ISOMETRIC IMMERSIONS OF $H^n_1$ INTO $H^{n+1}_1$

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The complete codimension 1 totally geodesic laminations of the hyperquadrics of constant curvature 1 or $-1$ are completely determined. Also the set of all codimension 1 isometric immersions of a hyperquadric into another of the same constant curvature are determined and characterized in terms of the naturally associated totally geodesic laminations and the curvature of the laminations. Note that the hyperquadrics of constant curvature 1 and $-1$ are often called the de Sitter space-time and the anti de Sitter space-time, respectively.

1. Introduction.

A classical problem in differential geometry is to characterize and determine all the possible submanifolds in space forms. To this end, there have been various geometrically sensible conditions imposed upon in an effort to make the problem more realistically viable. One of such is to restrict the submanifolds to being of codimension 1 and of the same constant curvature as the ambient space form.

In this restricted situation, the original problem has received much attention and, indeed, has seen much progress. In particular, the problem has fundamentally been settled in the positive definite case. In the indefinite case, Graves [5] gave the answer when the metric is Lorentzian and the constant curvature equals 0. For non-zero constant curvature cases, there have been a number of attempts made with limited success, see [6] for instance. One of the main causes hampering a further advance in the indefinite case was the lack of a completeness theorem on the leaves of the associated totally geodesic foliations. Now that the completeness theorem [1] is available, the time has come to address the problem in the indefinite case from a more comprehensive point of view.

In this paper, we restrict ourselves to the Lorentzian case with non-zero constant curvatures, namely, 1 or $-1$. The rationale for this restriction will become self-evident later in the paper.

Let $S_p^n$ and $H_p^n$ be the n-dimensional hyperquadrics of signature $p$ and of constant curvatures 1 and $-1$, respectively. Among others, we will show the following:
a) Only hyperquadrics which can support a complete totally geodesic foliation of codimension 1 in an open subset are either $H_0^n = H^n$, $H_1^n$, $n \geq 1$ or $S^n_{n-1}$, $2 \geq n \geq 1$.

b) For $n \geq 3$, only isometric immersions $f : S^n_1 \to S^{n+1}_1$ are totally geodesic ones; therefore, they are congruent to the standard imbedding of $S^n_1$ into $S^{n+1}_1$.

c) We determine all the complete degenerate totally geodesic foliations in any open subset of $H^n_1$, $n \geq 1$.

d) We give a characterization of the space of isometric immersions from $H^n_1$ into $H^{n+1}_1$ in terms of totally geodesic laminations and curvature functions.

In particular, (d) may be regarded as a form of answer to the problem posed at the beginning for the Lorentzian hyperbolic (or so-called anti-De Sitter) spaces.

2. Hyperquadrics.

Let $R^{n+1}_p$ be $R^{n+1}$ equipped with the quadratic form

\[ q(X) = -\sum_{j=1}^{p} x_j^2 + \sum_{j=p+1}^{n+1} x_j^2, \]

\[ \forall X = (x_1, \cdots, x_{n+1}) \in R^{n+1}_p. \]

The quadratic form $q$ then naturally induces an indefinite but non-degenerate symmetric bilinear functional $g$ by

\[ g(X, Y) = -\sum_{j=1}^{p} x_j y_j + \sum_{j=p+1}^{n+1} x_j y_j, \]

\[ \forall X = (x_1, \cdots, x_{n+1}), Y = (y_1, \cdots, y_{n+1}) \in R^{n+1}. \]

Set

\[ S^n_p = \{ \forall X \in R^{n+1}_p : q(X) = 1 \} \]

and

\[ H^n_p = \{ \forall X \in R^{n+1}_p : q(X) = -1 \}. \]

It is quite elementary to see that the map $\sigma : R^{n+1}_p \to R^{n+1}_{n-p+1}$ given by

\[ \sigma(x_1, \cdots, x_{n+1}) = (x_{p+1}, \cdots, x_{n+1}, x_1, \cdots, x_p), \]

\[ \forall X = (x_1, \cdots, x_{n+1}) \in R^{n+1}_p. \]
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is an anti-isometry from $S^n_p$ onto $H^{n-p}_n$. It is also equally elementary to see that $S^n_p$ is diffeomorphic to $R^p \times S^{n-p}$ and $H^n_p$ is diffeomorphic to $S^p \times R^{n-p}$, respectively. The first diffeomorphism $\phi$, for instance, is given by

$$\phi(x, y) = \left(x, \sqrt{1 + \left| x \right|^2 y} \right) \in R^p \times R^{n+1-p} \approx R^{n+1}_p.$$  

We will call $S^n_p$ and $H^n_p$ the (unit) hyperquadrics of dimension $n$ and the signature $p$. It is well-known that the quadratic form $q$ or the symmetric bilinear form $g$ induces an indefinite Riemannian structure, denoted by the same letter $g$, on the hyperquadrics, see [2], [7]. With the induced indefinite Riemannian structure, $S^n_p$ and $H^n_p$ have constant curvatures $1$ and $-1$, respectively. Note that $S^n_0 = S^n$, the ordinary unit n-sphere and $H^n_0 = H^n$, the hyperbolic space of dimension $n$. These are the only hyperquadrics which are Riemannian.

Among the hyperquadrics introduced above, only ones that are not simply connected are $H^n_1 \approx S^{n-1}_n$.

Define a map $F : R^n \to R^{n+1}_2$ by

$$F(x) = \left(\sqrt{1 + \sum_{j=2}^n (x_j)^2 \cos x_1}, \sqrt{1 + \sum_{j=2}^n (x_j)^2 \sin x_1, x_2, \ldots, x_n}\right),$$

where $x = (x_1, \ldots, x_n) \in R^n$. It is easy to see that $F$ actually induces a map $R^n$ onto $H^n_1$. Its derivative, for example for $n = 3$, is given by:

$$\frac{\partial F}{\partial x_1} = \left(\sqrt{1 + \sum_{j=2}^3 (x_j)^2 \sin x_1}, \sqrt{1 + \sum_{j=2}^3 (x_j)^2 \cos x_1, 0, 0}\right),$$

$$\frac{\partial F}{\partial x_2} = \left(\frac{x_2}{\sqrt{1 + \sum_{j=2}^3 (x_j)^2}} \cos x_1, \frac{x_2}{\sqrt{1 + \sum_{j=2}^3 (x_j)^2}} \sin x_1, 1, 0\right),$$

$$\frac{\partial F}{\partial x_3} = \left(\frac{x_3}{\sqrt{1 + \sum_{j=2}^3 (x_j)^2}} \cos x_1, \frac{x_3}{\sqrt{1 + \sum_{j=2}^3 (x_j)^2}} \sin x_1, 0, 1\right).$$

The metric with respect to the basis $\{\partial F/\partial x_1, \partial F/\partial x_2, \partial F/\partial x_3\}$ is given in the following matrix form:

$$\begin{bmatrix}
-\left(1 + \sum_{j=2}^n (x_j)^2\right) & 0 & 0 \\
0 & \frac{1+(x_3)^2}{1+\sum_{j=2}^n (x_j)^2} & -\frac{x_2x_3}{1+\sum_{j=2}^n (x_j)^2} \\
0 & -\frac{x_2x_3}{1+\sum_{j=2}^n (x_j)^2} & \frac{1+(x_2)^2}{1+\sum_{j=2}^n (x_j)^2}
\end{bmatrix}.$$
The derivative for the general case can be expressed in a similar way. Rank \( F^* \) is easily seen to be 3 in the above case and \( n \) in the general case. Hence, \( F \) is an immersion. Indeed, it is an isometric immersion with the above induced metric. Therefore, \( F \) is a covering projection. We denote by \( \hat{H}^n_1 \) the space \( R^n \) endowed with the above induced metric and call it the n-dimensional hyperbolic space form of constant curvature \(-1\) with signature 1. Note that \( \hat{H}^n_1 \) is simply connected. All other hyperquadrics except \( S^1 \) are simply connected.

Any geodesic in \( \hat{H}^n_1 \) is mapped under \( F \) onto a geodesic in \( H^1_1 \) and the preimage of a geodesic in \( H^1_1 \) is a disjoint union of geodesics in \( \hat{H}^n_1 \).

A totally geodesic submanifold in a hyperquadric is, in the spirit of the Kleinian approach, defined as follows: A complete totally geodesic submanifold of dimension \( k \) in a hyperquadric is the intersection of the hyperquadric and a \((k+1)\)-dimensional linear subspace of \( R^{n+1}_p \). Obviously, a totally geodesic submanifold of a hyperquadric is also a hyperquadric in the linear subspace used to define it. From now on in this paper, a totally geodesic submanifold always means a complete one unless otherwise specified.

Given an open subset \( U \) in a hyperquadric, a (complete) \( c^k \)-totally geodesic foliation of \( U \) is a \( c^k \)-foliation of \( U \) whose leaves are (complete) totally geodesic submanifolds of constant dimension.

Let \( \{U_\alpha\}, \alpha \in \mathcal{A} \) be a collection of mutually disjoint connected open subsets of the hyperquadric. Let \( \mathcal{F}_\alpha, \forall \alpha \in \mathcal{A} \) be a \( c^k \)-totally geodesic foliation of dimension \( q \) defined in \( U_\alpha \). Then the collection of the pairs \( \{U_\alpha, \mathcal{F}_\alpha\}, \alpha \in \mathcal{A} \), will be called a \( c^k \)-lamination of dimension \( q \) on the hyperquadric. We will often call them just a differentiable lamination. Note that the original definition of a lamination is due to Thurston.

Now let \( V(n, r) \) be the Stiefel manifold of ordered \( r \)-frames in \( R^n \). It is well known that \( V(n, r) \to V(n, 1) \) forms a principal fiber bundle in a natural way. Denote by \( \rho \) the largest integer such that the fibration \( V(n, \rho) \to V(n, 1) \) has a global section. Define by \( \nu(n) \) the largest number so that \( \rho(n - \nu(n)) \geq \nu(n) + 1 \). Some of the numerical values for \( \nu(n) \) are given as follows: \( \nu(1) = 0, \nu(2) = 0, \nu(3) = 1, \nu(4) = 0, \nu(5) = 1, \nu(6) = 2, \nu(7) = 3 \) etc.

**Theorem 1.** Let \( \mathcal{F} \) be a (geodesically) complete totally geodesic foliation of an open subset \( U \subset S^n_p \) of dimension \( q \), then \( q - p \leq \nu(n - p) \). In particular, a codimension 1 complete totally geodesic foliation on \( U \) can exist only when \( n - p = 0 \) or 1.

**Proof.** Decompose \( R^{n+1}_p = R^p_p \bigoplus R^{n+1-1}_0 \). Given a \((q+1)\)-dimensional subspace \( L \subset R^{n+1}_p \), \( L \cap R^{n+1-1}_0 \) has dimension at least \( q - p + 1 \). The totally geodesic submanifold of \( S^n_p \) defined by the linear subspace \( R^{n+1-1}_0 \)
is the ordinary \((n-p)\)-sphere. Thus the subspace \(L \cap R_0^{n+1-p}\) defines a complete totally geodesic submanifold of dimension \(q-p\) in \(S_0^{n-p}\). Hence, \(\mathcal{F}\) induces a complete totally geodesic foliation on \(U \cap S_0^{n-p}\). The desired inequality then follows from the arguments given in [1]. For the second half, if \(q = n - 1\), the induced foliation will have dimension \(n - p - 1\). From the above inequality, this is possible only when \(n = p\) or \(n = p + 1\).

In view of the above anti-isometry between \(S_p^n\) and \(H^n_{n-p}\), only hyper-quadrics that can support a complete totally geodesic foliation of codimension 1 in an open subset are either \(H^n_0\) or \(H^n_1\), \(n = 1, 2, \ldots\).

The totally geodesic foliations of codimension 1 in \(H^n\) were treated in [3]. We will, in this paper, concentrate on those in \(H^n_1\).

Let \(U\) be a connected open subset of \(H^n_1\). Let \(\mathcal{F}\) be a complete differentiable totally geodesic foliation of \(U\). Given a leaf \(L_x\) of the foliation passing through \(x\), one can find a regular curve \(\mu(s), s \in (-\epsilon, \epsilon), \epsilon > 0\), transversal to the foliation in a neighborhood of \(x\). If the leaf \(L_x\) is non-degenerate, take the unit normal vector field to the foliation along the curve \(\mu(s)\). Translate parallelly the vector field along the leaves passing through \(\mu(s)\). Denote by \(X\) the resulting vector field in a neighborhood of the leaf \(L\). If \(L\) happens to be a degenerate leaf, there is a light-like vector which is transversal to the leaf \(L\) at \(x\). Extend that vector to a vector field along \(\mu(s)\) through the parallel displacement in a neighborhood of \(x\). Then as before, translate it parallelly along the leaves. Denote the resulting vector field by \(X\). Also denote by \(\nabla\) the Levi-Civita connection in \(\tilde{H}^n_1\) (or in \(H^n_1\)), induced from the quadratic form \(q\) given in (1).

Given a vector field \(Z\) tangent to the leaves, define a field of linear operator \(c_Z(x)\) from \(\text{Span} \{X\}\) into itself at \(x\) by

\[
(3) \quad c_Z(x)(X) = -P(\nabla_X Z),
\]

where \(P\) is the natural projection of \(T_x H^n_1\) onto \(\text{Span} \{X\}\) with respect to the direct sum decomposition \(T_x H^n_1 = T_x L \oplus \text{Span} \{X\}\).

Let \(\gamma = \gamma(t), -\infty < t < \infty\), be the geodesic in \(L_x\) such that \(\gamma(0) = x\) and \(\dot{\gamma}(0) = Z\). It is then shown in [1] that the operator \(c_Z\) along \(L_x\) satisfies a Ricatti type differential equation

\[
(4) \quad \frac{dy}{dt} = y^2 - \langle Z(t), Z(t) \rangle.
\]

This differential equation has two types of global solutions

\[
c_Z(t) = -\frac{\sinh t - c_Z(0) \cosh t}{\cosh t - c_Z(0) \sinh t}, \quad \text{if } \langle Z, Z \rangle = -1
\]
or

\[ c_Z(t) = 0, \forall t \in R, \text{ if } \langle Z, Z \rangle = 0. \]

In the former case, \( c_Z(0) \) is a constant satisfying \( |c_Z(0)| \leq 1 \). As a consequence, we have that \( |c_Z(t)| \leq 1, \forall t \in R \) and \( |c_Z(0)| = 1 \) if and only if \( c_Z(t) \equiv 1 \) or \(-1 \) \( \forall t \in R \).

The laminations on \( H^n = H^n_0 \) have been taken up in various occasions from the point of recent developments in hyperbolic geometry. From differential geometric point of view, we have made use of the notion to describe the isometric immersions of codimension 1 of a hyperbolic space into another hyperbolic space of the same constant curvature \( -1 \) [3].

In what follows, we will describe all the codimension 1 laminations on \( H^n_0 = H^n \) and \( H^n_1 \approx S^{n-1}_0 \). From the Kleinian point of view, there is a striking dual nature between these two spaces.

In order to explain our construction most effectively, we will start out with the lowest dimension, that is, when \( n = 2 \). In this case, \( H^2_1 \) is anti-isometric to \( S^2_1 \). The anti-isometry maps the totally geodesic submanifolds of one to those of the other. The codimension 1 lamination is exactly a geodesic lamination in this case.

Denote by \( P^2 \) the real projective space of dimension 2. According to the Klein model of \( H^2 \), it is given as the unit disk in \( P^2 \) represented by the homogeneous coordinates \( \{(1, \xi_1, \xi_2) : \xi_1^2 + \xi_2^2 < 1\} \), where the first coordinate 1 represents the time like coordinate in \( R^3_1 \). It is well-known that the boundary of the unit disk is represented by the null cone. The complement of \( H^2 \) in \( P^2 \) then represents \( S^2_1 \). The geodesics of \( H^2 \), the chords in the disk, are given as the segments of lines inside the disk in \( R^2(\xi_1, \xi_2) \). Similarly, a geodesic in \( S^2_1 \) is represented by the portion of a line outside the disk in \( R^2(\xi_1, \xi_2) \). Two geodesics in \( S^2_1 \) do not meet in \( S^2_1 \) if and only if the lines representing them intersect inside the closed unit disk. Note that the lines are considered to be projective lines; hence they always intersect in \( P^2 \).

Now let us denote by \( S^1 \) the center circle in \( S^2_1 \), which is represented by the homogeneous coordinates \( (0, y_1, y_2) \) in \( P^2 \). Let \( \{U, \mathcal{F}\} \) be a complete totally geodesic foliation in an open subset \( U \subset S^2_1 \). Since all the leaves of the foliation meet \( .S^1 \) at a unique point (projectively), the foliation will be uniquely characterized by a function from \( U \cap S^1 \) into the set of all geodesics in \( S^2_1 \).

The set of all line in \( P^2 \) consists of the set of all lines in \( R^2(\xi_1, \xi_2) \) and the line of infinity. The set of all oriented lines in \( R^2(\xi_1, \xi_2) \) is parametrized by two parameters \( (a, \theta) \in R \times S^1 \). The actual correspondence is elementary.
and is given by

\[(a, \theta) \mapsto \xi_1 \cos \theta + \xi_2 \sin \theta = a.\]

\(\theta\) and \(\pi + \theta\) give the same line with the opposite orientations. With this parametrization the set of all oriented lines in \(\mathbb{R}^2(\xi_1, \xi_2)\) becomes a differentiable manifold, i.e., the cylinder built on the unit circle. For an open subset \(U \subset S^1_i\), \(U \cap S^1\) is a disjoint union of open intervals or the center circle itself. Since the center circle always intersects any other complete geodesics, it cannot be a leaf of a complete totally geodesic foliation of any open subset of \(S^2_i\). Thus any such a foliation is completely determined by a differentiable function from a connected open subset of the center circle into the cylinder. As any two lines in \(P^2\) always intersect, we can further restrict the range space of the function by requiring that \(0 \leq a \leq 1\). Thus, the range should be in \([0,1] \times S^1\). The function must be an immersion as we shall see later. The following lemma will be useful later.

**Lemma 1.** Let \(\mathcal{F}\) be a codimension one totally geodesic foliation defined in a connected open subset \(U \subset H^n_i\). Assume that the leaves of the foliation are degenerate. Let \(L_1\) and \(L_2\) be two leaves of the foliation. Then they contain a unique light like line and the tangent space to the leaves at the points in the center circle are parallel to each other along the center circle.

**Proof.** By definition \(L_1\) and \(L_2\) are given as the intersections of \(n\)-planes \(P_1\) and \(P_2\) with \(H^n_i\) in \(R^{n+1}\), respectively. Then \(P_1 \cap P_2\) has dimension at least \(n - 1\). Let \(L_1\) and \(L_2\) meet the center circle at \(x_1\) and \(x_2\), respectively. If we also denote the tangent planes of \(L_1\) and \(L_2\) at \(x_1\) and \(x_2\) by \(T_{x_1}L_1\) and \(T_{x_2}L_2\), respectively, there are unique degenerate lines \(l_1\) and \(l_2\) in \(T_{x_1}L_1\) and \(T_{x_2}L_2\), respectively. For, if there is another degenerate line, say \(l\), in \(T_{x_1}L_1\) then light-like vectors in these two lines will have non-vanishing inner product; hence contradicting the degeneracy.

Let \(\gamma = \gamma(s), a \leq s \leq b\) be the portion of the center circle given by \(\gamma = U \cap S^1\) such that \(\gamma(a) \in L_1\) and \(\gamma(b) \in L_2\). From the above argument one sees that each leaf along the curve \(\gamma(s)\) contains a unique light-like line which form a line bundle over \(\gamma(s)\).

At this point, we digress to discuss a special but useful choice of a basis for \(R^n_i\). Let \((x_1, x_2, \ldots, x_n)\) be the standard coordinates of \(R^n_i\) so that the inner product is given by \(q(x) = -x_1^2 + \sum_{j=2}^{n} x_j^2\). Given a null direction in \(R^n_i\), we may assume that the null direction is given by the vector of the form \((y, y, 0, \ldots, 0) \in R^n_i\) up to an isometry of \(R^n_i\). Now let \(e = (1/\sqrt{2})(1, -1, 0, \ldots, 0)\) and \(\bar{e} = (1/\sqrt{2})(1, 1, 0, \ldots, 0)\). Clearly these two vectors are null vectors and \(\langle e, \bar{e} \rangle = -1\). They span the space...
If we denote by \( e_j, 1 \leq j \leq n - 2 \), the standard orthonormal basis for the orthogonal complement of \( R^2_1 \) in \( R^{n+1}_1 \), our desired basis consists of \( \{ e, \bar{e}, e_j, 1 \leq j \leq n - 2 \} \). We call this basis an adapted basis for the degenerate space, which is the span of \( \{ e, e_j, 1 \leq j \leq n - 2 \} \). Conversely, given a degenerate subspace of codimension one, one can always obtain an adapted basis up to an isometry.

Going back to the line bundle of null vectors along \( \gamma = \gamma(s) \), we can choose a frame \( \{ e(s), \bar{e}(s), e_j(s), 1 \leq j \leq n - 2 \} \) along the curve \( \gamma(s), a \leq s \leq b \), such that \( e(s) \) spans the null direction of the leaf at \( \gamma(s) \) and the frame forms an adapted basis for the leaf.

As before, denote by \( \nabla \) the Levi-Civita connection in \( \tilde{H}_1^n \) induced from the quadratic form \( q \) in (1). Then we have

\[
\nabla_{\gamma(s)} e(s) = e_L(s) + c_{e(s)}(\dot{\gamma}(s)).
\]

Here \( c_{e(s)} \) is the conullity operator at \( \gamma(s) \) and \( e_L(s) \) is the leaf component of \( \nabla_{\gamma(s)} e(s) \). See [1] for more details about the conullity operator and its properties in the indefinite setting. Since \( \langle e(t), e(t) \rangle = 0, \forall t \), the Ricatti equation is \( dy/dt = y^2 \) and the trivial solution \( y(t) = 0 \) is the only global solution along the null line spanned by \( e(s) \). Thus \( c_{e(s)} = 0, a \leq s \leq b \). This tells us that the parallel displacement of \( e(s) \) along the curve \( \gamma = \gamma(s) \) always stays in the leaves. But the parallel displacement of a null vector is a null vector; hence, by the above uniqueness, \( e(s), a \leq s \leq b \) are parallel to each other along the curve \( \gamma = \gamma(s) \). Next let \( e(s) \) and \( \bar{e}(s) \) be the parallel displacement of \( e(a) \) and \( \bar{e}(a) \) along the curve \( \gamma = \gamma(s) \), respectively. Denote by \( V(s) \) the orthogonal complement of the span of \( e(s) \) and \( \bar{e}(s) \) in \( T_{\gamma(s)} \tilde{H}_1^n, s \in [a, b] \). Then we see that the span of \( e(s) \) and \( V(s) \) is precisely the tangent space of the leaf at the point \( \gamma(s) \). For, suppose that there is a vector in the tangent space of the leaf given in the form of \( u e(s) + v \bar{e}(s) + W, u, v \in R; W \in V(s) \). \( \langle e(s), u e(s) + v \bar{e}(s) + W \rangle = -v = 0 \) implies that the vector is in the span of \( e(s) \) and \( V(s) \). Since the span of \( e(s) \) and \( \bar{e}(s) \) is parallel along \( \gamma(s) \), \( V(s) \) is also parallel; hence, the span of \( e(s) \) and \( V(s) \) is parallel along the curve \( \gamma(s) \).

The parallel displacement in \( H_1^n \) along the center curve is given by the rotations in the plane spanned by two coordinates of \( R^{n+1}_2 \):

\[
(x_1, x_2, x_3, \cdots, x_{n+2}) \mapsto (x_1 \cos s - x_2 \sin s, x_1 \sin s + x_2 \cos s, x_3, \cdots, x_{n+2}).
\]

Let \( L \) be the tangent space to a leaf at a point in the center circle, say \( x = (x_1 \cos s - x_2 \sin s, x_1 \sin s + x_2 \cos s, x_3, \cdots, x_{n+2}) \). Then the leaf itself is given as the intersection of \( H_1^n \) and the n-plane spanned by \( L \) and \( x = (x_1 \cos s - x_2 \sin s, x_1 \sin s + x_2 \cos s, x_3, \cdots, x_{n+2}) \).
Remark. The arguments in the proof can easily be applied to more general situations. Indeed, we may choose any transversal curve to the foliation. We will still have the same conclusion as in Lemma 1.

We will give some useful examples of complete totally geodesic foliations.

**Example 1.** Note that any differentiable function \( f \) from a connected open subset \( \mathcal{O} \subset S^1 \) into the set of non-super parallel geodesics in \( H^2 \) will define a differentiable totally geodesic foliation of the open subset \( U = \bigcup_{s \in \mathcal{O}} f(s) \subset S^2_1 \). Let \( \mathcal{P}_w \) be the differentiable family of lines in \( R^2(\xi_1, \xi_2) \), where \( w \) is in the unit circle and is the common intersection point of all of the lines in the family. The family is otherwise known as the parabolic pencil in \( H^2 \), consisting of the parallel geodesics whose end points are \( w \). For a given point \( s \in \mathcal{O} \), there is a (projectively) unique line \( \ell_s \in \mathcal{P}_w \) which meets \( s \) in the line of infinity. We define \( f(s) = \ell_s, \forall s \in \mathcal{O} \). We see readily that \( f \) is an immersion.

Similarly, we may consider a family of lines in \( R^2(\xi_1, \xi_2) \), where \( w \) is in \( H^2 \). This family corresponds to the elliptic pencil. As in the above case, one can uniquely define a differentiable immersion from \( \mathcal{O} \) into the family, defining a complete totally geodesic foliation.

Among the examples, let us choose a very special one. Set \( w = (0,0) \in R^2(\xi_1, \xi_2) \). Then the elliptic pencil \( \mathcal{P}_{(0,0)} \) gives the complete totally geodesic foliation of \( S^2_1 \) consisting of the vertical time-like geodesics perpendicular to the center circle. This is the foliation studied in [6].

**Example 2.** Define \( \mathcal{L} \) to be the family of lines which are tangent to the unit circle in \( R^2(\xi_1, \xi_2) \). Each line in this family projectively represents two opposite generators in the well-known two rulings of \( S^2_1 \) by two families of null lines. Every pair in the family \( \mathcal{L} \) intersect in the complement of the closed unit disk in \( P^2 \), but this point of intersection projectively represents the intersection points of lines in the pair which belong to the opposite rulings. Thus \( \mathcal{L} \) projectively represents the two rulings which are obviously complete totally geodesic foliations. In fact it is easy to see that all the leaves are null lines of \( S^2_1 \).

Variations of this construction can be readily obtained. For example, consider the interval \( (-1,1) \subset R \). Then \( (a, \theta) \in (-1,1) \times S^1 \) represents a line in \( P^2 \) which intersects the unit disk. Rotate the line about the origin in \( R^2(\xi_1, \xi_2) \). The resulting one parameter family of lines gives rise a complete totally geodesic foliation of \( S^2_1 \) by time-like lines.

It is easy to see that there are families of lines which are not always given by a line under a one parameter family of rotations. Let \( \mathcal{L} = \ell_s, s \in \mathcal{O} \subset S^1 \) be a given differentiable one parameter family of lines in \( R^2(\xi_1, \xi_2) \). We
assume that the one parameter family is orientable, that is, all the line in the family can be oriented at least continuously. We also assume for an obvious reason that every line in the family intersects the closed unit disk in \( R^2(\xi_1, \xi_2) \). Furthermore, we require that any pair of lines in the family meet either in the closed unit disk or the point of intersection outside of the disk occurs in the incoming portion of one line toward the disk and in the outgoing portion of the other from the disk. There will be, indeed, infinitely many choices of such one parameter families. Any one of such families gives rise to a complete totally geodesic foliation of the open subset of \( S^2_1 \), which is basically given as the union of the geodesics in the family.

**Example 3.** In \( H^n_1 \) similar constructions as in the previous examples give higher dimensional cases of totally geodesic foliations. Obviously, there are many more totally geodesic foliations exist. We will give a totally geodesic foliation of \( H^n_1 \), which will be of particular interest later on. Let the the quadratic form be given by

\[
q(x) = -\sum_{j=1}^{2} (x_j)^2 + \sum_{j=3}^{n+1} (x_j)^2.
\]

Then \( P^n \) is divided into three disjoint components \( S^n_2 \), \( \partial H^n_1 \) and \( H^n_1 = S^n_{n-1} \). In particular, if \( (\xi_1, \cdots, \xi_n) \) is the inhomogeneous coordinate system for \( x_1 = 1 \), we have in the n-plane \( x_1 = 1 \) that

\[
S^n_2 = \left\{ (\xi_1, \cdots, \xi_n) : -\xi_1^2 + \sum_{j=2}^{n} \xi_j > 1 \right\},
\]

\[
\partial H^n_1 = \left\{ (\xi_1, \cdots, \xi_n) : -\xi_1^2 + \sum_{j=2}^{n} \xi_j = 1 \right\},
\]

\[
H^n_1 = \left\{ (\xi_1, \cdots, \xi_n) : -\xi_1^2 + \sum_{j=2}^{n} \xi_j < 1 \right\}.
\]

Now set

\[
e_1(t) = (1, \sinh t, \pm \cosh t, 0, \cdots, 0),
\]

\[
e_2(t) = (0, \cosh t, \pm \sinh t, 0, \cdots, 0),
\]

\[
e_3(t) = (0, 0, 0, 1, 0, \cdots, 0),
\]

\[
\vdots
\]

\[
e_n(t) = (0, 0, 0, 0, \cdots, 0, 1).
\]

Then \( \{e_j(t), \ t \in \mathbb{R}, \ 1 \leq j \leq n\} \) is a frame field along a branch of the hyperbola in \( H^2_1(x_1, x_2, x_3) \subset H^n_1 \).

\[
e_i(t) \perp e_j(t) \quad (1 \leq i, j \leq n)
\]

\[
\langle e_1(t), e_1(t) \rangle = 0, \quad \langle e_2(t), e_2(t) \rangle = -1, \quad \langle e_j(t), e_j(t) \rangle = 1, \ (3 \leq j \leq n).
\]
Set $V(t)$ to be the span of $e_j(t)$, $1 \leq j \leq n$. Then $V(t)$ is an $n$-dimensional subspace in $\mathbb{R}^{n+1}$. In terms of the original rectangular coordinates, an element $x = \sum_{j=1}^{n} a_j e_j(t) \in V(t)$ is expressed as

$$x = (a_1, a_1 \sinh t + a_2 \cosh t, a_1 \cosh t + a_2 \sinh t, a_3, \ldots, a_n).$$

Then we have

$$x \in \mathbb{H}_1^n \iff \langle x, x \rangle = -1 \iff -(a_2)^2 + \sum_{j=3}^{n} (a_j)^2 = -1.$$  

$V(t) \cap \mathbb{H}_1^n$ is a hyperboloid of two sheets in $\mathbb{H}_1^n$. Suppose that $V(t) \cap V(t') \cap \mathbb{H}_1^n \neq \emptyset$. Then for some pairs of $n$-tuples $(a_1, \ldots, a_n)$ and $(a_1', \ldots, a_n')$,

$$(a_1, a_1 \sinh t + a_2 \cosh t, a_1 \cosh t + a_2 \sinh t, a_3, \ldots, a_n)
= (a_1', a_1' \sinh t' + a_2' \cosh t', a_1' \cosh t' + a_2' \sinh t', a_3', \ldots, a_n').$$

This implies that

$$a_1 \sinh t + a_2 \cosh t = a_1' \sinh t + a_2' \cosh t,
\quad a_1 \cosh t + a_2 \sinh t = a_1' \cosh t + a_2' \sinh t,$$

$$(6) \quad a_j' = a_j, \; 1 \leq j \leq n, /; j \neq 2.$$

Combining (5) and (6), we get

$$(a_2')^2 = (a_2)^2 \iff a_2' = \pm a_2.$$  

But $a_2' = a_2$ implies that $t = t'$ and $V(t) \equiv V(t')$; hence, the subspaces $V(t)$ and $V(t')$ meet in the $(n-1)$-dimensional plane determined by $a_2' = -a_2$. This equality is the precisely the condition that the two sheets of the hyperboloids represented by the $n$-planes $V(t)$ and $V(t')$ meet each other in the opposite sheets. Thus, for instance, by choosing the sheets of the hyperboloid corresponding to the half plane determined by $a_2 > 0$, we obtain a foliation of codimension 1 in $\mathbb{H}_1^n$ consisting of degenerate leaves, as $t$ runs from $-\infty$ to $+\infty$ along the original hyperbola. This foliation is the higher dimensional version of the double rulings in $S_1^2$. Note that the above construction can be carried out starting with any geodesic in $\partial H_1^n$ instead of the particular hyperbolas of the above kind.

Any complete totally geodesic submanifold of codimension 1 intersects the center circle $S^1$ of $H_1^n = S_{n-1}^n$ once and only once in the projective sense. Thus again any completely totally geodesic foliation of codimension 1
of a connected open subset $U$ in $H^n = S^n_{n-1}$ is parametrized by the open interval $U \cap H^n \cap S^1 = U \cap S^n_{n-1} \cap S^1$ in the center circle $S^1$. Consequently, any differential lamination on $H^n_1 = S^n_{n-1}$ will be parametrized by a family of mutually disjoint connected open subsets in the center circle. As before the set of all oriented $(n-1)$-planes corresponds to the differentiable manifold $R \times S^{n-1}$, where $S^{n-1}$ is the unit sphere in $R^n$. A parametrization by a connected open subset of the center circle means an differentiable immersion of the open subset into the parameter space $R \times S^{n-1}$, which satisfies appropriate conditions to generate a complete totally geodesic foliation.

Unlike in $H^n$ the space of differentiable laminations on $H^n_1$ is rather simple. Since any complete totally geodesic foliation of an open subset is uniquely parametrized by one parameter in the intersection of the open subset and the center circle, any mutually disjoint family of connected open subsets of the center circle parametrizes a lamination on $H^n_1$ and vice versa. As seen in the above examples, a connected open subset of the center circle may parametrize, in general, infinitely many complete totally geodesics foliations. Thus, a given family of mutually disjoint connected open subsets in the center circle can parametrize infinitely many mutually distinct differentiable laminations.

### 3. Isometric immersions.

Let $M^n_p$ and $\tilde{M}^{n+1}_q$ be two (indefinite) Riemannian manifolds of dimensions $n$ and $n+1$ and signatures $p$ and $q$, respectively ($p = 0$ and $q = 0$ inclusive). We will often denote them and their differential geometric quantities without superscripts or subscripts when there is no fear of confusion. Thus, $TM_x$ and $T\tilde{M}_y$ denote the tangent spaces of $M^n_p$ and $\tilde{M}^{n+1}_q$ at the points $x \in M^n_p$ and $y \in \tilde{M}^{n+1}_q$, respectively. Denoted by $\langle , \rangle$ the (indefinite) Riemannian metrics on $M$ and $\tilde{M}$. As usual, $\nabla$ and $\tilde{\nabla}$ denote the corresponding Levi-Civita connections on $M$ and $\tilde{M}$.

Let $f : M \to \tilde{M}$ be an isometric immersion of $M$ into $\tilde{M}$. Given $X$, $Y$, and $Z \in TM_x$, and the unit normal $e_x$ at $x \in M$, there are two fundamental equations of the immersion:

\begin{align}
R(X, Y)Z &= \tilde{R}(X, Y)Z + (A_{e_x} X \wedge A_{e_x} Y)Z; \\
(\nabla_X A_{e_x})Y &= (\nabla_Y A_{e_x})X.
\end{align}

Here, $R$ and $\tilde{R}$ are the curvature tensors of $M$ and $\tilde{M}$ and $A_{e_x}$ is the shape operator of the immersion $f$ in the direction of $e_x$ at $x \in M$. The first equation (7) is called the Gauss equation and the second (8) the Codazzi equation.
The dimension of the null space of $A_{e^x}$ is called the conullity of the immersion and is denoted by $v$, $x \in M$. The open subset of $M$ where the conullity assumes the minimum $v$ is then foliated by (geodesically) complete totally geodesic submanifolds of dimension $v$ if $M$ is geodesically complete. See [1] for more details of these accounts.

In particular, if both $M$ and $\tilde{M}$ are space forms of the same constant curvature, then $v \geq n - 1$. Unless $f : M \to \tilde{M}$ is a totally geodesic immersion, $v = n - 1$, and the immersion naturally defines a differentiable lamination on $M$. In particular, non-totally geodesic isometric immersions from a space form into another space form of the same curvature can exist only if $M = H^n$ or $H^n_1$ by Theorem 1. When $M = H^n$, we have obtained in [3] that any differentiable lamination can be realized as a conullity lamination of an isometric immersion $f : H^n \to H^{n+1}$. Furthermore, any differentiable lamination on $H^n$ can be the conullity foliation of infinitely many mutually non-rigid isometric immersions.

Note that $H^n_1$ may possibly isometrically and non-totally geodesically immersed only into either $H^{n+1}_1$ or $H^{n+1}_2$.

In [6] Graves and Nomizu proved that there was no umbilic free isometric imbedding of $S^n_0$ into $S^{n+1}_0$ if $n \geq 4$. The following is an improvement of their result.

**Theorem 2.** Only isometric immersions of $S^n_1$ into $S^{n+1}_1$ for $n \geq 3$ are totally geodesic ones; hence they are all congruent to the standard imbedding induced from the imbedding $(x_1, \cdots, x_{n+1}) \mapsto (x_1, \cdots, x_{n+1}, 0)$ of $R^n_1$ into $R^{n+1}_1$.

**Proof.** Since $S^n_1$ is anti-isometric to $H^{n-1}_n$, $S^n_1$ can have a complete totally geodesic lamination of codimension 1 only if either $n - 1 = 0$ or 1. If $n = 1$, we have an isometric immersion of a curve; hence any regular and non-totally geodesic immersion of $R$ with arclength parametrization will do. When $n = 2$, there are many isometric immersions of $S^2_1$ into $S^3_1$, which are not totally geodesic. Some examples are given in [6].

**Lemma 2.** Let $\mathcal{F}$ be a complete totally geodesic foliation of a connected open subset $U \subset \tilde{H}^n_1$. Then $U$ admits a global Frobenius coordinate system with respect to $\mathcal{F}$.

**Proof.** Let $x_1$ and $x_2$ be the first and second coordinate functions of $R^{n+1}_2$. Let $L$ a leaf of $\mathcal{F}$. Then $L$ is the intersection of an $n$-plane $P$ and $\tilde{H}^n_1$ in $R^{n+1}_2$. Consider $P \cap R^2_2(x_1, x_2) \subset R^{n+1}_2$. The intersection is of either dimension 1 or dimension 2. In the latter case, $P \cap R^2_2(x_1, x_2) = R^2_2(x_1, x_2)$. Thus $L$ contains the entire $R^1_1 = \tilde{H}^1_1 \subset \tilde{H}^n_1$. Hence $L$ meets every totally geodesic submanifold in $\tilde{H}^n_1$, presenting a contradiction. In the former, the
line of intersection represents two distinct points in $R^n_1(\xi_1) = \tilde{H}^1_1 \subset \tilde{H}^n_1$. Suppose that these points lie in $L$. Then they can be joined by a geodesic segment in $L$. Hence, there is a 2-plane $P_2$ in $P$ which represents the geodesic segment. Given any leaf $L'$ of $\mathcal{F}$ nearby the leaf $L$, $L'$ and the geodesic represented by $P$ has non-empty intersection. Thus $L \cap L'$ is non-empty, contradicting that they are leaves of the same foliation. What we have seen so far is that every leaf intersects $\tilde{H}^1_1 \subset \tilde{H}^n_1$ once and only once. Denote by $\mathcal{O}_x$ the intersection $U \cap \tilde{H}^1_1$. $\mathcal{O}_x$ is an open interval, say $(a, b)$, $-\infty < a < b < \infty$, in $\tilde{H}^1_1 \approx R$. Now let $\{e_j, j = 1 \leq j \leq n-1\}$ be a differentiable frame field along the open interval so that the frame at each point in the open interval is a basis for the leaf passing through the point. Define a differentiable map $F : R^{n-1} \times (a, b) \rightarrow U$ by

\begin{equation}
F(t_1, \ldots, t_{n-1}) = \exp_s \left( \sum_{j=1}^{n-1} t_j e_j \right), \quad (t_1, \ldots, t_{n-1}) \in R^{n-1}, \quad s \in (a, b).
\end{equation}

Here $\exp_s$ means the exponential map at $s$. $F$ is clearly a diffeomorphism because of the curvature condition on $\tilde{H}^n_1$. This completes the proof.

A null curve $x = x(s)$, $s \in (a, b)$, $-\infty < a < b < \infty$, in $\tilde{H}^n_1$ is a differentiable curve such that $(dx/ds, dx/ds) = (\dot{x}, \dot{x}) = 0, \forall s \in (a, b)$. As before, we locally identify $\tilde{H}^n_1$ with $H^n_1 \subset R_2^{n+1}$ under the covering map in (2).

We will give here a brief account on Cartan frames on $\tilde{H}^3_1$. The idea can be applied to $\tilde{H}^n_1$ with minor modifications. The reader should find more detailed discussions about the Cartan frames in [4, 5] and [6].

A null frame in $\tilde{H}^3_1$ is an ordered quadruple

$$F = [A, B, C, x] = \begin{bmatrix}
a_1 & b_1 & c_1 & x_1 \\
a_2 & b_2 & c_2 & x_2 \\
a_3 & b_3 & c_3 & x_3 \\
a_4 & b_4 & c_4 & x_4
\end{bmatrix},$$

where $A, B, C$ and $x$ are vectors in $R^4_2$ with the following additional conditions:

- $\langle A, B \rangle = -1, \langle A, A \rangle = \langle B, B \rangle = 0$,
- $\langle C, C \rangle = 1, \langle A, C \rangle = \langle B, C \rangle = 0$,
- $\langle x, x \rangle = -1, \langle x, A \rangle = \langle x, B \rangle = \langle x, C \rangle = 0$,
- $\det F = \pm 1$.

To any null frame there is an associated orthonormal frame defined by

$$L(F) = \left[ \frac{1}{\sqrt{2}} (A + B), \frac{1}{\sqrt{2}} (A - B), C, x \right].$$
Let

\[ N = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \]

It is easy to see that \( L(F) = F \cdot N^{-1}, \forall F \). Given a null curve \( x = x(s), s \in (a, b), -\infty \leq a < b \leq \infty \), in \( \mathcal{H}_1^n \), we call the following system of differential equations the adapted Frenet equations on the null curve.

\[
\begin{align*}
\frac{dA}{ds} &= k_1(s)A(s) + k_2(s)C(s), \\
\frac{dB}{ds} &= -k_1(s)B(s) + k_3(s)C(s) + k_4(s)x(s), \\
\frac{dC}{ds} &= k_3(s)A(s) + k_2(s)B(s), \\
\frac{dx}{ds} &= k_4(s)A(s).
\end{align*}
\]

In a matrix form the adapted Frenet equations are stated as

\[
F^{-1}\frac{dF}{ds} = \begin{bmatrix} k_1 & 0 & k_3 & k_4 \\ 0 & -k_1 & k_2 & 0 \\ k_2 & k_3 & 0 & 0 \\ 0 & k_4 & 0 & 0 \end{bmatrix}.
\]

It is known that there is a unique framed curve \( F = F(s) \) which is a solution to the Frenet equations under given initial conditions \( F(0) = F_0 \). In particular, a Cartan framed curve is an adapted framed curve \( F = F(s) \) whose Frenet equations have the form

\[
F^{-1}\frac{dF}{ds} = \begin{bmatrix} 0 & 0 & k_3 & 1 \\ 0 & 0 & k_2 & 0 \\ k_2 & k_3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.
\]

Given initial conditions, Cartan framed fields uniquely exist. Given a Cartan framed curve \( F \), we define the B-scroll \( B_F : \mathbb{R}^2 \to \mathcal{H}_1^3 \) of \( F \) by

\[ B_F(u, s) = x(s) + uB(s). \]

We see easily that

\[
\begin{align*}
B_{F_*}(\partial/\partial u) &= B(s), \\
B_{F_*}(\partial/\partial s) &= A + uk_3C + ux.
\end{align*}
\]
Therefore we have
\[
\langle B_F(\partial/\partial u), B_F(\partial/\partial u) \rangle = 0,
\]
\[
\langle B_F(\partial/\partial u), B_F(\partial/\partial s) \rangle = -1,
\]
\[
\langle B_F(\partial/\partial s), B_F(\partial/\partial s) \rangle = u^2(1 + k_3^2).
\]

$R^2$ with this induced metric has signature $(-, +)$. The shape operator of the B-scroll relative to the basis $\{\partial/\partial u, \partial/\partial s\}$ will then be calculated to be

\[
\begin{bmatrix}
-k_3 -k_2 - udk_3/ds \\
0 -k_3
\end{bmatrix}.
\]

Thus the Lorentz surface $B_F : R^2 \to H^3_1$ has the same constant curvature $-1$ as $H^3_1$ if and only if $k_3 = 0$. This surface is often referred to as a generalized null cubic. If, in addition, $k_2 \equiv 1$ everywhere, the B-scroll takes especially simple form with the adapted Cartan framed curve given by

\[
F(s) = \frac{1}{\sqrt{2}} \begin{bmatrix}
\cosh s & \cosh s & \sinh s & \sinh s \\
\cos s & -\cos s & -\sin s & \sin s \\
\sin s & -\sin s & \cos s & -\cos s \\
\sinh s & \sinh s & \cosh s & \cosh s
\end{bmatrix}.
\]

The induced metric and the shape operator relative to the basis $\{\partial/\partial u, \partial/\partial s\}$ are respectively given as

\[
\begin{bmatrix}
0 & -1 \\
-1 & u^2
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
0 & -1 \\
0 & 0
\end{bmatrix}.
\]

The $u$ coordinate curves are the precisely the geodesics in the relative nullity foliation and are imbedded isometrically onto the the geodesics in the direction of $B$ vector field.

Let $\mathcal{F}$ be a complete totally geodesic foliation of a connected open subset $U \subset \tilde{H}^n_i$. Let $f : U \to H^{n+1}_i$ be an isometric immersion such that its relative nullity foliation is $\mathcal{F}$.

**Lemma 3.** The leaves of $\mathcal{F}$ are either all non-degenerate or all degenerate.

**Proof.** If a leaf $L$ is non-degenerate, there is a neighborhood of $L$ where all the leaves are non-degenerate. Suppose that there is a maximal open subset $V \subset U$ such that all the leaves in $V$ are non-degenerate. Then the boundary $\partial V$ of $V$ consists of two degenerate complete totally geodesic submanifolds of codimension 1. According to the classification of the shape
operator [8], there are four possible matrix representations of the shape operator relative to an pseudo orthonormal basis \( \{ \hat{e}, e, e_1, \cdots, e_{n-2} \} \). In our case the rank of the shape operator is known to be less than or equal to 1. If the shape operator \( A_{e_x} \) is diagonalized relative to the above basis, assuming \( A_{e_x} \hat{e} \neq 0 \), we have that \( \langle A_{e_x} \hat{e}, e \rangle = \langle \hat{e}, e \rangle = -1 \). On the other hand, \( \langle A_{e_x} \hat{e}, e \rangle = \langle \hat{e}, A_{e_x} e \rangle = 0 \). This is a contradiction. Therefore, the shape operator has rank \( n \). This also implies that the ranks of the shape operators in the leaf are \( n \) everywhere, contradiction. Only other possible case is that it is represented by the matrix:

\[
\begin{pmatrix}
0 & 0 \\
1 & 0 \\
& & \ddots \\
& & & 0 \\
\end{pmatrix}
\]

Now suppose that one of the totally geodesic submanifolds in \( \partial V \) is the limit set of non-degenerate leaves \( \{ L_j, j = 1, 2, \cdots \} \) in \( U \). The shape operator at a point in a non-degenerate leaf is always diagonalizable. For, if not, the shape operator will have the above matrix representation relative to a pseudo orthonormal basis. It implies that the vectors \( \{ e, e_1, \cdots, e_{n-2} \} \) span the null space, which will consequently be degenerate, a contradiction.

Let \( \{ x_j \in L_j, j = 1, 2, \cdots \} \) be a sequence of points which converges to a point \( x \in L \subset \partial V \). Since the correspondence \( x \mapsto A_{e_x} \) is differentiable, the correspondence \( x \mapsto (A_{e_x})^2 \) is also differentiable, which equals the zero matrix at any point on the leaf in the boundary \( \partial V \). By differentiability of the correspondence \( x \mapsto (A_{e_x})^2 \), \( \{ A_{e_{x_j}}, j = 1, 2, \cdots \} \) converges to the zero matrix. Consequently, the shape operator at any point in the leaf must also be represented by the zero matrix. This is a contradiction. Hence, the rank of the shape operator at any point in the leaf in \( \partial V \) must equal \( n \), if the leaf is a limit leaf of non-degenerate leaves.

This proves that under the assumption in the lemma \( U \) is either completely foliated by non-degenerated leaves or else completely foliated by degenerated leaves.

Suppose that a connected open set \( U \subset \tilde{H}_1^n \) is foliated by the degenerated leaves of the complete relative nullity foliation \( F \) which is induced by an isometric immersion \( f : U \subset \tilde{H}_1^n \rightarrow H_1^{n+1} \). Let \( \beta = \beta(s), s \in (a, b) \subset R \), the portion of the center circle which is in \( U \). Let \( e(s) \) and \( \hat{e}(s) \) be the parallel null vector fields along \( \beta \) such as described in the proof of Lemma 1. Recall that \( e(s) \) generate the unique null line and \( \langle e(s), e(s) \rangle = \langle \hat{e}(s), \hat{e}(s) \rangle = 0 \) and \( \langle e(s), \hat{e}(s) \rangle = -1 \) in the leaf at each \( s \in (a, b) \). Then \( \hat{e} \) and \( e \) span
a Lorentzian plane in $T_x\tilde{H}_1^n$, \( \forall x \in U \subset \tilde{H}_1^n \). The orthogonal complement of the plane is in the null space of the immersion. Since each leaf intersects the curve $\beta$ once and only once, we can extend the vector fields $e(s)$ and $\dot{e}(s)$ to the entire $U$ by parallelly displacing them along the radial geodesic rays staring at $\beta(s)$, $s \in (a, b)$ in each leaf passing through $\beta(s)$. We denote the resulting vector fields by the same letter $e$ and $\dot{e}$, respectively. Denote by $\gamma_s = \gamma_s(t)$, $t \in (c, d)$ the maximal integral curve of the vector field $\dot{e}$ passing through $\beta(s)$, $s \in (a, b)$. As before, we denote by $e_j(s)$, $1 \leq j \leq n - 3$, the parallel displacement along the curve $s$ of an orthonormal basis for the orthogonal complement of the plane spanned by $e(s')$ and $\dot{e}(s')$ at a point $s' \in (a, b)$. The leaf passing through a point $\beta(s)$ is given as the intersection of $H_1^n$ and the $n$-plane spanned by $e(s), e_j(s)$, $1 \leq j \leq n - 2$, and the position vector $\beta(s)$. In particular, the plane $P(s)$ spanned by $e(s)$ and $\beta(s)$ gives a geodesic in the leaf. The geodesic has $e(s)$ as its velocity vector at $\beta(s)$; hence, it has to be a light-like geodesic. On the other hand, $P(s)$ is degenerate and contains the line spanned by $e(s)$ as a unique null line. This implies that the geodesic must be the straight null line itself. Let us consider the $(n-1)$-plane in $R_2^{n+1}$ which is perpendicular to the 2-plane that contains the center circle. Call the $(n-1)$-plane $P_V$. A dimensional argument then yields that the the tangent plane to the leaf at $\beta(s)$ contains a unique $(n - 2)$-plane in $P_V$. Call the unique $(n - 2)$-plane $\Theta$. Since $P_V$ is pointwisely fixed under the rotation in the plane containing the center circle, the tangent spaces to the leaves passing through $\beta(s)$, which are parallel along $\beta$, contain $\Theta$ as a unique common subspace for all $s \in (a, b)$. Denote by $\Theta^\perp(s)$ the orthogonal complement of $\Theta$ in the tangent space $T_sH_1^n$ to $H_1^n$ at $\beta(s)$. Since $\Theta$ is a space-like space, $\Theta^\perp(s)$ is a Lorentzian space of dimension 2. $e(s)$ is in $\Theta^\perp(s)$; hence, there is a unique light-like vector $\dot{e}(s) \in \Theta^\perp(s)$; such that $\langle e(s), \dot{e}(s) \rangle = -1$. The vector field $\dot{e}(s)$ is parallel along $\beta = \beta(s)$. The vector $\dot{e}(s)$ generates a unique light-like geodesic, denoted by $\gamma = \gamma(t)$, $t \in R$. The geodesic $\gamma$ meets all the leaves of the foliation transversally except possibly one leaf which is exactly at the antipodal position in the center circle. As is proved in Lemma 1, one can come up with a parallel null vector field $e(t)$, $t \in R$ along $\gamma$ such that $\langle e(t), \dot{e}(t) \rangle = -1$, $t \in R$ and $e(t)$ determines the unique light-like direction in each leaf at $\gamma(t)$. The parallel displacement $\Theta^\perp(t)$ of $\Theta^\perp(s)$ along $\gamma(t)$ is tangential to the leaf at $t \in R$. Denote by $e_j(t)$, $1 \leq j \leq n - 2$, the parallel displacement along $\gamma(t)$ of an orthonormal basis $e_j(s)$, $1 \leq j \leq n - 2$, for $\Theta^\perp(s)$. Denote by $x = x(t) = f \circ \gamma(t)$, $t \in R$. Denote also $f_\ast(\dot{e}(t)), f_\ast(e(t)), f_\ast(e_j(t))$, $1 \leq j \leq n - 2$, by $A(t), B(t), B_j(t)$, $1 \leq j \leq n - 2$, respectively. Then the $(n + 2)$-tuple of vector fields $\{\gamma(t), A(t), B(t), C(t), B_j(t), 1 \leq j \leq n - 2\}$ form what might
be called a generalized Cartan framed curve in $H^{n+1}_1$. Here $C(t)$ is a unit normal field to the immersion $f$ along $x = x(t), t \in \mathbb{R}$. Each leaf passing through $\gamma(t)$ is mapped onto the totally geodesic submanifold given as the intersection of $H^{n+1}_1$ and the span of $\{x(t), B(t), B_j(t), 1 \leq j \leq n-2\}$. It can be expressed as a mapping $\tilde{f} : R^n \to H^{n+1}_1$ in the following manner:

$$\tilde{f}(t, u_1, \cdots, u_{n-2}, u) = x(t) + \sum_{j=1}^{n-2} u_j B_j(t) + uB(t),$$

with

$$-u^2 + \sum_{j=1}^{n-2} (u_j)^2 = -1.$$

The induced metric in $R^n$ under $\tilde{f}$ gives the same constant curvature $-1$ as $H^{n+1}_1$; hence, making $R^n$ into $\tilde{H}^n_1$. The shape operator of the immersion $\tilde{f}$ is given in terms of the matrix representation relative to the basis $\{e(t), e(t), e_j(t), 1 \leq j \leq n-2\}$ by

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \text{where} \quad A_{11} = \begin{bmatrix} 0 & -k(s) \\ 0 & 0 \end{bmatrix}$$

and the remaining $A_{ij}$'s are the zero matrices of appropriate sizes, where $k = k(t)$ is a differentiable function of $t \in \mathbb{R}$.

Conversely, suppose there is given a generalized Cartan framed curve $\{x(t), A(t), B(t), B_j(t), 1 \leq j \leq n-2\}$ in $H^{n+1}_1$. We have the following conditions satisfied:

$$\langle x(t), x(t) \rangle = -1, \quad \langle A(t), A(t) \rangle = 0, \quad \langle B(t), B(t) \rangle = 0,$$

$$\langle C(t), C(t) \rangle = 1, \quad \langle A(t), B(t) \rangle = -1, \quad \langle A(t), B_j(t) \rangle = 0,$$

$$\langle B(t), B_j(t) \rangle = 0, \quad \langle A(t), C(t) \rangle = 0, \quad \langle B(t), C(t) \rangle = 0,$$

$$\langle B_i(t), B_j(t) \rangle = \delta_{ij}.$$

Furthermore, assume the following hold:

$$\frac{dx}{dt} = A,$$

$$\hat{\nabla}_{A(t)} A(t) = k(t)C(t), \quad k = k(t),$$

$$\hat{\nabla}_{A(t)} B(t) = w(t)C(t), \quad w = w(t),$$

$$\hat{\nabla}_{A(t)} C(t) = w(t)A(t) + k(t)B(t),$$

$$\hat{\nabla}_{A(t)} B_j(t) = 0, \quad 1 \leq j \leq n-2.$$
Here $\tilde{\nabla}$ is the connection in $H_1^{n+1}$. If, in general, $Y = Y(t)$ is a vector field tangent to $H_1^{n+1}$ along a curve $x = x(t)$, then we have:

$$dY/dt = \tilde{\nabla}_{dx/dt}Y(t) + \lambda(t)x(t),$$

where $\lambda(t) = \langle dY/dt, Y(t) \rangle = -\langle Y(t), A(t) \rangle$. In terms of the differentiation in $R_2^{n+2}$, the above equations can be written as follows:

$$\frac{dx}{dt} = A(t),$$
$$\frac{dA(t)}{dt} = k(t)C(t),$$
$$\frac{dB(t)}{dt} = x(t) + w(t)C(t),$$
$$\frac{dC(t)}{dt} = w(t)A(t) + k(t)B(t),$$
$$\frac{dB_j}{dt} = 0.$$

Assuming that $x = x(t), t \in R$, is a null curve in $H_1^{n+1}$, we define a mapping $\tilde{f}: R^n \to H_1^{n+1}$ by

$$\tilde{f}(t, u_1, \cdots, u_{n-2}, u) = x(t) + \sum_{j=1}^{n-2} u_j B_j(t) + uB(t)$$

with

$$-u^2 + \sum_{j=1}^{n-2} (u_j)^2 = -1.$$

We have

$$\tilde{f}_*(\partial/\partial u) = B(t),$$
$$\tilde{f}_*(\partial/\partial t) = dx/dt + udB(t)/dt = A(t) + uw(t)C(t) + ux(t),$$
$$\tilde{f}_*(\partial/\partial u_j) = B_j(t), 1 \leq j \leq n-2;$$

so that

$$\langle \tilde{f}_*(\partial/\partial u), \tilde{f}_*(\partial/\partial u) \rangle = 0,$$
$$\langle \tilde{f}_*(\partial/\partial u), \tilde{f}_*(\partial/\partial t) \rangle = -1,$$
$$\langle \tilde{f}_*(\partial/\partial t), \tilde{f}_*(\partial/\partial t) \rangle = u^2(w^2 - 1),$$
$$\langle \tilde{f}_*(\partial/\partial u), \tilde{f}_*(\partial/\partial u_j) \rangle = \langle \tilde{f}_*(\partial/\partial t), \tilde{f}_*(\partial/\partial u_j) \rangle = 0,$$
$$\langle \tilde{f}_*(\partial/\partial u_i), \tilde{f}_*(\partial/\partial u_j) \rangle = \delta_{ij}, 1 \leq i, j \leq n - 2.$$

Clearly, the metric in $R^n$ induced by $\tilde{f}$ is Lorentzian and of signature 1. We may call the immersion $\tilde{f}: R^n \to H_1^{n+1}$ the generalized B-scoll of the generalized Cartan framed curve

$$\{x(t), A(t), B(t), C(t), B_j(t), 1 \leq j \leq n - 2\}.$$
In order to compute the shape operator of the generalized B-scroll, we choose a space-like unit normal field $\xi = C + uwB$. Then,

$$\nabla_{\partial/\partial u}\xi = \frac{d\xi}{du} = wB = f_*(w(\partial/\partial u))$$

and

$$\nabla_{\partial/\partial t}\xi = \frac{d\xi}{dt} = \frac{dC}{dt} + u\left(\frac{dw}{dt}\right)B + uw\left(\frac{dB}{dt}\right) = (k + ud\frac{w}{dt})B + w(A + uwC + ux) = (k + ud\frac{w}{dt})f_*(w\partial/\partial u) + wf_*(w\partial/\partial t).$$

Furthermore,

$$\nabla_{\partial/\partial u_j}\xi = 0, \ 1 \leq j \leq n - 2.$$

Denoting by $A_\xi$ the shape operator of the map in the direction of $\xi$ relative to the basis $\{\partial/\partial u, \partial/\partial s, \partial/\partial u_j\}$, $1 \leq j \leq n - 2$, we get

$$A_\xi = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Here,

$$A_{11} = \begin{bmatrix} -w - k - ud\frac{w}{dt} \\ 0 & -w \end{bmatrix}.$$

And the remaining $A_{ij}$'s are the zero matrices of appropriate sizes.

The metric induced by the immersion $\tilde{f} : R^n \to H^{n+1}_1$ gives rise to the same constant curvature $-1$ in $R^n$ if and only if $w = 0$. Now we have seen that

**Theorem 3.** Let $\mathcal{F}$ be a complete totally geodesic foliation of codimension 1 by degenerate leaves in a connected open subset of $H^n_1$. Then $\mathcal{F}$ occurs as the relative nullity foliation of the generalized B-scroll of a generalized Cartan framed curve in $H^{n+1}_1$ and vice versa. Such a foliation can have more than one immersions in the form of different B-scrolls, but they are completely determined by the curvature function $k = k(t)$, $t \in R$, of the null curve $x = x(t)$, $t \in R$. 
4. Foliations with non-degenerate leaves.

Let $U$ be a connected open subset of $\mathbb{H}^n_1$. Let $\mathcal{F}$ be a complete totally geodesic foliation of codimension 1 in $U$. Given a tangent vector $Z$ at $x$ to the leaf $L_x$ of $\mathcal{F}$ passing through the point $x$. We denote by $c_Z(x)$ the conullity operator of $\mathcal{F}$ at $x$. Along the geodesic $\gamma = \gamma(t)$ determined by the initial conditions $\gamma(0) = x$ and $d\gamma/dt(0) = Z$ in $L_x$, $c_Z(t)$ satisfies the Ricatti type differential equation (4), i.e.,

$$dy/dt = y^2 - 1.$$ 

The only global solution is given by

$$c_Z(t) = -\frac{\sinh t - c_Z(0) \cosh t}{\cosh t - c_Z(0) \sinh t}.$$

If we denote by $\sigma = \sigma(s), s \in (a, b)$, the maximal integral curve through $x$ of the unit orthogonal vector fields to the foliation $\mathcal{F}$, the global coordinate mapping $\Sigma: (a, b) \times \mathbb{R}^{n-1} \to U \subset \mathbb{H}^n_1 \subset \mathbb{R}_2^{n+1}$, which is given as $F$ by (9) in the proof of Lemma 2 can be more specifically expressed as follows: Denote by $T_x L_x$ the tangent space of the leaf $L_x$ passing through $x \in U$. The induced inner product in $T_x L_x$ is positive definite. Denote by $S_x$ the unit $(n-2)$-sphere in the tangent space. Given a unit vector $Z(s) \in S_{\sigma(s)}$, define a map $\Sigma_Z: (a, b) \times \mathbb{R} \to U \subset \mathbb{H}^n_1 \subset \mathbb{R}_2^{n+1}$ by

(10) \quad $\Sigma_Z(s, t) = \sigma(s) \cosh t + Z(s) \sinh t$.

Our global coordinate map $\Sigma$ coincides with $\Sigma_Z$ in the plane

$$\{s\dot{\sigma}(s) + tZ(s)\}, \quad \text{where} \quad \dot{\sigma}(s) = \sigma_*(d/ds)(s).$$

We have

$$\Sigma_*(\partial/\partial s)(s, t) = \Sigma_{Z(s)_*} (\partial/\partial s)(s, t) = \dot{\sigma}(s) \cosh t + \dot{Z}(s) \sinh t$$

$$\Sigma_*(\partial/\partial t)(s, t) = \Sigma_{Z(s)_*} (\partial/\partial t)(s, t) = \sigma(s) \sinh t + Z(s) \cosh t.$$

Hence, we have that

$$\langle \Sigma_*(\partial/\partial s)(s, t), \Sigma_*(\partial/\partial t)(s, t) \rangle = 0,$$

$$\langle \Sigma_*(\partial/\partial t)(s, t), \Sigma_*(\partial/\partial t)(s, t) \rangle = 1,$$

$$\Sigma_*(\partial/\partial s)(s, t) = (\cosh t - c(s) \sinh t) \dot{\sigma}(s).$$
Here, \( c(s) \) is the curvature of \( \sigma(s) \) in \( H^n \), i.e., \( \langle d^2/ds^2, Z(s) \rangle = c(s) = -\langle \dot{\sigma}(s), \nabla_{\dot{\sigma}}(s)Z(s) \rangle \). As \( |c(s)| \leq 1 \), \( \Sigma_*(\partial/\partial s) \) never vanishes. This implies that \( \Sigma \) is an immersion. It is obvious that \( \Sigma \) is one-to one and onto.

**Lemma 4.** Let \( \{a_i\} \) and \( \{b_i\} \), \( i = 1,2, \cdots \), be two sequences of positive real numbers such that \( (a_i, b_i) \subset (a_{i+1}, b_{i+1}) \subset (a, b) \) and \( \lim_{i \to \infty} a_i = a \) and \( \lim_{i \to \infty} b_i = b \). Then there exists a countable family

\[
\{g = g(s), \quad g_i = g_i(s), \; i = 1,2, \cdots \}
\]

of \( \mathcal{C}^\infty \)-realvalued functions defined in \( R \) satisfying the following conditions:

i) \( g(s) \geq 0 \), \( s \in (a, b) \) and 0 elsewhere;

ii) \( g_i(s) \geq 0 \), \( s \in (a_i, b_i) \) and 0 elsewhere;

iii) \( d^jg_i/ds^j \to d^jg/ds^j \), \( i, j = 1,2, \cdots \), uniformly in any compact subset of \( R \).

**Proof.** It is well-known that there exists a \( \mathcal{C}^\infty \)-function \( h_i \) such that \( 0 < h_i \leq 1 \) in \( (a_i, b_i) \) and \( h_i \equiv 0 \) outside. Each \( h_i \) has compact support \( [a_i, b_i] \). Set \( m_i = \max \{ |d^jh_i/ds^j(s)|, \; s \in [a_i, b_i], \; 0 \leq j \leq i \} \). Then \( 0 < m_i < \infty \). Let \( \rho_i \) be a positive real number such that \( m_i \leq \rho_i(1/2)^i, \; i = 1,2, \cdots \). Define \( g_i(s) = \sum_{i=1}^{i} h_i(s)/\rho_i, \; i = 1,2, \cdots \). Clearly \( g_i(s) \) is a \( \mathcal{C}^\infty \)-function and satisfies (ii). From the choice of \( \rho_i \)'s, \( d^jg_i/ds^j \) converges to \( d^jg/ds^j \) uniformly in any compact subset of \( R \), where \( g(s) = \lim_{i \to \infty} g_i(s) \). Hence, \( g = g(s) \) is a \( \mathcal{C}^\infty \)-function. With this choice of \( g \) and \( g_i \)'s, (i), (ii) and (iii) are satisfied.

**Lemma 5.** Let \( \mathcal{F} \) be a codimension one complete totally geodesic foliation of a connected open subset \( U \subset H^n_1 \). Furthermore, assume that all the leaves are non-degenerate. Let \( c(s) \) be the curvature of a maximal orthogonal trajectory of the foliation \( \mathcal{F} \). Then there is a \( \mathcal{C}^\infty \)-function \( G \) defined in \( H^n_1 \) of the following form:

\[
G \circ \Sigma_{Z(s)}(s, t) = g(s)k(s)/(\cosh t - c(s) \sinh t), \; \forall (s, t) \in (a, b) \times R;
\]

\[
G(x) = 0, \; \forall x \in \tilde{H}^n_1 - U.
\]

Here \( \Sigma_{Z(s)}(s, t) \) is the restriction of the global coordinate mapping to the span of \( \sigma(s) \) and \( Z(s) \), \( s \in (a, b) \) and \( k: (a, b) \to R \) is a \( \mathcal{C}^\infty \)-function.

**Proof.** Define \( G_i \) to be the \( \mathcal{C}^\infty \)-function defined in \( \tilde{H}^n_1 \) by

\[
G_i \circ \Sigma_{Z(s)}(s, t) = g_i(s)k(s)/(\cosh t - c(s) \sinh t), \; (s, t) \in (a_i, b_i) \times R;
\]

\[
G_i(x) = 0, \; \forall x \in \tilde{H}^n_1 - U_i.
\]
The \(g_i\)'s are given in Lemma 4 and \(U_i = \Sigma((a_i, b_i) \times R^{n-1}), i = 1, 2, \ldots\).
Since \(\tilde{H}_1^n\) is diffeomorphic to \(R^n\), there are global coordinate systems in \(\tilde{H}_1^n\). Choose one of them and denote it by \((u_1, \cdots, u_n), -\infty < u_i < \infty\). Indeed, we might as well assume that it is the Cartesian coordinate system in \(R^n\). Then a function in \(\tilde{H}_1^n\) is of \(c^\infty\) if all the partials in the \(u_i, 1 \leq i \leq n\), exist and continuous. Define a positive real number \(B_i\) by

\[
B_i = \max \left\{|\partial^j G_i/\partial u^\alpha \partial u^\beta (\Sigma(s, Z))| : \alpha + \beta = j, 0 \leq j \leq i, |Z| \leq i, (s, Z) \in (a, b) \times R^{n-1}\right\}.
\]

These \(B_i\)'s are bounded by positive constants. Let \(\tau_i\) be positive number such that \(B_i \leq \tau_i(1/2)^i, i = 1, 2, \cdots\). Then define \(G\) as follows:

\[
(11) \quad G(x) = \sum_{i=1}^{\infty} \left(1/\tau_i\right) G_i(x), \quad x \in \tilde{H}_1^n.
\]

Expressing \(g = \sum_{i=1}^{\infty} \left(1/\tau_i\right) g_i(x), \quad G\) has the desired form.

Next let \(A_i\) be a symmetric tensor field of type (1,1) in \(\tilde{H}_1^n\) defined as follows:

\[
A_i(\Sigma(s, t))(\Sigma_*(Z)) = 0,
\]

\[
A_i(\Sigma(s, t))(\Sigma_*(\partial/\partial s)) = G_i(s, Z) \Sigma_*(\partial/\partial s), \quad \Sigma(s, Z) \in U;
\]

\[
A_i(x) = 0, \quad x \in \tilde{H}_1^n - U.
\]

\(A_i\) is obviously a \(c^\infty\)-symmetric tensor field of type (1,1). Let \(\{e_1, \cdots, e_n\}\) be the orthonormal frame field in \(\tilde{H}_1^n\) obtained from the global coordinate vector fields \(\partial/\partial u_1, \cdots, \partial/\partial u_n\) through the Gram-Schmidt orthonormalization process. Then \(A_i\) can be expressed as a \(c^\infty\)-matrix-valued function of \(\tilde{H}_1^n\), which we denote by

\[
(12) \quad A_i(x) = \left[a_{pq}^i(x)\right], \quad 1 \leq p, q \leq n, \quad x \in \tilde{H}_1^n.
\]

The components \(A_{pq}^i(x)\)'s of \(A_i\) are \(c^\infty\)-functions of \(\tilde{H}_1^n\) and vanish outside \(U_i\). Just as in the proof of Lemma 4, we make the infinite sum of these \(A_{pq}^i\)'s with appropriate weights if necessary. The sum will be denoted by \(A = A(x)\). \(A\) can be expressed in terms of the base \(\Sigma_*(\partial/\partial s)\) and \(\Sigma_*(Z_1), \cdots, \Sigma_*(Z_{n-1})\), where \(\{Z_1, \cdots, Z_{n-1}\}\) forms an orthonormal basis for \(R^{n-1}\),

\[
A(x)(\Sigma_*(\partial/\partial s)) = \Psi(x) \Sigma_*(\partial/\partial s);
\]

\[
A(x)(Z_j) = 0, \quad \forall x = \Sigma(s, Z) \in U \quad (1 \leq j \leq n - 1);
\]

\[
A(x) = 0, \quad \forall x \in \tilde{H}_1^n - U.
\]
Here, $\Psi$ is a $\mathcal{C}^\infty$-function of the form given in (11) with an appropriate function $\psi$ replacing $g$.

**Lemma 6.** $A$ satisfies the Gauss and Codazzi equations in $\tilde{H}_1^n$.

**Proof.** We first show that each $A_i$ satisfies these equations. Since rank $A_i \leq 1$, the Gauss equation is automatically satisfied. Since $A_i$ is a tensor field, it suffices to show that in $U$

\begin{equation}
(\nabla_{\Sigma_*(\partial/\partial s)} A_i)(Z_j) = (\nabla_{\Sigma_*(Z_i)} A_i)(\partial/\partial s).
\end{equation}

Note that $A_i = 0$ outside $U_i$; hence, the equations are trivially satisfied there. We calculate the terms in the above equation (12) by restricting them to the totally geodesic surfaces $\Sigma Z(s,t)$ of $\tilde{H}_1^n$ given by (10). Since $\Sigma_*(\partial/\partial s) = \Sigma Z(s)_*(\partial/\partial s)$ and $\Sigma_*(\partial/\partial t) = \Sigma Z(s)_*(\partial/\partial t)$ form a coordinate frame fields for $R^2$, the Lie bracket between them equals 0. This fact together with $A_i(\Sigma_*(\partial/\partial t)) = 0$ implies that the equation (12) is equivalent to the following:

\begin{equation}
(\Sigma_*(\partial/\partial t)G_i - G_i c(\Sigma(s,t))) = 0.
\end{equation}

$c(\Sigma(s,t))$ is the conullity operator defined in (3), which can be expressed as

\[ c(\Sigma(s,t)) = c_Z(t) = \frac{\sinh t - c(\Sigma(\sigma(s))) \cosh t}{\cosh t - c(\Sigma(\sigma(s))) \sinh t}, \]

where the $\sigma(s)$ is the orthogonal trajectory of the foliation. Equation (13) implies that $dG_i/dt - G_i c(\Sigma(s,t)) = 0$. But

\[ G_i c(\Sigma(s,t)) = \frac{g_i(s)k(s)}{\cosh t - c(\sigma(s)) \sinh t}, \quad (s,t) \in (a_i, b_i) \times R. \]

Hence,

\begin{equation}
(dG_i/dt)(\Sigma(s,t)) = \frac{g_i(s)k(s)(\sinh t - c(\sigma(s)) \cosh t)}{(\cosh t - c(\sigma(s)) \sinh t)^2}. \end{equation}

Substituting this expression (15) in (14), we get the desired equation. Using the expression (12) for $A_i$, we see that the Codazzi equation (13) holds for $A_i$, $i = 1, 2, \cdots$. Since by the definition of $A$, the convergence of all partials is uniform in any compact subset of $\tilde{H}_1^n$, $A$ also satisfies the Codazzi equation (13).

**Theorem 4.** Given a $\mathcal{C}^\infty$-lamination $\{U, F\}$, where $U$ is a a connected open subset of $\tilde{H}_1^n$, there is an isometric immersion of $\tilde{H}_1^n$, into $\tilde{H}_1^{n+1}$.
so that the associated lamination is precisely \( \{U, \mathcal{F}\} \). In fact, there are infinitely many such immersions, which are parametrized by the space of differentiable functions \( k = k(s) \)’s.

**Proof.** Obvious from the proof of the previous lemmas and the fundamental theorem of hypersurfaces in the ambient space of constant curvature.

Now let \( \{U, \mathcal{F}\} \) be a \( c^\infty \)-lamination of \( \tilde{H}^n_1 \). This means that \( U \) is given as a disjoint union of countable connected open subset \( U_i, i = 1, 2, \cdots \), of \( \tilde{H}^n_1 \) such that each \( U_i \) is endowed with a \( c^\infty \)-complete totally geodesic foliation \( \mathcal{F}_i \). By Lemma 6 the lamination gives rise to a \( c^\infty \)-symmetric tensor field \( A_i \) of type (1,1), which satisfies the Gauss and Codazzi equations. Let us express each \( A_i \) as a matrix-valued function

\[
A_i(u_1, \cdots, u_n) = (A^i_{pq}(u_1, \cdots, u_n)),
\]

\( \forall (u_1, \cdots, u_n) \in \tilde{H}^n_1, 1 \leq p, q \leq n \)

relative the orthogonal frame field \( \{e_1, \cdots, e_n\} \) in \( \tilde{H}^n_1 \). Let \( M_i, i = 1, 2, \cdots, \) be the positive real number given by

\[
M_i = \text{Max} \left\{ \left| \frac{\partial^\alpha a^i_{pq}}{\partial u_j^\alpha \partial u_k^\beta} (u_1(\Sigma_i(s, Z(s))), \cdots, u_n(\Sigma_i(s, Z(s)))) \right| : 1 \leq p, q \leq n; \ 0 \leq l \leq i; \ \alpha + \beta = l; s \in [a_i, b_i]; \ \langle Z(s), Z(s) \rangle \leq i \right\}.
\]

Here \( \Sigma_i : (a_i, b_i) \times R^{n-1} \to U_i \) is the global parametrization of \( U_i \) in (9). Now set \( \eta_i \) to be a positive number such that \( M_i \leq \eta_i(1/2)^i \). Define \( A \) by the following series:

\[
A = \sum_{i=1}^{\infty} (1/\eta_i) A_i.
\]

The series converges uniformly in any compact subset of \( \tilde{H}^n_1 \). The components of \( A \) are \( c^\infty \)-functions. Moreover, it is easy to see that \( A \) never vanishes in \( U \) but vanishes everywhere outside \( U \). Since the \( A_i \)'s satisfy the Gauss and Codazzi equations, so must \( A \). Hence, we have reached

**Theorem 5.** Given a \( c^\infty \)-lamination \( \{U, \mathcal{F}\} \) there exist infinitely many isometric immersions of \( \tilde{H}^n_1 \) into \( \tilde{H}^{n+1}_1 \) so that the associated laminations are precisely the given lamination. The space of such immersions is parametrized by the set of countably many \( c^\infty \)-functions \( k_i = k_i(s) : I_i \to \).
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$R$, where $I_i$ is an open interval of real numbers determined by the maximal integral curves of the orthogonal trajectory to the foliation in $\mathcal{F}_i$ of $U_i$, $i = 1, 2, \cdots$.

Proof. From Lemma 6, we have a $c^\infty$-symmetric tensor field $A$ of type (1,1) defined in $\tilde{H}^n_1$. $A$ satisfies the Gauss (7) and Codazzi (8) equations. By the fundamental theorem of hypersurfaces in $H^{n+1}_1$, there is an isometric immersion whose shape operator is precisely $A$. Rank $A = 1$ in $U$ and $A \equiv 0$ elsewhere. This completes the proof.

Remark. Let $R = R^1_1$ be the preimage of the center circle under the covering map in (2). Given a totally geodesic lamination $\{U, \mathcal{F}\}$, let $U = \cup_{j=1}^{\infty} U_j$, where $U_j$'s are mutually disjoint connected open subsets, each of which is endowed with a complete totally geodesic foliation of codimension 1. Then $U_j \cap R$ is an open interval $I_j$, $1 \leq j$, in $R$. If the totally geodesic foliation of $U_j$ is non-degenerate, we may consider the curvature function $k_j = k_j(s)$ defined in a maximal integral curve of the orthogonal distribution to the foliation as a differentiable function defined in $I_j$ in a natural way. If the foliation in $U_j$ happens to be degenerate, then the curvature function $k_j$ defined in the complementary null curve can be regarded as a differentiable function in $I_j$. Thus, we may state that the space of isometric immersions $f : \tilde{H}^n_1 \rightarrow \tilde{H}^{n+1}_1$ is parametrized by the family of triples $\{U_j, \mathcal{F}_j, k_j = k_j(s)\}$, $1 \leq j$.

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Received September 22, 1993.
UNIVERSITY OF CONNECTICUT
STORRS, CT 06269-3009
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