$. If $\{z_0, w_0\}$ is an extra pair of points consider the extended problem

$$(*) \qquad f(z_n) = w_n, \qquad n = 0, 1, 2, \dots, \qquad f \in U.$$

One can assume $z_0 = 0$. The sets $F = \{f \in H^\infty : \ ||f||_\infty \leq 1, \ f(z_n) = w_n, \ n \geq 1\}$ and $B = \{f(0) : \ f \in F, \ ||f||_\infty < 1\}$ are convex. Suppose B is non-empty and that the only functions in F with $f(0) = w_0$ have norm 1. We will show that such f are inner. Since the average of two inner functions is not inner, this will also prove uniqueness.

If $||f||_\infty \leq 1$, $||g||_\infty < 1$ and $0 < \epsilon < 1$, then $||\epsilon g + (1-\epsilon) f||_\infty < 1$, and hence $\overline{B} = \{f(0) : \ f \in F\}$. The assumptions mean that $w_0 \in \overline{B} \setminus B$. The proof in [**2**, p. 152] works verbatim, and shows that any $f \in F$ with $f(z_0) \in \partial B$ must be inner. □

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SOBOLEV SPACES ON LIPSCHITZ CURVES

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We study Sobolev spaces on Lipschitz graphs Γ, by means of a square function of a geometric second difference. Given a function in the Sobolev space $W^{1,p}(\Gamma)$ we show that the geometric square function is also in $L^p(\Gamma)$. For $p = 2$ we prove a dyadic analogue of this result, and a partial converse.

1. Introduction.

The Sobolev space on the real line, $W^{1,p}(\mathbf{R})$, is the set of functions in $L^p(\mathbf{R})$ whose distributional derivatives are also functions in $L^p(\mathbf{R})$.

There are several characterizations of these spaces. In the early 80's Dorronsoro (see [**Do**]) gave a mean oscillation characterization of potential spaces, extending earlier results due to R.S. Stritchartz. In the late 80's, Semmes showed that the Sobolev spaces $W^{1,2}(M)$ have many of the properties of $W^{1,2}(\mathbf{R}^n)$ when M is a chord-arc surface (see [**Se**]). Dorronsoro and Semmes used square functions closely related to the square functions we use.

There is a characterization, due to E. Stein (see [**St1**] Ch.V) that involves the second differences of the given function. More precisely, let

$$\Delta_t f(x) = f(x+t) + f(x-t) - 2f(x),$$

and define the square function

$$Sf(x) = \left(\int_0^\infty |\Delta_t f(x)|^2 \frac{dt}{t^3}\right)^{1/2}.$$

Then the following result is true (see [**St1**]):

Theorem A [Stein]. *For $1 < p < \infty$, $f \in W^{1,p}(\boldsymbol{R})$ if and only if f, $Sf \in L^p(\boldsymbol{R})$. Moreover $\|Sf\|_p \sim \|f'\|_p$.*

For $p = 2$ the proof of this theorem is just an application of Plancherel's theorem. In this case $\|Sf\|_2 = \|f'\|_2$.

It is important for applications (eg. boundary problems for PDE's) to obtain similar results when $\mathbf{R}$ is replaced by a curve Γ. Smooth curves can be treated reducing to the case $\Gamma = \mathbf{R}$ after a suitable change of variables.

Difficulties appear when the curve is merely Lipschitz, as it often happens in harmonic analysis (eg. boundedness of the Cauchy integral on Lipschitz curves, see [**Ch**], [**M**], [**CJS**]).

Let Γ be a Lipschitz graph:

$$\Gamma = \{z = x + iA(x) : \ \|A'\|_\infty < \infty\}.$$

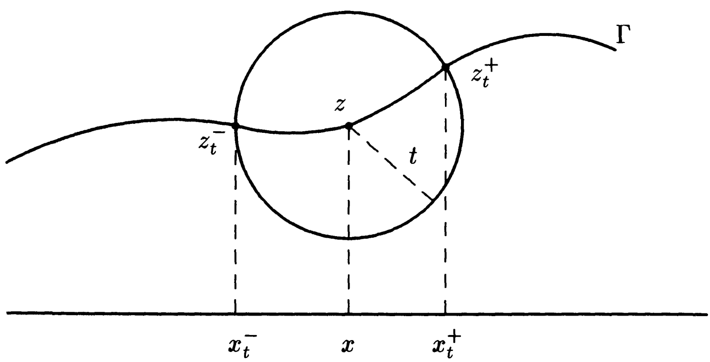
We define the Sobolev space on the curve just pulling back to the line,

$$W^{1,p}(\Gamma) = \{f \in L^p(\Gamma) : f(\tilde{A}) \in W^{1,p}(\mathbf{R}), \quad \tilde{A}(x) = x + iA(x)\}. \tag{1}$$

We introduce a *geometric second difference*, to do it we must restrict our attention to Lipschitz graphs with Lipschitz constant less than one. From now on Γ is always a Lipschitz graph, with $\|A'\|_\infty < 1$. For any $z \in \Gamma$, let

$$\tilde{\Delta}_t f(z) := f(z_t^+) + f(z_t^-) - 2f(z), \tag{2}$$

where $z_t^\pm$ are the *unique* points on Γ at distance t from z. It is clear that one point lies on the right and the other on the left of z, denoted respectively z_t^+ and z_t^-. Let us denote the corresponding x-coordinates x, $x_t^\pm$, see figure below,



We define the *geometric square function*, $\tilde{S}f$, by analogy with Stein's square function Sf; just replacing the second difference by the geometric one,

$$\tilde{S}f(z) = \left(\int_0^\infty |\tilde{\Delta}_t f(z)|^2 \frac{dt}{t^3}\right)^{1/2}, \qquad z \in \Gamma.$$

We can prove the following result,

Theorem 1. *Let Γ be a Lipschitz graph with Lipschitz constant less than one. Assume $f \in W^{1,p}(\Gamma)$ then $\tilde{S}f \in L^p(\Gamma)$ for $1 < p < \infty$. Moreover*

$$\|\tilde{S}f\|_{L^p(\Gamma)} \leq C\|f'\|_{L^p(\Gamma)}.$$

We can prove dyadic analogues of Theorem 1, and a partial converse. We assume the reader is familiar with the dyadic intervals on the line, and with the Haar basis (see definitions in Section 3).

Let us consider the case $\Gamma = \mathbf{R}$.

Denote by $\mathcal{D}$ the collection of dyadic intervals on the line. Let χ_I denote the characteristic function of the interval I.

Define the *dyadic square function* by:

$$S_d f(x) = \left(\sum_{I \in \mathcal{D}} \frac{|\Delta_I f|^2}{|I|^2} \chi_I(x) \right)^{1/2}, \tag{3}$$

where $\Delta_I f$ denotes the second difference of f associated to the interval $I = [x_I^-, x_I^+]$ centered at x_I, namely:

$$\Delta_I f = f(x_I^+) + f(x_I^-) - 2f(x_I).$$

The square function S_d is a dyadic analogue of the square function defined in the begining of the paper.

In this case, the analogue of Theorem 1 is very simple. The main observation being that the second difference $\Delta_I f$ of an absolutely continuous function f is, up to a scaling factor, the Haar coefficient of the derivative f' corresponding to the interval I. More precisely:

$$\Delta_I f = \langle f', h_I \rangle |I|^{1/2},$$

where the Haar function h_I is the step function supported on I that takes the values $\pm 1/|I|^{1/2}$ on the right and left halves of I, respectively.

The Haar functions indexed on $\mathcal{D}$ form a basis of $L^2(\mathbf{R})$. Hence if $f \in W^{1,2}(\mathbf{R})$, an application of Plancherel's Theorem for orthonormal systems implies:

$$\|f'\|_2^2 = \sum_{I \in \mathcal{D}} |\langle f', h_I \rangle|^2 = \sum_{I \in \mathcal{D}} \frac{|\Delta_I f|^2}{|I|}.$$

The right hand side coincides with the L^2 norm of the dyadic square function, hence:

$$f \in W^{1,2}(\mathbf{R}) \Rightarrow \|S_d f\|_2 = \|f'\|_2.$$

We also get a partial converse.

Define the *dyadic derivative*, Df, of $f \in L^2(\mathbf{R})$, as the L^2 limit (when it exists) of the sequence:

$$D_n f(x) = \frac{f(x_I^+) - f(x_I^-)}{|I|}, \qquad x \in I \in \mathcal{D}_n;$$

where $I = [x_I^-, x_I^+]$, and $\mathcal{D}_n$ denotes the nth-generation of dyadic intervals.

In this case if f and $S_d f$ are in $L^2(\mathbf{R})$, then the limit exists, so Df is in $L^2(\mathbf{R})$. Moreover, $\|Df\|_2 = \|S_d f\|_2$. This is another application of Plancherel's Theorem, once we observe that:

$$Df(x) = \sum_{I \in \mathcal{D}} \frac{\Delta_I f}{|I|^{1/2}} h_I(x).$$

We are ready now to describe the results for Lipschitz curves. We will replace the dyadic square function S_d by a *geometric dyadic square function* $\tilde{S}_d$.

We construct a family $\mathcal{F}$ of intervals related to the geometry of the problem. $\mathcal{F}$ is what we call a *regular dyadic grid.* It preserves the nesting properties of the standard dyadics, but the scaling is more involved. (For the precise definitions see Section 3.2.)

Let Γ be a Lipschitz graph with Lipschitz constant less than one. For a function f on Γ define the *geometric second difference* corresponding to the interval I by:

$$\tilde{\Delta}_I f = f(z_I^+) + f(z_I^-) - 2f(z_I);$$

where $z_I^{\pm}$ are the points on the curve Γ whose projections coincide with the endpoints, $x_I^{\pm}$ of I. And z_I is the unique point in Γ which is equidistant to both $z_I^{\pm}$.

Define now the *geometric dyadic square function*:

$$\tilde{S}_d(z) = \left(\sum_{I \in \mathcal{F}} \frac{|\tilde{\Delta}_I f|^2}{|I|^2} \chi_I(\pi(z)) \right)^{1/2};$$

where $\pi(z)$ is the X-coordinate of z.

We can then prove an analogue of Theorem 1 (for $p = 2$):

Theorem 1′. *Let Γ be a Lipschitz graph with Lipschitz constant smaller than one. Assume $f \in W^{1,2}(\Gamma)$ then $\tilde{S}_d f \in L^2(\Gamma)$. Moreover*

$$\|\tilde{S}_d f\|_2 \leq C\|f'\|_2.$$

We also get a partial converse, which is the main result of this paper.

Define the *dyadic derivative of f associated to the grid $\mathcal{F}$*, $D_{\mathcal{F}} f$, for $f \in L^2(\Gamma)$, as the limit in $L^2(\Gamma)$ (when it exists) of the sequence:

$$D_n f(z) = \frac{f(z_I^+) - f(z_I^-)}{z_I^+ - z_I^-}, \quad \pi(z) \in I \in \mathcal{F}_n,$$

where $\mathcal{F}_n$ is the n$^t h$-generation of $\mathcal{F}$ (see Section 3.2).

Theorem 2. *Let* Γ *be a Lipschitz graph with Lipschitz constant smaller than one. Assume both* f *and* $\tilde{S}_d f$ *are in* $L^2(\Gamma)$. *Then* $D_{\mathcal{F}} f$ *exists as a limit in* $L^2(\Gamma)$. *Moreover,* $\|D_{\mathcal{F}} f\|_2 \leq C\|\tilde{S}_d\|_2$.

It should be clear that if we know *a priori* that $f \in W^{1,2}(\Gamma)$, then $f' = D_{\mathcal{F}} f$, and hence $\|f'\|_2 \leq C\|\tilde{S}_d f\|_2$.

To prove these theorems we try to mimic the argument described in the case $\Gamma = \mathbf{R}$. We build a Haar basis adjusted to the Lipschitz curve Γ and supported on the grid $\mathcal{F}$ which itself is related to the geometry of the problem. This can be done without great difficulty, we will not get a basis but a frame, exactly as in [**CJS**] for the study of Cauchy integrals on Lipschitz curves.

In this setting the *Haar coefficients* of the derivative will not be exact multiples of $\tilde{\Delta}_I f$. There will be an error that can be controlled by the geometry of the problem.

The proof of Theorem 2 is not as straightforward as in the case of the line. Surprisingly enough it is here where operators like the ones studied in [**P**] appeared first. We will use the techniques developed there. For more details see the introduction to the third section.

The norm $\|\tilde{S}_d f\|_2^2 = \sum_{I\in\mathcal{F}} \frac{|\tilde{\Delta}_I f|^2}{|I|}$ can be regarded as a Riemann sum for

$$\int_R \int_0^\infty |\tilde{\Delta}_t f(z)|^2 \frac{dt}{t^3} dx = \|\tilde{S} f\|_2^2.$$

In the case $\Gamma = \mathbf{R}$ we could use Theorem 2 to prove the full converse of Stein's theorem, averaging over translations and dilations of the dyadic intervals. In the general case it is not clear how to do the averaging, since we no longer have the group structure of the line available. (See [**GJ**] for examples on how to go from dyadic to continuous situations.)

The paper is organized as follows: We will prove Theorem 1 in the next section; we will use a result of Dorronsoro and some Carleson type estimates. This proof, suggested by the referee, greatly simplifies the original proof of the author. In Section 3 we will prove Theorems 1′ and 2, together with all the discrete ingredients (see the introduction to Section 3 for more details).

Throughout this paper C is a constant that might change from line to line. We will use the notation $a \sim b$, for positive numbers a and b, whenever there exists a positive and finite constant C such that $C^{-1}b \leq a \leq Cb$; we will say, in that case, that a and b are *comparable*.

These results are part of my PhD thesis. I would like to thank my advisor P.W. Jones for suggesting the problem and guiding me through the completion of this work. I extend my warmest thanks to R.R. Coifman and Stephen Semmes for very helpful conversations. Finally, I am grateful to the referee who carefully read this paper, and made a lot of valuable suggestions.

2. Proof of Theorem 1.

We are going to prove in this section the necessity of the boundedness of the geometric square function $\tilde{S}f$ for a function f to be in the Sobolev space of a Lipschitz curve. The idea is to control the geometric square function by Stein's square function. There will be some left overs that can be controlled in turn by Dorronsoro's mixed norm estimate on the approximation of these functions by affine functions. Further errors can be handled by Carleson-type estimates given by the geometry of the curve.

Let us state some geometric lemmas that we will prove at the end of this section.

Recall that $x_t^{\pm}$ are the projections onto the real line of the points on the curve Γ which are at distance t from a given point $z \in \Gamma$ whose projection is x.

Lemma 1. *Let* $u_x^+(t) := x_t^+ - x := t_x^+$, *for* $t > 0$; *then* $u_x^+ > 0$ *is an increasing homeomorphism of* t. *Moreover it is uniformly bilipschitz on* x, *i.e.*

$$\frac{1}{C} \leq \frac{d\, u_x^+(t)}{dt} \leq C \quad \forall x, t.$$

Similarly for $\quad u_x^-(t) := x - x_t^- := t_x^- > 0.$

Let us define the following quantities, as they are defined by Peter Jones [**J**] in the Traveling Salesman Problem.

For a point $z \in K$, K a subset of the plane; and $t > 0$, let

$$\beta(z,t) = \inf_L \sup_{w \in K, |w-z|<2t} t^{-1} \operatorname{dist}(w, L)$$

where L is any line in the plane. This quantity measures how close is the set $K \bigcap \{w : |w - z| < 2t\}$ to a line.

In our case $K = \Gamma$ and, since it is a graph, we will talk indistinctly about $z \in \Gamma$ or its projection $x \in \mathbf{R}$.

In general $t_x^+ \neq t_x^-$. This assymetry is what causes most of the problems. Since the curve is flat enough, we can control the difference

Lemma 2. $\quad |t_x^+ - t_x^-| \leq C\beta(x,t)\, t.$

We will prove Lemmata 1 and 2 at the end.

Recall that μ is a *Carleson measure* on the upper half plane if

$$\int_{\hat{I}} d\mu(x,t) \leq C|I| \quad \forall I \subset \mathbf{R}$$

where I is any interval of the line and $\hat{I}$ is the cube lifted above I.

Finally we can control the β's in the sense that

Lemma 3 (P. Jones' Geometric Lemma). *The measure given by*

$$d\mu(x,t) = \beta^2(x,t)\frac{dt}{t}dx,$$

is a Carleson measure on the upper half plane $\boldsymbol{R}^2_+$.

For a proof of this result see [**J**] and also [**Do**].

We will need the following facts concerning Carleson measures:

Carleson's Lemma. *Given a Carleson measure μ in the upper half plane, and a positive function $F(x,t)$ then*

$$\int_{\boldsymbol{R}}\int_0^\infty [F(x,t)]^p d\mu(x,t) \leq C\int_{\boldsymbol{R}} [F^*(x)]^p dx, \quad 0 < p < \infty$$

where $F^*(x) = \sup_{t>0,|y-x|<t} F(y,t)$.

For a proof of this lemma and the next see [**St2**], Corollary 2.4 in Ch.II.

As an immediate consequence of Carleson's Lemma and the Hardy-Littlewood Maximal Theorem, we conclude that for the case $F(x,t) = |m_{x,t}f|$, where $m_{x,t}f = \frac{1}{2t}\int_{x-t}^{x+t} f(y)dy$ the following inequality is true:

Lemma 4. *Given a Carleson measure μ in the upper half plane, and $f \in L^p(\boldsymbol{R})$ for $1 < p < \infty$, then:*

$$\int_{\boldsymbol{R}}\int_0^\infty |m_{x,t}f|^p d\mu(x,t) \leq C\int_{\boldsymbol{R}} |f(x)|^p dx.$$

We can deduce from this lemma the following mixed norm estimate; here the β's, are the ones given by the geometry, which in particular are bounded by a constant.

Lemma 5. *Given the Carleson measure in the upper half plane,*

$$d\mu(x,t) = \beta^2(x,t)\frac{dt}{t}dx$$

and $f \in L^p(\boldsymbol{R})$ for $1 < p < \infty$, then:

$$\int_{\boldsymbol{R}}\left(\int_0^\infty |m_{x,t}f|^2\beta^2(x,t)\frac{dt}{t}\right)^{p/2} dx \leq C\int_{\boldsymbol{R}} |f(x)|^p dx.$$

We will prove this result at the end of the section.

We are going to use the following result due to Dorronsoro:

Theorem [Dorronsoro]. *Let $f \in W^{1,p}(\boldsymbol{R})$ be given, with $1 < p < \infty$. Then for each $x \in \boldsymbol{R}$ and $t > 0$ there is an affine function $a_{x,t}$ with the following properties:*

$$|a'_{x,t}| \leq Ct^{-1} \int_{x-t}^{x+t} |f'(y)|dy; \tag{4}$$

$$\int_{\mathbf{R}} \left(\int_0^\infty \left(t^{-1} \sup_{|x-y|\leq t} |f(y) - a_{x,t}(y)| \right)^2 \frac{dt}{t} \right)^{p/2} dx \leq C \int_{\mathbf{R}} |f'(x)|^p dx. \tag{5}$$

If we drop the condition (4) this is a special case of Theorem 6 (i) in [**Do**]. The affine function $a_{x,t}$ used by Dorronsoro is the unique one such that:

$$\int_{x-t}^{x+t} [f(y) - a_{x,t}(y)] y^k dy = 0, \qquad k = 0, 1.$$

It can be computed explicitly. It is not hard to see that:

$$|a'_{x,t}| \leq \frac{C}{t} \left[\frac{1}{t} \int_{x-t}^{x+t} |f(y) - m_{x,t} f| dy \right].$$

The following inequality is true for absolutely continuous functions:

$$\frac{1}{t} \int_{x-t}^{x+t} |f(y) - m_{x,t} f| dy \leq C \int_{x-t}^{x+t} |f'(y)| dy;$$

(it is a calculus exercise to check it). Since functions $f \in W^{1,p}(\mathbf{R})$ are absolutely continuous after modifications on a set of measure zero, we see that condition (4) holds in Dorronsoro's Theorem.

Proof of Theorem 1. We want to bound with a constant times the L^p norm of the derivative of a function $f \in W^{1,p}(\mathbf{R})$ the following expression

$$\left(\int_{\mathbf{R}} \left(\int_0^\infty |f(x_t^+) + f(x_t^-) - 2f(x)|^2 \frac{dt}{t^3} \right)^{p/2} dx \right)^{1/p}. \tag{6}$$

Recall that $x_t^+ = x + t_x^+$. To get a symmetric second difference, add and subtract $f(x - t_x^+)$, we can bound (6) by Minkowski's inequality, up to a constant by:

$$(7)\quad \left(\int_{\mathbf{R}}\left(\int_0^\infty |f(x+t_x^+)+f(x-t_x^+)-2f(x)|^2\frac{dt}{t^3}\right)^{p/2}dx\right)^{1/p} + \left(\int_{\mathbf{R}}\left(\int_0^\infty |f(x-t_x^-)-f(x-t_x^+)|^2\frac{dt}{t^3}\right)^{p/2}dx\right)^{1/p}.$$

The first summand can be reduced to the euclidean case. Let us do the change of variable $s = t_x^+ = u_x^+(t)$; by Lemma 1, $s \sim t$, $ds \sim dt$. We can bound the first term by:

$$C\left(\int_{\mathbf{R}}\left(\int_0^\infty |f(x+s)+f(x-s)-2f(x)|^2\frac{ds}{s^3}\right)^{p/2}dx\right)^{1/p},$$

which is bounded by $C\|f'\|_p$ by Theorem A.

We are left with the second integral in (7). This time we will add and subtract $a_{x,t}(x-t_x^-)$ and $a_{x,t}(x-t_x^+)$; where $a_{x,t}$ is the affine function given in Dorronsoro's theorem. Certainly:

$$|f(x-t_x^\pm)-a_{x,t}(x-t_x^\pm)| \le \sup_{|y-x|\le t} |f(y)-a_{x,t}(y)|.$$

We can then bound (7) by a constant times:

$$(8)\quad \left(\int_{\mathbf{R}}\left(\int_0^\infty \left(t^{-1}\sup_{|y-x|\le t}|f(y)-a_{x,t}(y)|\right)^2\frac{dt}{t}\right)^{p/2}dx\right)^{1/p} + \left(\int_{\mathbf{R}}\left(\int_0^\infty |a_{x,t}(x-t_x^-)-a_{x,t}(x-t_x^+)|^2\frac{dt}{t^3}\right)^{p/2}dx\right)^{1/p}$$

The first term is bounded by $C\|f'\|_p$ by Dorronsoro's theorem. The second can be rewritten as:

$$\left(\int_{\mathbf{R}}\left(\int_0^\infty |a'_{x,t}|^2|t_x^+-t_x^-|^2\frac{dt}{t^3}\right)^{p/2}dx\right)^{1/p};$$

and using Dorronsoro's estimate (4) and Lemma 2, we can bound this by

$$\left(\int_{\mathbf{R}}\left(\int_0^\infty \left[\frac{1}{2t}\int_{x-t}^{x+t}|f'|\right]^2\beta^2(x,t)\frac{dt}{t}\right)^{p/2}dx\right)^{1/p};$$

which in turn is bounded by Carleson's mixed norm lemma (Lemma 5), and P. Jones geometric lemma (Lemma 3) by $C\|f'\|_p$.

This finishes the proof of Theorem 1 except for the geometric lemmas, and Carleson's mixed norm lemma. □

Proof of Lemma 1. We want to prove that $u_x^+(t) = t_x^+ = x_t^+ - x$ is an increasing bilipschitz homeomorphism. Clearly u_x^+ is increasing (because Γ is a Lipschitz graph with Lipschitz constant less than one). The inverse of this mapping is given by the distance between the images on the curve Γ of x and $y = x+s$, namely $(u_x^+)^{-1}(s) = |\tilde{A}(x+s) - \tilde{A}(x)|$, where A is the Lipschitz map defining Γ and $\tilde{A}(y) = y + iA(y)$. By hypothesis, $|A(y+h) - A(y)| \leq \eta h$ where $\eta < 1$.

Showing that u_x^+ is bilipschitz is equivalent to show that its inverse is bilipschitz. To show this it is enough to show that there exists a constant C such that $\forall x,\ s \geq 0,\ h > 0$

$$\frac{1}{C} \leq \frac{(u_x^+)^{-1}(s+h) - (u_x^+)^{-1}(s)}{h} \leq C.$$

We can assume without loss of generality that $x = A(x) = 0$. We want to bound $(|\tilde{A}(y+h)| - |\tilde{A}(y)|)/h$, from above and below.

The upper bound is trivial by the triangle inequality and by the fact that the map $\tilde{A}$ is bilipschitz, since

$$h \leq |\tilde{A}(y+h) - \tilde{A}(y)| = |h + i(A(y+h) - A(y))| \leq h\sqrt{1+\eta^2}.$$

Note that for all z and y,

$$|\tilde{A}(z)|^2 - |\tilde{A}(y)|^2 = z^2 - y^2 + (A^2(z) - A^2(y)).$$

It is not hard to check that for every $0 < y < z$

$$A^2(z) - A^2(y) \geq -\eta^2(z^2 - y^2),$$

therefore $|\tilde{A}(z)|^2 - |\tilde{A}(y)|^2 \geq (1-\eta^2)(z^2 - y^2)$.

Since $|\tilde{A}(z)| \leq |z|\sqrt{1+\eta^2}$, then

$$\frac{|z| + |y|}{|\tilde{A}(z)| + |\tilde{A}(y)|} \geq \frac{1}{\sqrt{1+\eta^2}},$$

hence, choosing $z = y + h,\ \ h > 0$ we get that

$$\frac{|\tilde{A}(y+h)| - |\tilde{A}(y)|}{h} \geq \frac{1-\eta^2}{\sqrt{1+\eta^2}},$$

which is certainly larger than zero, since $\eta < 1$.

This finishes the proof of the lemma. □

Proof of Lemma 2. We want to show that there exists a constant C independent of x and t such that

$$|t_x^+ - t_x^-| \leq C\, t\beta(x,t),$$

where the β's were defined for $z = \tilde{A}(x)$, by

$$\beta(x,t) = \inf_L \sup_{w\in\Gamma,\, |w-z|<2t} t^{-1} \operatorname{dist}(w, L),$$

and L is any line in the plane.

Notice that the height h of the isosceles triangle drawn through the images on the curve of x, $x_t^+ = x + t_x^+$ and $x_t^- = x - t_x^-$ (which we will denote respectively by z, z_t^+ and z_t^-) is certainly bounded by $t\beta(x,t)$.

Therefore it is enough to show that $|t_x^+ - t_x^-| \leq C\, h$.

Let $\alpha = \alpha(x,t)$ be the common angle in the isosceles triangle. Let $\theta = \theta(x,t)$ be the angle between the horizontal and the chord through z_t^+ and z_t^-. We can assume without loss of generality that $\theta \geq 0$ and that $\arg z \geq \arg z_t^-$. Then high school geometry shows that

$$t_x^- = t\cos(\alpha+\theta), \quad h = t\sin\alpha,$$

$$t\cos\alpha\cos\theta = \frac{x_t^+ - x_t^-}{2} = \tilde{x}_t, \qquad t_x^+ = (x_t^+ - x_t^-) - t_x^-.$$

Therefore $t_x^- = \tilde{x}_t - h\sin\theta$. and $t_x^+ = \tilde{x}_t + h\sin\theta$.

Hence

$$|t_x^+ - t_x^-| = 2h\sin\theta < 2h.$$

We can have a better bound if we notice that $\sin\theta \leq \frac{\eta}{1+\eta^2}$.

This finishes the proof of Lemma 2.

□

Proof of Lemma 5. The case $p = 2$ is an immediate consequence of Lemma 4. We will get the inequality for $1 < p < 2$ using the atomic decomposition of the tent spaces T_∞^q for $q \leq 1$ (see [**CMS**]), as suggested by the referee. For $2 < p < \infty$ we will get the result interpolating between a mixed L^2 norm space and the space of Carleson measures.

Case $1 < p < 2$**:** Denote by $\Gamma(x)$ the standard cone whose vertex is x, i.e., $\Gamma(x) = \{(y,t) : |y-x| < t\}$. For a function G on $\mathbf{R}_+^2$, define $A_\infty(G)(x) = \sup_{(y,t)\in\Gamma(x)} |G(y,t)|$.

The *tent space* T^q_∞ consists of exactly those functions G continuous in $\mathbf{R}^2_+$, so that $A_\infty(G) \in L^q(\mathbf{R})$, and for which $G(x,t)$ has non-tangential limits at the boundary almost everywhere. We define $\|G\|_{T^q_\infty} = \|A_\infty(G)\|_q$.

A T^q_∞-atom is a function $a(x,t)$ supported on a tent $\hat{I}$, and such that $\sup_{(x,t)} |a(x,t)| \leq 1/|I|^{1/q}$; where I is an interval centered at x_I, and $\hat{I} = \{(x,t) \in \mathbf{R}^2_+ : x \in I, t < |I|/2 - |x - x_I|\}$. Clearly $\|a\|_{T^q_\infty} \leq 1$. The atomic decomposition for T^q_∞ when $q \leq 1$ given in Proposition 5 on p. 326 of [**CMS**], says that if $G \in T^q_\infty$, $q \leq 1$, then $G(x,t) = \sum \lambda_j a_j(x,t)$, where a_j are T^q_∞-atoms. Moreover $\sum |\lambda_j|^q \leq \|G\|^q_{T^q_\infty}$.

Let $f \in L^p(\mathbf{R})$ be given and set

$$F(x,t) = |m_{x,t} f|^2.$$

Then F lies in the tent space T^q_∞ of [**CMS**] with $q = p/2 < 1$. Moreover, as an application of the Hardy-Littlewood Theorem, $\|F\|^{p/2}_{T^{p/2}_\infty} \leq C\|f\|^p_p$.

It is simple to check for $T^{p/2}_\infty$-atoms, $a(x,t)$, that the quantity:

$$\int_{\mathbf{R}} \left(\int_0^\infty a(x,t)\beta^2(x,t)\frac{dt}{t} \right)^{p/2} dx, \tag{9}$$

is bounded by a constant C independent of the atom a. More precisely, using the support and size properties of the atom we see that (9) is bounded by:

$$\frac{1}{|I|}\int_I \left(\int_0^{|I|/2} \beta^2(x,t)\frac{dt}{t} \right)^{p/2} dx \leq \left(\frac{1}{|I|}\int_I \int_0^{|I|/2} \beta^2(x,t)\frac{dt}{t} dx \right)^{p/2} \leq C;$$

the first inequality by the Cauchy-Schwartz inequality with $p' = 2/p > 1$, the last one by P. Jones' geometric lemma.

Finally, writing an atomic decomposition for $F(x,t) = \sum \lambda_j a_j(x,t)$, using the above estimate for atoms, and the fact that $p/2 < 1$, we conclude that

$$\int_{\mathbf{R}} \left(\int_0^\infty F(x,t)\beta^2(x,t)\frac{dt}{t} \right)^{p/2} dx \leq \sum C\lambda_j^{p/2} \leq C\|F\|^{p/2}_{T^{p/2}_\infty} \leq C\|f\|^p_p.$$

Case $2 < p < \infty$: Let us introduce the mixed norm spaces, $1 < p < \infty$

$$L^{2,p} = \left\{ f : \mathbf{R}^2_+ \to \mathbf{R};\ \|f\|_{2,p} = \left(\int_{\mathbf{R}} \left(\int_0^\infty |f(x,t)|^2 \frac{dt}{t} \right)^{p/2} dx \right)^{1/p} < \infty \right\}.$$

Define the Carleson measure space by:

$$CM = \left\{ g : \mathbf{R}^2_+ \to \mathbf{R};\ \|g\|_{CM} = \sup_{I \subset \mathbf{R}} \frac{1}{|I|}\int_I \left(\int_0^{|I|} g^2(x,t)\frac{dt}{t} \right)^{1/2} dx < \infty \right\}.$$

These are Banach spaces with the corresponding norms. We can interpolate between mixed norm spaces and Carleson measure space. In the sense that, given a linear operator T bounded simultaneously from L^2 into $L^{2,2}$, and from L^∞ into CM, it is also bounded from L^p into $L^{2,p}$, for $2 < p < \infty$. See [**CMS**] and [**AM**].

Define the linear operator T for integrable functions by:

$$Tf(x,t) = \beta(x,t) m_{x,t} f.$$

T is bounded from L^2 into $L^{2,2}$, it only remains to check that is bounded from L^∞ into CM. We want to show that:

$$\frac{1}{|I|}\int_I \left(\int_0^{|I|} |m_{x,t}f|^2 \beta^2(x,t)\frac{dt}{t}\right)^{1/2} dx < C\|f\|_\infty.$$

Certainly $|m_{x,t}f| \leq \|f\|_\infty$; substituting it into the integral, applying the Cauchy-Schwartz inequality, and using once more P. Jones' geometric lemma we get the desired inequality.

As it was pointed out by the referee, the result for $p > 2$ is related to Remark b on p. 320 of [**CMS**]. This remark addresses essentially the same point, but with integrals in t replaced by integrals over cones.

This finishes the proof of the mixed norm Carleson's lemma. □

3. Dyadic Version.

3.1. Introduction. Let Γ be a Lipschitz graph, $\Gamma = \{z = x + iA(x) : \|A'\|_\infty < \infty\}$. We will assume that $\|A'\|_\infty < 1$, as before.

When $\Gamma = \mathbf{R}$ it is not difficult to see that

$$f \in W^{1,2}(\mathbf{R}) \iff f, \tilde{S}f \in L^2(\mathbf{R}).$$

As we pointed out in the introduction of the paper, in this case this result can be regarded as a continuous version of Plancherel's theorem for the Haar basis. The key observation being that the Haar coefficients of the derivative f' of an absolutely continuous function f are, up to a scaling factor, the second difference of f at the corresponding interval.

We will take advantage of this natural *dyadic* interpretation in order to develop a discrete approach to the problem.

In Section 3.2 we will introduce the *regular dyadic grids* (substitutes for an ordinary dyadic grid). We will construct some Haar systems associated to these grids and to a *nice* complex measure $d\sigma$ (by nice we mean absolutely continuous with respect to Lebesgue measure, and such that $|\sigma(I)| \sim |I|$ for all intervals I in the grid, where $\sigma(I) = \int_I d\sigma$).

In Section 3.3 we will construct a regular dyadic grid $\mathcal{F}$ adjusted to the geometry of the problem and the corresponding Haar system $\{h_I^\sigma\}_{I\in\mathcal{F}}$, associated to the measure $d\sigma = (1+iA'(x))dx$ (this measure is certainly *nice*). We will show that this particular Haar system is a *frame*, i.e. it behaves almost like an orthonormal basis (see [**CJS**].) The deviation from the standard basis is controlled by a geometric quantity estimated in a *Geometric Lemma* (dyadic version of P. Jones Geometric Lemma 3, which in this case is very easy to prove; see [**J**]), and a discrete version of Carleson's Lemma.

Define the *geometric second difference* associated to the interval $I = (x_I^-, x_I^+)$ by

$$\tilde{\Delta}_I f = f(z_I^+) + f(z_I^-) - 2f(z_I),$$

where $z_I^\pm = x_I^\pm + iA(x_I^\pm)$, and $z_I \in \Gamma$ and is equidistant to $z_I^\pm$.

Define the *geometric dyadic square function*

$$\tilde{S}_d f(z) = \left(\sum_{I\in\mathcal{F}} \frac{|\tilde{\Delta}_I f|^2}{|I|^2}\chi_I(\pi(z))\right)^{1/2};$$

where $\pi(z)$ is the X-coordinate of z.

We can prove the dyadic analogue of Theorem 1, for $p = 2$,

Theorem 1′. *Given $f \in W^{1,2}(\Gamma)$ then*

$$\|\tilde{S}_d f\|_{L^2(\Gamma)}^2 = \sum_{I\in\mathcal{F}} \frac{|\tilde{\Delta}_I f|^2}{|I|} \le C\|f'\|_{L^2(\Gamma)}^2.$$

If we do not know *a priori* that $f \in W^{1,2}(\Gamma)$ we can still show a partial converse. Let $\mathcal{F}_n$ denotes the nth generation of the grid $\mathcal{F}$. Define the *dyadic derivative of f associated to the grid $\mathcal{F}$, $D_{\mathcal{F}} f$*, as the limit in $L^2(\Gamma)$, when it exists, of the sequence:

$$D_k f(z) = \frac{f(z_I^+) - f(z_I^-)}{z_I^+ - z_I^-}; \qquad \pi(z) \in I \in \mathcal{F}_k(J).$$

Theorem 2. *Assume that $f \in L^2(\Gamma)$ and that*

$$\|\tilde{S}_d f\|_{L^2(\Gamma)}^2 = \sum_{I\in\mathcal{F}} \frac{|\tilde{\Delta}_I f|^2}{|I|} < \infty.$$

Then $D_{\mathcal{F}} f$ exists and is in $L^2(\Gamma)$. Moreover

$$\|D_{\mathcal{F}} f\|_{L^2(\Gamma)}^2 \le C\|\tilde{S}_d f\|_{L^2(\Gamma)}^2.$$

It will be enough to prove local versions of these theorems. By this we mean to replace $\mathbf{R}$ by an interval J and prove the corresponding statements uniformly on J.

In Section 3.4 we will prove a local version of Theorem 1′. To do this we will use the orthogonality of the *Haar system* constructed and Carleson's Lemma for regular dyadic grids.

In Section 3.5 we will prove a local version of Theorem 2. We will reduce the problem to the boundedness of an operator, $P_{b,\sigma}$, that formally looks like the operator defined in [**P**] by,

$$P_b g = \sum_{n=0}^{\infty} \Delta_n g \prod_{j=n+1}^{\infty} (1 + \Delta_j b);$$

where g is a square integrable function, b comes from the geometry and is in the space of bounded mean oscillation functions (BMO), and $\Delta_n f$ is the projection onto the subspace generated by the Haar functions corresponding to the nth generation of the dyadics.

In Section 3.6 the operator $P_{b,\sigma}$ is analized. The strategy is the same as in [**P**]. We can rewrite the paraseries P_b in terms of the weight $\omega = \prod_{j=0}^{\infty}(1 + \Delta_j b)$ (see p. 581). The necessary and sufficient conditions for the boundedness of the operator P_b in L^2 are described in [**P**], and they reduce to a reverse Hölder condition on the weight. In our case the grid will be the regular dyadic grid $\mathcal{F}$; the Haar functions will not be the standard ones either. Nevertheless, we can mimic what we did in [**P**]. As we could expect, the boundedness of the operator will depend upon the boundedness of a weighted maximal operator, and this will be so provided the weight ω satisfies a Reverse Hölder condition on the grid. The proof in this case is simpler than in [**P**]; after a minute of reflexion we see that both the weight and the grid come from the geometry and some of the difficulties are cancelled out.

3.2. Dyadic grids and Haar functions. Consider a fix interval J. A *dyadic grid associated to* J is a collection of nested intervals $\mathcal{F}(J)$ such that $\mathcal{F}(J) = \bigcup_{n=0}^{\infty} \mathcal{F}_n(J)$. The *generations* $\mathcal{F}_n$ are defined inductively by $\mathcal{F}_{n+1}(J) = \bigcup_{I\in\mathcal{F}_n(J)} \mathcal{F}_1(I)$, and given any interval I, its first generation $\mathcal{F}_1(I) = \{I_l, I_r\}$ is a partition of I into two disjoint intervals that we will call the *children* of I.

A *regular dyadic grid associated to* J is a dyadic grid such that there is a constant $\frac{1}{2} \le C < 1$, such that given any interval $\tilde{I} \in \mathcal{F}(J)$ and I a child of $\tilde{I}$ then

$$(1 - C)|\tilde{I}| \le |I| \le C|\tilde{I}|.$$

If $C = \frac{1}{2}$ we get the ordinary dyadic decomposition of J. In this case given any $I \in \mathcal{F}_n(J)$, $|I| = 2^{-n}|J|$.

If $C > \frac{1}{2}$ then we can only say that for any $I \in \mathcal{F}_n(J)$

$$(1-C)^n|J| \leq |I| \leq C^n|J|.$$

This implies that given any point $x \in J$, if I_n is the unique interval in the nth generation that contains x then

$$\bigcap_{n=0}^{\infty} I_n = \{x\}; \quad \lim_{n\to\infty} |I_n| = 0.$$

It also implies that intervals of a given generation are comparable, but the comparison bounds are not independent of the generation.

We say that $\mathcal{F}$ is a *dyadic grid on* $\boldsymbol{R}$ if there exists a sequence of intervals $\{J_n\}_{n\geq 0}$ such that:

(i) $J_n \in \mathcal{F}_1(J_{n+1})$,

(ii) $\mathbf{R} = \cup_{n\geq 0} J_n$;

in that case $\mathcal{F} = \cup_{n\geq 0}\mathcal{F}(J_n)$. The generations can be defined by:

$$\mathcal{F}_k = \begin{cases} \cup_{n\geq 0}\mathcal{F}_{n+k}(J_n) & \text{for } k \geq 0 \\ \cup_{n\geq -k}\mathcal{F}_{n+k}(J_n) & \text{for } k < 0. \end{cases}$$

$\mathcal{F}$ is a *regular* dyadic grid on $\mathbf{R}$ if there exists a constant $1/2 < C < 1$ such that $(1-C)|\tilde{I}| \leq |I| \leq C|\tilde{I}|$, for all $I \in \mathcal{F}$, $\tilde{I}$ parent of I.

Given any regular dyadic grid associated to an interval J, $\mathcal{F}(J)$, and an absolutely continuous measure σ, such that $|\sigma(I)| \sim |I|$, for all $I \in \mathcal{F}(J)$; there is a *Haar system* associated to them. More precisely for each $I \in \mathcal{F}(J)$, let I_r, I_l be the right and left children of I respectively, define

$$h_I^\sigma(x) = \left(\frac{\sigma(I_r)\sigma(I_l)}{\sigma(I)}\right)^{1/2} \left(\frac{1}{\sigma(I_r)}\chi_{I_r}(x) - \frac{1}{\sigma(I_l)}\chi_{I_l}(x)\right); \tag{10}$$

and

$$h_o^\sigma(x) = \frac{1}{\sigma^{1/2}(J)}\chi_J(x), \tag{11}$$

where χ_I is the characteristic function of I.

Clearly each h_I^σ is supported on I and is constant on each child. Moreover its mean value with respect to $d\sigma$ is zero. Therefore, if we denote by $\langle .,. \rangle_\sigma$ the bilinear operation $\langle f, g\rangle_\sigma = \int fg\, d\sigma$ (notice that there is no conjugation), the system $\{h_I^\sigma\}_{I\in\mathcal{F}(J)}$ behaves like an orthonormal system with respect to this pseudo inner product, i.e. $\langle h_I^\sigma, h_{I'}^\sigma\rangle_\sigma$ is zero if $I \neq I'$, and one if $I = I'$. The function h_o^σ is certainly "orthogonal" with respect to the bilinear form $\langle .,.\rangle_\sigma$ to all the h_I^σ's and $\langle h_o^\sigma, h_o^\sigma\rangle_\sigma = 1$. Let us state this result as the first part of the next lemma.

Lemma 6. *The* Haar system associated *to the regular dyadic grid $\mathcal{F}$ and the measure σ as defined above satisfies the following properties:*

- *"orthonormality" with respect to the bilinear form $\langle .,.\rangle_\sigma$.*
- *"reconstruction formula" for functions $f \in L^2_{\text{Loc}}(J, d\sigma)$:*

$$f(x) = \sum_{I\in\mathcal{F}'(J)} \langle f, h_I^\sigma\rangle_\sigma h_I^\sigma(x), \qquad \sigma - a.e.x \tag{12}$$

where $\mathcal{F}'(J)$ is the grid $\mathcal{F}(J)$ with a second copy J_o of J and we agree that $h_{J_o}^\sigma := h_o^\sigma$.

The proof of this lemma is an standard application of Lebesgue's Differentiation Theorem (see for example [**P**] p. 631), replacing by the corresponding expectation and difference operators as defined next.

Define E_n^σ the *expectation operator* with respect to $d\sigma$, associated to the grid, by

$$E_n^\sigma f(x) = \frac{1}{\sigma(I)} \int_I f(y)d\sigma(y) \qquad x \in I \in \mathcal{F}_n(J). \tag{13}$$

Define the *difference operator*,

$$\Delta_n^\sigma f = E_{n+1}^\sigma f - E_n^\sigma f. \tag{14}$$

Observe that $E_o^\sigma f(x) = \langle f, h_o^\sigma\rangle_\sigma h_o^\sigma(x)$, and for $n > 0$,

$$\Delta_n^\sigma f(x) = \sum_{I\in\mathcal{F}_n(J)} \langle f, h_I^\sigma\rangle_\sigma h_I^\sigma(x). \tag{15}$$

We can use Plancherel's Theorem for orthogonal systems if the measure $d\sigma$ is positive (in that case we have an honest inner product); to get that

$$\|f\|^2_{L^2(J,d\sigma)} = \sum_{I\in\mathcal{F}'(J)} |\langle f, h_I^\sigma\rangle_\sigma|^2.$$

In particular, if $d\sigma = dx$ we have the *standard Haar basis* associated to the grid $\mathcal{F}$, that we will denote by $\{h_I\}_{I\in\mathcal{F}}$. for the record, note that,

$$h_I(x) = \left(\frac{|I_r||I_l|}{|I|}\right)^{1/2} \left(\frac{1}{|I_r|}\chi_{I_r}(x) - \frac{1}{|I_l|}\chi_{I_l}(x)\right). \tag{16}$$

We want to deal with complex measures, and we want to say something about the function being in ordinary $L^2(J)$. That is we would like to know under which conditions the system $\{h_I^\sigma\}_{I\in\mathcal{F}(J)}$ is a *frame* in $L^2(J)$. By this we mean that we can reconstruct the functions as in (12), and we can also recover the L^2 norm. More precisely, there exists a constant $C > 0$ such that

$$\frac{1}{C}\sum_{I\in\mathcal{F}'(J)} |\langle f, h_I^\sigma\rangle_\sigma|^2 \le \|f\|^2_{L^2(J)} \le C\sum_{I\in\mathcal{F}'(J)} |\langle f, h_I^\sigma\rangle_\sigma|^2. \tag{17}$$

In [**CJS**] a Haar system adjusted to a Lipschitz curve is built. There the grid is the ordinary dyadic grid and the measure involved is $d\sigma = z'(x)dx$, where z is the arclength parametrization. It turns out that in this case the system is a frame.

In the next section we will construct a Haar system associated to a regular dyadic grid $\mathcal{F}$ and to a measure $d\sigma$ related to the given Lipschitz curve. We will show that this particular system is a *frame*.

Carleson's lemma is still valid in this context. A *Carleson sequence with respect to* $\mathcal{F}(J)$ is a sequence of complex numbers $\{b_I\}_{I\in\mathcal{F}(J)}$ such that there exists a constant C (*Carleson's constant*) such that

$$\sum_{I\in\mathcal{F}(I_o)} |b_I| \le C|I_o|, \quad \forall I_o \in \mathcal{F}(J).$$

Lemma 7 (Carleson's Lemma). *Given* $\{b_I\}$ *a Carleson sequence with respect to* $\mathcal{F}(J)$ *and any sequence of positive numbers* $\{\lambda_I\}$ *then*

$$\sum_{I\in\mathcal{F}(J)} \lambda_I |b_I| \le C\int_J \lambda^*(x)dx,$$

where C *is the Carleson constant of the* $\{b_I\}$ *and* $\lambda^*(x) = \sup_{x\in I\in\mathcal{F}(J)} \lambda_I$.

A proof for the standard dyadic grid can be found in [**M**] p. 273. The proof for regular dyadic grids is essentially the same.

3.3. Our Grid. Given the Lipschitz graph $\Gamma = \{z = x + iA(x); \|A'\|_\infty$ $\eta < \infty\}$. We assume, as before, that $\eta < 1$.

Fix an interval J, let $\Gamma_J = \tilde{A}(J)$, i.e. the piece of the graph Γ whose projection is J.

We will construct a Haar system, adjusted to the Lipschitz graph Γ_J, but also to the geometry of our problem. In general the supporting dyadic grid will not be the ordinary dyadics (except in the trivial case when Γ_J is a line) but it will be a regular dyadic grid. The measure will be

$$d\sigma = (1 + iA'(x))dx. \tag{18}$$

To define the grid it is enough to indicate how to produce the children of a given interval. Let I be any interval, let us denote its left and right endpoints by x_I^- and x_I^+ respectively. Let $z_I^+ = x_I^+ + iA(x_I^+)$ and similarly z_I^-. Let z_I be the point on the curve Γ which is equidistant from z_I^+ and z_I^- (it is well defined because $\|A'\|_\infty < 1$). Let x_I be the point in I such that $z_I = x_I + iA(x_I)$. The children of I will then be

$$I_l = (x_I^-, x_I), \quad I_r = (x_I, x_I^+).$$

Lemma 8. *The grid $\mathcal{F}(J)$ defined by this procedure is a regular dyadic grid.*

Proof. Clearly, the vector $z_I^+ - z_I^- = \int_I d\sigma(x) = \sigma(I)$.

Let $\theta_I = \arg\sigma(I)$. Notice that by construction, $|\sigma(I_l)| = |\sigma(I_r)| := t_I$. Therefore $\alpha_I := \theta_{I_l} - \theta_I = \theta_I - \theta_{I_r}$ (here α_I is the common angle in the isosceles triangle defined by z_I, z_I^+ and z_I^-). Since the curve is a Lipschitz graph, then certainly both θ_I and α_I are bounded in absolute value by $\theta := \arctan\|A'\|_\infty < \pi/4$. In particular, since $|I| = |\sigma(I)|\cos\theta_I$ and by construction $|\sigma(\tilde{I})| = 2|\sigma(I)|\cos\alpha_{\tilde{I}}$ (where I is a kid of $\tilde{I}$) then

$$(1-C)|\tilde{I}| \le |I| \le C|\tilde{I}|; \quad \text{for } C = \frac{1+\eta^2}{2}.$$

Since $0 \le \eta < 1$ clearly $\frac{1}{2} \le C < 1$. □

The *Haar system* associated to $\mathcal{F}(J)$ and to $d\sigma = (1+iA'(x))dx$ is, as we can see by (10) and the fact that $\sigma(I_r)/\sigma(I_l) = e^{2i\alpha_I}$, given by:

$$h_I^\sigma(x) = \frac{1}{\sigma^{1/2}(I)}\left(e^{i\alpha_I}\chi_{I_r}(x) - e^{-i\alpha_I}\chi_{I_l}(x)\right). \tag{19}$$

Proposition 1. *The Haar system defined above is a frame on $L^2(J)$.*

Proof. The proof is essentially the same as the one in [**CJS**].

Let us compare the standard Haar basis, $\{h_I\}_{I\in\mathcal{F}(J)}$, associated to the grid $\mathcal{F}(J)$ (see (16)), and the new system. It is not hard to see that

$$h_I^\sigma(x) = c_I h_I(x) + d_I|I|^{1/2}\sin\alpha_I \frac{\chi_{I_l}(x)}{|I_l|},$$

where $|c_I| \sim 1$, $|d_I| \sim 1$, uniformly on I.

Therefore,

$$\langle f, h_I^\sigma\rangle_\sigma = c_I \int f h_I d\sigma + d_I|I|^{1/2}\sin\alpha_I \frac{1}{|I_l|}\int_{I_l} f d\sigma.$$

Recall that $d\sigma = (1 + iA'(x))dx$ and let us denote the mean value with respect to the Lebesgue measure by $m_I g = \frac{1}{|I|}\int_I g\,dx$, and recall that $\langle .,. \rangle$ denotes the ordinary inner product in L^2. Then we can rewrite the right hand side in the last equality as

$$c_I\langle f(1+iA'), h_I\rangle + d_I|I|^{1/2}\sin\alpha_I\, m_{I_l} f(1+iA').$$

Also notice that,

$$\langle f, h_o^\sigma\rangle_\sigma = c_J\langle f(1+iA'), h_o\rangle,$$

where $|c_J| \sim 1$ as well.

Since $|c_I| \sim 1$ and $|d_I| \sim 1$ independently of I; then

$$\sum_{I\in\mathcal{F}'(J)} |\langle f, h_I^\sigma\rangle|^2 \le C \sum_{I\in\mathcal{F}'(J)} |\langle f(1+iA'), h_I\rangle|^2 + C\sum_{I\in\mathcal{F}(J)} |I|\sin^2\alpha_I\, |m_{I_l} f(1+iA')|^2. \tag{20}$$

The first term on the right hand side of this inequality is clearly bounded by a multiple of $\|f\|_{L^2(J)}$, since $\{h_I\}_{I\in\mathcal{F}'(J)}$ is a basis on $L^2(J)$ and $\|1+iA'\|_\infty < 2$.

The second term can be controlled by Carleson's Lemma on regular dyadic grids, provided we can show that

Lemma 9 (Geometric Lemma). *The sequence $b_I = |I|\sin^2\alpha_I$, $I \in \mathcal{F}(J)$ satisfies Carleson's condition with Carleson's constant independent of the base interval J.*

We will prove this lemma at the end of the section. Assume it is true, and let $\lambda_I = |m_{I_l} f(1+iA')|^2$. Clearly

$$\lambda^*(x) \le CM^2|f|,$$

where M is the ordinary Hardy-Littlewood maximal operator.

By Carleson's Lemma and the boundedness on L^2 of M, we get that

$$\sum_{I\in\mathcal{F}(J)} |I|\sin^2\alpha_I |m_{I_l} f(1+iA')|^2 \le C\|f\|^2_{L^2(J)}.$$

Therefore, for all $f \in L^2(J)$

$$\sum_{I\in\mathcal{F}'(J)} |\langle f, h_I^\sigma\rangle_\sigma|^2 \le C\|f\|^2_{L^2(J)}. \tag{21}$$

The converse now follows from a standard polarization argument (see [**CJS**]).

This finishes the proof of the proposition. □

We will say that a locally integrable function b is in $\mathrm{BMO}(\mathcal{F},\sigma,J)$ if there exists a constant C such that

$$(22) \qquad \sum_{I\in\mathcal{F}_k(I_o)} |\langle b, h_I^\sigma\rangle_\sigma|^2 \le C|I_o| \quad \forall I_o \in \mathcal{F}(J).$$

Remark. Since the square of the absolute value of the sequence $b_I = i\sigma^{1/2}(I)\sin\alpha_I$ is a Carleson sequence with respect to $d\sigma$ and $\mathcal{F}(J)$ (Geometric Lemma 9), the function

$$b(x) = \sum_{I\in\mathcal{F}(J)} b_I h_I^\sigma(x)$$

is a well defined $L^2(J)$ function and is in $\mathrm{BMO}(\sigma,\mathcal{F},J)$; moreover, there exists constant $0<\epsilon<1$ such that for all I, $|b_I h_I^\sigma(x)| \le 1-\epsilon$.

Proof of Lemma 9. This proof is the same as the proof of the Lipschitz case in the Travelling Salesman Problem (see [**J**].)

Since $|I| \sim |\sigma(I)|$, it is enough to show that the sequence $|\sigma(I)|\sin^2\alpha_I$ satisfies Carleson's condition.

Denote by Γ_{I_o} the image curve of the interval I_o.

Certainly the arclength of Γ_{I_o} is comparable to $|I_o|$. We can compute this length $l(\Gamma_{I_o})$, by successive polygonal approximations to Γ_{I_o}.

Let $\sigma_I = \{z = z_I^- + t\sigma(I) : 0 \le t \le 1\}$, be the chord built joining the images of the endpoints of I on Γ. Clearly, $|\sigma_I| = |\sigma(I)|$.

Let $\Gamma_o = \sigma_{I_o}$ and define for $n>0$

$$\Gamma_n = \bigcup_{I\in\mathcal{F}_n(I_o)} \sigma_I.$$

Clearly $\Gamma_n \longrightarrow \Gamma_{I_o}$ and $l(\Gamma_n) \longrightarrow l(\Gamma_{I_o})$.

Therefore

$$l(\Gamma_{I_o}) - l(\Gamma_o) = \sum_{n=0}^{\infty} l(\Gamma_{n+1}) - l(\Gamma_n).$$

It is easy to compare the lengths of two succesive polygonals,

$$l(\Gamma_{n+1}) - l(\Gamma_n) = \sum_{I\in\mathcal{F}_n(I_o)} \left(|\sigma(I_r)| + |\sigma(I_l)| - |\sigma(I)|\right).$$

By definition of the grid, $|\sigma(\tilde{I})| = 2|\sigma(I)|\cos\alpha_{\tilde{I}}$, for $\tilde{I}$ parent of I; hence, since Γ is Lipschitz, $|\sigma(I_r)| + |\sigma(I_l)| - |\sigma(I)| \sim |\sigma(I)|\sin^2\alpha_I$.

Therefore

$$l(\Gamma_{I_o}) - l(\Gamma_o) \sim \sum_{I \in \mathcal{F}_n(I_o)} |\sigma(I)| \sin^2 \alpha_I.$$

Finally since $l(\Gamma_o) = |\sigma(I_o)| \sim |I_o|$ and $l(\Gamma_{I_o}) \sim |I_o|$ we see that for all $I_o \in \mathcal{F}(J)$

$$\sum_{I \in \mathcal{F}_n(I_o)} |\sigma(I)| \sin^2 \alpha_I \leq C|I_o|.$$

This finishes the proof of Lemma 9. □

The bilinear form $\langle .,. \rangle_\sigma$ is not an honest inner product. We would like to study the boundedness in L^2 of certain operators and their *adjoints* with respect to the bilinear form. Let us state here a lemma that we will use later. The proof of the lemma is an exercise in functional analysis left to the reader.

Lemma 10. *Given T and T^* linear operators in $L^2(J)$ such that*

$$\langle Tf, g \rangle_\sigma = \langle f, T^* g \rangle_\sigma, \quad \forall f, g \in L^2(J),$$

then T is bounded in $L^2(J)$ if an only if T^* is bounded in $L^2(J)$.

3.4. Proof of Theorem 1′. Suppose $f \in W^{1,2}(\Gamma)$, where $\Gamma = \{x + iA(x) : \|A'\|_\infty = \eta < 1\}$. Let $\tilde{A}(x) = x + iA(x)$.

By definition of the Sobolev space on the curve, $f(\tilde{A})$ and $(f(\tilde{A}))'$ are in $L^2(\mathbf{R})$. We can assume that $f(\tilde{A})$ is absolutely continuous.

Let $d\sigma = (1 + iA'(x))dx$, be the measure used in the previous section. There we showed that given an interval I then (see (19))

$$h_I^\sigma(x) = \frac{1}{\sigma^{1/2}(I)} \left(e^{i\alpha_I} \chi_{I_r}(x) - e^{-i\alpha_I} \chi_{I_l}(x) \right).$$

Clearly $(f(\tilde{A}))' = f'(\tilde{A})(1 + iA')$, and by the fundamental theorem of calculus,

$$\langle f'(\tilde{A}), h_I^\sigma \rangle_\sigma = \frac{1}{\sigma^{1/2}(I)} \left[e^{i\alpha_I} f(z_I^+) + e^{-i\alpha_I} f(z_I^-) - 2\cos\alpha_I f(z_I) \right],$$

where $I_r = [x_I, x_I^+]$, $I_l = [x_I^-, x_I]$ and $z_I^\pm = \tilde{A}(x_I^\pm)$.

The right hand side is almost the geometric second difference that we associated to I, namely $\tilde{\Delta}_I f = f(z_I^+) + f(z_I^-) - 2f(z_I)$.

Let us introduce an *adjusted geometric second difference*

$$\Delta_I f = e^{i\alpha_I} f(z_I^+) + e^{-i\alpha_I} f(z_I^-) - 2\cos\alpha_I f(z_I). \tag{23}$$

Observe that when $\Gamma = \mathbf{R}$, the two differences $\Delta_I f$ and $\tilde{\Delta}_I f$ coincide with the ordinary second difference.

Remark. This *adjusted* second difference is, in some sense, a better behaved object. If we define

$$\Delta_t f(z) = e^{i\alpha(z,t)} f(z_t^+) + e^{-i\alpha(z,t)} f(z_t^-) - 2\cos\alpha(z,t) f(z);$$

then Δ_t will annihilate linear holomorphic functions. This is something that an ordinary second difference does but ours does not!! The nonlinearity introduced in the construction of $z_t^{\pm}$ is compensated in Δ_t by the introduction of the correction factors $e^{\pm i\alpha(z,t)}$ and $\cos\alpha(z,t)$.

Fix an interval J. We just showed that if $f \in W^{1,2}(\Gamma)$ then for all $I \in \mathcal{F}(J)$,

$$\langle f'(\tilde{A}), h_I^\sigma \rangle_\sigma = \frac{\Delta_I f}{\sigma^{1/2}(I)}.$$

Also recall that

$$\langle f'(\tilde{A}), h_o^\sigma \rangle_\sigma = \frac{1}{\sigma^{1/2}(J)} \int_J f'(\tilde{A}) d\sigma = \frac{f(z_J^+) - f(z_J^-)}{\sigma^{1/2}(J)}.$$

Let $\Gamma_J = \tilde{A}(J)$. Notice that $\|f'\|_{L^2(\Gamma_J, dz)} = \|(f(\tilde{A}))'\|_{L^2(J)} \sim \|f'(\tilde{A})\|_{L^2(J)}$, therefore by Proposition 1 it follows that

$$\|f'\|^2_{L^2(\Gamma_J)} \sim \sum_{I \in \mathcal{F}(J)} \frac{|\Delta_I f|^2}{|\sigma(I)|} + \frac{|f(z_J^+) - f(z_J^-)|^2}{|\sigma(J)|}. \tag{24}$$

If we replace Δ_I by $\tilde{\Delta}_I$ we can still show a local version of Theorem 1′. Since $|I| \sim |\sigma(I)|$, we can use either of them in the estimates.

Theorem 1′ (Local version). *Given $f \in W^{1,2}(\Gamma)$, then for every interval $J \in \mathcal{F}$*

$$\sum_{I \in \mathcal{F}(J)} \frac{|\tilde{\Delta}_I f|^2}{|\sigma(I)|} + \frac{|f(z_J^+) - f(z_J^-)|^2}{|\sigma(J)|} \leq C \|f'\|^2_{L^2(\Gamma_J)},$$

uniformly on J.

Remark. This local version implies Theorem 1′. Since it holds uniformly on J, and clearly $f \in W^{1,2}(\Gamma)$ implies that $\frac{|f(z_J^+) - f(z_J^-)|^2}{|\sigma(J)|} \leq \|f'\|^2_{L^2(\Gamma_J)}$ (more is actually true: $f \in W^{1,2}(\Gamma)$, f absolutely continuous, implies that $\frac{|f(z_J^+) - f(z_J^-)|^2}{|\sigma(J)|} \to 0$, as $J \to \mathbf{R}$, this is a consequence of the elementary fact that for any function $f \in L^2(\mathbf{R})$, $\frac{1}{|I|}(\int_I f)^2 \to 0$ as $|I| \to \infty$.) Denote by $\mathcal{F} = \cup_{n\geq 0}\mathcal{F}(J_n)$ where $J_n \in \mathcal{F}_1(J_{n+1})$ and $\mathbf{R} = \cup_{n\geq 0} J_n$, then certainly

$$\sum_{I\in\mathcal{F}} \frac{|\tilde{\Delta}_I f|^2}{|\sigma(I)|} \leq C\|f'\|^2_{L^2(\Gamma)};$$

which is the conclusion we were seeking.

Proof of Theorem 1′ (Local version). After observation (24), we see that it is enough to compare $\sum_{I\in\mathcal{F}(J)} |\tilde{\Delta}_I f|^2/|\sigma(I)|$ and $\sum_{I\in\mathcal{F}(J)} |\Delta_I f|^2/|\sigma(I)|$.

In particular

$$\Delta_I f = \cos\alpha_I \tilde{\Delta}_I f + i\sin\alpha_I \left(f(z_I^+) - f(z_I^-)\right). \tag{25}$$

Since $f \in W^{1,2}(\Gamma)$, we can assume that $f(\tilde{A})$ is absolutely continuous; i.e. $f(\tilde{A})(b) - f(\tilde{A})(a) = \int_a^b (f(\tilde{A}))'(x)dx = \int_a^b f'(\tilde{A})d\sigma$. Hence if we denote the mean value of g with respect to σ on I by $m_I^\sigma g$, then

$$\frac{f(z_I^+) - f(z_I^-)}{\sigma(I)} = m_I^\sigma f'(\tilde{A}).$$

Therefore

$$\sum_{I\in\mathcal{F}(J)} \frac{|\tilde{\Delta}_I f|^2}{|\sigma(I)|} \leq C \sum_{I\in\mathcal{F}(J)} \frac{|\Delta_I f|^2}{|\sigma(I)|} + C \sum_{I\in\mathcal{F}(J)} |\sigma(I)| \sin^2\alpha_I |m_I^\sigma f'(\tilde{A})|^2.$$

The second summand on the right hand side is bounded by $\|f'(\tilde{A})\|_2^2 \sim \|f'\|^2_{L^2(\Gamma)}$ by Carleson's Lemma and the same argument with the maximal function that we used at the end of Proposition 1.

This finishes the proof of the local version of Theorem 1′. □

3.5. Proof of Theorem 2. If we do not know *a priori* that $f \in W^{1,2}(\Gamma)$ but only that $f \in L^2(\Gamma)$ and that for a fixed interval J,

$$\frac{|f(z_J^+) - f(z_J^-)|^2}{|\sigma(J)|} + \sum_{I\in\mathcal{F}(J)} \frac{|\tilde{\Delta}_I f|^2}{|\sigma(I)|} < \infty, \tag{26}$$

we can still say something. Certainly (26) does not carry enough information about the smoothness of f. for instance it only considers the values of f at a countable number of points which is negligible. Nevertheless, if (26) is true the sequence

$$D_k^J f(z) = \frac{f(z_I^+) - f(z_I^-)}{\sigma(I)}, \qquad \pi(z) \in I \in \mathcal{F}_k(J)$$

will converge to a function $D^J f$ in $L^2(\Gamma_J)$, that we will call the *dyadic derivative of f on J with respect to the grid $\mathcal{F}(J)$* (clearly if we start with a differentiable function f then the sequence converges pointwise to f' in J). More precisely, we can prove the following:

Theorem 2 (Local version). *Let $f \in L^2(\Gamma_J)$ and assume* (26). *Then, the sequence $D_k^J f$ defined above converges to a function $D^J f \in L^2(\Gamma_J)$. Moreover,*

$$\|D^J f\|_{L^2(\Gamma_J)}^2 \leq C \left(\frac{|f(z_J^+) - f(z_J^-)|^2}{|\sigma(J)|} + \sum_{I \in \mathcal{F}(J)} \frac{|\tilde{\Delta}_I f|^2}{|\sigma(I)|} \right),$$

where the constant C is independent of the base interval J.

Remark. To get the global estimate, denote by $\mathcal{F} = \cup_{n \geq 0} \mathcal{F}_1(J_n)$ where $J_n \in \mathcal{F}(J_{n+1})$ and $\mathbf{R} = \cup_{n\geq 0} J_n$, as in the remark right after the local version of Theorem 1$'$. Clearly, $\mathcal{F}(J_n) \subset \mathcal{F}(J_{n+1}) \subset \dots \subset \mathcal{F}$, assume that $\sum_{I\in\mathcal{F}} \frac{|\tilde{\Delta}_I f|^2}{|\sigma(I)|} < \infty$. This implies that (26) holds uniformly on J_n (since $|f(z_J^+) - f(z_J^-)|^2/|\sigma(J)| \leq c \sum_{J \subset I \in \mathcal{F}} |\tilde{\Delta}_I f|^2/|\sigma(I)|$). Given $f \in L^2(\Gamma)$, we will get a sequence of functions $D^n f$ defined by $D^{J_n} f$ on Γ_{J_n} and zero otherwise, uniformly bounded in L^2. By construction $D^{n+1} f|_{J_n} = D^n f|_{J_n}$, hence $D^n f \to D_{\mathcal{F}} f$ in the L^2 sense as $n \to \infty$, and

$$\|D_{\mathcal{F}} f\|_{L^2(\Gamma)}^2 \leq C \sum_{I\in\mathcal{F}} \frac{|\tilde{\Delta}_I f|^2}{|\sigma(I)|}.$$

Hence, Theorem 2 is proved, up to the local version.

Proof of Theorem 2 (Local version). Fix an interval J. Let us drop the superscripts J in the notation for dyadic derivative (i.e. D_k and D will be used instead of D_k^J and D^J).

We do not know *a priori* that f' exists, so we cannot use Carleson's Lemma straight away as we did in the previous section.

Nevertheless, notice that for every $x_I \in I \in \mathcal{F}_k(J)$ we can write by (25)

$$\Delta_I f = \cos\alpha_I \tilde{\Delta}_I f + i\sigma(I) \sin\alpha_I D_k f(x_I), \tag{27}$$

by an abuse of language, we are identifying $D_k f$ with $D_k f(\tilde{A})$, and we are writing $D_k f(x)$ instead of $D_k f(\tilde{A}(x))$.

It is not hard to see that

$$D_{k+1} f(x) - D_k f(x) = \frac{\Delta_I f}{\sigma^{1/2}(I)} h_I^\sigma(x), \quad x \in I \in \mathcal{F}_k(J).$$

Therefore, multiplying (27) by $h_I^\sigma/\sigma^{1/2}(I)$ and using the last equality we get for every $x \in I \in \mathcal{F}_k(J)$

(28)
$$D_{k+1}f(x) = \cos\alpha_I \frac{\tilde{\Delta}_I f}{\sigma^{1/2}(I)} h_I^\sigma(x) + (1 + i\sigma^{1/2}(I)\sin\alpha_I h_I^\sigma(x))D_k f(x).$$

By hypothesis and Proposition 1, the function

$$g(x) = \sum_{I\in\mathcal{F}(J)} \cos\alpha_I \frac{\tilde{\Delta}_I f}{\sigma^{1/2}(I)} h_I^\sigma(x), \tag{29}$$

is in $L_o^2(J)$.

Let $b(x) = \sum_{I\in\mathcal{F}(J)} b_I h_I^\sigma(x)$, where $b_I = i\sigma^{1/2}(I)\sin\alpha_I$. By the remark on p. 573, b is in $\mathrm{BMO}(\mathcal{F}, \sigma, J)$.

Moreover, with the notation of Section 3.2 p. 569,

$$\Delta_k^\sigma g(x) = \sum_{I\in\mathcal{F}_k(J)} \cos\alpha_I \frac{\tilde{\Delta}_I f}{\sigma^{1/2}(I)} h_I^\sigma(x), \qquad E_o^\sigma g = 0,$$

and similarly for $\Delta_k^\sigma b(x)$.

With this notation we can rewrite (28) for all $k \geq 0$ as

$$D_{k+1}f(x) = \Delta_k^\sigma g(x) + (1 + \Delta_k^\sigma b(x))\, D_k f(x). \tag{30}$$

This is the recurrence equation that we solved in [**P**] under some conditions on b.

Let us replace $D_k f$ by the corresponding sum and continue down until we reach $k = 0$. We get

$$D_{k+1}f = \Delta_k^\sigma g + \sum_{n=0}^{k-1} \Delta_n^\sigma g \prod_{j=n+1}^{k} (1 + \Delta_j^\sigma b) + D_o f \prod_{j=0}^{k} (1 + \Delta_j^\sigma b). \tag{31}$$

The last summand on the right hand side of this equation is a multiple of $D_o f = \frac{f(z_J^+) - f(z_J^-)}{\sigma(J)}$ which is not necessarily zero.

Lemma 11. *The sequence* $\omega_k = \prod_{j=0}^k (1 + \Delta_j^\sigma b)$ *converges in* $L^2(J)$ *and a.e. to the function* $\omega = \prod_{j=0}^\infty (1 + \Delta_j^\sigma b)$. *Moreover* $\|\omega\|_{L^\infty(J)} \leq 1$.

We will prove this lemma at the end of the section. These products had been studied in [**FKP**].

The first two summands in the right hand side of (31) look formally like the finite sum operator P_b^k in [**P**]. The only differences are that here the supporting grid is not the standard dyadic grid and the measure is $d\sigma$ instead of the Lebesgue measure. The function b comes from the geometry, just as the measure $d\sigma$ and the grid do. All the algebra is still valid, including the algebra to pass to the corresponding finite *paraseries*.

Let us define the analogous finite sum operators, for $b \in \mathrm{BMO}(\sigma, \mathcal{F}, J)$ and $g \in L^2_o(J, d\sigma)$ (the space of functions in $L^2(J)$ with mean value zero on J with respect to $d\sigma$)

$$P^k_{b,\sigma} g := \sum_{n=0}^{k-1} \Delta^\sigma_n g \prod_{j=n+1}^{k} (1 + \Delta^\sigma_j b) + \Delta^\sigma_k g(x). \tag{32}$$

Proposition 2. *The operators $P^k_{b,\sigma}$ converge to a bounded operator in $L^2(J)$.*

To show the convergence of the martingale $D_k f$ (see (31)), it is enough to show that $P^k_{b,\sigma} g$ converges to a function in $L^2(J)$ since the other term converges to $\omega D_o f$, a multiple of $\omega \in L^2(J)$ (by Lemma 11), where $D_o f = \frac{f(z_J^+)-f(z_J^-)}{\sigma(J)}$. As a consequence of Proposition 2,

$$\|P^k_{b,\sigma} g\|^2_{L^2(J)} \leq C\|g\|^2_{L^2(J)} \leq C \sum_{I\in\mathcal{F}(J)} \frac{|\Delta_I f|^2}{|\sigma(I)|}.$$

It is clear that $\|\omega D_o f\|^2_{L^2(J)} \leq C\frac{|f(z_J^+)-f(z_J^-)|^2}{|\sigma(J)|}$, because by Lemma 11, $|\omega| \leq 1$. Therefore, in the limit, the function $Df = \lim_{k\to\infty} D_k f$, will be in $L^2(J)$ and moreover,

$$\|Df\|^2_{L^2(J)} \leq C \sum_{I\in\mathcal{F}(J)} \frac{|\Delta_I f|^2}{|\sigma(I)|} + C\frac{|f(z_J^+) - f(z_J^-)|^2}{|\sigma(J)|},$$

where C is a constant independent of J. The local version of Theorem 2 is proved up to the study of the operators $P^k_{b,\sigma}$, and the weight ω (Lemma 11). □

3.6. Convergence of the operators $P^k_{b,\sigma}$. Since formally the operators $P^k_{b,\sigma}$ look exactly like the ones treated in [**P**], we want to analize them in a similar way.

In this setting we can define the *paraproduct*

$$\Pi^\sigma_b g = \sum_{j=0}^{\infty} E^\sigma_j g \Delta^\sigma_j b \tag{33}$$

and its *adjoint* with respect to $\langle .,. \rangle_\sigma$

$$(\Pi_b^\sigma)^* g = \sum_{j=0}^{\infty} \Delta_j^\sigma g \Delta_j^\sigma b. \tag{34}$$

(It is easy to check that $\langle \Pi_b^\sigma g, f\rangle_\sigma = \langle g, (\Pi_b^\sigma)^* f\rangle_\sigma$.)

For $b \in \mathrm{BMO}(\mathcal{F}, \sigma, J)$ the paraproduct is bounded in $L^2(J)$ by Carleson's Lemma and so is its adjoint by Lemma 10.

The basic product and composition rules for the expectation and difference operators are true (see Definitions (13), (14), and see [**P**], and [**Ga**]), namely

$$E_n^\sigma \Delta_j^\sigma = \begin{cases} \Delta_j^\sigma & \text{if } n > j \\ 0 & \text{otherwise} \end{cases};$$

$$\Delta_n^\sigma f \times \Delta_j^\sigma g = \Delta_n^\sigma (f \times \Delta_j^\sigma g) \text{ when } n > j.$$

Therefore for all $i_1 < i_2 < \ldots < i_M$ and $n \le i_M$

$$E_n^\sigma (\Delta_{i_1}^\sigma f_1 \times \ldots \times \Delta_{i_M}^\sigma f_M) = 0; \tag{35}$$

and for all $M \geq n$

$$E_n^\sigma \left(\sum_{k \geq M} \Delta_k^\sigma \right) = 0. \tag{36}$$

Let $b = \sum_{I \in \mathcal{F}(J)} b_I h_I^\sigma$, where $b_I = i\sigma^{1/2}(I) \sin \alpha_I$. By the remark on p. 573 $b \in \mathrm{BMO}(\mathcal{F}, \sigma, J)$.

We can now reproduce word by word what we did in [**P**], except for

Proposition 3. *The operator*

$$P_b^\sigma g = \sum_{n=0}^{\infty} \Delta_n^\sigma g \prod_{j=n+1}^{\infty} (1 + \Delta_j^\sigma b), \tag{37}$$

is well defined and is bounded on $L^2(J)$.

Nevertheless we can do similar computations to the ones done in [**P**] to prove the analogous result. Let us assume that it is true for a moment, and let us go back to our problem. We want to study the convergence of $P_{b,\sigma}^k g$ as $k \to \infty$. Let $b_k = \sum_{n=0}^k \Delta_n^\sigma b$.

Then clearly

$$P_{b_k}^\sigma g = P_{b,\sigma}^k g + (g - g_k).$$

Therefore $P_{b,\sigma}^k g$ will converge simultaneously with $P_{b_k}^\sigma g$ (since $(g - g_k) \to 0$). But reproducing the proof of the corresponding theorem in [**P**], we see that $P^\sigma_{b_k} g$ converges to $P^\sigma_b g = (I - \Pi^\sigma_b)^{-1} g$. Therefore $P^k_{b,\sigma} g$ converges to $P^\sigma_b g$, which is a function in L^2, by Proposition 3.

Proof of Proposition 3. As in the proof of the analogous result in [**P**], the weight ω (see Lemma 11) can be used to rewrite the operator so that it will now look like the operators P_ω treated in [**P**].

Recall that

$$\omega(x) = \prod_{n=0}^{\infty} (1 + \Delta^\sigma_n b(x)). \tag{38}$$

As a byproduct of the proof of Lemma 11 we will get (see (56)) that

$$E^\sigma_n \omega = \prod_{j=0}^{n-1} (1 + \Delta^\sigma_j b), \tag{39}$$

which is equivalent to

$$m^\sigma_I \omega = \prod_{I' \supset I} (1 + b_{I'} h^\sigma_{I'}(x_I)), \quad x_I \in I. \tag{40}$$

With this in mind we can rewrite the operator P^σ_b as

$$\begin{aligned} P^\sigma_b g(x) &= \sum_{n=0}^{\infty} \frac{\omega(x) \Delta^\sigma_n g(x)}{E^\sigma_{n+1} \omega(x)} \\ &= \sum_{I \in \mathcal{F}(J)} \frac{\omega(x) \langle g, h^\sigma_I \rangle_\sigma h^\sigma_I(x)}{m^\sigma_I \omega (1 + b_I h^\sigma_I(x))}. \end{aligned} \tag{41}$$

Written in this way the operator looks formally like what we called P_ω in [**P**]. The main step over there was to study the boundedness of the adjoint operator.

Let

$$(P^\sigma_b)^* g(x) = \sum_{I \in \mathcal{F}(J)} \left[\frac{1}{m^\sigma_I \omega} \int \frac{\omega g h^\sigma_I}{1 + b_I h^\sigma_I} d\sigma \right] h^\sigma_I(x). \tag{42}$$

It is easy to check that for all $f, g \in L^2(J)$

$$\langle P^\sigma_b f, g \rangle_\sigma = \langle f, (P^\sigma_b)^* g \rangle_\sigma .$$

Therefore by Lemma 10 it is enough to show the boundedness of the operator $(P^\sigma_b)^*$. Since $\{h^\sigma_I\}$ is a frame, it is enough to show that there exists a constant C such that for every $g \in L^2(J)$

$$\sum_{I \in \mathcal{F}(J)} \left| \frac{1}{m^\sigma_I \omega} \int \frac{\omega g h^\sigma_I}{1 + b_I h^\sigma_I} d\sigma \right|^2 \leq C \|g\|_2^2. \tag{43}$$

We can rewrite the operator in a simpler form.

Let $I \in \mathcal{F}(J)$ be a child of $\tilde{I}$. Then by (40), for any $x_I \in I$,

$$m_I^\sigma \omega = m_{\tilde{I}}^\sigma \omega (1 + b_{\tilde{I}} h_{\tilde{I}}^\sigma (x_I)). \tag{44}$$

Therefore, recalling that $\sigma(I_l) = e^{i\alpha_I}\sigma(I)/2\cos\alpha_I$ and $\sigma(I_r) = e^{-2i\alpha_I}\sigma(I_l)$, we get

$$\begin{aligned} \frac{1}{m_I^\sigma \omega} \int \frac{\omega g h_I^\sigma}{1 + b_I h_I^\sigma} d\sigma &= \frac{1}{\sigma^{1/2}(I)} \left[\frac{e^{i\alpha_I} \int_{I_r} \omega g \, d\sigma}{m_I^\sigma \omega (1 + b_I h_I^\sigma (x_{I_r}))} - \frac{e^{-i\alpha_I} \int_{I_l} \omega g \, d\sigma}{m_I^\sigma \omega (1 + b_I h_I^\sigma (x_{I_l}))} \right] \\ &= 2 \cos \alpha_I \sigma^{1/2}(I) \left[\frac{m_{I_r}^\sigma \omega g}{m_{I_r}^\sigma \omega} - \frac{m_{I_l}^\sigma \omega g}{m_{I_l}^\sigma \omega} \right]. \end{aligned}$$

Let $d\mu = \omega \, d\sigma$. With this notation (43) is equivalent to

$$\sum_{I \in \mathcal{F}(J)} |\sigma(I)| |m_{I_r}^\mu g - m_{I_l}^\mu g|^2 \leq C \|g\|_2^2, \tag{45}$$

where $m_I^\mu g$ denotes the mean value of g on I with respect to μ.

Remark. The left hand side of (45) resembles the $L^2(d\sigma)$ norm of the standard dyadic square function $Sf(x) = (\sum_{x \in I} (m_{I_r} f - m_{I_l} f)^2)^{1/2}$, namely,

$$\|Sf\|_{L^2(d\sigma)} = \sum_{I \in \mathcal{D}} \sigma(I) |m_{I_r} f - m_{I_l} f|^2.$$

It is known that such an operator is bounded in $L^2(d\sigma)$ for $d\sigma = v dx$ if and only if the weight v is in the Muckenhoup class A_2 (see [**GC-Rf**] for the general weight theory). There is a very nice proof of this result in [**B**]. Our proof follows the ideas in that paper.

Lemma 12. *The measure μ restores dyadicity to $\mathcal{F}$. More precisely, for every $I \in \mathcal{F}(J)$, I child of $\tilde{I}$, $\mu(\tilde{I}) = 2\mu(I)$.*

Proof. By definition and using (44) for any $x_I \in I$

$$\frac{\mu(I)}{\mu(\tilde{I})} = \frac{\sigma(I)}{\sigma(\tilde{I})} (1 + b_{\tilde{I}} h_{\tilde{I}}^\sigma (x_I)).$$

It is not hard to see that for $x \in I$

$$1 + b_{\tilde{I}} h_{\tilde{I}}^\sigma (x) = \begin{cases} e^{i\alpha_{\tilde{I}}} \cos \alpha_{\tilde{I}} & x \in \tilde{I}_r \\ e^{-i\alpha_{\tilde{I}}} \cos \alpha_{\tilde{I}} & x \in \tilde{I}_l \end{cases}. \tag{46}$$

Also recall that $\sigma(I) = e^{i\theta_I}|\sigma(I)|$, $\theta_{I_r} = \theta_I - \alpha_I$, $\theta_{I_l} = \theta_I + \alpha_I$ and $|\sigma(I_r)| = |\sigma(I_l)| = |\sigma(I)|/2\cos\alpha_I$. Therefore for $x \in I$

$$\frac{\sigma(I)}{\sigma(\tilde{I})} = \frac{1}{2(1 + b_{\tilde{I}} h^{\sigma}_{\tilde{I}}(x_I))}.$$

This finishes the proof of the lemma. □

It is not hard to see, after the last lemma, that

$$\frac{m^{\mu}_{I_l} g + m^{\mu}_{I_r} g}{2} = m^{\mu}_I g.$$

We recall that for all complex numbers z, w the following identity holds,

$$\frac{|z-w|^2}{2} = \left(|z|^2 - \left|\frac{z+w}{2}\right|^2\right) + \left(|w|^2 - \left|\frac{z+w}{2}\right|^2\right).$$

Let $z = m^{\mu}_{I_l} g$, $w = m^{\mu}_{I_r} g$ and $(z+w)/2 = m^{\mu}_I g$. Then (45) is equal, up to a constant, to

$$\sum_{I \in \mathcal{F}(J)} |\sigma(\tilde{I})| \left(|m^{\mu}_I g|^2 - |m^{\mu}_{\tilde{I}} g|^2\right). \tag{47}$$

Adding and subtracting $2|\sigma(I)||m^{\mu}_I g|^2$ we get

(48)
$$\sum_{I \in \mathcal{F}(J)} \left(|\sigma(\tilde{I})| - 2|\sigma(I)|\right) |m^{\mu}_I g|^2 + \sum_{I \in \mathcal{F}(J)} \left(2|\sigma(I)||m^{\mu}_I g|^2 - |\sigma(\tilde{I})||m^{\mu}_{\tilde{I}} g|^2\right).$$

The first summand in the last expression can be bounded by

$$C \sum_{I \in \mathcal{F}(J)} \sin^2 \alpha_I |\sigma(I)||m^{\mu}_I g|^2, \tag{49}$$

because $\left||\sigma(\tilde{I})| - 2|\sigma(I)|\right| = 2|\cos\alpha_I - 1||\sigma(I)| \le C\sin^2\alpha_I|\sigma(I)|$.

This last expression can be bounded in turn using Carleson's Lemma by

$$C \int_J |M^{\mu} g(x)|^2 dx,$$

where

$$M^{\mu} g(x) = \sup_{x \in I \in \mathcal{F}(J)} |m^{\mu}_I g|. \tag{50}$$

Let

$$a_m = \sum_{I\in\mathcal{F}_m(J)} 2|\sigma(I)||m_I^\mu g|^2$$
$$= \sum_{I\in\mathcal{F}_{m+1}(J)} |\sigma(\tilde{I})||m_{\tilde{I}}^\mu g|^2.$$

Clearly the second term in (48) is a telescopic sum for this sequence, hence it equals to $\sum_{n=1}^{\infty}(a_m - a_{m-1}) = \lim_{m\to\infty} a_m - a_0$.

But

$$a_m \le C\int_J g_m^2(x)\,dx;$$

where

$$g_m(x) = \sum_{I\in\mathcal{F}_m(J)} |m_I^\mu g|\chi_I(x).$$

Clearly for all m

$$g_m(x) \le M^\mu g(x),$$

and therefore $a_m \le \|M^\mu g\|_{L^2(J)}^2$.

Finally we can bound (47) by a constant times the L^2 norm of $M^\mu g$, and we will be done as soon as we can show that this maximal function is bounded on $L^2(J)$.

Lemma 13. *The maximal operator M^μ is bounded on $L^2(J)$.*

Proof. By definition

$$m_I^\mu g = \frac{\int_I \omega g\, d\sigma}{\int_I \omega d\sigma}.$$

It is enough to show that ω satisfies a *weighted Reverse Hölder* $(2+\epsilon)$ *condition*; namely, that there exists $\epsilon > 0$ such that for all $I \in \mathcal{F}(J)$

$$\left(\frac{1}{|I|}\int_I |\omega|^{2+\epsilon}dx\right)^{1/2+\epsilon} \le C\frac{1}{|I|}\left|\int_I \omega d\sigma\right|. \tag{51}$$

Let us assume that (51) is true. for $I \in \mathcal{F}(J)$, $g \in L^2(J)$, and by Hölder's inequality with $p = 2+\epsilon$, $q = \frac{2+\epsilon}{1+\epsilon}$ we get

$$|m_I^\mu g|^2 \le \frac{1}{|\frac{1}{|I|}\int_I \omega d\sigma|^2}\left(\frac{1}{|I|}\int_I |\omega|^p|d\sigma|\right)^{2/p}\left(\frac{1}{|I|}\int_I |g|^q|d\sigma|\right)^{2/q}.$$

Since $|d\sigma| \sim dx$ and by (51) we can bound this by

$$C\left[\frac{1}{|I|}\int_I |g(x)|^q dx\right]^{2/q} \le C(M\,|g|^q(y))^{2/q}, \quad y \in I$$

where M is now the ordinary Hardy-Littlewood maximal operator which is bounded in L^s for all $s > 1$. In particular, since $g \in L^2$ then $|g|^q \in L^{2/q}$, where $2/q > 1$ by hypothesis, and therefore,

$$\int_J |M^\mu g|^2 dx \leq C \int_J |M(|g|^q)|^{2/q} dx \leq C \int_J |g|^2 dx.$$

This proves the lemma; the only missing step is (51). □

It is enough to show that ω satisfies (51) for $\epsilon = 0$. This resembles the classical result of Gehring (see [**Ge**]), that says that if a weight satisfies a Reverse Hölder condition of order p, it does satisfy a condition of order $p+\epsilon$ for some positive ϵ.

Lemma 14. *There exists a constant C such that*

$$\frac{1}{|I|}\int_I |\omega|^2 dx \leq C|m_I^\sigma \omega|^2, \quad \forall I \in \mathcal{F}(J).$$

We will prove this lemma at the end, and as a corollary of it and of the precise description of ω, we will conclude that,

Lemma 15. *There exist $\epsilon > 0$ such that* (51) *is true for all $I \in \mathcal{F}(J)$.*

Proof of Lemma 11: Let

(52) $$\omega_k(x) = \prod_{n=0}^{k} \prod_{I \in \mathcal{F}_n(J)} (1 + b_I h_I^\sigma(x)).$$

Notice that by (46)

(53) $$\omega_k(x) = e^{i\sum_{n=0}^{k} s_n(x)\alpha_n(x)} \prod_{n=0}^{k} \cos \alpha_n(x),$$

where for $x \in I \in \mathcal{F}_n(J)$ we define $\alpha_n(x) = \alpha_I$, $\theta_n(x) = \theta_I$, and $s_n(x) = s_I(x) = \begin{cases} 1 & x \in I_r \\ -1 & x \in I_l \end{cases}$. Recall that $\theta_{I_r} = \theta_I - \alpha_I$, $\theta_{I_l} = \theta_I + \alpha_I$; therefore $\theta_{n+1}(x) = \theta_n(x) - s_n(x)\alpha_n(x)$ and $\sum_{n=0}^{k} s_n \alpha_n = \theta_0 - \theta_{k+1}$.

Hence

(54) $$\omega_k(x) = e^{i(\theta_J - \theta_{k+1}(x))} \prod_{n=0}^{k} \cos \alpha_n(x).$$

Clearly $|\omega_k(x)| \leq 1$, therefore $\omega_k \in L^2(J)$ and $\|\omega_k\|_{L^2(J)} \leq C|J|^{1/2}$, $\forall k$. Moreover, $|\omega_{k+1}| \leq |\omega_k|$, hence it is a decreasing sequence. Therefore there is a subsequence convergent to a function $\omega \in L^2(J)$.

We can also say something about a.e. convergence. Since Γ is a Lipschitz graph parametrized by A, then A is differentiable a.e. Let $x \in J$ be a point where $A'(x)$ exists. Clearly $\theta_k(x) \to \arctan A'(x)$. On the other hand, the infinite product $\prod_{n=0}^{\infty} \cos\alpha_n(x)$ converges for each fixed x simultaneously with $\sum_{n=0}^{\infty}(1-\cos\alpha_n(x)) \sim \sum_{n=0}^{\infty} \sin^2\alpha_n(x)$.

But

$$\int_J \left(\sum_{n=0}^{\infty} \sin^2\alpha_n(x)\right) dx = \sum_{n=0}^{\infty} \int_J \left(\sum_{I\in\mathcal{F}_n(J)} \sin^2\alpha_I\, \chi_I(x)\right) dx$$
$$= \sum_{I\in\mathcal{F}(J)} |I| \sin^2\alpha_I;$$

this last expression is bounded by $C|J|$ by the geometric lemma (Lemma 9). Therefore $\sum_{n=0}^{\infty} \sin^2\alpha_n(x) < \infty$ for a.e. $x \in J$.

Hence for a.e $x \in J$

$$\lim_{k\to\infty} \omega_k(x) = e^{i(\theta_J - \arctan A'(x))} \prod_{n=0}^{\infty} \cos\alpha_n(x).$$

In conclusion, ω is well defined as the L^2 limit of the ω_k and also as a pointwise limit; for a.e. x ,

$$\omega(x) = e^{i\theta_J}(1+iA'(x))^{-1} \prod_{n=0}^{\infty} \cos\alpha_n(x). \tag{55}$$

This finishes the proof of the lemma. □

We can safely write

$$\omega(x) = \prod_{n=0}^{\infty}(1+\Delta_n^\sigma b(x)).$$

It is not hard to see that

$$E_j^\sigma \omega(x) = \prod_{n=0}^{j-1}(1+\Delta_n^\sigma b(x)), \tag{56}$$

which is equivalent for $x_I \in I$ to

$$m_I^\sigma \omega = \prod_{I'\supset I}(1+b_{I'}h_{I'}^\sigma(x_I)). \tag{57}$$

To prove this last statement, observe that $\omega = \prod_{n=0}^{j-1}(1+\Delta_n^\sigma b)\prod_{n=j}^{\infty}(1+\Delta_n^\sigma b)$. The first factor is constant for all $x \in I \in \mathcal{F}_j(J)$ and the second factor looks like 1+ sums of products of $\Delta_k^\sigma b$ where $k \geq j$. When we compute the mean value on intervals $I \in \mathcal{F}_j(J)$ we pick the value of the first factor at a point $x_I \in I$ times the mean value of just the function $f(x) = 1$, because all the other summands have mean value zero by (35).

Now (56) implies that $1 + \Delta_n^\sigma b = E_{n+1}^\sigma \omega / E_n^\sigma \omega$, which in turn implies that

$$\Delta_n^\sigma b = \frac{E_{n+1}^\sigma \omega - E_n^\sigma \omega}{E_n^\sigma \omega} = \frac{\Delta_n^\sigma \omega}{E_n^\sigma \omega}.$$

Therefore

$$b_I = \langle b, h_I^\sigma \rangle_\sigma = \frac{\langle \omega, h_I^\sigma \rangle_\sigma}{m_I^\sigma \omega}. \tag{58}$$

Proof of Lemma 14. Because the system $\{h_I^\sigma\}_{I \in \mathcal{F}'(I_o)}$ is a frame for $L^2(I_o)$ and $\omega \in L^2(I_o)$ for all $I_o \in \mathcal{F}(J)$ then

$$\int_{I_o} |\omega|^2 dx \sim \sum_{I \in \mathcal{F}(I_o)} |\langle \omega, h_I^\sigma \rangle_\sigma|^2 + |m_{I_o}^\sigma \omega|^2 |\sigma(I_o)|.$$

But by (58), $\langle \omega, h_I^\sigma \rangle_\sigma = b_I m_I^\sigma \omega$.

Therefore to prove the lemma, it is enough to check that for every $I_o \in \mathcal{F}(J)$

$$\sum_{I \in \mathcal{F}(I_o)} |b_I|^2 |m_I^\sigma \omega|^2 \leq C |m_{I_o}^\sigma \omega|^2 |I_o|.$$

But for $I, I' \in \mathcal{F}(I_o)$ and $x_I \in I$, by (57), and (54)

$$m_I^\sigma \omega = m_{I_o}^\sigma \omega \prod_{I \subset I' \subseteq I_o} (1 + b_{I'} h_{I'}(x_I)) = m_{I_o}^\sigma \omega \, e^{i(\theta_{I_o} - \theta_I)} \prod_{I \subset I' \subseteq I_o} \cos \alpha_{I'}.$$

Hence $|m_I^\sigma \omega| \leq |m_{I_o}^\sigma \omega|$ and since $b_I = i\sigma^{1/2}(I) \sin \alpha_I$ then

$$\sum_{I \in \mathcal{F}(I_o)} |b_I|^2 |m_I^\sigma \omega|^2 \leq |m_{I_o}^\sigma \omega|^2 \sum_{I \in \mathcal{F}(I_o)} |\sigma(I)| \sin^2 \alpha_I.$$

But the second factor on the right hand side is bounded by $C|I_o|$ by the geometric lemma (Lemma 9.)

Notice that the constants involved are independent of the base interval J. This finishes the proof of the lemma. □

Proof of (51) (Lemma 15). We conclude immediately from Lemma 14 that for all $I \in \mathcal{F}(J)$,

$$|m_I^\sigma \omega| \sim m_I |\omega|. \tag{59}$$

This observation and Lemma 14 imply that the weight $|\omega|$ satisfies a Reverse Hölder condition of order two on the intervals of the grid. Namely, for all $I \in \mathcal{F}(J)$,

$$\left(\frac{1}{|I|}\int_I |\omega|^2 dx\right)^{1/2} \le C m_I|\omega|. \tag{60}$$

This is enough to ensure that the weight $|\omega|$ satisfies a Reverse Hölder condition of order $2+\epsilon$, for some $\epsilon > 0$. Namely, for $I \in \mathcal{F}(J)$,

$$\left(\frac{1}{|I|}\int_I |\omega|^{2+\epsilon} dx\right)^{1/2+\epsilon} \le C m_I|\omega|. \tag{61}$$

Since $|m_I^\sigma \omega| \sim m_I|\omega|$ (see (59)), we then get the desired result.

That condition (60) implies condition (61) for some $\epsilon > 0$ is Gehring's Theorem. One can follow word by word the proof in [**G**] p. 260; you need the RH_2 condition to be true on a lot of subintervals of the starting interval J, enough so that a Calderon-Zygmund decomposition argument can be used. Usually the intervals used are those that come from a standard dyadic decomposition of J, but it is straightforward to check that it can also be done if the intervals are given by a regular dyadic grid associated to J.

This finishes the proof of (51). □

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A TYPE OF UNIQUENESS FOR THE DIRICHLET PROBLEM ON A HALF-SPACE WITH CONTINUOUS DATA

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Dedicated to Professor F.-Y. Maeda on his 60th birthday

In this paper, we shall prove a property of the harmonic function H defined on a half-space T which is represented by the generalized Poisson integral with a slowly growing continuous function f on the boundary ∂T of T. Then we shall investigate the difference between H and more general harmonic functions having the same boundary value f on ∂T. These give a kind of positive answer to a question asked by Siegel.

1. Introduction.

Let $\mathbb{R}$ and $\mathbb{R}_+$ be the sets of all real numbers and of all positive real numbers, respectively. We introduce the spherical coordinate (r, Θ), $\Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, in the n-dimensional Euclidean space $\mathbb{R}^n$ $(n \geq 2)$ which are related to the cartesian coordinates (X, y), $X = (x_1, x_2, \ldots, x_{n-1}, y)$ by the formulas

$$x_1 = r\left(\prod_{j=1}^{n-1} \sin\theta_j\right), \qquad y = r\cos\theta_1,$$

and if $n \geq 3$,

$$x_{n+1-k} = r\left(\prod_{j=1}^{k-1} \sin\theta_j\right)\cos\theta_k \qquad (2 \leq k \leq n-1),$$

where

$$0 \leq r < +\infty,\ -2^{-1}\pi \leq \theta_{n-1} < 2^{-1}3\pi$$

and if

$$n \geq 3,\ 0 \leq \theta_j \leq \pi\ (1 \leq j \leq n-2).$$

The unit sphere (the unit circle, if $n = 2$) and the upper half unit sphere $\{(1, \theta_1, \theta_2, \ldots, \theta_{n-1}) \in \mathbb{R}^n;\ 0 \leq \theta_1 < \frac{\pi}{2}\}$ (the upper half circle $\{(1, \theta_1) \in \mathbb{R}^2;$

$-2^{-1}\pi < \theta_1 < 2^{-1}\pi\}$ if $n = 2$) in $\mathbb{R}^n$ $(n \geq 2)$ are denoted by $\mathbb{S}^{n-1}$ and $\mathbb{S}_+^{n-1}$, respectively. The half-space

$$\{(X, y) \in \mathbb{R}^n;\ X \in \mathbb{R}^{n-1}, y > 0\} = \{(r, \Theta) \in \mathbb{R}^n;\ \Theta \in \mathbb{S}_+^{n-1},\ 0 < r < +\infty\}$$

is denoted by $\mathbb{T}_n$. Then the boundary $\partial\mathbb{T}_n$ of $\mathbb{T}_n$ in $\mathbb{R}^n$ $(n \geq 2)$ is identified with $\mathbb{R}^{n-1}$, which is represented as

$$\{Q = (t, \xi) \in \mathbb{R}^{n-1};\quad |Q| = t \geq 0,\ \xi \in \partial\mathbb{S}_+^{n-1}\}$$

by the spherical coordinates, where $\partial\mathbb{S}_+^{n-1}$ is the boundary of $\mathbb{S}_+^{n-1}$ in $\mathbb{S}^{n-1}$ (if $n \geq 3$, then $\partial\mathbb{S}_+^{n-1} = \mathbb{S}^{n-2}$ and if $n = 2$, then $\partial\mathbb{S}_+^1 = \{-\frac{\pi}{2}, \frac{\pi}{2}\}$, $(t, \frac{\pi}{2}) = t \in \mathbb{R}$ and $(t, -\frac{\pi}{2}) = -t \in \mathbb{R}$ $(t \geq 0)$).

Given a continuous function f on $\partial\mathbb{T}_n$, we say that h is a solution of the (classical) Dirichlet problem on $\mathbb{T}_n$ with f, if h is harmonic in $\mathbb{T}_n$ and

$$\lim_{P \in \mathbb{T}_n, P \to Q} h(P) = f(Q)$$

for every $Q \in \partial\mathbb{T}_n$.

Helms [**4**, p. 42 and p. 158] states that even if f is a bounded continuous function on $\partial\mathbb{T}_n$, the solution of the Dirichlet problem on $\mathbb{T}_n$ with f is not unique and to obtain the unique solution $H(P)$ $(P = (X, y) \in \mathbb{T}_n)$ we must specify the behavior of $H(P)$ as $y \to +\infty$. With respect to this fact, Siegel [**6**, Theorems 1] proved the following result. *Let F_ℓ $(\ell \geq 0)$ be the set of continuous functions $f(x)$ on $\mathbb{R}$ such that*

$$\int_{-\infty}^{+\infty} \frac{|f(x)|}{1 + |x|^{2+\ell}}\, dx < +\infty.$$

If $f \in F_\ell$, then there exists a solution $H_{\ell,2}(f)(P)$ of the Dirichlet problem on $\mathbb{T}_2$ with f satisfying

$$H_{\ell,2}(f)(P) = o(r^{\ell+1}/\cos\theta_1) \qquad (r \to +\infty)$$
$$(P = (r\sin\theta_1,\ r\cos\theta_1) \in \mathbb{T}_2).$$

If $h(P)$ is a solution of the Dirichlet problem on $\mathbb{T}_2$ with this f such that

$$h(P) = o(r^{\ell+1}/\cos\theta_1) \qquad (r \to +\infty)\,(P = (r\sin\theta_1,\ r\cos\theta_1) \in \mathbb{T}_2),$$

then

$$h(P) = H_{\ell,2}(f)(P) + U(h)(P)$$

for every $P \in \mathbb{T}_2$, where $U(h)(P)$ is a harmonic polynomial (of $P = (x, y) \in \mathbb{R}^2$) of degree at most ℓ vanishing on $\partial\mathbb{T}_2 = \{(x, 0) \in \mathbb{R}^2;\ x \in \mathbb{R}\}$. Further he stated the following result without proof (Siegel [**6**, Theorem 3]). *Let* ℓ *be a non-negative integer. If* f *is a continuous function on* $\partial\mathbb{T}_n$ $(n \geq 2)$ *such that*

$$(1.1) \qquad |f(Q)| \leq F(x) \qquad (Q \in \partial\mathbb{T}_n = \mathbb{R}^{n-1}, \quad |Q| = x)$$

for some $F(x) \in F_\ell$, $F(x) = F(-x)$ $(x \in \mathbb{R})$, *then there exists a solution* $H_{\ell,n}(f)(P)$ *of the Dirichlet problem on* $\mathbb{T}_n$ *with* f *satisfying*

$$(1.2) \quad H_{\ell,n}(f)(P) = o(r^{\ell+1}/\cos\theta_1) \qquad (r \to +\infty)$$
$$(P = (r, \Theta) \in \mathbb{T}_n, \ \Theta = (\theta_1, \theta_2, \dots, \theta_{n-1})).$$

If $h(P)$ *is a solution of the Dirichlet problem on* $\mathbb{T}_n$ *with this* f *satisfying*

$$(1.3) \quad h(P) = o(r^{\ell+1}/\cos\theta_1) \qquad (r \to +\infty)$$
$$(P = (r, \Theta) \in \mathbb{T}_n, \ \Theta = (\theta_1, \theta_2, \dots, \theta_{n-1})),$$

then

$$h(P) = H_{\ell,n}(f)(P) + U(h)(P) \qquad (P \in \mathbb{T}_n),$$

where $U(h)(P)$ *is a harmonic polynomial of* $P = (x_1, x_2, \dots, x_{n-1}, y) \in \mathbb{R}^n$ *of degree at most* ℓ *vanishing on* $\partial\mathbb{T}_n = \{(X, 0) \in \mathbb{R}^n;\ X \in \mathbb{R}^{n-1}\}$.

In connection with these results, Siegel [**6**, p. 8] asked whether the condition (1.1) of $f(Q)$ can be replaced by more natural condition

$$(1.4) \qquad \int_{\mathbb{R}^{n-1}} \frac{|f(Q)|}{1 + |Q|^{n+\ell}}\, dX < +\infty \qquad (\ell \geq 0),$$

under which $H_{\ell,n}(f)(P)$ exists.

A special case of the following result of Yoshida shows that this question is solved affirmatively in the case where $\ell = 0$. To state it, we need the following notations. Let $\Phi(r, \Theta)$ be a function on $\mathbb{T}_n$. We put

$$N(\Phi)(r) = \int_{\mathbb{S}_+^{n-1}} \Phi(r, \Theta) \cos\theta_1\, d\sigma_\Theta \quad (\Theta = (\theta_1, \theta_2, \dots, \theta_{n-1}))$$

and

$$\mu_0(\Phi) = \lim_{r \to \infty} r^{-1} N(\Phi)(r),$$

if they exist, where $d\sigma_\Theta$ is the surface element on $\mathbb{S}^{n-1}$. Let $G_n(P_1, P_2)(P_1, P_2 \in \mathbb{T}_n)$ be the Green function of $\mathbb{T}_n$. By $K_{0,n}(P, Q)$ $(P \in \mathbb{T}_n,\ Q \in \partial\mathbb{T}_n)$, we denote the ordinary Poisson kernel of $\mathbb{T}_n$

$$c_n^{-1} \frac{\partial}{\partial\nu} G_n(P, Q) = \frac{2y}{s_n}|P - Q|^{-n} \qquad c_n = \begin{cases} 2\pi, & (n = 2) \\ (n-2)s_n, & (n \geq 3) \end{cases},$$

where $\dfrac{\partial}{\partial\nu}$ denotes the differentiation at Q along the inward normal into $\mathbb{T}_n$ and s_n is the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbb{S}^{n-1}$.

Theorem A. (Yoshida [**8**, Theorem 3 and Lemma 3]). *Let $f(Q)$ be a continuous function on $\mathbb{T}_n$ $(n \geq 2)$ satisfying*

$$\int^{+\infty} t^{-2}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\, d\sigma_\xi\right) dt < +\infty, \tag{1.5}$$

where $d\sigma_\xi$ is the surface element of $\partial\mathbb{S}_+^{n-1} = \mathbb{S}^{n-2}$ $(n \geq 3)$ and

$$\int_{\partial\mathbb{S}_+^1} |f(t,\xi)|\, d\sigma_\xi = \left|f\left(t, \frac{\pi}{2}\right)\right| + \left|f\left(t, -\frac{\pi}{2}\right)\right| \quad (n = 2).$$

Then the Poisson integral

$$H_{0,n}(f)(P) = \int_{\partial\mathbb{T}_n} f(Q)K_{0,n}(P,Q)\, d\sigma_Q$$

is a solution of the classical Dirichlet problem on $\mathbb{T}_n$ with f such that

$$\mu_0\left(H_{0,n}(|f|)\right) = 0.$$

If $h(P)$ is a solution of the classical Dirichlet problem on $\mathbb{T}_n$ with this f, then two limits $\mu_0(h)$ $(-\infty < \mu_0(h) \leq +\infty)$ and $\mu_0(|h|)$ $(0 \leq \mu_0(|h|) \leq +\infty)$ exist, and if

$$\mu_0(|h|) < +\infty, \tag{1.6}$$

then

$$h(P) = H_{0,n}(f)(P) + 2ns_n^{-1}\mu_0(h)y \tag{1.7}$$

for any $P = (X, y) \in \mathbb{T}_n$.

We remark that (1.5) is equivalent to

$$\int_{\mathbb{R}^{n-1}} \frac{|f(Q)|}{1 + |Q|^n}\, dQ < +\infty.$$

If h is a solution of the Dirichlet problem on $\mathbb{T}_n$ with this f such that $h = o(r/\cos\theta_1)$ $(r \to \infty)$, then $\mu_0(|h|) = 0$, $\mu_0(h) = 0$ and hence $h(P) = H_{0,n}(f)(P)$. This shows that Theorem A gives a positive answer to Siegel's question in the case where $\ell = 0$. However Theorem A gives a form of h not only in the case where $\mu_0(|h|) = 0$ but also in the case where $0 < \mu_0(|h|) < +\infty$.

In this paper we shall show that a solution of the Dirichlet problem on $\mathbb{T}_n$ with f satisfying (1.4) satisfies a natural condition weaker than (1.2) (Theorem 1) and other solutions with this f satisfying some growth condition different from (1.3) are specified in a certain sense (Theorem 2), which contains a positive answer to Siegel's question in every case (Corollary 1) and gives a generalized form of Theorem A (Corollary 2). We shall also state Theorem 2 in more general form (Theorem 3).

I would like to thank the referee for suggesting a much simpler proof of Lemma 3.

2. Statement of results.

We denote the origin of $\mathbb{R}^n$ by O. Let k $(k \geq 0)$ and n $(n \geq 2)$ be two integers and let $L_{k,n+2}$ be the $(n+2)$-dimensional Legendre polynomial of degree k, where $L_{0,n+2} = 1$. We also put

$$c_{k,n+2} = \binom{k+n-1}{k}.$$

We note that $c_{k,n+2}L_{k,n+2}(t)$ is equal to the ultraspherical (or Gegenbauer) polynomial $P_k^{n/2}$ of degree k associated with $\frac{n}{2}$ (see Stein and Weiss [**7**, p. 148]).

The following theorem gives the Fourier expansion of $K_{0,n}(P,Q)$.

Theorem B. (Armitage [**1**, Theorem E] and Gardiner [**3**, Theorem B]). *Let* $Q = (Z) = (t,\xi) \in \mathbb{R}^{n-1} - \{O\}$, $|Q| = t$, $\xi \in \mathbb{S}^{n-2}$ $(n \geq 2)$. *The function* $J_{k,n,Q}$ *of* $P = (X,y) = (r,\Theta) \in \mathbb{R}^n$, $\Theta = (\theta_1, \theta_2, \dots, \theta_{n-1})$, *given by*

$$(2.1) \quad J_{k,n,Q}(P) = r^{k+1}\cos\theta_1 L_{k,n+2}(\sin\theta_1\cos\gamma)$$

$$(\gamma \quad \textit{is the angle between} \quad (X,0) \quad \textit{and} \quad (Z,0))$$

is a homogeneous harmonic polynomial of degree $k+1$. *Further the function independent of* t *and* r

$$I_{k,n,\xi}(\Theta) = r^{-k-1}J_{k,n,Q}(P)$$

(*which is the restriction to the surface* $\mathbb{S}^{n-1}$ *of* $J_{k,n,Q}(P)$ *and hence a spherical harmonic of degree* $k+1$) *satisfies*

$$(2.2) \qquad |I_{k,n,\xi}(\Theta)| \leq \cos\theta_1$$

for each $P = (r,\Theta) \in \mathbb{R}^n$. *If* $r < t$ *and* $\Theta \in \mathbb{S}_+^{n-1}$ *then* $K_{0,n}(P,Q)$ *is given by*

$$K_{0,n}(P,Q) = \frac{2}{s_n}\sum_{k=0}^{\infty} c_{k,n+2}t^{-k-n}r^{k+1}I_{k,n,\xi}(\Theta).$$

For an integer $\ell \geq 1$ and two points $P = (r, \Theta) \in \mathbb{T}_n$, $Q = (t, \xi) \in \partial\mathbb{T}_n$, we put

$$V_{\ell,n}(P,Q) = \frac{2}{s_n} \sum_{k=0}^{\ell-1} c_{k,n+2} t^{-k-n} r^{k+1} I_{k,n,\xi}(\Theta).$$

We see from Theorem B that for any fixed $Q \in \partial\mathbb{T}_n$ the function $V_{\ell,n}(P,Q)$ of $P \in \mathbb{T}_n$ is harmonic on $\mathbb{T}_n$ and vanishes on $\partial\mathbb{T}_n$. We define another function

$$W_{\ell,n}(P,Q) = \begin{cases} V_{\ell,n}(P,Q) & (P \in \mathbb{T}_n, Q = (t,\xi) \in \partial\mathbb{T}_n,\ 1 \leq t < +\infty) \\ 0 & (P \in \mathbb{T}_n, Q = (t,\xi) \in \partial\mathbb{T}_n,\ 0 \leq t < 1). \end{cases}$$

In addition to $K_{0,n}(P,Q)$, the *Poisson kernel* $K_{\ell,n}(P,Q)$ $(P \in \mathbb{T}_n,\ Q \in \partial\mathbb{T}_n)$ of order $\ell\,(\ell \geq 1)$ is defined by

$$K_{\ell,n}(P,Q) = K_{0,n}(P,Q) - W_{\ell,n}(P,Q)$$

(see Siegel [**6**, p. 7] and also see Armitage [**1**, p. 56]).

Let ℓ be a non-negative integer. Given a function $\Phi(r,\Theta)$ on $\mathbb{T}_n$, we set

$$\mu_\ell(\Phi) = \lim_{r\to\infty} r^{-\ell-1} N(\Phi)(r),$$

if it exists. By $F_{\ell,n}$ we denote the set of continuous functions $f(Q)$ on $\partial\mathbb{T}_n = \mathbb{R}^{n-1}$ $(n \geq 2)$ such that

$$\int_{\mathbb{R}^{n-1}} \frac{|f(Q)|}{1+|Q|^{n+\ell}}\, dQ < +\infty, \tag{2.3}$$

which is equivalent to

$$\int^{+\infty} t^{-\ell-2} \left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\, d\sigma_\xi \right) dt < +\infty.$$

Hence $F_{\ell,2}$ is equal to F_ℓ.

Theorem 1. *Let $\ell\,(\ell \geq 0)$, $n\,(n \geq 2)$ be two integers and $f \in F_{\ell,n}$. Then*

$$H_{\ell,n}(f)(P) = \int_{\partial\mathbb{T}_n} f(Q) K_{\ell,n}(P,Q)\, d\sigma_Q$$

is a solution of the classical Dirichlet problem on $\mathbb{T}_n$ with f satisfying

$$\mu_\ell\left(|H_{\ell,n}(f)|\right) = 0. \tag{2.4}$$

Remark 1. Further, suppose in Theorem 1 that $f \in F_{\ell',n}$ for some ℓ' less than ℓ. Then

$$H_{\ell',n}(f)(P) - H_{\ell,n}(f)(P) = \frac{2}{s_n} \sum_{k=\ell'}^{\ell-1} c_{k,n+2} J^*_{k,n}(f)(P),$$

where

$$J_{k,n}^*(f)(P) = r^{k+1} \int_1^{+\infty} t^{-k-2} \left(\int_{\partial \mathbb{S}_+^{n-1}} I_{k,n,\xi}(\Theta) f(t,\xi)\, d\sigma_\xi \right) dt \quad P = (r,\Theta).$$

We note from (2.2) that

$$\left| J_{k,n}^*(f)(P) \right| \le r^{k+1} \cos\theta_1 \int_1^{+\infty} t^{-k-2} \left(\int_{\partial \mathbb{S}_+^{n-1}} |f(t,\xi)|\, d\sigma_\xi \right) dt < +\infty.$$

Put $J_{k,n,Q}(P) = y\Upsilon_{k,n,Q}(P)$, and observe from (2.1) that $\Upsilon_{k,n,Q}(P)$ is a polynomial of $P = (x_1, x_2, \dots, x_{n-1}, y) \in \mathbb{R}^n$ of degree at most k and even with respect to the variable y. Hence, if we set $J_{k,n}^*(f)(P) = y\Upsilon_{k,n}^*(f)(P)$, then $\Upsilon_{k,n}^*(f)(P)$ is a polynomial of $P = (x_1, x_2, \dots, x_{n-1}, y)$ of degree at most k and even with respect to y ($k = \ell', \ell'+1, \ell'+2, \dots, \ell-1$). Thus

$$H_{\ell,n}(f)(P) = H_{\ell',n}(f)(P) + yL(f)(P),$$

where $L(f)(P)$ is a polynomial of $P = (x_1, x_2, \dots, x_{n-1}, y) \in \mathbb{R}^n$ of degree at most $\ell - 1$ and even with respect to y.

Remark 2. If (1.2) is satisfied, then (2.4) also holds. Since Siegel assumed (1.1) which is stronger than (2.3), he could obtain (1.2). It is interesting to ask whether (1.2) follows under (2.3) or not.

The following result is just a generalization of Picard's theorem stating that a positive harmonic function in the Euclidean space is a constant. *Let $H(r,\Theta)$ be harmonic on $\mathbb{R}^m$ ($m \ge 2$). If, for some positive $t > 1$,*

$$r^{-t-1}\mathcal{M}(H^+)(r) \to 0 \quad (r \to +\infty), \quad \mathcal{M}(H^+)(r) = \int_{\mathbb{S}_+^{m-1}} H^+(r,\Theta)\, d\sigma_\Theta,$$

then for some positive integer ℓ less than t

$$H(r,\Theta) = C + \sum_{k=1}^{\ell} \Xi_k(r,\Theta) \qquad ((r,\Theta) \in \mathbb{R}^m),$$

where C is a constant and $\Xi_k(r,\Theta) = r^k Y_k(\Theta)$ is a homogeneous harmonic polynomial of order k ($Y_k(\Theta)$ is a spherical harmonic function) (see e.g. Brelot [**2**, Appendix; §26]).

It is well known that many results on harmonic functions in $\mathbb{R}^n$ can easily obtained by a passage to $\mathbb{R}^{n+2}$. By using this fact and the result with $m = n+2$ stated above, Kuran proved the following Theorem C. To state it, for a function $\Phi(r,\Theta)$ on $\mathbb{T}_n$ we define

$$\mathcal{D}(y\Phi, r) = (\sigma_r^+)^{-1} \int_{\mathbb{S}_r^+} y\Phi(r,\Theta)\, d\mathbb{S}_r^+,$$

if it exists, where $\mathbb{S}_r^+ = \{(r,\Theta) \in \mathbb{T}_n;\ \Theta \in \mathbb{S}_+^{n-1}\}$, σ_r^+ is the surface area of the spherical part of $\mathbb{S}_r^+$ and $d\mathbb{S}_r^+$ is the surface element of $\mathbb{S}_r^+$.

Theorem C. (Kuran [**5**, Theorem 10]). *Let $h(X,y)$ $(= h(r,\Theta))$ be a harmonic function on $\mathbb{T}_n$ such that h vanishes continuously on $\partial\mathbb{T}_n$.*

If, for some positive t,

$$\lim_{r\to\infty} r^{-t-2}\mathcal{D}(yh^+, r) = 0, \tag{2.5}$$

then

$$h = y\Pi(h)$$

in $\mathbb{T}_n$, where $\Pi(h)$ is a polynomial of $(x_1, x_2, \dots, x_{n-1}, y) \in \mathbb{R}^n$ of degree less than t and even with respect to the variable y.

Remark 3. Let $\Phi(r,\Theta)$ be a function on $\mathbb{T}_n$. Then

$$\mathcal{D}(y\Phi, r) = 2s_n^{-1} r N(\Phi)(r), \tag{2.6}$$

if they exist. Hence (2.5) is equivalent to

$$\lim_{r\to\infty} r^{-(t+1)} N(h^+)(r) = 0.$$

The following theorem answers affirmatively Siegel's question in the case where ℓ is a positive integer.

Theorem 2. *Let ℓ $(\ell \ge 1)$, n $(n \ge 2)$ be two integers and*

$$f \in F_{\ell,n}. \tag{2.7}$$

If $h(r,\Theta)$ is a solution of the Dirichlet problem on $\mathbb{T}_n$ with f satisfying

$$\mu_\ell(h^+) = 0, \tag{2.8}$$

then

$$h(P) = H_{\ell,n}(f)(P) + y\Pi(h)(P) \tag{2.9}$$

for every $P = (X,y) \in \mathbb{T}_n$, where $\Pi(h)(P)$ is a polynomial of $P = (x_1, x_2, \dots, x_{n-1}, y) \in \mathbb{R}^n$ of degree at most $\ell - 1$ and even with respect to the variable y.

The result obtained by Siegel immediately follows from the remark following Theorem A (the case $\ell = 0$) and Theorem 2 (the case $\ell \ge 1$).

Corollary 1. *Let ℓ be a non-negative integer and $f(Q)$ be a continuous function on $\partial\mathbb{T}_n = \mathbb{R}^{n-1}$ $(n \geq 2)$ satisfying*

$$|f(Q)| \leq F(x) \qquad (Q \in \mathbb{R}^{n-1},\ |Q| = x > 0)$$

for some $F(x) \in F_\ell$ $(\ell \geq 0)$, $F(x) = F(-x)$ $(x \in \mathbb{R})$. If $h(P)$ is a solution of the Dirichlet problem on $\mathbb{T}_n$ with f such that

$$h(P) = o(r^{\ell+1}/\cos\theta_1) \quad (r \to \infty) \quad (P = (r, \Theta) \in \mathbb{T}_n)\,,$$

then

$$h(P) = H_{\ell,n}(f)(P) + U(h)(P) \qquad (P = (r, \Theta) \in \mathbb{T}_n)\,,$$

where $U(h)(P)$ is a harmonic polynomial of $P = (x_1, x_2, \dots, x_{n-1}, y) \in \mathbb{R}^n$ of degree at most ℓ vanishing on $\partial\mathbb{T}_n$.

Theorems 1, 2 and Remark 1 also give a generalized form of Theorem A.

Corollary 2. *Let ℓ be a positive integer and $f(Q)$ be a continuous function on $\partial\mathbb{T}_n$ $(n \geq 2)$ satisfying $f \in F_{\ell-1,n}$. Then the Poisson integral*

$$H_{\ell-1,n}(f)(P) = \int_{\partial\mathbb{T}_n} f(Q) K_{\ell-1,n}(P, Q)\, d\sigma_Q$$

is a solution of the classical Dirichlet problem on $\mathbb{T}_n$ with f satisfying

$$\mu_{\ell-1}\left(|H_{\ell-1,n}(f)|\right) = 0. \tag{2.10}$$

If $h(P)$ is any solution of the classical Dirichlet problem on $\mathbb{T}_n$ with this f satisfying

$$\mu_\ell(h^+) = 0,$$

then

$$h(P) = H_{\ell-1,n}(f)(P) = y\Pi^*(h)(P) \tag{2.11}$$

for every $P = (X, y) \in \mathbb{T}_n$, where $\Pi^*(h)(P)$ is a polynomial of P with degree at most $\ell - 1$ and even with respect to the variable y.

Remark 4. Since

$$\mu_{\ell-1}(h) = \mu_{\ell-1}(y\Pi^*(h))$$

from (2.10) and (2.11) and

$$y\Pi^*(h)(P) = r^\ell \varphi(h)(\Theta)\{1 + o(1)\} \qquad (r \to +\infty) \quad (P = (r, \Theta) \in \mathbb{T}_n)$$

for some $\varphi(h)(\Theta)$ on $\mathbb{S}_{n-1}^+$, it follows that

$$\mu_{\ell-1}(h) = \int_{\mathbb{S}_+^{n-1}} \varphi(h)(\Theta) \cos\theta_1 \, d\sigma_\Theta$$

exists. Put $\ell = 1$ in Corollary 2. Then $\Pi^*(h)(P)$ is a constant C and $\mu_0(h) = C\mu_0(y) = \dfrac{C}{2n} s_n$. Thus we obtain (1.7) under the weaker condition $\mu_1(h^+) = 0$ than (1.6).

It may be more desirable to restate Theorem 2 in the following form.

Theorem 3. *If $h(r,\Theta)$ is a solution of the Dirichlet problem on $\mathbb{T}_n$ ($n \geq 2$) with some $f \in F_{\ell,n}$ ($\ell \geq 0$) satisfying*

$$\overline{\lim}_{r\to\infty} \frac{\log N(h^+)(r)}{\log r} < +\infty,$$

then

$$h(P) = H_{\ell,n}(f)(P) + y\Lambda(h)(P)$$

for every $P = (X,y) \in \mathbb{T}_n$, where $\Lambda(h)(P)$ is a polynomial of $P = (x_1, x_2, \dots, x_{n-1}, y) \in \mathbb{R}^n$ and even with respect to the variable y.

3. Proofs of the Theorems 1, 2, 3 and Corollary 2.

For a set E, $E \subset \mathbb{R}_+ \cup \{0\}$, we denote $\{(r,\Theta) \in \mathbb{T}_n;\ r \in E\}$ and $\{(r,\Theta) \in \partial\mathbb{T}_n;\ r \in E\}$ by $\mathbb{T}_n E$ and $\partial\mathbb{T}_n E$, respectively.

Lemma 1. *For a positive integer ℓ we have*

$$|K_{0,n}(P,Q) - V_{\ell,n}(P,Q)| \leq C_1 r^{\ell+1} t^{-n-\ell} \cos\theta_1$$

for any $P = (r,\Theta) \in \mathbb{T}_n$, $\Theta = (\theta_1, \theta_2, \dots, \theta_{n-1})$ and any $Q = (t,\xi) \in \partial\mathbb{T}_n - \{O\}$ ($n \geq 2$) satisfying $0 < \dfrac{2r}{t} \leq 1$, where C_1 is a constant depending only on ℓ and n.

Proof. Take any $P = (r,\Theta) \in \mathbb{T}_n$ and any $Q = (t,\xi) \in \partial\mathbb{T}_n - \{O\}$. Put $R_1 = \dfrac{2r}{t}$, $a = \dfrac{t}{2}$ and $\Theta_1 = \Theta$ in

$$a^{n-2} G_n\left((aR_1, \Theta_1), (aR_2, \Theta_2)\right) = G_n\left((R_1, \Theta_1), (R_2, \Theta_2)\right)$$
$$(a \in \mathbb{R}_+,\ (R_1, \Theta_1),\ (R_2, \Theta_2) \in \mathbb{T}_n).$$

When (R_2, Θ_2) approach to $(2,\xi) \in \partial\mathbb{T}_n$ along the inward normal, we obtain

$$\text{(3.1)} \qquad \left(\frac{1}{2}t\right)^{n-1} K_{0,n}((r,\Theta),\ (t,\xi)) = K_{0,n}\left(\left(\frac{2r}{t}, \Theta\right),\ (2,\xi)\right).$$

Suppose that $0 < \dfrac{2r}{t} \leq 1$. From Theorem B and (2.2) we have that

(3.2)

$$\begin{aligned}
&\left| K_{0,n}\left(\left(\frac{2r}{t}, \Theta\right), (2,\xi)\right) - 2s_n^{-1} \sum_{k=0}^{\ell-1} c_{k,n+2} 2^{-k-n} \left(\frac{2r}{t}\right)^{k+1} I_{k,n,\xi}(\Theta) \right| \\
&\leq s_n^{-1} 2^{-n+1} \sum_{k=\ell}^{\infty} c_{k,n+2} 2^{-k} \left(\frac{2r}{t}\right)^{k+1} |I_{k,n,\xi}(\Theta)| \\
&\leq s_n^{-1} 2^{-n+1} \left(\frac{2r}{t}\right)^{\ell+1} \cos\theta_1 \sum_{k=\ell}^{\infty} c_{k,n+2} 2^{-k}.
\end{aligned}$$

Since

$$\sum_{k=\ell}^{\infty} c_{k,n+2} 2^{-k} = \frac{(n+\ell-1)!}{(n-1)!(\ell-1)!} \int_0^{1/2} \left(\frac{1}{2} - u\right)^{\ell-1} (1-u)^{-n-\ell} \, du = C_1'$$

is finite, we immediately have

$$\begin{aligned}
&\left| K_{0,n}((r,\Theta), (t,\xi)) - 2s_n^{-1} \sum_{k=0}^{\ell-1} c_{k,n+2} t^{-n-k} r^{k+1} I_{k,n,\xi}(\Theta) \right| \\
&\leq C_1 t^{-n-\ell} r^{\ell+1} \cos\theta_1 \qquad (C_1 = 2^{\ell+1} s_n^{-1} C_1')
\end{aligned}$$

from (3.1) and (3.2), which is the conclusion. □

Lemma 2. *Let ℓ be any positive integer. Let $f(Q)$ be a locally integrable function on $\partial\mathbb{T}_n$ $(n \geq 2)$ satisfying (2.3). Then $H_{\ell,n}(f)(P)$ is a harmonic function on $\mathbb{T}_n$.*

Proof. For any fixed $P = (r,\Theta) \in \mathbb{T}_n$, take a number R satisfying $R \geq \max(1, 2r)$. Then from Lemma 1 we have

(3.3)

$$\begin{aligned}
&\int_{\partial\mathbb{T}_n[R,+\infty)} |f(Q)| \, |K_{\ell,n}(P,Q)| \, d\sigma_Q \\
&\qquad = \int_{\partial\mathbb{T}_n[R,+\infty)} |f(Q)| \, |K_{0,n}(P,Q) - V_{\ell,n}(P,Q)| \, d\sigma_Q \\
&\qquad \leq C_1 r^{\ell+1} \cos\theta_1 \int_R^{+\infty} t^{-\ell-2} \left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)| \, d\sigma_\xi \right) dt < +\infty.
\end{aligned}$$

Thus $H_{\ell,n}(f)(P)$ is finite for any $P \in \mathbb{T}_n$. Since $K_{\ell,n}(P,Q)$ is a harmonic function of $P \in \mathbb{T}_n$ for any fixed $Q \in \partial\mathbb{T}_n$, $H_{\ell,n}(f)(P)$ is also a harmonic function of $P \in \mathbb{T}_n$. □

Lemma 3. *Let ℓ be any positive integer. Let $f(Q)$ be a locally integrable and finite-valued upper semicontinuous function on $\partial\mathbb{T}_n$ $(n \geq 2)$ satisfying* (2.3). *Then*

$$\overline{\lim}_{P\to Q^*,\, P\in\mathbb{T}_n} H_{\ell,n}(f)(P) \leq f(Q^*)$$

for any $Q^* \in \partial\mathbb{T}_n$.

Proof. Let $Q^* = (t^*, \xi^*)$ be any fixed point of $\partial\mathbb{T}_n$ and ε be any positive number. Take a positive number δ, $\delta < 1$, such that

$$f(Q) < f(Q^*) + \varepsilon \tag{3.4}$$

for any $Q \in \partial\mathbb{T}_n \cap U_\delta(Q^*)$, where $U_\delta(Q^*) = \{P \in \mathbb{R}^n;\ |P - Q^*| < \delta\}$. From (3.3), we can choose a number R^*, $R^* > 2(t^* + 1)$, such that

$$\int_{\partial\mathbb{T}_n[R^*,+\infty)} |f(Q)|\,|K_{\ell,n}(P,Q)|\, d\sigma_Q < \varepsilon, \tag{3.5}$$

for any $P \in \mathbb{T}_n \cap U_\delta(Q^*)$. Now we write

$$\begin{aligned}
H_{\ell,n}(f)(P) &= \int_{\partial\mathbb{T}_n\cap U_\delta(Q^*)} f(Q)K_{\ell,n}(P,Q)\, d\sigma_Q \\
&\quad + \int_{\partial\mathbb{T}_n[0,R^*)-U_\delta(Q^*)} f(Q)K_{\ell,n}(P,Q)\, d\sigma_Q \\
&\quad + \int_{\partial\mathbb{T}_n[R^*,+\infty)} f(Q)K_{\ell,n}(P,Q)\, d\sigma_Q \\
&= I_1(P) + I_2(P) + I_3(P), \\
I_1(P) &= \int_{\partial\mathbb{T}_n\cap U_\delta(Q^*)} f(Q)K_{0,n}(P,Q)\, d\sigma_Q \\
&\quad - \int_{\partial\mathbb{T}_n\cap U_\delta(Q^*)} f(Q)W_{\ell,n}(P,Q)\, d\sigma_Q \\
&= I_{1,1}(P) + I_{1,2}(P)
\end{aligned}$$

and

$$\begin{aligned}
I_2(P) &= \int_{\partial\mathbb{T}_n[0,R^*)-U_\delta(Q^*)} f(Q)K_{0,n}(P,Q)\, d\sigma_Q \\
&\quad - \int_{\partial\mathbb{T}_n[0,R^*)-U_\delta(Q^*)} f(Q)W_{\ell,n}(P,Q)\, d\sigma_Q \\
&= I_{2,1}(P) + I_{2,2}(P).
\end{aligned}$$

First we see from (3.5) that

$$|I_3(P)| < \varepsilon \tag{3.6}$$

for any $P \in \mathbb{T}_n \cap U_\delta(Q^*)$. Since

$$\begin{aligned} 1 - \int_{\partial\mathbb{T}_n \cap U_\delta(Q^*)} K_{0,n}(P,Q)\, d\sigma_Q &= \int_{\partial\mathbb{T}_n - U_\delta(Q^*)} K_{0,n}(P,Q)\, d\sigma_Q \\ &= \frac{2y}{s_n} \int_{\partial\mathbb{T}_n - U_\delta(Q^*)} |P-Q|^{-n}\, d\sigma_Q \end{aligned}$$

for any $P = (X,y) \in \mathbb{T}_n$, we have

$$\lim_{P\in\mathbb{T}_n,\ P\to Q^*} \int_{\partial\mathbb{T}_n \cap U_\delta(Q^*)} K_{0,n}(P,Q)\, d\sigma_Q = 1$$

and hence from (3.4)

$$\overline{\lim}_{P\in\mathbb{T}_n,\ P\to Q^*} I_{1,1}(P) \le f(Q^*) + \varepsilon. \tag{3.7}$$

Also observe that

$$|I_{2,1}(P)| \le \frac{2y}{s_n}\left(\frac{\delta}{2}\right)^{-n} \int_0^{R^*} t^{n-2}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\, d\sigma_\xi\right) dt \tag{3.8}$$

for any $P = (X,y) \in \mathbb{T}_n \cap U_{\delta/2}(Q^*)$. Since

$$\int_{\partial\mathbb{T}_n[0,R^*)} |f(Q)|\ |W_{\ell,n}(P,Q)|\ d\sigma_Q \le C_2 \cos\theta_1$$

for any $P = (r,\Theta) \in \mathbb{T}_n \cap U_\delta(Q^*)$, $\Theta = (\theta_1, \theta_2, \dots, \theta_{n-1})$, where

$$C_2 = 2s_n^{-1} \sum_{k=0}^{\ell-1} c_{k,n+2}(t^*+1)^{k+1} \int_1^{R^*} t^{-k-2}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\, d\sigma_\xi\right) dt,$$

we obtain that

$$\begin{aligned} |I_{1,2}(P)| &\le \int_{\partial\mathbb{T}_n \cap U_\delta(Q^*)} |f(Q)|\,|W_{\ell,n}(P,Q)|\ d\sigma_Q \\ &\le C_2\cos\theta_1 \to 0 \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} |I_{2,2}(P)| &\le \int_{\partial\mathbb{T}_n[0,R^*) - U_\delta(Q^*)} |f(Q)|\,|W_{\ell,n}(P,Q)|\ d\sigma_Q \\ &\le C_2\cos\theta_1 \to 0, \end{aligned} \tag{3.10}$$

as $P=(r,\Theta)\to Q^*$. All (3.6), (3.7), (3.8), (3.9) and (3.10) give

$$\overline{\lim}_{P\in\mathbb{T}_n,\ P\to Q^*} H_{\ell,n}(f)(P)\le f(Q^*)+2\varepsilon,$$

from which the conclusion immediately follows. □

Proof of Theorem 1. If $\ell=0$, then Theorem 1 is included in Theorem A. Hence we can assume that $\ell\ge 1$. It immediately follows from Lemma 2 and Lemma 3 that $H_{\ell,n}(f)(P)$ is a harmonic function on $\mathbb{T}_n$ and

$$\lim_{p\in\mathbb{T}_n,\ P\to Q} H_{\ell,n}(f)(P)=f(Q^*)$$

for any $Q^*\in\partial\mathbb{T}_n$.

To prove (2.4), we see first that

$$\begin{aligned} N\left(|H_{\ell,n}(f)|\right)(r) &\le \int_{\mathbb{S}_+^{n-1}}\left(\int_{\partial\mathbb{T}_n}|f(Q)|\,|K_{\ell,n}(P,Q)|\,d\sigma_Q\right)\cos\theta_1\,d\sigma_\Theta \\ &= I_1(r)+I_2(r) \end{aligned} \tag{3.11}$$

for any $P=(r,\Theta)\in\mathbb{T}_n$, $\Theta=(\theta_1,\theta_2,\dots,\theta_{n-1})$, where

$$I_1(r)=\int_{\mathbb{S}_+^{n-1}}\left(\int_{\partial\mathbb{T}_n[2r,+\infty)}|f(Q)|\,|K_{\ell,n}(P,Q)|\,d\sigma_Q\right)\cos\theta_1\,d\sigma_\Theta$$

and

$$I_2(r)=\int_{\mathbb{S}_+^{n-1}}\left(\int_{\partial\mathbb{T}_n[0,2r)}|f(Q)|\,|K_{\ell,n}(P,Q)|\,d\sigma_Q\right)\cos\theta_1\,d\sigma_\Theta.$$

Let ε be any positive number. Take a sufficiently large number r_0 such that

$$\int_{2r_0}^{+\infty}t^{-\ell-2}\left(\int_{\partial\mathbb{S}_+^{n-1}}|f(t,\xi)|\,d\sigma_\xi\right)dt<n\left(C_1s_n\right)^{-1}\varepsilon,$$

where C_1 is the constant in Lemma 1. Since

$$\int_{\mathbb{S}_+^{n-1}}\cos^2\theta_1\,d\sigma_\Theta=(2n)^{-1}s_n, \tag{3.12}$$

we have from (3.3)

$$I_1(r)\le\frac{\varepsilon}{2}r^{\ell+1} \tag{3.13}$$

for any $P = (r, \Theta) \in \mathbb{T}_n$, $r \geq r_0$.

Suppose $P = (r, \Theta) \in \mathbb{T}_n[\frac{1}{2}, +\infty)$. For any $Q = (t, \xi) \in \partial\mathbb{T}_n$ $(0 < t \leq 2r)$ we obtain

$$\begin{aligned} |V_{\ell,n}(P,Q)| &\leq 2s_n^{-1} t^{-n} r \cos\theta_1 \sum_{k=0}^{\ell-1} 2^{-k} c_{k,n+2} (2r/t)^k \\ &\leq C_3 t^{-n-\ell+1} r^\ell \cos\theta_1 \qquad \Theta = (\theta_1, \theta_2, \dots, \theta_{n-1}) \end{aligned}$$

from (2.2) and hence

$$|K_{\ell,n}(P,Q)| \leq \begin{cases} K_{0,n}(P,Q) + C_3 r^\ell t^{-n-\ell+1} \cos\theta_1, & (t \geq 1) \\ K_{0,n}(P,Q), & (0 < t < 1), \end{cases}$$

where

$$C_3 = \ell 2^\ell s_n^{-1} \max_{0 \leq k \leq \ell-1} 2^{-k} c_{k,n+2}.$$

Hence we have

$$I_2(r) \leq I_{2,1}(r) + I_{2,2}(r) \tag{3.14}$$

from (3.12), where

$$I_{2,1}(r) = \int_{\partial\mathbb{T}_n[0,2r)} |f(Q)| \left(\int_{\mathbb{S}_+^{n-1}} K_{0,n}(P,Q) \cos\theta_1 \, d\sigma_\Theta \right) d\sigma_Q$$

and

$$I_{2,2}(r) = C_3 (2n)^{-1} s_n r^\ell \int_1^{2r} t^{-\ell-1} \left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)| \, d\sigma_\xi \right) dt.$$

Here, consider the function $K_{0,n}(P,Q)$ of $P = (r, \Theta) \in \mathbb{T}_n$ for any fixed $Q = (t, \xi) \in \partial\mathbb{T}_n$. Then we see from (2.5) that

$$N\left(K_{0,n}\right)(r) = \frac{s_n}{2r} \mathcal{D}\left(yK_{0,n}, r\right)$$

and from Kuran [**5**, Lemma 2] and Helms [**4**, p. 109; Example 2] that

$$n\mathcal{D}\left(yK_{0,n}, r\right) = \begin{cases} 2r^2 s_n^{-1} r^{-n}, & (t \leq r) \\ 2r^2 s_n^{-1} t^{-n}, & (r \leq t) \end{cases},$$

which gives

$$\int_{\mathbb{S}_+^{n-1}} K_{0,n}(P,Q) \cos\theta_1 \, d\sigma_\Theta = \begin{cases} n^{-1} r^{1-n}, & (t \leq r) \\ n^{-1} r t^{-n}, & (r \leq t) \end{cases} \leq n^{-1} r^{1-n}$$

$$(\Theta = (\theta_1, \theta_2, \dots, \theta_{n-1})).$$

Hence we obtain

(3.15)

$$\begin{aligned}I_{2,1}(r) &\leq n^{-1}r^{1-n}\int_0^{2r} t^{n-2}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\,d\sigma_\xi\right)dt\\ &= n^{-1}r^{1-n}\int_0^{1} t^{n-2}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\,d\sigma_\xi\right)dt\\ &\quad + n^{-1}r^{1-n}\int_1^{2r} t^{n-2}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\,d\sigma_\xi\right)dt\\ &\leq C_4 n^{-1}r^{1-n} + n^{-1}r^{1-n}\int_1^{2r} t^{-\ell-1}(2r)^{n+\ell-1}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\,d\sigma_\xi\right)dt\\ &= C_4 n^{-1}r^{1-n} + n^{-1}2^{n+\ell-1}r^{\ell}\psi(r),\end{aligned}$$

where

$$C_4 = \int_0^1 t^{n-2}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\,d\sigma_\xi\right)dt$$

and

$$\psi(r) = \int_1^{2r} t^{-\ell-1}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\,d\sigma_\xi\right)dt.$$

Then

$$I_{2,2}(r) = C_3(2n)^{-1}s_n r^{\ell}\psi(r). \tag{3.16}$$

Thus if we can show

$$\psi(r) = o(r) \qquad (r \to \infty), \tag{3.17}$$

then we have

$$I_{2,1}(r) = o(r^{\ell+1}) \qquad (r \to \infty)$$

from (3.15),

$$I_{2,2}(r) = o(r^{\ell+1}) \qquad (r \to \infty)$$

from (3.16) and hence from (3.14) we can find a number r_1 such that

$$I_2(r) < \frac{\varepsilon}{2}r^{\ell+1} \tag{3.18}$$

for any $r \geq r_1$.

To see (3.17), we note that $\psi(r)$ is increasing,

$$\int_1^{+\infty} \frac{\psi'(r)}{r}\, dr = 2\int_2^{+\infty} t^{-\ell-2}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\, d\sigma_\xi\right) dt \leq 2C_5$$

and

$$\frac{\psi(r)}{r} \leq 2\int_1^{2r} t^{-\ell-2}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\, d\sigma_\xi\right) dt < 2C_5,$$

where

$$C_5 = \int_1^{\infty} t^{-\ell-2}\left(\int_{\partial\mathbb{S}_+^{n-1}} |f(t,\xi)|\, d\sigma_\xi\right) dt.$$

From these we see

$$\int_1^{+\infty} r^{-2}\psi(r)\, dr < +\infty$$

by the integration by parts. Since

$$\frac{\psi(r)}{r} = \psi(r)\int_r^{+\infty} x^{-2}\, dx \leq \int_r^{+\infty} x^{-2}\psi(x)\, dx,$$

this gives (3.17).

If we put $r_2 = \max(r_0, r_1)$, then we finally have from (3.11), (3.13) and (3.18)

$$r^{-\ell-1}N\left(|H_{\ell,n}(f)|\right)(r) < \varepsilon$$

for any r, $r \geq r_2$, which gives (2.14). □

Proof of Theorem 2. Consider the function $h - H_{\ell,n}(f)$. Then it follows from Theorem 1 that this is harmonic in $\mathbb{T}_n$ and vanishes continuously on $\partial\mathbb{T}_n$. Since

$$0 \leq \{h - H_{\ell,n}(f)\}^+(P) \leq h^+(P) + \{H_{\ell,n}(f)\}^-(P) \tag{3.19}$$

for any $P \in \mathbb{T}_n$ and

$$\mu_\ell\left(\{H_{\ell,n}(f)\}^-\right) = 0$$

from (2.4) of Theorem 1, (2.8) gives that

$$\mu_\ell\left(\{h - H_{\ell,n}(f)\}^+\right) = 0.$$

From Remark 3 and Theorem C we see that

$$h(P) - H_{\ell,n}(f)(P) = y\Pi(h)(P)$$

for every $P = (X, y) \in \mathbb{T}_n$, where $\Pi(h)$ is a polynomial in $\mathbb{R}^n$ of degree at most $\ell - 1$ and even with respect to the variable y, which gives the conclusion of Theorem 2. □

Proof of Corollary 2. The first part follows from Theorem 1. Since $f \in F_{\ell,n}$, Theorem 2 gives

$$h(P) = H_{\ell,n}(f)(P) + y\Pi(h)(P)$$

for every $P = (X, y) \in \mathbb{T}_n$, where $\Pi(h)(P)$ is a polynomial of $P \in \mathbb{R}^n$ with degree at most $\ell - 1$ and even with respect to the variable y. Remark 1 also gives

$$H_{\ell,n}(f)(P) = H_{\ell-1,n}(f)(P) + yL(f)(P)$$

for every $P = (X, y) \in \mathbb{T}_n$, where $L(f)(P)$ is a polynomial of $P \in \mathbb{R}^n$ with degree at most $\ell - 1$ and even with respect to the variable y. From these, we evidently obtain (2.11). □

Proof of Theorem 3. Put

$$\overline{\lim}_{r\to\infty} \frac{\log N(h^+)(r)}{\log r} = \alpha.$$

It immediately follows that $\mu_{[\alpha]+1}(h^+) = 0$. Take an integer ℓ^* satisfying $\ell^* \geq \max(\ell, [\alpha] + 1)$. Since $f \in F_{\ell^*,n}$ and $\mu_{\ell^*}(h^+) = 0$, Theorem 2 gives that

$$h(P) = H_{\ell^*,n}(f)(P) + y\Pi(h)(P), \tag{3.20}$$

where $\Pi(h)(P)$ is a polynomial of P and even with respect to y. If $\ell = \ell^*$, then (3.20) gives the conclusion. Suppose that $\ell^* > \ell$. From Remark 1 we also see

$$H_{\ell^*,n}(f)(P) = H_{\ell,n}(f)(P) + yL(f)(P), \tag{3.21}$$

where $L(f)(P)$ is a polynomial of P and even with respect to y. From (3.20) and (3.21) we have

$$h(P) = H_{\ell,n}(f)(P) + y\Lambda(h)(P), \quad \Lambda(h)(P) = \Pi(h)(P) + L(h)(P),$$

which is also the conclusion of Theorem 3. □

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