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Let X be an n -fold loop space. Working with an auxiliary space $P_p^n X$ analogous to the projective plane $P_2 X$, we show that the existence of certain Steenrod connections in $H^*(P_p^n X; \mathbf{F}_p)$ (p odd) implies the vanishing of certain corresponding Dyer-Lashof operations in $H^*(X; \mathbf{F}_p)$, and vice versa.

1. Introduction.

A useful property of the projective plane of an H-space is that the vanishing of certain cup products in $H^*(P_2 X)$ implies the existence of certain related nonzero reduced coproducts in $H^*(X)$ (see [BT]). In [KSW], Kuhn, Slack, and Williams introduce the necessary theory to construct a space $P_p^n X$ with the property that the Steenrod action on its cohomology bears a similar relation to the Dyer-Lashof action on the homology of X . We develop this relation at odd primes in this paper. In particular, our main theorem can be summarized as follows:

Main Theorem. *Suppose X is an n -fold loop space, $1 \leq n \leq \infty$. Then there is a certain cofibration sequence*

$$\Sigma^n \tilde{C}_{n,p} X \xrightarrow{h} \Sigma^n X \xrightarrow{i} P_p^n X \xrightarrow{j} \Sigma^{n+1} \tilde{C}_{n,p} X$$

such that

1. For each $x \in \tilde{H}^q(X)$ there exist elements $\tilde{Q}_r x \in \tilde{H}^{pq+r}(\tilde{C}_{n,p} X)$ such that $Q_r^* w = x$ whenever $h^*(\sigma^n w) = \sigma^n \tilde{Q}_r x$, where σ is the suspension isomorphism and

$$Q_r^*: \tilde{H}^{pq+r}(X) \rightarrow H^q(X)$$

is an element of the opposite algebra to the Dyer-Lashof algebra.

2. If $\bar{x} \in \tilde{H}^{2s+1}(P_p^n X)$ and $i^*(\bar{x}) = \sigma^n x$ then

$$\begin{aligned} \mathcal{P}^s \bar{x} &= u \cdot j^* \left(\sigma^{n+1} \tilde{Q}_{(n-1)(p-1)-1} x \right), \\ \beta \mathcal{P}^s \bar{x} &= u \cdot j^* \left(\sigma^{n+1} \tilde{Q}_{(n-1)(p-1)} x \right), \end{aligned}$$

where u is some undetermined unit in \mathbf{F}_p .

The analogous result at the prime 2 was proved in [KSW].

Using this theorem and the long exact sequence of a cofibration it is easy to see how the triviality of certain Steenrod operations in $\tilde{H}^*(P_p^n X)$ can imply the existence of nontrivial Dyer-Lashof operations in $\tilde{H}_*(X)$, and vice versa. In [Sl], the second author exploits this relation (in the stable case, see Corollary 4.2) to prove the following result.

Theorem (Slack). *If X is a connected infinite loop space (of finite type), and all of the Dyer-Lashof operations are trivial on the mod p homology of X , then X is mod p homotopy equivalent to a product of Eilenberg-MacLane spaces.*

The definitions of the spaces $P_p^n X$ and $\tilde{C}_{n,p} X$ are briefly summarized in Section 2. In Section 3 we define the elements $\tilde{Q}_r x$ and show how they relate to the homology Dyer-Lashof operations, and in Section 4 we prove Part 2 of the Main Theorem. We conclude with an appendix giving the Nishida relations as they apply to the external operations \tilde{Q}_r , which are useful in applications.

We summarize here some of the notational conventions used in this paper. Let p be an odd prime, and take all homology and cohomology with coefficients in the field \mathbf{F}_p of p elements. Recall ([CLM]) that the homology of an n -fold loop space X admits certain Dyer-Lashof operations $Q_r: \tilde{H}_q(X) \rightarrow \tilde{H}_{pq+r}(X)$. (We are using the lower notation of [CPS] in which $Q_{s(p-1)v}$ is defined to be $Q^{(s+\deg v)/2} v$.) In this paper, all spaces will be compactly generated Hausdorff spaces with non-degenerate basepoint $*$, and all maps will be based maps.

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2. The construction of the projective planes.

In this section we recall relevant information from [KSW]. Let $C_n X$ denote the standard approximation to $\Omega^n \Sigma^n X$ of [Ma2, Mi]. It admits a filtration

$$X \simeq C_{n,1} X \subset C_{n,2} X \subset \cdots \subset C_n X.$$

If there is a map $\theta_n^X: C_{n,p} X \rightarrow X$ extending the identity on X , then X is called an H_p^n -space, and there is a category \mathcal{H}_p^n for which the objects are H_p^n -spaces and the morphisms can be thought of as homotopy classes of maps preserving the H_p^n -structure.¹ From now on X will generally be assumed

¹Actually, they only preserve the H_p^n -structure up to homotopy, but they come equipped with prescribed classes of homotopies making the appropriate diagram commute.

to be an H_p^n -space, and, in the main theorem, the hypothesis that X is an n -fold loop space may be replaced by the somewhat weaker hypothesis that $X \in \text{Ob } \mathcal{H}_p^n$.

There is a functor $\Omega^n: \mathcal{H}_p^0 \rightarrow \mathcal{H}_p^n$ that, on objects, is the usual n -fold loop space functor. It admits a left adjoint $P_p^n: \mathcal{H}_p^n \rightarrow \mathcal{H}_p^0$ that is constructed (on objects) as follows. If X is an H_p^n -space, there are two natural retractions $\Sigma^n C_{n,p} X \rightarrow \Sigma^n X$; one is $\Sigma^n \theta_n^X$ and the other is the adjoint ε_n of the composition

$$C_{n,p} X \hookrightarrow C_n X \longrightarrow \Omega^n \Sigma^n X.$$

If $\tilde{C}_{n,p} X$ is the cofiber of the inclusion $X \hookrightarrow C_{n,p} X$, then there is a map $h: \Sigma^n \tilde{C}_{n,p} X \rightarrow \Sigma^n X$, making the diagram

$$\begin{array}{ccccc} \Sigma^n X & \longrightarrow & \Sigma^n C_{n,p} X & \longrightarrow & \Sigma^n \tilde{C}_{n,p} X \\ \downarrow & & \downarrow \Sigma^n \theta_n^X - \varepsilon_n & & \downarrow h \\ * & \longrightarrow & \Sigma^n X & \xlongequal{\quad} & \Sigma^n X \end{array}$$

homotopy commute; h is unique up to homotopy by the cofiber mapping sequence. The higher projective plane $P_p^n X$ is defined to be the cofiber of h , and we thus have the cofibration sequence

$$\Sigma^n \tilde{C}_{n,p} X \xrightarrow{h} \Sigma^n X \xrightarrow{i} P_p^n X \xrightarrow{j} \Sigma^{n+1} \tilde{C}_{n,p} X$$

of the Main Theorem.

By way of comparison, the ordinary projective plane $P_2 X$ can be defined as the cofiber of the map h determined by the commutative diagram

$$\begin{array}{ccccc} \Sigma(X \vee X) & \longrightarrow & \Sigma(X \times X) & \longrightarrow & \Sigma X \wedge X \\ \downarrow & & \downarrow -\Sigma\pi_1 + \Sigma\mu - \Sigma\pi_2 & & \downarrow h \\ * & \longrightarrow & \Sigma X & \xlongequal{\quad} & \Sigma^n X. \end{array}$$

3. The external operations.

We now start to define the external operations \tilde{Q}_r . We will begin by using the calculation of $H_*(C_n X)$ in [CLM] to observe that $H_*(\tilde{C}_{n,p} X)$ can be viewed as a particular direct summand of $H_*(C_n X)$. We then use this information to produce a specific basis for $H_*(\tilde{C}_{\infty,p} K(\mathbf{Z}/p, s+q-\gamma))$, in terms of which we define a map

$$G: \tilde{C}_{\infty,p} K(\mathbf{Z}/p, s+q-\gamma) \longrightarrow K(\mathbf{Z}/p, p(s+q)-2\gamma).$$

(The roles of s , q , and γ , will be explained shortly.) Then, after recalling another map

$$\tilde{\varepsilon}_k: \Sigma^k \tilde{C}_{n+k,p} X \rightarrow \tilde{C}_{n,p} \Sigma^k X,$$

we define the external operations (Definition 3.3) in terms of $\tilde{\varepsilon}_k$, G , and a map f representing $x \in H^*(X)$. We end the section by demonstrating some of their properties.

In this section, r will denote $s(p-1) - \gamma$, where $\gamma \in \{0, 1\}$, so that $Q_r: \tilde{H}_q(X) \rightarrow \tilde{H}_{pq+r}(X)$ will be defined on an H_p^n -space provided $s < n$ and $q + s$ is even. Recall ([CLM, III]) that, along with the Dyer-Lashof operations Q_r , the homology of $C_n X$ admits Browder operations

$$\lambda_{n-1}: \tilde{H}_q(C_n X) \otimes \tilde{H}_m(C_n X) \longrightarrow \tilde{H}_{q+m+n-1}(C_n X).$$

We will say that a *Browder product of weight l* is a composition of Browder operations in which l (not necessarily distinct) variables appear; for instance, $\lambda_{n-1}(x, \lambda_{n-1}(x, y))$ is a Browder product of weight 3. We recall that the Browder operations are neither associative nor commutative.

Let $\eta = \eta_1: X \rightarrow C_n X$ and $\eta_k: C_{n,k} X \rightarrow C_n X$ be the respective inclusions. We then have the following proposition, which summarizes information contained in Cohen's proof of the structure theorem for $\tilde{H}_*(C_n X)$ ([CLM, III, Theorem 3.1]).

Proposition 3.1. *The map $(\eta_k)_*: \tilde{H}_*(C_{n,k} X) \rightarrow \tilde{H}_*(C_n X)$ is one-to-one, and the filtration on homology defined by $F_k \tilde{H}_*(C_n X) = \text{im}(\eta_k)_*$ is identical to the following algebraic filtration: If λ is a Browder product of weight l applied to elements of $\eta_*(\tilde{H}_*(X))$, then the filtration of $Q_{r_1} \cdots Q_{r_l} \lambda$ is $p^t l$, and, if $a \in F_j \tilde{H}_*(C_n X)$ and $b \in F_k \tilde{H}_*(C_n X)$ then $a * b \in F_{j+k} \tilde{H}_*(C_n X)$, where $a * b$ is the Pontryagin product of a and b .*

From this, along with the structure theorem for $\tilde{H}_*(C_n X)$, it follows that $\text{im}(\eta_p)_* \subset \tilde{H}_*(C_n X)$ can be written as $A \oplus B \oplus C$, where A , B , and C are defined as follows.

$$\begin{aligned} A &= \eta_* \tilde{H}_*(X), \\ B &= \{Q_{s(p-1)-\gamma} \eta_*(x) \mid s > 0, (s + \deg x) \text{ is even}\}, \end{aligned}$$

and C is the set $\{\eta_*(x_1) * \cdots * \eta_*(x_k) * \lambda_1 * \cdots * \lambda_t\}$, where $x_1, \dots, x_k \in \tilde{H}_*(X)$, $\lambda_1, \dots, \lambda_t$ are Browder products of weight l_1, \dots, l_t respectively, and $2 \leq k + l_1 + \cdots + l_t \leq p$. Since η_* and $(\eta_p)_*$ are one-to-one, we can by abuse of notation consider $B \oplus C$ as a decomposition of $\tilde{H}_*(\tilde{C}_{n,p} X)$. We can then make

Definition 3.2. If $v \in \tilde{H}_q(X)$ and $q + s$ is even then the external homology Dyer-Lashof operation $\bar{Q}_{s(p-1)-\gamma} v \in \tilde{H}_*(\tilde{C}_{n,p} X) \subset \tilde{H}_*(C_{n,p} X)$ is defined as $Q_{s(p-1)-\gamma} \eta_*(v)$.

If $(X, \theta_n^X) \in \text{Ob } \mathcal{H}_p^n$, then the structure map θ_n^X characterizes the internal Dyer-Lashof operations in $\tilde{H}_*(X)$ by the formula

$$Q_r v = (\theta_n^X)_*(\overline{Q}_r v).$$

We also have

$$h_*(\sigma^n \overline{Q}_r v) = \sigma^n Q_r v.$$

To see this, first note that $(\varepsilon_n)_*(\overline{Q}_r v) = 0$ for otherwise the formula $(\varepsilon_n)_*(\overline{Q}_r v)$ would define a natural positive-dimensional homology operation on an arbitrary space, which is impossible. The formula then follows by a simple diagram chase on the diagram from Section 2 defining h .

Now we come to the maps G and $\tilde{\varepsilon}_k$. From the point of view of the theory of spectra, one way to think about G is that, when $\gamma = 0$, the existence of the G we want is essentially equivalent to the fact that the mod p Eilenberg-MacLane spectrum admits the structure of an H_∞^2 ring spectrum (as defined by Bruner et. al. [BMMS]). In [KSW], the analogous key fact is the existence of an H_∞^1 ring spectrum structure on the mod 2 Eilenberg-MacLane spectrum. In fact, using the methods of this paper and of [KSW] it should be possible to generalize our main theorem to a theorem on arbitrary H_∞^d ring spectrums E , which would relate power operations in the E^* cohomology of a generalized projective plane to power operations in the E_* homology of the original space.

For our purposes, it is convenient to have an explicit description of G . The main technical difficulty in the odd primary case which does not cause a problem in the 2 primary case in [KSW] is the fact that $\tilde{C}_{\infty,p} X$ is not homotopy equivalent to the p -adic construction, $D_p X$, when p is odd (stably it contains $D_p X$ as a wedge summand), whereas it is when $p = 2$. This leads to many possible choices for the definition of G in the odd primary case, and it is important to carefully define the ‘‘correct’’ one.

To simplify notation let $K = K(\mathbf{Z}/p, s + q - \gamma)$. Then the fundamental class $\iota = \iota_{s+q-\gamma}$, its images under the Steenrod algebra, and their respective products form a standard basis $\mathcal{B}\tilde{H}^*(K)$ for $\tilde{H}^*(K)$; let its dual basis be $\mathcal{B}\tilde{H}_*(K)$, and let the element of $\mathcal{B}\tilde{H}_*(K)$ dual to ι be ν .

By Theorem 3.1, $\tilde{H}_*(\tilde{C}_{\infty,p} K) \cong F_p \tilde{H}_*(C_\infty K) / F_1 \tilde{H}_*(C_\infty K)$. If we then apply the structure theorem for $\tilde{H}_*(C_\infty X)$ ([CLM, I, 4.1]), it is not hard to show that one gets a basis for $\tilde{H}_*(\tilde{C}_{\infty,p} K)$ by totally ordering the elements of $\mathcal{B}\tilde{H}_*(K)$ and then defining the set $\mathcal{B}\tilde{H}_*(\tilde{C}_{\infty,p} K)$ to be $A \cup B$, where

$$\begin{aligned} A &= \{\eta_*(v_1) * \cdots * \eta_*(v_k) | v_1, \dots, \\ &\quad v_k \in \mathcal{B}\tilde{H}_*(K), 2 \leq k \leq p, v_1 \leq v_2 \leq \cdots \leq v_k\}, \\ B &= \{\overline{Q}_{s(p-1)-\gamma} v | v \in \mathcal{B}\tilde{H}_*(K), (s + \deg v) \text{ even}, s > 0, \gamma = 0 \text{ or } 1\}. \end{aligned}$$

We emphasize that the notation $\eta_*(v_1) * \cdots * \eta_*(v_k)$ for an element of $\tilde{H}_*(\tilde{C}_{\infty,p}K)$ only makes sense via the isomorphism with

$$F_p \tilde{H}_*(C_\infty K) / F_1 \tilde{H}_*(C_\infty K),$$

since $\tilde{C}_{\infty,p}K$ itself is not an H-space and thus admits no Pontryagin products.

We now let $\mathcal{B}\tilde{H}^*(\tilde{C}_{\infty,p}K)$ be the dual basis to $\mathcal{B}\tilde{H}_*(\tilde{C}_{\infty,p}K)$, and, for each v in $\mathcal{B}\tilde{H}_*(\tilde{C}_{\infty,p}K)$, we denote the dual basis element by $(v)^{\text{dual}}$. If $\gamma = 0$ we then define $G: \tilde{C}_{\infty,p}K(\mathbf{Z}/p, s+q) \rightarrow K(\mathbf{Z}/p, p(s+q))$ to be the map representing the cohomology class

$$\sum (\eta_*(\nu^{t_1}) * \cdots * \eta_*(\nu^{t_k}))^{\text{dual}}$$

where the sum runs over sequences (t_1, \dots, t_k) such that

1. $k \leq p$,
2. $t_1 + \cdots + t_k = p$, and
3. $\nu^{t_1} \leq \nu^{t_2} \leq \cdots \leq \nu^{t_k}$ in the ordering on $\mathcal{B}\tilde{H}_*(K)$.

This definition makes sense since we have required that $s+q$ be even.

If $\gamma = 1$ then we define $G: \tilde{C}_{\infty,p}K(\mathbf{Z}/p, s+q-1) \rightarrow K(\mathbf{Z}/p, p(s+q)-2)$ to be the map representing the cohomology class

$$(\overline{Q}_{p-2}\nu)^{\text{dual}}.$$

Now we consider the map $\tilde{\varepsilon}_k$. It is constructed in [KSW], and it fits into a commutative diagram

$$\begin{array}{ccccc} \Sigma^k \tilde{C}_{n+k,p}X & \longleftarrow & \Sigma^k C_{n+k,p}X & \longrightarrow & \Sigma^k \Omega^{n+k} \Sigma^{n+k} X \\ & & \downarrow \varepsilon_k & & \downarrow \varepsilon \\ \tilde{C}_{n,p} \Sigma^k X & \longleftarrow & C_{n,p} \Sigma^k X & \longrightarrow & \Omega^n \Sigma^{n+k} X \end{array}$$

where ε_k is a map constructed in [Ma2] and $\varepsilon: \Sigma^k \Omega^k Y \rightarrow Y$ is the evaluation map. If X is connected, so that $C_n X \simeq \Omega^n \Sigma^n X$, then ε_k may be seen as “filtering” ε .

One important fact about $\tilde{\varepsilon}_k$ is that it commutes with the (homology) Dyer-Lashof operations in the sense that

$$(\tilde{\varepsilon}_k)_* \left(\sigma^k \overline{Q}_{s(p-1)-\gamma} v \right) = \begin{cases} \overline{Q}_{(s-k)(p-1)-\gamma} \sigma^k v & \text{if } s-k > \gamma; \\ 0 & \text{otherwise.} \end{cases}$$

This follows from the above commutative diagram combined with the standard fact that the Dyer-Lashof operations commute with the loop homomorphism induced by ε (see, e.g., [CLM, III, Theorem 1.4]). We also note that $\tilde{\varepsilon}_k$ is still defined when $n = \infty$.

We now define the external cohomology operations.

Definition 3.3. Suppose γ is an element of the set $\{0, 1\}$, and $s \geq \gamma$ and $q \geq 0$ are integers such that $s + q$ is even. Let $r = s(p - 1) - \gamma$. For every $1 \leq n \leq \infty$ and any space X , let

$$\tilde{Q}_r: \tilde{H}^q(X) \longrightarrow \tilde{H}^{pq+r}(\tilde{C}_{n,p}X)$$

be defined as follows. If $x \in \tilde{H}^q(X)$ is represented by $f: X \rightarrow K(\mathbf{Z}/p, q)$, let $\sigma^{s-\gamma}(\tilde{Q}_r x)$ be represented by the composite

$$\begin{aligned} \Sigma^{s-\gamma} \tilde{C}_{n,p}X &\rightarrow \Sigma^{s-\gamma} \tilde{C}_{\infty,p}X \xrightarrow{\tilde{\epsilon}_{s-\gamma}} \tilde{C}_{\infty,p} \Sigma^{s-\gamma} X \xrightarrow{\tilde{C}_{\infty,p} \Sigma^{s-\gamma} f} \tilde{C}_{\infty,p} \Sigma^{s-\gamma} K(\mathbf{Z}/p, q) \\ &\xrightarrow{\tilde{C}_{\infty,p} \epsilon} \tilde{C}_{\infty,p} K(\mathbf{Z}/p, s+q-\gamma) \xrightarrow{G} K(\mathbf{Z}/p, p(s+q)-2\gamma). \end{aligned}$$

The element $\tilde{Q}_r x$ for $r > 0$ is sometimes denoted $e_r \otimes x^p$ or $e_r \int x$ in the literature (the way our operations are defined, $\tilde{Q}_0 x \neq e_0 \int x$). We remark for future reference that when $s \geq n$ above, the operation \tilde{Q}_r is trivial.

The next proposition relates our external cohomology operations \tilde{Q}_r to the external homology operations, via the Kronecker pairing $\langle \cdot, \cdot \rangle$. This will make it reasonable to see \tilde{Q}_r as a sort of “dual external Dyer-Lashof operation”.

Proposition 3.4. *Suppose $x \in \tilde{H}^q(X)$ and $v \in \tilde{H}_q(X)$. Then*

$$\langle \tilde{Q}_r x, \overline{Q}_{r'} v \rangle = \begin{cases} \langle x, v \rangle & \text{if } r = r'; \\ 0 & \text{if } r \neq r'. \end{cases}$$

Furthermore, if $r = s(p - 1) - \gamma > 0$, then the element $\tilde{Q}_r x$ paired with any Browder or Pontryagin product operation yields zero.

Proof. If $r > 0$ we assume by naturality that $n = \infty$ and make the following calculation. By the definition of $\tilde{Q}_r x$,

$$\begin{aligned} \langle \tilde{Q}_r x, \overline{Q}_{r'} v \rangle &= \langle \sigma^{s-\gamma} \tilde{Q}_{s(p-1)-\gamma} x, \sigma^{s-\gamma} \overline{Q}_{s'(p-1)-\gamma'} v \rangle \\ &= \langle \tilde{\epsilon}_{s-\gamma}^* (\tilde{C}_{\infty,p} \Sigma^{s-\gamma} f)^* (\tilde{C}_{\infty,p} \epsilon)^* [G], \sigma^{s-\gamma} \overline{Q}_{s'(p-1)-\gamma'} v \rangle \\ &= \langle [G], (\tilde{C}_{\infty,p} \epsilon)_* (\tilde{C}_{\infty,p} \Sigma^{s-\gamma} f)_* (\tilde{\epsilon}_{s-\gamma})_* \sigma^{s-\gamma} \overline{Q}_{s'(p-1)-\gamma'} v \rangle. \end{aligned}$$

Now, we have already observed how $(\tilde{\epsilon}_{s-\gamma})_*$ commutes with the \overline{Q}_r . In addition, for any map $g: X \rightarrow Y$, we have $(C_{\infty} g)_*(Q_r \eta_* v) = Q_r g_*(\eta_* v)$ [CLM], and hence

$$(\tilde{C}_{\infty,p} g)_*(\overline{Q}_r v) = \overline{Q}_r g_*(v).$$

Thus,

$$\begin{aligned} & \langle [G], (\tilde{C}_{\infty,p}\epsilon)_*(\tilde{C}_{\infty,p}\Sigma^{s-\gamma}f)_*(\tilde{\epsilon}_{s-\gamma})_*\sigma^{s-\gamma}\overline{Q}_{s'(p-1)-\gamma'}v \rangle \\ &= \langle [G], \overline{Q}_{(s'-s+\gamma)(p-1)-\gamma'}\epsilon_*\sigma^{s-\gamma}f_*(v) \rangle. \end{aligned}$$

Here we let $\overline{Q}_r x = 0$ if $r < 0$. Now, by the construction of G , the last quantity can be nonzero only if $\overline{Q}_{(s'-s+\gamma)(p-1)-\gamma'}\epsilon_*\sigma^{s-\gamma}f_*(v)$ equals either $u \cdot \overline{Q}_0\nu_{s+q}$ or $u \cdot \overline{Q}_{p-2}\nu_{s+q-1}$, where $u \in \mathbf{F}_p$ is some unit. In the former case, $\gamma = \gamma' = 0$ and $s - s' = 0$, and in the latter case, $\deg(\epsilon_*\sigma^{s-\gamma}f_*(v)) = s + q - \gamma$ must be odd for the Dyer-Lashof operation to be nontrivial, implying (since $s + q$ is even) that $\gamma = 1$. Thus $\gamma = \gamma' = 1$ and $s - s' = 0$, and so in either case $s' = s$ and $\gamma' = \gamma$, i.e., $r' = r$.

Now let us write r'' for $(s' - s + \gamma)(p - 1) - \gamma' = \gamma(p - 1) - \gamma$. Then $\langle [G], u \cdot \overline{Q}_{r''}\nu_{s+q-\gamma} \rangle = u$, so we need to show that

$$\overline{Q}_{r''}\epsilon_*\sigma^{s-\gamma}f_*(v) = \langle x, v \rangle \cdot \overline{Q}_{r''}\nu_{s+q-\gamma}.$$

By linearity of the Dyer-Lashof operations it suffices to show that $\epsilon_*\sigma^{s-\gamma}f_*(v) = \langle x, v \rangle \cdot \nu_{s+q-\gamma}$, and, since $H_{s+q-\gamma}(K(\mathbf{Z}/p, s + q - \gamma))$ is one-dimensional, this amounts to showing that $\langle \iota_{s+q-\gamma}, \epsilon_*\sigma^{s-\gamma}f_*(v) \rangle = \langle x, v \rangle$. This is easily seen to be true.

For the other formulas we are assuming $r > 0$, which means that $\tilde{Q}_r x$ is in the kernel of the map $\tilde{H}^*(\tilde{C}_{n,p}X) \rightarrow \tilde{H}^*(\tilde{C}_{1,p}X)$ induced by inclusion. Thus the pairing of $\tilde{Q}_r x$ with any Pontryagin product operation applied to (v, w) must be zero. Similarly, since $\tilde{Q}_r x$ comes from the cohomology of $\tilde{C}_{\infty,p}X$, it must pair trivially with any Browder operation. \square

It follows from Proposition 3.4, along with the formula for $h_*(\sigma^n \overline{Q}_r v)$, that if $h^*(\sigma^n y) = \sigma^n \tilde{Q}_r x$ for some $x, y \in \tilde{H}^*(X)$, then $\langle x, v \rangle = \langle y, \overline{Q}_r v \rangle$ for all $v \in \tilde{H}_*(X)$. Equivalently, if $h^*(\sigma^n y) = \sigma^n \tilde{Q}_r x$, then $\overline{Q}_r^* y = x$ as in Part 1 of the Main Theorem.

We close this section with the following lemma, which shows that the \tilde{Q}_r commute with $\tilde{\epsilon}_k^*$ just as one would want.

Lemma 3.5. *Let $k > 0$ be an integer, let X be an H_p^{n+k} -space, and choose $x \in \tilde{H}^q(X)$. Then*

$$\tilde{\epsilon}_k^*(\tilde{Q}_{s(p-1)-\gamma}\sigma^k x) = \sigma^k \tilde{Q}_{(s+k)(p-1)-\gamma} x$$

for any $\gamma \leq s < n + k$.

Proof. As with Proposition 3.4, we assume $n = \infty$ by naturality, and we will show that

$$\sigma^{q-\gamma} \tilde{\varepsilon}_k^* (\tilde{Q}_{s(p-1)-\gamma} \sigma^k x) = \sigma^{q-\gamma+k} \tilde{Q}_{(s+k)(p-1)-\gamma} x.$$

Let x be represented by the map $f: X \rightarrow K(\mathbf{Z}/p, q)$, in which case $\sigma^k x$ will be represented by the map $\epsilon \circ \Sigma^k f: \Sigma^k X \rightarrow K(\mathbf{Z}/p, q+k)$. Then, by definition

$$\sigma^{q-\gamma} \tilde{\varepsilon}_k^* (\tilde{Q}_{s(p-1)-\gamma} \sigma^k x) = (\Sigma^{s-\gamma} \tilde{\varepsilon}_k)^* \tilde{\varepsilon}_{s-\gamma}^* (\tilde{C}_{\infty, p} \Sigma^{s-\gamma} (\epsilon \circ \Sigma^k f))^* (\tilde{C}_{\infty, p} \epsilon)^* [G].$$

Now, $\tilde{\varepsilon}_j \circ \Sigma^j \tilde{\varepsilon}_k = \tilde{\varepsilon}_{j+k}$ by construction (see [KSW]), and

$$\tilde{C}_{\infty, p} \Sigma^{s-\gamma} (\epsilon \circ \Sigma^k f) = \tilde{C}_{\infty, p} \Sigma^{s-\gamma} \epsilon \circ \tilde{C}_{\infty, p} \Sigma^{k+s-\gamma} f$$

by functoriality, so we actually have

$$\sigma^{q-\gamma} \tilde{\varepsilon}_k^* (\tilde{Q}_{s(p-1)-\gamma} \sigma^k x) = \tilde{\varepsilon}_{k+s-\gamma}^* (\tilde{C}_{\infty, p} \Sigma^{k+s-\gamma} f)^* (\tilde{C}_{\infty, p} \Sigma^{s-\gamma} \epsilon)^* (\tilde{C}_{\infty, p} \epsilon)^* [G].$$

But then $\epsilon \circ \Sigma^{s-\gamma} \epsilon = \epsilon$, so in fact

$$\begin{aligned} \sigma^{q-\gamma} \tilde{\varepsilon}_k^* (\tilde{Q}_{s(p-1)-\gamma} \sigma^k x) &= \tilde{\varepsilon}_{k+s-\gamma}^* (\tilde{C}_{\infty, p} \Sigma^{k+s-\gamma} f)^* (\tilde{C}_{\infty, p} \epsilon)^* [G] \\ &= \sigma^{q-\gamma+k} \tilde{Q}_{(s+k)(p-1)-\gamma} x. \end{aligned}$$

□

4. The Main Theorem.

Part 1 of the Main Theorem has already been proved; here we restate Part 2 in slightly altered form. First we introduce the following convention. Suppose x_1 and x_2 are two elements of an \mathbf{F}_p -algebra (e.g. the mod p cohomology of a space). Then we say $x_1 \doteq x_2$ if $x_1 = u \cdot x_2$ where $u \in \mathbf{F}_p$ is a unit. This is easily seen to be an equivalence relation.

Theorem 4.1. *Suppose p is an odd prime and $X \in \text{Ob } \mathcal{H}_p^n$, with basic cofibration sequence*

$$\Sigma^n \tilde{C}_{n,p} X \xrightarrow{h} \Sigma^n X \xrightarrow{i} P_p^n X \xrightarrow{j} \Sigma^{n+1} \tilde{C}_{n,p} X.$$

Let $n = 2s + 1 - \delta$ for some $\delta \in \{0, 1\}$ and $s \geq \delta$. If $\bar{x} \in \tilde{H}^{2q+\delta+n}(P_p^n X)$, then

$$\begin{aligned} \mathcal{P}^{q+s} \bar{x} &\doteq j^* \left(\sigma^{n+1} \tilde{Q}_{(n-1)(p-1)-1} x \right), \\ \beta \mathcal{P}^{q+s} \bar{x} &\doteq j^* \left(\sigma^{n+1} \tilde{Q}_{(n-1)(p-1)} x \right), \end{aligned}$$

where $i^* \bar{x} = \sigma^n x$ defines $x \in \tilde{H}^{2q+\delta}(X)$.

We first prove Theorem 4.1 for the universal example $X = K(\mathbf{Z}/p, 2q + \delta)$ and $x = \iota_{2q+\delta}$; the universal example is then used to prove the theorem for general X . We will prove the formulas for $\beta\mathcal{P}^{q+s}\bar{\iota}_{2q+\delta}$ and $\mathcal{P}^{q+s}\bar{\iota}_{2q+\delta}$ separately.

For the $\beta\mathcal{P}^{q+s}\bar{\iota}_{2q+\delta}$ formula we first consider the special case in which $s = \delta = 0$. Let $\bar{\iota}_{2q} \in \tilde{H}^{2q+1}(P_p^1 K(\mathbf{Z}/p, 2q + \delta))$ be (the unique class) such that $i^*(\bar{\iota}_{2q}) = \iota_{2q}$; we know such a class exists because ι_{2q} is representable by an infinite loop map, and hence $h^*(\iota_{2q}) = 0$. We use the following commutative diagram, where $K(\mathbf{Z}/p, t)$ is written K_t , κ represents $\beta\mathcal{P}^q \iota_{2q+1}$, and E is the homotopy fiber of κ .

$$\begin{array}{ccccccc} K_{2pq+1} & \xrightarrow{\lambda} & E & \xrightarrow{\pi} & K_{2q+1} & \xrightarrow{\kappa} & K_{2pq+2} \\ \uparrow f_2 & & \uparrow f_1 & & \uparrow f & & \uparrow f'_2 \\ \Sigma \tilde{C}_{1,p} K_{2q} & \xrightarrow{h} & \Sigma K_{2q} & \xrightarrow{i} & P_p^1 K_{2q} & \xrightarrow{j} & \Sigma^2 \tilde{C}_{1,p} K_{2q} \end{array} .$$

Here the map f represents the class $\bar{\iota}_{2q}$, and the map f_1 exists by the lifting property of a fibration since $\kappa f i$ represents

$$\beta\mathcal{P}^q i^*(\bar{\iota}_{2q}) = \beta\mathcal{P}^q \iota_{2q} = 0$$

and is thus homotopically trivial. There is of course some indeterminacy in the choice of f_1 ; we will begin with an arbitrary choice and then alter it as necessary by maps factoring through the fiber.

We will give a specific construction of f_2 making the leftmost square commute, and then we can choose f'_2 to be the adjoint of f_2 . This will make the rightmost square commute as well. (We will in general let g' denote the adjoint of g , where g may either be a map $X \rightarrow \Omega Y$ or $\Sigma X \rightarrow Y$.) The construction of f_2 will in turn be as the adjoint of a map $\tilde{C}_{1,p} K_{2q} \rightarrow K_{2pq}$ factoring through ΩE , and the key to the proof is that the image of ι_{2pq} in $H^*(\Omega E)$ is connected by the opposite Dyer-Lashof operation Q_0^* to $(\Omega\pi)^*(\iota_{2q})$, and that this connection is preserved by the map $\tilde{C}_{1,p} K_{2q} \rightarrow \Omega E$.

First let us construct f_2 . Because $\Omega\kappa$ is homotopically trivial, $\Omega E \simeq K_{2pq} \times K_{2q}$. Hence there is a map $\rho: \Omega E \rightarrow K_{2pq}$ such that $\rho\Omega\lambda$ is homotopic to the identity. We then define f_2 to be the adjoint of the composition

$$\tilde{C}_{1,p} K_{2q} \xrightarrow{h'} \Omega \Sigma K_{2q} \xrightarrow{\Omega f_1} \Omega E \xrightarrow{\rho} K_{2pq} .$$

Thus, if we write $\rho^*(\iota_{2pq})$ as just $\iota_{2pq} \in \tilde{H}^*(\Omega E)$, then f_2 represents $\sigma(h')^*(\Omega f_1)^* \iota_{2pq}$. Using the definitions of the various maps and the relationships between them, we then see that

$$\beta\mathcal{P}^q \bar{\iota} = j^*(\sigma^2(h')^*(\Omega f_1)^* \iota_{2pq}) .$$

Hence to prove the theorem in this case we need only show that

$$(h')^*(\Omega f_1)^* \iota_{2pq} \doteq \tilde{Q}_0 \iota_{2q}.$$

We will do this by showing that

$$\langle (h')^*(\Omega f_1)^* \iota_{2pq}, v \rangle \doteq \begin{cases} 1 & \text{if } v = \eta_*(\nu_{2q}^{t_1}) * \cdots * \eta_*(\nu_{2q}^{t_k}) \\ & \text{with } t_1 + \cdots + t_k = p, \\ 0 & \text{otherwise,} \end{cases}$$

for any v in the basis

$$\begin{aligned} & \mathcal{B}\tilde{H}_*(\tilde{C}_{1,p}K_{2q}) \\ & = \left\{ \eta_*(v_1) * \cdots * \eta_*(v_k) \mid v_1, \dots, v_k \in \mathcal{B}\tilde{H}_*(K_{2q}), 2 \leq k \leq p \right\}. \end{aligned}$$

It is easy to see that this characterizes $\tilde{Q}_0 \iota_{2q}$.

We will be using the fact that $\langle (h')^*(\Omega f_1)^* \iota_{2pq}, v \rangle = \langle \iota_{2pq}, (\Omega f_1)_*(h')_* v \rangle$, and so we begin by considering ι_{2pq} . We claimed above that ι_{2pq} was connected to ι_{2q} by Q^* . More precisely, let $\mathcal{B}\tilde{H}^*(\Omega E)$ be the obvious basis consisting of products of elements of $\mathcal{B}\tilde{H}^*(K(\mathbf{Z}/p, 2q))$ with elements of $\mathcal{B}\tilde{H}^*(K(\mathbf{Z}/p, 2pq))$, and let $\mathcal{B}\tilde{H}_*(\Omega E)$ be the dual basis. Then

$$(\nu_{2q}^p)^{\text{dual}} \doteq \iota_{2pq}.$$

This follows from the well-known formula for the coproduct of ι_{2pq} in $H^*(E)$; see [Za, §3]. We cannot simply choose a representation of ΩE as $K_{2q} \times K_{2pq}$ to make the equality exact as does Zabrodsky, since the projection $\rho: \Omega E \rightarrow K_{2pq}$ is constrained by the requirement that it be a retraction of the map $\Omega\lambda$.

Since $(\nu_{2q}^p)^{\text{dual}} \doteq \iota_{2pq}$, it will suffice to show, for $v \in \mathcal{B}\tilde{H}_*(\tilde{C}_{1,p}K_{2q})$, that $(\Omega f_1)_*(h')_* v \doteq \nu_{2q}^p$ if and only if $v \doteq \eta_*(\nu_{2q}^{t_1}) * \cdots * \eta_*(\nu_{2q}^{t_k})$ with $t_1 + \cdots + t_k = p$. We will prove the following formula:

$$(\Omega f_1)_*(h')_*(\eta_*(v_1) * \cdots * \eta_*(v_k)) = -v_1 * \cdots * v_k,$$

which does the trick.

We begin with $(h')_*$. By considering how h was constructed (Section 2, see [KSW] for details), we see that its adjoint h' fits into the following commutative diagram.

$$\begin{array}{ccccc} K_{2q} & \longrightarrow & C_{1,p}K_{2q} & \longrightarrow & \tilde{C}_{1,p}K_{2q} \\ \downarrow & & \downarrow (\Sigma\theta_p^1)' - \varepsilon'_1 & & \downarrow h' \\ * & \longrightarrow & \Omega\Sigma K_{2q} & \xlongequal{\quad} & \Omega\Sigma K_{2q} \end{array} .$$

Here we note that ε'_1 is the map

$$C_{1,p}K_{2q} \hookrightarrow C_1K_{2q} \xrightarrow{\cong} \Omega\Sigma K_{2q},$$

where the second arrow is the usual equivalence of May [Ma2].

Let $\eta_*(v_1) * \cdots * \eta_*(v_k) \in \mathcal{B}\tilde{H}_*(\tilde{C}_{1,p}K_{2q})$ be given, and observe, since $(\varepsilon'_1)_*$ is operation preserving, that

$$(h')_*(\eta_*(v_1) * \cdots * \eta_*(v_k)) = (\Sigma\theta_p^1)'_*(\eta_*(v_1) * \cdots * \eta_*(v_k)) - \eta_*(v_1) * \cdots * \eta_*(v_k).$$

Now $(\Sigma\theta_p^1)'$ is the composition

$$C_{1,p}K_{2q} \xrightarrow{\eta} \Omega\Sigma C_{1,p}K_{2q} \xrightarrow{\Omega\Sigma\theta_p^1} \Omega\Sigma K_{2q},$$

implying that $(\Sigma\theta_p^1)'_*(\eta_*(v_1) * \cdots * \eta_*(v_k)) = \eta_*(v_1 * \cdots * v_k)$ and hence

$$(h')_*(\eta_*(v_1) * \cdots * \eta_*(v_k)) = \eta_*(v_1 * \cdots * v_k) - \eta_*(v_1) * \cdots * \eta_*(v_k).$$

We will handle the two terms on the right hand separately, first altering f_1 by a map factoring through K_{2pq} so that $\langle \iota_{2pq}, (\Omega f_1)_*\eta_*(v_1 * \cdots * v_k) \rangle = 0$. For each $v \in \mathcal{B}\tilde{H}_{2pq}(K_{2q})$, let $f_v: \Sigma K_{2q} \rightarrow K_{2pq+1}$ be the map representing the cohomology class $\langle (\Omega f_1)^*\iota_{2pq}, \eta_*(v) \rangle \cdot \sigma(v^{\text{dual}})$, and take f_1 now to be the map obtained from the original choice of f_1 by subtracting the sum

$$\sum_{v \in \mathcal{B}\tilde{H}_{2pq}(K_{2q})} \lambda \circ f_v$$

using the H-structure of E . Redefine f_2 accordingly. Then

$$\langle \iota_{2pq}, (\Omega f_1)_*\eta_*(v) \rangle = 0$$

for any v in $\mathcal{B}\tilde{H}_{2pq}(K_{2q})$ by the construction of f_1 , and in particular,

$$\langle \iota_{2pq}, (\Omega f_1)_*\eta_*(v_1 * \cdots * v_k) \rangle = 0.$$

We now have to consider $\langle \iota_{2pq}, (\Omega f_1)_*(-\eta_*(v_1) * \cdots * \eta_*(v_k)) \rangle$. By construction,

$$(\Omega\pi)(\Omega f_1)\eta \simeq (\Omega f)(\Omega i)\eta \simeq \text{id},$$

and hence $(\Omega f_1)_*\eta_*(v) = v$ for any $v \in H_*(K_{2q})$, where, on the right hand side, v denotes $1 \otimes v$ in $H_*(\Omega E)$. Since Ωf_1 is an H-map, we see that $(\Omega f_1)_*(-\eta_*(v_1) * \cdots * \eta_*(v_k)) = -v_1 * \cdots * v_k \in \tilde{H}^{2pq}(\Omega E)$. Thus, putting together our calculations of $(\Omega f_1)_*$ and $(h')_*$, we have $(\Omega f_1)_*(h')_*(\eta_*(v_1) * \cdots * \eta_*(v_k)) = -v_1 * \cdots * v_k$ for any $\eta_*(v_1) * \cdots * \eta_*(v_k) \in \mathcal{B}\tilde{H}_*(\tilde{C}_{1,p}K_{2q})$, as we wanted.

Consider now the case in which $\delta = 0$ or 1 and $s > 1$. Theorem 4.1 of [KSW] implies that the following diagram commutes up to homotopy.

$$\begin{array}{ccc}
P_p^n K(\mathbf{Z}/p, 2q + \delta) & \xrightarrow{j} & \Sigma^{n+1} \tilde{C}_{n,p} K(\mathbf{Z}/p, 2q + \delta) \\
\downarrow \epsilon & & \downarrow \Sigma^2 \tilde{\epsilon}_{n-1} \\
& & \Sigma^2 \tilde{C}_{1,p} \Sigma^{n-1} K(\mathbf{Z}/p, 2q + \delta) \\
& & \downarrow \Sigma^2 \tilde{C}_{1,p} \epsilon \\
P_p^1 K(\mathbf{Z}/p, 2(q + s)) & \xrightarrow{j} & \Sigma^2 \tilde{C}_{2,p} K(\mathbf{Z}/p, 2(q + s))
\end{array}$$

Here, ϵ represents one of two natural transformations, running either

$$P_p^{n+k} \Omega^n \rightarrow P_p^k \quad \text{or} \quad \Sigma^{n+k} \Omega^n \rightarrow \Sigma^k,$$

which are determined by the adjointness of P_p^n and Σ^n with Ω^n , respectively. Since the choices of $\bar{\iota}_{2q+1}$ and $\bar{\iota}_{2(q+s-1)+1}$ are unique, one sees that $\epsilon^*(\bar{\iota}_{2(q+s)}) = \bar{\iota}_{2q+\delta}$, and hence $\epsilon^*(\beta \mathcal{P}^{q+s} \bar{\iota}_{2(q+s)}) = \beta \mathcal{P}^{q+s} \bar{\iota}_{2q+\delta}$. By Lemma 3.5,

$$\left(\Sigma^2 (\tilde{C}_{1,p} \epsilon \circ \tilde{\epsilon}_{n-1}) \right)^* (\sigma^2 \tilde{Q}_0 \iota_{2(q+s)}) = \sigma^{n+1} \tilde{Q}_{(n-1)(p-1) \iota_{2q+\delta}};$$

thus

$$\begin{aligned}
j^* \left(\sigma^{n+1} \tilde{Q}_{(n-1)(p-1) \iota_{2q+\delta}} \right) &= j^* \left(\Sigma^2 (\tilde{C}_{1,p} \epsilon \circ \tilde{\epsilon}_{n-1}) \right)^* \left(\sigma^2 \tilde{Q}_0 \iota_{2(q+s)} \right) \\
&= \epsilon^* j^* \left(\sigma^2 \tilde{Q}_0 \iota_{2(q+s)} \right) \\
&\doteq \epsilon^* (\beta \mathcal{P}^{q+s} \bar{\iota}_{2(q+s)}) \\
&= \beta \mathcal{P}^{q+s} \bar{\iota}_{2q+\delta}.
\end{aligned}$$

In order to prove the formula $\mathcal{P}^{q+s} \bar{\iota}_{2q+\delta} \doteq j^* \left(\sigma^{n+1} \tilde{Q}_{(n-1)(p-1) \iota_{2q+\delta}} \right)$ for $s > 0$, we need only show the case in which $n = 2$ (and hence $s = \delta = 1$). Then the case for general n will follow by exactly the same argument as for $\beta \mathcal{P}^{q+s} \bar{\iota}_{2q+\delta}$.

Because of the commutative diagram

$$\begin{array}{ccc}
\Sigma^{-2} \Sigma^\infty P_p^2 K(\mathbf{Z}/p, 2q + 1) & \xrightarrow{j} & \Sigma \Sigma^\infty \tilde{C}_{2,p} K(\mathbf{Z}/p, 2q + 1) \\
\downarrow & & \downarrow \\
P_p K(\mathbf{Z}/p, 2q + 1) & \xrightarrow{j} & \Sigma \Sigma^\infty \tilde{C}_{\infty,p} K(\mathbf{Z}/p, 2q + 1)
\end{array}$$

it suffices to show $\mathcal{P}^{q+1} \bar{\iota}_{2q+1} \doteq j^* (\sigma \tilde{Q}_{p-2} \iota_{2q+1})$ holds in $\tilde{H}^*(P_p K(\mathbf{Z}/p, 2q+1))$.

The class $\bar{\iota}_{2q+1} \in \tilde{H}^{2q+1}(P_p K(\mathbf{Z}/p, 2q+1))$ is the unique element for which $i^* \bar{\iota}_{2q+1} = \iota_{2q+1}$. Since we have computed that $\beta \mathcal{P}^{q+1} \bar{\iota}_{2q+1} \neq 0$ in

$\tilde{H}^*(P_p^2 K(\mathbf{Z}/p, 2q+1))$, the above diagram shows us that $\mathcal{P}^{q+1} \bar{\iota}_{2q+1} \neq 0$. However, $i^*(\mathcal{P}^{q+1} \bar{\iota}_{2q+1}) = \mathcal{P}^{q+1} \iota_{2q+1} = 0$ by the unstable condition; therefore, $0 \neq \mathcal{P}^{q+1} \bar{\iota}_{2q+1} \in \text{im } j^*$. On the other hand, because the space $P_p^1 K(\mathbf{Z}/p, 2q+1)$ has L-S category 2, all threefold cup products vanish in its cohomology, and hence the image of

$$\mathcal{P}^{q+1} \bar{\iota}_{2q+1} = \bar{\iota}_{2q+1}^p \quad \text{in} \quad \tilde{H}^{2p(q+1)}(P_p^1 K(\mathbf{Z}/p, 2q+1))$$

must be zero.

In order to simplify notation, let $K = K(\mathbf{Z}/p, 2q+1)$. Consider the following commutative diagram.

$$\begin{array}{ccc} \Sigma^\infty K & \longrightarrow & \Sigma^{-1} \Sigma^\infty P_p^1 K \\ \downarrow i & & \downarrow \\ P_p K & \xlongequal{\quad} & P_p K \\ \downarrow j & & \downarrow \\ \Sigma \Sigma^\infty \tilde{C}_{\infty,p} K & \xrightarrow{p} & \Sigma \Sigma^\infty (\tilde{C}_{\infty,p} K / \tilde{C}_{1,p} K) \end{array} .$$

The discussion above implies that there is an element

$$v \in \tilde{H}^{2p(q+1)-1}(\tilde{C}_{\infty,p} K / \tilde{C}_{1,p} K)$$

for which $j^* p^*(v) = \mathcal{P}^{q+1} \bar{\iota}_{2q+1} \neq 0$. Since all (external) Pontryagin product elements in the mod p homology of $\tilde{C}_{\infty,p} K$ must come from the mod p homology of $\tilde{C}_{1,p} K$, $p^*(v)$ must be equal (up to a unit in \mathbf{F}_p) to $\tilde{Q}_{p-2\iota_{2q+1}}$, since it is the only element in that degree which is in the image of p^* . Thus $\mathcal{P}^{q+1} \bar{\iota}_{2q+1} \doteq j^*(\tilde{Q}_{p-2\iota_{2q+1}})$ as desired.

For the general case, let $f: X \rightarrow K(\mathbf{Z}/p, 2q+\delta)$ represent x . Then Theorem 6.1 of [KSW] implies that, for every choice of element \bar{x} such that $i^*(\bar{x}) = \sigma^n x$, there is a corresponding (unique) H_p^n -structure on f (note that if $i^*(\bar{x}) = 0$, then $\bar{x} \in \text{im } j^*$, and $\mathcal{P}^{q+s} \bar{x} = 0$ by the unstable condition). Now consider the following commutative diagram.

$$\begin{array}{ccccc} \Sigma^n X & \xrightarrow{i} & P_p^n X & \xrightarrow{j} & \Sigma^{n+1} \tilde{C}_{n,p} X \\ \downarrow f & & \downarrow P_p^n f & & \downarrow \Sigma^{n+1} \tilde{C}_{n,p} f \\ \Sigma^n K(\mathbf{Z}/p, 2q+\delta) & \xrightarrow{i} & P_p^n K(\mathbf{Z}/p, 2q+\delta) & \xrightarrow{j} & \Sigma^{n+1} \tilde{C}_{n,p} K(\mathbf{Z}/p, 2q+\delta) \end{array}$$

By construction, $f^*(\iota_{2q+\delta}) = x$, $(P_p^n f)^*(\bar{\iota}_{2q+\delta}) = \bar{x}$ and $(\tilde{C}_{n,p} f)^*(\tilde{Q}_r \iota_{2q+\delta}) \doteq \tilde{Q}_r x$, and the result follows. QED

The following corollary is the stable version of Theorem 4.1. Recall from [KSW] that there is a natural map $\Sigma^{-n} \Sigma^\infty P_p^n X \rightarrow P_p X$ for every

$X \in \text{Ob } \mathcal{H}_p^\infty$. The geometric filtration determined by these maps yields an algebraic filtration of $\tilde{H}^*(P_p X)$, and we define

$$F^{k+1} = \ker\{\tilde{H}^*(P_p X) \rightarrow \tilde{H}^*(\Sigma^{-k}\Sigma^\infty P_p^k X)\}.$$

Corollary 4.2. *Suppose p is an odd prime and $X \in \text{Ob } \mathcal{H}_p^\infty$, with basic cofibration sequence*

$$\Sigma^\infty \tilde{C}_{\infty,p} X \xrightarrow{h} \Sigma^\infty X \xrightarrow{i} P_p X \xrightarrow{j} \Sigma \Sigma^\infty \tilde{C}_{\infty,p} X.$$

Let $n = 2s + 1 - \delta$ for some $\delta \in \{0, 1\}$ and $s \geq \delta$. If $\bar{x} \in \tilde{H}^{2q+\delta}(P_p X)$, and $s > 0$, then

$$\begin{aligned} \mathcal{P}^{q+s} \bar{x} &\doteq j^*(\sigma^\infty \tilde{Q}_{(n-1)(p-1)-1} x) \pmod{F^{n+1}}, \\ \beta \mathcal{P}^{q+s} \bar{x} &\doteq j^*(\sigma^\infty \tilde{Q}_{(n-1)(p-1)} x) \pmod{F^{n+1}}, \end{aligned}$$

where $i^* \bar{x} = \sigma^\infty x$ defines $x \in \tilde{H}^{2q+\delta}(X)$.

Appendix: The Nishida relations.

The Nishida relations [NI, CLM], are formulas which relate the action of Dyer-Lashof operations to the (adjoint) action of the Steenrod algebra on $\tilde{H}_*(\tilde{C}_{n,p} X)$. In computations using the main theorem it is helpful to have the analogous formulas in cohomology. They are given in the next proposition.

Recall from Section 3 that $\tilde{H}_*(\tilde{C}_{n,p} X) \cong B \oplus C$, for certain submodules B and C . Let $\text{Ann}(B) \subset \tilde{H}^*(\tilde{C}_{n,p} X)$ be the submodule of elements which pair trivially with every element of B under the Kronecker pairing.

Let $\delta \in \{0, 1\}$ and define a function λ by $\lambda(2j + \delta) = \delta$. Finally, let $u \in \mathbf{F}_p$ denote an undetermined unit.

Proposition A.1. *The following formulas (where (i) is taken mod $\text{Ann}(B)$ in the case that $r = 0$) hold for every $x \in \tilde{H}^q(X)$.*

(i)

$$\begin{aligned} \mathcal{P}^k \tilde{Q}_r x &\doteq \sum_{i=0}^{\lfloor k/p \rfloor} \binom{\lfloor r/2 \rfloor + (p-1)(q/2 - i)}{k - pi} \tilde{Q}_{r+2(p-1)(k-pi)} \mathcal{P}^i x \\ &+ u \lambda(r-1) \sum_{i=0}^{\lfloor k/p \rfloor} \binom{\lfloor r/2 \rfloor + (p-1)(q/2 - i) - 1}{k - pi - 1} \tilde{Q}_{r+2(p-1)(k-pi)-p} \beta \mathcal{P}^i x. \end{aligned}$$

(ii)

$$\beta \tilde{Q}_{2r-1} x \doteq \tilde{Q}_{2r} x.$$

Proof. It suffices to prove the formulas in $\tilde{H}^*(D_p K(\mathbf{Z}/p, q))$, with $x = \iota_q$, where they are dual to the usual Nishida formulas. \square

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