(A₂)-CONDITIONS AND CARLESON INEQUALITIES IN BERGMAN SPACES

Takahiko Nakazi and Masahiro Yamada
(A_2)-CONDITIONS AND CARLESON INEQUALITIES IN BERGMAN SPACES

Takahiko Nakazi and Masahiro Yamada

Let \( \nu \) and \( \mu \) be finite positive measures on the open unit disk \( D \). We say that \( \nu \) and \( \mu \) satisfy the \((\nu, \mu)\)-Carleson inequality, if there is a constant \( C > 0 \) such that

\[
\int_D |f|^2 \, d\nu \leq C \int_D |f|^2 \, d\mu
\]

for all analytic polynomials \( f \). In this paper, we study the necessary and sufficient condition for the \((\nu, \mu)\)-Carleson inequality. We establish it when \( \nu \) or \( \mu \) is an absolutely continuous measure with respect to the Lebesgue area measure which satisfy the \((A_2)\)-condition. Moreover, many concrete examples of such measures are given.

§1. Introduction.

Let \( D \) denote the open unit disk in the complex plane. For \( 1 \leq p \leq \infty \), let \( L^p \) denote the Lebesgue space on \( D \) with respect to the normalized Lebesgue area measure \( m \), and \( \| \cdot \|_p \) represents the usual \( L^p \)-norm. For \( 1 \leq p < \infty \), let \( L^p_0 \) be the collection of analytic functions \( f \) on \( D \) such that \( \| f \|_p \) is finite, which are so called the Bergman spaces. For any \( z \) in \( D \), let \( \phi_z \) be the Möbius function on \( D \), that is

\[
\phi_z (w) = \frac{z - w}{1 - \bar{z}w} \quad (w \in D),
\]

and put,

\[
\beta(z, w) = 1/2 \log(1 + |\phi_z(w)|)(1 - |\phi_z(w)|^{-1} \quad (z, w \in D).
\]

For \( 0 < r < \infty \) and \( z \) in \( D \), set

\[
D_r(z) = \{ w \in D; \beta(z, w) < r \}
\]

be the Bergman disk with “center” \( z \) and “radius” \( r \), and we define an average of a finite positive measure \( \mu \) on \( D_r(a) \) by

\[
\hat{\mu}_r(a) = \frac{1}{m(D_r(a))} \int_{D_r(a)} d\mu \quad (a \in D),
\]
and if there exists a non-negative function \( u \) in \( L^1 \) such that \( d\mu = ud\, m \), then we may write it \( \hat{u}_r \), instead of \( \hat{\mu}_r \).

Let \( \nu \) and \( \mu \) be finite positive measures on \( D \), and let \( P \) be the set of all analytic polynomials. We say that \( \nu \) and \( \mu \) satisfy the \((\nu,\mu)\)-Carleson inequality, if there is a constant \( C > 0 \) such that

\[
\int_D |f|^2 d\nu \leq C \int_D |f|^2 d\mu
\]

for all \( f \) in \( P \). Our purpose of this paper is to study conditions on \( \nu \) and \( \mu \) so that the \((\nu,\mu)\)-Carleson inequality is satisfied. If \( \nu \leq C\mu \) on \( D \), then the \((\nu,\mu)\)-Carleson inequality is true. However it is clear that this sufficient condition for the \((\nu,\mu)\)-Carleson inequality is too strong. A reasonable and natural condition is the following: there exist \( r > 0 \) and \( \gamma > 0 \) such that

\[
(*) \quad \hat{\nu}_r(a) \leq \gamma \hat{\mu}_r(a) \quad (a \in D).
\]

The average \( \hat{\nu}_r(a) \) are sometimes computable. If \( \mu = m \), then \( \hat{\mu}_r(a) = 1 \) on \( D \). If \( d\mu = (1 - |z|^2)^\alpha d\, m \) for \( \alpha > -1 \), then \( \hat{\mu}_r(a) \) is equivalent to \( (1 - |a|^2)^\alpha \) on \( D \).

When \( d\mu = (1 - |z|^2)^\alpha d\, m \) for \( \alpha > -1 \), Oleinik-Pavlov [7], Hastings [2], or Sittegenga [8] showed that \( \nu \) and \( \mu \) satisfy the Carleson inequality if and only if they satisfy \( (*) \). In §3 of this paper, when \( d\mu = ud\, m \) and \( u \) satisfies the \((A_2)^\circ\)-condition (the definition is in §3), we obtain that the \((\nu,\mu)\)-Carleson inequality is satisfied if and only if they satisfy \( (*) \). We show that if both \( u \) and \( u^{-1} \) are in \( BMO \) ( see [9, p. 127] ), then \( u \) satisfies the \((A_2)^\circ\)-condition. We give some concrete examples which satisfy the \((A_2)^\circ\)-condition.

When \( \nu = m \) and \( d\mu = \chi_G d\, m \), where \( \chi_G \) is a characteristic function of a measurable subset \( G \) of \( D \), Luecking [4] showed the equivalence between the \((\nu,\mu)\)-Carleson inequality and the condition \( (*) \). If we do not put any hypotheses on \( \mu \), the problem is very difficult. The equivalence between the \((\nu,\mu)\)-Carleson inequality and the condition \( (*) \) is not known even if \( \nu = m \). Luecking [5] showed the following:

1. If there exists \( \gamma > 0 \) such that \( \hat{\nu}_r(a) \leq \gamma \hat{\mu}_r(a) \) for all \( r > 0 \) and \( a \) in \( D \), then the \((m,\mu)\)-Carleson inequality is satisfied.

2. Suppose the \((\mu,\mu)\)-Carleson inequality is valid (equivalently \( \hat{\mu}_r \) is bounded on \( D \)). Then the \((m,\mu)\)-Carleson inequality implies the condition \( (*) \).

In §2 of this paper, we give a sufficient condition (close to that of (1)) for the \((\nu,\mu)\)-Carleson inequality when \( \nu \) is not necessarily \( m \). Moreover, using the idea of Luecking’s proof of (2), a generalization of (2) is given. In §4, when \( d\nu = \nu d\, m \) and \( \nu \) satisfies the \((A_2)\)-condition (the definition is in
§3), we establish a more natural extension of (2) under some condition of a quantity $\varepsilon_r(\nu)$ (the definition is in §2), that is $\varepsilon_r(\nu) \to 0$ as $r \to \infty$. The $(A_2)$-condition is weaker than the $(A_2)_\beta$-condition. We give some concrete examples which satisfy the $(A_2)$-condition or the above condition of $\varepsilon_r(\nu)$.

§2. $(\nu, \mu)$-Carleson inequality.

Let $G$ be a measurable subset of $D$ and $u$ be a non-negative function in $L^1$, and put

$$(u_G^{-1})^\wedge_r(a) = \frac{1}{m(D_r(a))} \int_{D_r(a)} u^{-1} \chi_G \, dm.$$

Particular, when $G = D$, we will omit the letter $D$ in the above notation. The following Proposition 1 gives a general sufficient condition on $\nu$ and $\mu$ which satisfy the $(\nu, \mu)$-Carleson inequality. In order to prove it we use ideas in [5] and [9, p. 109]. Since $(u^{-1})_r^\wedge(a)^{-1} \leq \hat{u}_r(a)$ for all $a$ in $D$, Proposition 1 is also related with (1) of §1 (cf. [5, Theorem 4.2]).

**Proposition 1.** Suppose that $d \mu = ud \, m$. Put $E_r = \{z \in D; \text{there is a } w \in \text{supp}\, \nu \text{ such that } \beta(z, w) < r/2\}$. If there exist $r > 0$ and $\gamma > 0$ such that $u > 0$ a.e. on $E = E_r$, and $\hat{\nu}(a) \times (u_E^{-1})_r^\wedge(a) \leq \gamma$ for all $a$ in $D$, then there is a constant $C > 0$ such that

$$\int_D |f|^2 \, d\nu \leq C \int_E |f|^2 \, d\mu$$

for all $f \in P$.

**Proof.** Suppose that $\hat{\nu}_{2r}(a) \times (u_E^{-1})_r^\wedge(a) \leq \gamma$ for all $a$ in $D$, and put $E = \{z \in D; \text{there is a } w \in \text{supp}\, \nu \text{ such that } \beta(z, w) < r\}$. By an elementary theory for Bergman disks, there is a positive integer $N = N_r$ such that there exists $\{\lambda_n\} \subset D$ satisfying that $D = \bigcup D_r(\lambda_n)$ and any $z$ in $D$ belongs to at most $N$ of the sets $D_{2r}(\lambda_n)$ (cf. [9, p. 62]) therefore

$$\int_{\text{supp}\, \nu} |f|^2 \, d\nu \leq \sum \int_{D_r(\lambda_n) \cap \text{supp}\, \nu} |f|^2 \, d\nu \leq \sum \nu(D_r(\lambda_n)) \times \sup\{|f(z)|^2; \, z \in D_r(\lambda_n) \cap \text{supp}\, \nu\}.$$ 

By Proposition 4.3.8 in [9, p. 62], there is a constant $C = C_r > 0$ such that

$$|f(z)| \leq \frac{C}{m(D_r(z))} \int_{D_r(z)} |f(w)| \, dm(w)$$

for all $f$ analytic, $z$ in $D$. If $z$ in $D_r(\lambda_n) \cap \text{supp}\, \nu$, then $D_r(z)$ is contained in $D_{2r}(\lambda_n) \cap E$, and there exists a constant $K = K_r > 0$ such that $m(D_{2r}(\lambda_n)) \leq$
$Km(D_r(z))$ for all $n \geq 1$ (cf. [9, p. 61]). Hence the Cauchy-Schwarz’s inequality implies that

$$
\int_D |f|^2 d\nu \leq \sum \nu(D_r(\lambda_n)) \times \left( \frac{KC}{m(D_{2r}(\lambda_n))} \int_{D_{2r}(\lambda_n) \cap E} |f| d\mu \right)^2
\leq \sum \nu(D_r(\lambda_n)) \times K^2C^2
\times \left( \frac{1}{m(D_{2r}(\lambda_n))} \int_{D_{2r}(\lambda_n)} |f|^2 u\chi_E d\mu \right)
\times \left( \frac{1}{m(D_{2r}(\lambda_n))} \int_{D_{2r}(\lambda_n)} u^{-1}\chi_E d\mu \right)
\leq K^2C^2 \sum \hat{\nu}_{2r}(\lambda_n) \times (uE^{-1})_{2r}(\lambda_n)
\times \left( \int_{D_{2r}(\lambda_n) \cap E} |f|^2 ud\mu \right).
$$

By the hypothesis and a choice of disks, it follows that

$$
\int_D |f|^2 d\nu \leq K^2C^2\gamma_N \int_E |f|^2 d\mu.
$$

This completes the proof.

Let $\mu$ be a finite nonzero positive measure on $D$. For any $a$ in $D$, put

$$k_a(z) = (1 - |a|^2)/(1 - \bar{a}z)^2 \quad (z \in D),$$

and a function $\tilde{\mu}$ on $D$ is defined by

$$\tilde{\mu}(a) = \int_D |k_a|^2 d\mu.$$

Moreover, for any fixed $r < \infty$, put

$$\varepsilon_r(\mu) = \sup_{a \in D} \left( \int_{D \setminus D_r(a)} |k_a|^2 d\mu \right) \times \left( \int_D |k_a|^2 d\mu \right)^{-1}.$$

If there exists a non-negative function $u$ in $L^1$ such that $d\mu = ud\mu$, then making a change of variable, it is easy to see that

$$\varepsilon_r(\mu) = \sup_{a \in D} \left( \int_{D \setminus D_r(0)} u \circ \phi_a d\mu \right) \times \left( \int_D u \circ \phi_a d\mu \right)^{-1}.$$

In general $0 < \varepsilon_r(\mu) \leq 1$. In this section and §4, this quantity $\varepsilon_r$ is important. The following Proposition 2 gives two general necessary conditions on $\nu$
and μ which satisfy the (ν, μ)-Carleson inequality. In order to prove (2) of Proposition 2 we use ideas in [5, Theorem 4.3]. Since ε_r(m) < 1 and ε_r(m) → 0 (r → ∞), (2) of Proposition 2 is related with (2) of §1.

**Lemma 1.** Let μ be a finite positive measure on D and 0 < r < ∞, then the following (1) ~ (3) are equivalent.

1. ε_r(μ) < 1.
2. There is a δ = δ_r < ∞ such that 
   \[ \int_{D \setminus D_r(a)} |k_a|^2 dμ \leq \delta \int_{D_r(a)} |k_a|^2 dμ \]
   for all a in D.
3. There is a ρ = ρ_r < ∞ such that 
   \[ \tilde{μ}(a) \leq \rho \tilde{μ_r}(a) \]
   for all a in D.

**Proof.** The implication (1) ⇒ (2) is trivial. (2) ⇒ (3) and (3) ⇒ (1) follow from Lemma 4.3.3 in [9, p. 60]. In fact, by Lemma 4.3.3, there exist \( L = L_r > 0 \) and \( M = M_r > 0 \) such that 

\[ L \leq m(D_r(a)) \times \inf\{|k_a(z)|^2; z \in D_r(a)\} \]

and 

\[ m(D_r,(a)) \times \sup\{|k_a(z)|^2; z \in D_r(a)\} \leq M \]

for all a in D. Thus remainder implications are obtained. \( \square \)

**Proposition 2.** Suppose that ν and μ satisfy the (ν, μ)-Carleson inequality, then the following are true.

1. If there exists r < ∞ such that ε_r(μ) < 1, then there exists γ > 0 such that \( \tilde{ν}_r(a) \leq \gamma \tilde{μ}_r(a) \) for all a in D.
2. If d ν = νd m, ν > 0 a.e. on D, ε_t(ν) → 0 (t → ∞), and there are l > 0 and \( \gamma' > 0 \) such that \( \tilde{ν}_l(a) \times (ν^{-1})^{(a)} \leq \gamma' \) for all a in D, then there are r > 0 and γ = γ_r > 0 such that \( \tilde{ν}_r(a) \leq \gamma \tilde{μ}_r(a) \) for all a in D.

**Proof.** Since \( k_a(z) \) is uniformly approximated by polynomials, the inequality is valid for \( f = k_a \), that is 

\[ \int_D |k_a|^2 dν \leq C \int_D |k_a|^2 dμ. \]

Firstly, we show that (1) is true. The above inequality and Lemma 1 imply that 

\[ \tilde{ν}(a) \leq C \tilde{μ}(a) \leq Cρ \tilde{μ_r}(a) \]
for all \( a \) in \( D \). Moreover, by Lemma 4.3.3 in [9, p. 60], there exists a constant \( L > 0 \) such that
\[
\hat{\nu}_r(a) \leq L^{-1} \hat{\nu}(a)
\]
for all \( a \) in \( D \). Hence we have that
\[
\hat{\nu}_r(a) \leq C \rho L^{-1} \hat{\mu}_r(a).
\]

Next, we prove that (2) is true. For any \( a \) in \( D \) and \( r \geq l \), put \( d \mu_{a,r} = (1 - \chi_{D_r(a)}) d \mu \). By the latter half of the hypothesis in (2), we have that
\[
(\mu_{a,r})^{\wedge}(\lambda) \times (v^{-1})^{\wedge}(\lambda) \leq \gamma'
\]
for all \( a, \lambda \) in \( D \), and \( r \geq l \). Set \( E_{a,r,l} = \{ z \in D; \text{there is a } w \in \text{supp} \mu_{a,r}, \text{such that } \beta(w, z) < l/2 \} \). By Proposition 1, there exists a constant \( C'' > 0 \) such that
\[
\int_{D \setminus D_{r/2}(a)} |f|^2 d \mu \leq C'' \int_{E_{a,r,l}} |f|^2 d \nu
\]
for all \( a \) in \( D \), \( r \geq l \) and \( f \) in \( P \). Here we claim that \( E_{a,r,l} \) is contained in \( D \setminus D_{r/2}(a) \). In fact, since \( D \setminus D_r(a) \) contains \( \text{supp} \mu_{a,r} \) and \( r \geq l \), if \( z \) belongs to \( E_{a,r,l} \) then there exists \( w \) in \( D \) such that \( \beta(w, a) \geq r \) and \( \beta(w, z) < r/2 \). Therefore,
\[
r \leq \beta(w, a) \leq \beta(w, z) + \beta(z, a) < r/2 + \beta(z, a),
\]
thus we have that \( z \) is contained in \( D \setminus D_{r/2}(a) \). Particularly put \( f = k_a \) in the above inequality, then
\[
\int_{D \setminus D_r(a)} |k_a|^2 d \mu \leq C'' \int_{D \setminus D_{r/2}(a)} |k_a|^2 d \nu
\]
for all \( a \) in \( D \) and \( r \geq l \). It follows that
\[
\int_{D_r(a)} |k_a|^2 d \mu = \int_{D} |k_a|^2 d \mu - \int_{D \setminus D_r(a)} |k_a|^2 d \mu \geq C^{-1} \int_{D} |k_a|^2 d \nu - C'' \int_{D \setminus D_{r/2}(a)} |k_a|^2 d \nu.
\]
By the definition of \( \varepsilon_t(\nu) \), the above inequality implies that
\[
\int_{D_r(a)} |k_a|^2 d \mu \geq \left( C^{-1} - C'' \varepsilon_{r/2}(\nu) \right) \int_{D} |k_a|^2 d \nu
\]
for all \( a \) in \( D \) and \( r \geq l \). Here let \( r \) be sufficiently large, then by the hypothesis on \( \varepsilon_r(\nu) \), \( C^{-1} - C'' \varepsilon_{r/2}(\nu) > 0 \), and by Lemma 4.3.3 in [9, p. 60], we conclude that
\[
\hat{\mu}_r(a) \geq [M^{-1}C^{-1}C'' \varepsilon_{r/2}(\nu)L] \hat{\nu}_r(a)
\]
for all \( a \) in \( D \). \( \square \)
§3. \((A_2)\)-condition.

For a complex measure \(\mu\) on \(D\), recall that a function \(\tilde{\mu}\) on \(D\) is defined by

\[
\tilde{\mu}(a) = \int_D |k|^2 d\mu.
\]

Particularly, if there exists a complex valued \(L^1\)-function \(u\) such that \(d\mu = ud\,m\), then we denote the function by \(\tilde{u}\) instead of \(\tilde{\mu}\), and say that \(\tilde{u}\) is the Berezin transform of the function \(u\).

Let \(v\) and \(u\) be non-negative functions in \(L^1\), put \(d\,v = vd\,m\) and \(d\mu = ud\,m\). Suppose that there is a constant \(\gamma > 0\) such that

\[
\tilde{v}(a) \times (u^{-1})^\sim(a) \leq \gamma
\]

for all \(a\) in \(D\), then Lemma 4.3.3 in [9, p. 60] implies that there exist \(r > 0\) and \(\gamma' > 0\) such that

\[
\tilde{v}_r(a) \times (u^{-1})^\sim_r(a) \leq \gamma'
\]

for all \(a\) in \(D\), and hence by Proposition 1, we obtain that the \((\nu, \mu)\)-Carleson inequality is satisfied. In the above two inequalities, if we put \(u = v\), then such a function \(u\) is interesting for us.

A non-negative function \(u\) in \(L^1\) is said to satisfy an \((A_2)\)-condition, if there exists a constant \(A > 0\) such that

\[
\tilde{u}(a) \times (u^{-1})^\sim(a) \leq A
\]

for all \(a\) in \(D\). If there exist \(r > 0\) and \(A_r > 0\) such that

\[
\tilde{u}_r(a) \times (u^{-1})^\sim_r(a) \leq A_r
\]

for all \(a\) in \(D\), then we say that \(u\) satisfies an \((A_2)\)-condition. In [6], the \((A_2)\)-condition is called Condition \(C_2\). It is known that \(u\) satisfies the \((A_2)\)-condition for some \(0 < r < \infty\) if and only if \(u\) satisfies the \((A_2)\)-condition for all \(0 < r < \infty\) [6]. Hence it shows that the definition of the \((A_2)\)-condition is independent of \(r\). In general, Lemma 4.3.3 in [9, p. 60] and the familiar inequality between the harmonic and arithmetic means imply that for any \(0 < r < \infty\) there exists a constant \(M = M_r > 0\) such that

\[
M^{-1}(u^{-1})^\sim^{-1} \leq (u^{-1})^\sim_r^{-1} \leq \tilde{u}_r \leq M\tilde{u}.
\]

Therefore, if \(u\) satisfies the \((A_2)\)-condition, then \((u^{-1})^\sim^{-1}, (u^{-1})^\sim_r^{-1}, \tilde{u}_r, \tilde{u}\) are equivalent. Similarly, if \(u\) satisfies the \((A_2)\)-condition, then \((u^{-1})^\sim_r^{-1}, \tilde{u}_r\), and \(\tilde{u}\) are equivalent. When \(u\) is in \(L^1(\partial D)\) (\(L^1\) is a usual Lebesgue space on the unit circle and \(k_a(z)\) is a normalized reproducing kernel of a Hardy space), the \((A_2)\)-condition has been studied in [3, (c) of Theorem 2].
The following Theorem 3 gives a necessary and sufficient condition in order to satisfy the \((\nu, \mu)\)-Carleson inequality when \(d \mu = u \, d \, m\) and \(u\) satisfies the \((A_{2})_{\varphi}\)-condition.

**Theorem 3.** Suppose that \(u\) satisfies the \((A_{2})_{\varphi}\)-condition, then the following are equivalent.

1. There is a constant \(C > 0\) such that
   \[
   \int_{D} |f|^{2} \, d\nu \leq C \int_{D} |f|^{2} u \, d\, m
   \]
   for all \(f\) in \(P\).
2. There exist \(r > 0\) and \(\gamma > 0\) such that
   \[
   \hat{\nu}_{r}(a) \leq \gamma \hat{u}_{r}(a)
   \]
   for all \(a\) in \(D\).
3. For any \(r > 0\), there exists \(\gamma = \gamma_{r} > 0\) such that
   \[
   \hat{\nu}_{r}(a) \leq \gamma \hat{u}_{r}(a)
   \]
   for all \(a\) in \(D\).

**Proof.** Suppose that (1) holds. Since \(u\) satisfies the \((A_{2})_{\varphi}\)-condition, by (1) of Proposition 8, \(u\) satisfies a relation in (3) of Lemma 1 for all \(r > 0\). Therefore, (3) follows from (1) of Proposition 2. The implication (3) \(\Rightarrow\) (2) is obvious. We will show that (2) \(\Rightarrow\) (1). Since \(u\) satisfies the \((A_{2})_{\varphi}\)-condition, \(u^{-1}\) is integrable, hence \(u > 0\) a.e. on \(D\). Moreover, by (5) of Proposition 4, \(u\) satisfies the \((A_{2})\)-condition for all \(r > 0\) and therefore (2) implies that

\[
\hat{\nu}_{r}(a) \times (u^{-1})_{r}^{\varphi}(a) \leq A_{r} \gamma
\]

for all \(a\) in \(D\). In the statement of Proposition 1, put \(E = D\), then the above fact shows that the inequality in (1) is satisfied. This completes the proof. \(\square\)

For any \(u\) in \(L^{2}\), \(a\) in \(D\), we put

\[
MO(u)(a) = \{|u|^{2\varphi}(a) - |\bar{u}(a)|^{2}\}^{1/2},
\]

and let \(BMO_{\varphi}\) be the space of functions \(u\) such that \(MO(u)(a)\) is bounded on \(D\) (cf. [9, p. 127]). We give several simple sufficient conditions.

**Proposition 4.** Let \(u\) be a non-negative function in \(L^{1}\), then the following are true.
(1) If both \( \bar{u} \) and \((u^{-1})^\sim\) are in \( L^\infty \), then \( u \) satisfies the \((A_2)_\theta\)-condition.

(2) If both \( u \) and \( u^{-1} \) are in \( BMO_\theta \), then \( u \) satisfies the \((A_2)_\theta\)-condition.

(3) Let \( 1 < p, q < \infty \) and \( 1/p + 1/q = 1 \). If \( u^p_1 \) and \( u^q_2 \) satisfy the \((A_2)_\theta\)-condition, then \( u = u_1u_2 \) satisfies the \((A_2)_\theta\)-condition.

(4) Suppose that \( f \) is a complex valued function in \( L^1 \) such that \( f \neq 0 \) on \( D \), \( f^{-1} \) is in \( L^1 \), \( \tilde{f} \times (f^{-1})^\sim \) is in \( L^\infty \), and \( |\arg f| \leq \pi/2 - \varepsilon \) for some \( \varepsilon > 0 \). If \( u = |f| \), then \( u \) satisfies the \((A_2)_\theta\)-condition.

(5) If \( u \) satisfies the \((A_2)_\theta\)-condition, then \( u \) satisfies the \((A_2)_\theta\)-condition.

**Proof.** (1) is trivial. By Proposition 6.1.7 in [9, p. 108], we have that

\[
\tilde{u}(a) \times (u^{-1})^\sim(a) \leq MO(u)(a) \times MO(u^{-1})(a) + 1.
\]

This implies that (2) is true. The Hölder’s inequality implies that (3) is true. (5) follows from Lemma 4.3.3 in [9, p. 60].

We show that (4) is true. Suppose that \( u = |f| \) and there exists \( \varepsilon > 0 \) such that \( |\arg f| \leq \pi/2 - \varepsilon \) on \( D \). Since \( |\arg f| \leq \pi/2 - \varepsilon \) on \( D \), there exists \( \delta > 0 \) such that \( \cos(\arg f) \geq \delta \) on \( D \). Therefore, we have that

\[
Re f = |f| \times \cos(\arg f) \geq |f| \cdot \delta = \delta u.
\]

For any \( a \) in \( D \), it follows that

\[
\delta \tilde{u}(a) \leq \int Re f \cdot |k_a|^2 d m \leq |\tilde{f}(a)|.
\]

Similarly, we have that

\[
\delta (u^{-1})^\sim(a) \leq |(f^{-1})^\sim(a)|.
\]

Thus,

\[
\tilde{u}(a) \times (u^{-1})^\sim(a) \leq \delta^{-2} \times |\tilde{f}(a)| \times |(f^{-1})^\sim(a)|
\]

for all \( a \) in \( D \), and hence (4) follows. \( \Box \)

We exhibit some concrete examples which satisfy the \((A_2)_\theta\)-condition.

**Proposition 5.** If \( u \) is a function that is given by (1), (2), or (3), then \( u \) satisfies the \((A_2)_\theta\)-condition.

(1) For any \(-1 < \alpha < 1\), put \( u(z) = (1 - |z|^2)^\alpha \).

(2) Let \( \{b_j\} \) be a finite sequence of complex numbers in \( D \cup \partial D \) with \( b_i \neq b_j (i \neq j) \), and let \( 0 \leq \alpha(j) < 2 \) for all \( j \) or \(-2 < \alpha(j) \leq 0 \) for all \( j \). Put \( u = \prod p_j^{\alpha(j)} \) where \( p_j(z) = |z - b_j| \).
(3) Let \( \{b_j\}, \{p_j\} \) as in (2) and \(-1 < \alpha(j) < 1 \) for all \( j \). Put \( u = \prod_{j} p_{j}^{\alpha(j)} \).

Proof. We suppose that \( u \) has the form of (1). For any \( a \) in \( D \), making a change of variable, we have that

\[
\tilde{u}(a) \times (u^{-1})\sim(a) = \int (1 - |a|^2)^{\alpha}(1 - |z|^2)^{\alpha}|1 - \bar{a}z|^{2\alpha} \, dm(z)
\]

\[
\times \int (1 - |a|^2)^{-\alpha}(1 - |z|^2)^{-\alpha}|1 - \bar{a}z|^{-2\alpha} \, dm(z)
\]

\[
= \int (1 - |z|^2)^{\alpha}|1 - \bar{a}z|^{-2\alpha} \, dm(z)
\]

\[
\times \int (1 - |z|^2)^{-\alpha}|1 - \bar{a}z|^{2\alpha} \, dm(z).
\]

Since \(-1 < \alpha < 1\), Rudin’s lemma (cf. [9, p. 53]) implies that both factors of the right hand side in the above equality are bounded. Hence satisfies the \((A_2)_{\partial}\)-condition.

We show that \( u \) satisfies the \((A_2)_{\partial}\)-condition when \( u \) has the form of (2). Let \( \alpha \) be a real number such that \( 0 < \alpha < 2 \). For any fixed \( b \) in \( D \), put \( p(z) = |z - b| \). Firstly, we show that the Berezin transform of \( p^{-\alpha} \) is bounded. In fact, making a change of variable, elementary calculations show that

\[
(p^{-\alpha})\sim(a) \leq |1 - \bar{a}b|^{-\alpha} \cdot \|1 - \bar{a}z\|_{\infty}^{\alpha} \times \int |\phi_{a}(b) - z|^{-\alpha} \, dm(z).
\]

Since \( \phi_{a}(b) - z \) lies in \( 2D = \{2z; z \in D\} \) for any \( a, z \) in \( D \) and an area measure is translation invariant, we have that

\[
(p^{-\alpha})\sim(a) \leq (1 - |b|)^{-\alpha} \cdot \|1 - \bar{a}z\|_{\infty}^{\alpha} \times \int_{2D} |w|^{-\alpha} \, dm(w)
\]

for all \( a \) in \( D \). Hence we obtain that the Berezin transform of \( p^{-\alpha} \) is bounded. Next, let \( b \) be in \( \partial D \) and put \( p(z) = |z - b| \). Then, as in the proof of the above case, we have that

\[
(p^{\alpha})\sim(a) \leq |a - b|^{\alpha} \cdot \|\phi_{a}(b) - z\|_{\infty}^{\alpha} \times \int |1 - \bar{a}z|^{-\alpha} \, dm(z),
\]

and

\[
(p^{-\alpha})\sim(a) \leq |a - b|^{-\alpha} \cdot \|1 - \bar{a}z\|_{\infty}^{\alpha} \times \int_{2D} |w|^{-\alpha} \, dm(w).
\]

Therefore, Rudin’s lemma implies that \( p^{\alpha} \) satisfies the \((A_2)_{\partial}\)-condition. For any \( b_1 \) in \( D \) and \( b_2 \) in \( \partial D \), put \( p_1(z) = |z - b_1| \) and \( p_2(z) = |z - b_2| \). Fix \( 0 < \alpha(j) < 2 \) for \( j = 1, 2 \) and \( \varepsilon > 0 \). Because \( b_1 = b_2 \), there exist measurable
subsets $B_j$ of $D$ such that $B_1 \cap B_2 = \phi$ and $p_j \geq \epsilon$ on $B_j^c$ for $j = 1, 2$. Set $B_0 = D \setminus B_1 \cup B_2$, then

\[(p_1^{(1)} \cdot p_1^{(2)})^{-1}(a) \times (p_1^{-\alpha(1)} \cdot p_2^{-\alpha(2)})^{-1}(a) \leq (p_1^{(1)} \cdot p_2^{(2)})^{-1}(a) \times \left(\epsilon^{-\alpha(1)-\alpha(2)} \int_{B_0} |k_a|^2 d m + \epsilon^{-\alpha(2)} \int_{B_1} p_1^{-\alpha(1)}|k_a|^2 d m + \epsilon^{-\alpha(1)} \int_{B_2} p_2^{-\alpha(2)}|k_a|^2 d m\right) \leq M_0 \times \epsilon^{-\alpha(1)-\alpha(2)} + M_0 \times \epsilon^{-\alpha(2)} \cdot (p_1^{-\alpha(1)})^{-1}(a) + M_1 \times \epsilon^{-\alpha(1)} \cdot (p_2^{-\alpha(2)})^{-1}(a),\]

where $M_0 = \|p_1^{(1)} \cdot p_2^{(2)}\|_{\infty}$ and $M_1 = \|p_1^{(1)}\|_{\infty}$. Hence we have that $p_1^{(1)} \cdot p_2^{(2)}$ satisfies the $(A_2)^{\theta}$-condition. If $u$ has the form of (2), then applying the same argument for finitely many factors of $u$ and $u^{-1}$, we obtain that $u$ satisfies $(A_2)^{\theta}$-condition.

Apparently, (3) follows from (2) of this proposition and (3) of Proposition 4. In fact, we let $-1 < \alpha(j) < 1$ for all $j$, and set

\[j(+) = \{j; \alpha(j) \geq 0\}, \quad j(-) = \{j; \alpha(j) < 0\}.\]

Put $u_1 = \prod_{j(+)} p_j^{\alpha(j)}$ and $u_2 = \prod_{j(-)} p_j^{\alpha(j)}$, then $u_1^2$ and $u_2^2$ satisfy the $(A_2)^{\theta}$-condition. Hence, (3) of Proposition 4 implies that $u = u_1 \times u_2$ satisfies the $(A_2)^{\theta}$-condition. \hfill \Box

Corollary 1 is a partial result of [2], [7] and [8].

**Corollary 1.** Oleinik-Pavlov-Hastings-Stegenga. Let $\nu$ be a finite positive measure on $D$. For any $-1 < \alpha < 1$, there is a constant $C > 0$ such that

\[
\int_D |f|^2 d \nu \leq C \int_D |f|^2 (1 - |z|^2)^\alpha d m
\]

for all $f$ in $P$ if and only if there exist $r > 0$ and $\gamma > 0$ such that

\[
\hat{\nu}_r(a) \leq \gamma (1 - |a|^2)^\alpha
\]

for all $a$ in $D$.

**Proof.** Since $[(1 - |z|^2)^\alpha]_r(a)$ is comparable to $(1 - |a|^2)^\alpha$, by Theorem 3 and (1) of Proposition 5 the corollary follows. \hfill \Box
Lemma 2. Let \( \{b_j\} \) be a finite sequence of complex numbers in \( D \cup \partial D \) with \( b_i \neq b_j (i \neq j) \), and let \( \{\alpha(j)\} \) be a finite sequence of real numbers such that \(-2 < \alpha(j) \) when \( j \) is in \( \Lambda^c \) (the definition of \( \Lambda \) is below). Put \( p_j(z) = |z - b_j| \) and \( u = \prod p_j^{\alpha(j)} \), and let \( 0 < r < \infty \), then there are constants \( \gamma_1 > 0 \) and \( \gamma_2 > 0 \) such that

\[
\gamma_1 \hat{u}_r(a) \leq \prod_{j \in \Lambda} |a - b_j|^{\alpha(j)} \leq \gamma_2 \hat{u}_r(a)
\]

for all \( a \) in \( D \), here \( \Lambda = \{j; b_j \text{ is in } \partial D\} \).

Proof. For any fixed \( 0 < r < \infty \), in general, Lemma 4.3.3 in [9, p. 60] implies that there are constants \( L > 0 \) and \( M > 0 \) such that

\[
L \hat{u}_r(a) \leq \int_{D_r(0)} u \circ \phi_a d \, m \leq M \hat{u}_r(a)
\]

for all \( a \) in \( D \), where \( u \) is a non-negative integrable function on \( D \). Let \( u = \prod |z - b_j|^{\alpha(j)} \), \( \{b_j\} \subset D \cup \partial D \), \( b_i \neq b_j (i \neq j) \), and \( \alpha(j) \) be real numbers. Then, by the same calculations in the proof of (2) of Proposition 5, we have that

\[
\int_{D_r(0)} u \circ \phi_a d \, m = \prod |1 - \bar{a} b_j|^{\alpha(j)} \int_{D_r(0)} \prod |\phi_a(b_j) - z|^{\alpha(j)} \cdot |1 - \bar{a} z|^{-\Sigma \alpha(j)} d \, m(z).
\]

Put

\[
I(a) = \int_{D_r(0)} \prod |\phi_a(b_j) - z|^{\alpha(j)} d \, m(z),
\]

then it is easy to see that \( \int_{D_r(0)} u \circ \phi_a d \, m \) is equivalent to

\[
I(a) \times \prod_{j \in \Lambda} |a - b_j|^{\alpha(j)}.
\]

Firstly, we show that the lemma is true when \( 0 \leq \alpha(j) \) for all \( j \). By the above facts, it is enough to prove that the integration

\[
I(a) = \int_{D_r(0)} \prod |\phi_a(b_j) - z|^{\alpha(j)} d \, m(z)
\]

is bounded below for all \( a \) in \( D \), because \( 0 \leq \alpha(j) \). Conversely, suppose that there exists \( \{a_n\} \subset D \) such that \( I(a_n) < 1/n \). Here we can choose a subsequence \( \{a_k\} \subset \{a_n\} \) such that \( a_k \rightarrow a' (k \rightarrow \infty) \), where \( a' \) may be in \( D \cup \partial D \). Therefore, Fatou's lemma implies that \( I(a') = 0 \), thus it follows
that $\prod |\phi_a'(b_j) - z|^{\alpha(j)} = 0$ on $D_r(0)$. This contradiction implies that the assertion is true when $0 \leq \alpha(j)$ for all $j$.

Next, we prove that the lemma is true when $-2 < \alpha(j) < 0$ for all $j$ in $\Lambda^c$ and $-\infty < \alpha(j) < 0$ for all $j$ in $\Lambda$. In fact, we claim that $I(a)$ is bounded for all $a$ in $D$. If $j$ is in $\Lambda$, then $|\phi_a(b_j)| = 1$ for all $a$ in $D$, therefore $|\phi_a(b_j) - z|^{-1}$ is bounded, because $z$ belongs to $D_r(0)$. Analogously, if $j$ is in $\Lambda^c$, then $|\phi_a(b_j)| \to 1$ ($|a| \to 1$), therefore $|\phi_a(b_j) - z|^{-1}$ is bounded when $a$ is nearby $\partial D$, because $z$ belongs to $D_r(0)$. Thus, it is sufficient to prove that

$$J(a) = \int_{D_r(0)} \prod_{j \in \Lambda^c} |\phi_a(b_j) - z|^{\alpha(j)} d m(z)$$

is bounded for all $a$ in $U_\eta(0) = \{ a \in D; |a| \leq \eta \}$, where $0 < \eta < 1$ is a constant which is close to 1. Put

$$\Phi_{i,j}(a) = |\phi_a(b_i) - \phi_a(b_j)| \ (i, j \in \Lambda^c, a \in U_\eta(0)).$$

For any fixed $i, j \in \Lambda^c$, since $\Phi_{i,j}$ is a continuous function on $U_\eta(0)$ and Möbius functions are one-to-one correspondence on $D$, there exists $\varepsilon(i, j) > 0$ such that $\Phi_{i,j}(a) \geq \varepsilon(i, j)$ for all $a$ in $U_\eta(0)$ when $i \neq j$. Put $\varepsilon = \min \{ \varepsilon(i, j)/2; i, j \in \Lambda^c \text{ such that } i \neq j \}$,

$$B_j(a) = \{ z \in D_r(0); |\phi_a(b_j) - z| < \varepsilon \}$$

and $B_0(a) = D_r(0) \setminus \cup B_j(a)$. For any $j$ in $\Lambda^c \cup \{0\}$, since $|\phi_a(b_i) - z| \geq \varepsilon$ when $z$ belongs to $B_j(a)$ and $i$ belongs to $\Lambda^c$ such that $i \neq j$, therefore we have that

$$J(a) \leq \sum_{j \in \Lambda^c} \varepsilon^{\alpha - \alpha(j)} \int_{B_j(a)} |\phi_a(b_j) - z|^{\alpha(j)} d m(z) + \varepsilon^\alpha \int_{B_0(a)} d m(z)$$

$$\leq \sum_{j \in \Lambda^c} \varepsilon^{\alpha - \alpha(j)} \int_{2D} |w|^{\alpha(j)} d m(w) + \varepsilon^\alpha$$

where

$$\alpha = \sum_{j \in \Lambda^c} \alpha(j).$$

Therefore, $J$ is bounded on $D_\eta(0)$, and hence we obtain that $I$ is bounded on $D$.

Using the above facts, we can show that the assertion is true when $u$ has the general form of the statement of this lemma. Let $\{ \alpha(j) \}$ be a finite sequence of real numbers such that $-2 < \alpha(j) < \infty$ when $j$ is in $\Lambda^c$ and $-\infty < \alpha(j) < \infty$ when $j$ is in $\Lambda$. As in the proof of Proposition 5, set
\[ j(+) = \{ j; \alpha(j) \geq 0 \} \text{ and } j(-) = \{ j; \alpha(j) < 0 \}, \text{ then we have that} \]
\[ I(a) \leq 2^{\sum_{j(+) \alpha(j)}} \int_{D_r(0)} \prod_{j(-)} \left| \phi_a(b_j) - z \right|^{\alpha(j)} d m(z) \]

and
\[ I(a) \geq 2^{\sum_{j(-) \alpha(j)}} \int_{D_r(0)} \prod_{j(+)} \left| \phi_a(b_j) - z \right|^{\alpha(j)} d m(z). \]

Therefore, we obtain that \( I \) is bounded and bounded below on \( D \). Hence, this completes the proof. \( \square \)

Corollary 2. Let \( u \) be a non-negative function in \( L^1 \) that is given by (2), or (3) of Proposition 5 and \( \nu \) be a finite positive measure on \( D \), then there is a constant \( C > 0 \) such that
\[ \int_D |f|^2 d \nu \leq C \int_D |f|^2 u d m \]
for all \( f \) in \( P \) if and only if there exist \( r > 0 \) and \( \gamma = \gamma_r > 0 \) such that
\[ \hat{\nu}_r(a) \leq \gamma \prod_{j \in \Lambda} |a - b_j|^{\alpha(j)} \]
for all \( a \) in \( D \), here \( \Lambda = \{ j; b_j \text{ is in } \partial D \} \).

Proof. The corollary follows from Theorem 3, Proposition 5 and Lemma 2. \( \square \)

We give a characterization of \( u \) which satisfies the \((A_2)\)-condition or the \((A_2)_\theta\)-condition when \( u \) is a modulus of a rational function or a modulus of a polynomial, respectively. Let \( u \) be a non-negative integrable function on \( D \), then it is easy to see that if \( u \) satisfies the \((A_2)_\theta\)-condition then \( u^{-1} \) is integrable on \( D \). But, we claim that the converse is true, when \( u \) is a modulus of a polynomial. As the result, we show that the \((A_2)_\theta\)-condition is properly contained in the \((A_2)\)-condition. The essential part of the following theorem is proved in Proposition 5 and Lemma 2.

Theorem 6. Let \( \{ b_j \} \) be a finite sequence of complex numbers such that \( b_i \neq b_j (i \neq j) \) and \( \{ \alpha(j) \} \) be a finite sequence of real numbers. Put \( p_j(z) = |z - b_j| \) and \( u = \prod_j p_j^{\alpha(j)} \), then the following are true.

1. If \( \alpha(j) \geq 0 \) for all \( j \) or \( \alpha(j) \leq 0 \) for all \( j \), then \( u \) satisfies the \((A_2)_\theta\)-condition if and only if \( \alpha(j) < 2 \) or \( \alpha(j) > -2 \) when \( b_j \) is in \( D \cup \partial D \) respectively.
(2) \( u \) satisfies the \((A_2)\)-condition if and only if \(-2 < \alpha(j) < 2\) when \(b_j\) is in \(D\).

**Proof.** (1) By (2) of Proposition 5 and the remark above this theorem, it is enough to prove that \(u^{-1}\) is not integrable on \(D\) when \(\alpha(j) \geq 2\) for some \(b_j\) in \(D \cup \partial D\). Suppose that there is a \(j\) such that \(b_j\) in \(D \cup \partial D\) and \(\alpha(j) \geq 2\), then there exists a \(L^\infty\)-function \(h\) such that \(u(z) = |z - b_j|^2 \cdot h(z)\). It is easy to see that \(u^{-1}\) is not integrable on \(U = \{z \in D; |z - b| < \text{dist}(b_j, \partial D)\}\) when \(b_j\) is in \(D\), therefore we consider the case when \(b_j = 1\). Put \(M_2 = \|h\|_\infty\), then

\[
\int u^{-1} \, dm \geq M_2^{-1} \int_0^1 2r \int_0^{2\pi} |1 - re^{i\theta}|^2 \, d\theta / 2\pi \, dr
= M_2^{-1} \int_0^1 2r(1 - r^2)^{-1} \, dr = M_2^{-1} \int_0^1 t^{-1} \, dt.
\]

Hence we obtain that \(u^{-1}\) is not integrable.

(2) Suppose that \(-2 < \alpha(j) < 2\) when \(b_j\) is in \(D\), then apparently Lemma 2 implies that \(u\) satisfies the \((A_2)\)-condition. Conversely, suppose that there exist \(r > 0\) and \(A_r > 0\) such that

\[
\hat{u}_r(a) \times (u^{-1})_r^\land(a) \leq A_r
\]

for all \(a\) in \(D\). Since \(\hat{u}_r\) is non-zero on \(D\), therefore \((u^{-1})_r^\land(a) < \infty\) for all \(a\) in \(D\). By the same argument in (1), we have that \(\alpha(j)\) must be less than 2 when \(b_j\) is in \(D\). In fact, if \(\alpha(j) \geq 2\) for some \(b_j\) in \(D\), then there exists a function \(h\) such that \(u(z) = |z - b_j|^2 \cdot h(z)\). Put

\[
\varepsilon = \min\{\text{dist}(b_i, b_j)/2; i \neq j\}
\]

and

\[
U(j) = \{z \in D; |z - b_j| < \varepsilon\},
\]

then obviously \(h\) is bounded on \(U(j)\). Since there exists \(a_j\) such that a center of the Bergman disk \(D_r(a_j)\) is just equal to \(b_j\), therefore we have that \(u^{-1}\) is not integrable on \(D_r(a_j) \cap U(j)\), and thus, it follows that the average of \(u^{-1}\) on \(D_r(a_j)\) is infinite. This contradicts the above fact. Consequently, we obtain that \(\alpha(j)\) must lie in \((-\infty, 2)\) when \(b_j\) is in \(D\). Applying the same argument to \(u^{-1}\), we have that \(\alpha(j)\) must lie in \((-2, \infty)\) when \(b_j\) is in \(D\). Therefore, we conclude that \(-2 < \alpha(j) < 2\) when \(b_j\) is in \(D\). \(\square\)
§4. Uniformly absolutely continuous.

Recall that
\[ \varepsilon_r(\mu) = \sup_{a \in D} \left( \int_{D \setminus D_r(a)} |k_a|^2 \, d\mu \right) \times \left( \int_D |k_a|^2 \, d\mu \right)^{-1}, \]
where \( \mu \) is a finite positive measure on \( D \) (see Lemma 1 and Proposition 2). Using the quantity \( \varepsilon_r \) we give a necessary condition on \( \nu \) and \( \mu \) which satisfy the \( (\nu, \mu) \)-Carleson inequality.

**Theorem 7.** Suppose that \( d\nu = \nu d\mu, \varepsilon_t(\nu) \to 0 \ (t \to \infty) \), and that \( \nu \) satisfies the \( (A_2) \)-condition, furthermore \( \mu \) and \( \nu \) satisfy the \( (\mu, \nu) \)-Carleson inequality. If there is a constant \( C > 0 \) such that
\[ I \frac{|f|}{|x|} |f|^2 d\nu \leq C \int_D |f|^2 d\mu \]
for all \( f \) in \( P \), then there exist \( r > 0 \) and \( \gamma > 0 \) such that
\[ \dot{\nu}_r(\alpha) \leq \gamma \dot{\mu}_r(\alpha) \]
for all \( \alpha \) in \( D \).

**Proof.** By hypotheses on \( \nu \) and Lemma 1, there exist \( t > 0, \rho > 0 \) and \( A > 0 \) such that
\[ \nu \leq \rho \cdot \dot{\nu}_t \leq A\rho \cdot \nu^{-1} \]
Moreover, Lemma 4.3.3 in [9, p. 60] and the \( (\mu, \nu) \)-Carleson inequality imply that there exist \( L > 0 \) and \( C' > 0 \) such that
\[ L \cdot \dot{\mu}_t \leq \dot{\mu} \leq C' \cdot \dot{\nu}. \]
Thus, a desired result follows from (2) of Proposition 2. \( \square \)

Luecking [5] shows the above theorem when \( \nu \) is the Lebesgue area measure \( m \). It is clear that \( \varepsilon_r(m) \to 0 \ (r \to \infty) \) and \( m \) satisfies the \( (A_2) \)-condition. Now, we are interested in measures \( \mu \) such that \( \varepsilon_r(\mu) < 1 \) or \( \varepsilon_r(\mu) \to 0 (r \to \infty) \).

**Proposition 8.** Suppose that \( d\mu = ud\, m \), and \( u \) is a non-negative function in \( L^1 \). If \( u \) is the function such that (1) or (2), then there exists \( 0 < r < \infty \) such that \( \varepsilon_r(\mu) < 1 \).

1. \( u \) satisfies the \( (A_2)_\partial \)-condition.
2. \( u(z) = (1 - |z|^2)^\alpha \) for some \( 1 \leq \alpha < 2 \).

**Proof.** If \( u \) has the property in (1), then by the remark above Theorem 3, for any \( r > 0 \) there is a positive constant \( \rho = \rho_r \) such that \( \dot{\mu}(\alpha) \leq \rho \dot{\nu}_r(\alpha) \) for
all \(a\) in \(D\) and hence \(\varepsilon_r(\mu) < 1\) by Lemma 1. Suppose that \(u\) has the form of (2). For any fixed \(1 \leq \alpha < 2\), put \(u(z) = (1 - |z|^2)^\alpha\), Then, Rudin’s lemma (cf. [9, p. 53]) shows that

\[
\tilde{u}(a) = (1 - |a|^2)^\alpha \int_D (1 - |z|^2)^\alpha |1 - \bar{a}z|^{-2\alpha} d m(z) \leq \gamma(1 - |a|^2)^\alpha,
\]

where \(\gamma > 0\) is finite. On the other hand, Lemma 4.3.3 in [9, p. 60] implies that

\[
\tilde{u}_r(a) \geq M^{-1} \times (1 - |a|^2)^\alpha \int_{D_r(0)} (1 - |z|^2)^\alpha |1 - \bar{a}z|^{-2\alpha} d m(z)
\]

\[
\geq M^{-1} \times (1 - |z|^2)^\alpha (1 - \tanh^2 r)^\alpha \times 2^{-2\alpha},
\]

therefore, by (3) of Lemma 1, we obtain that \(\varepsilon_r(\mu) < 1\). 

**Proposition 9.** Suppose that \(d \mu = ud m\), and \(u\) is a non-negative function in \(L^1\). If \(u\) is one of the following functions (1) \(\sim\) (7), then \(\varepsilon_r(\mu) \to 0(r \to \infty)\).

1. There exists \(\varepsilon_0 > 0\) such that \(\tilde{u} \geq \varepsilon_0\) on \(D\), and \(\{u \circ \phi_a d m; a \in D\}\) is uniformly absolutely continuous with respect to the Lebesgue area measure \(m\).

2. There exists \(\varepsilon_0 > 0\) such that \(\tilde{u} \geq \varepsilon_0\) on \(D\), and there is a constant \(C > 0\) such that \((u^{1+\beta})^\sim \leq C\) on \(D\) for some \(\beta > 0\).

3. \(u\) is in \(L^\infty\), and there exist \(r > 0\) and \(\delta > 0\) such that \(u \geq \delta\) on \(D \setminus D_r(0)\).

4. \(u = |p|\), where \(p\) is an analytic polynomial which has no zeros on \(\partial D\).

5. \(u(z) = (1 - |z|^2)^\alpha\) for some \(-1 < \alpha \leq 1\).

6. \(u = \prod p_j^{\alpha(j)}\), where \(p_j(z) = |z - \beta_j|, b_i \neq b_j(i \neq j)\), and \(0 < \alpha(j) < 2\) for \(b_j\) in \(D \cup \partial D\), or \(-2 < \alpha(j) < 0\) for \(b_j\) in \(D \cup \partial D\).

7. \(u = \prod p_j^{\alpha(j)}\) where \(p_j(z) = |z - b_j|, b_i \neq b_j(i \neq j)\), and \(-1 < \alpha(j) < 1\) for \(b_j\) in \(D \cup \partial D\).

**Proof.** Firstly, we show that the assertion is true when \(u\) has the property of (1). Since \(\{u \circ \phi_a d m; a \in D\}\) is uniformly absolutely continuous, for any \(\varepsilon > 0\) there exists \(r > 0\) such that \(\int_{D_r(0)} \varepsilon u \circ \phi_a d m < \varepsilon_0 \cdot \varepsilon\) for all \(a\) in \(D\). Therefore, making a change of variable, let \(r\) be sufficiently large, then \(\varepsilon_r(\mu) < \varepsilon_0^{-1} \cdot \varepsilon_0 \cdot \varepsilon = \varepsilon\). Hence, we obtain that \(\varepsilon_r(\mu) \to 0(r \to \infty)\).

Next, we prove the implications (2) \(\Rightarrow\) (1), (3) \(\Rightarrow\) (2), and (4) \(\Rightarrow\) (3). Then \(\varepsilon_r(\mu) \to 0\) when \(u\) is a function such that (2), (3) or (4). In fact, suppose that there exists \(\beta > 0\) such that the Berezin transform of the function \(u^{1+\beta}\) is bounded, then a set of functions \(\{u \circ \phi_a; a \in D\}\) is uniformly integrable (cf. [1, p. 120]), therefore it follows that \(\{u \circ \phi_a d m; a \in D\}\) is uniformly
absolutely continuous with respect to \( m \). Hence, (2) implies (1). If there exist \( r > 0 \) and \( \delta > 0 \) such that \( u \geq \delta \) on \( D \setminus D_r(0) \), then

\[ \tilde{u}(a) \geq \delta - \delta \int_{D_r(0)} |k_a|^2 \, d \, m = \delta[1 - m(D_r(a))] \geq \delta(1 - \tanh^2 r) > 0. \]

Hence (3) implies (2) because \( (u^{1 + \beta})^\sim(a) \leq \|u\|^{1 + \beta}_\infty \) for all \( a \) in \( D \) and any \( \beta > 0 \). Next, let \( p \) be an analytic polynomial which has no zeros on \( \partial D \), then there are \( r > 0 \) and \( \delta > 0 \) such that \( u = |p| \geq \delta \) on \( D \setminus D_r(0) \), therefore (4) \( \Rightarrow \) (3).

We prove that the assertion is true when \( u \) has the form of (5). For any fixed \( -1 < \alpha \leq 1 \), put \( u(z) = (1 - |z|^2)^\alpha \) and making a change of variable, then

\[ \varepsilon_r(\mu) = \sup \left( \int_D (1 - |z|^2)^\alpha |1 - \bar{a}z|^{2\alpha} \, d \, m(z) \right) \times \left( \int_{D \setminus D_r(0)} (1 - |z|^2)^\alpha |1 - \bar{a}z|^{-2\alpha} \, d \, m(z) \right). \]

When \( 0 \leq \alpha \leq 1 \), since \( 0 < 1 - |z|^2 \leq 1 \), we have that

\[ \int_D (1 - |z|^2)^\alpha |1 - \bar{a}z|^{-2\alpha} \, d \, m \geq 2^{-2\alpha} \int_D (1 - |z|^2) \, d \, m = \text{constant}. \]

If \( -1 < \alpha < 0 \), then the familiar inequality between the harmonic and arithmetic means shows that

\[ \int_D (1 - |z|^2)^\alpha |1 - \bar{a}z|^{-2\alpha} \, d \, m \geq \left( \int_D (1 - |z|^2)^{-\alpha} |1 - \bar{a}z|^{2\alpha} \, d \, m \right)^{-1} \geq \text{constant}. \]

Here, the last inequality follows from Rudin's lemma (cf. [9, p. 53]). Again using Rudin's lemma, since \( -1 < \alpha \leq 1 \), there exists \( \beta > 0 \) such that a set of functions \( \{(1 - |z|^2)^\alpha |1 - az|^{-2\alpha} \}^\beta, a \in D \) is bounded in \( L^1 \). This implies that the set of these functions are uniformly integrable (cf. [1, p. 120]), therefore it follows that \( \varepsilon_r(\mu) \to 0(r \to \infty) \).

We show that \( \varepsilon_r(\mu) \to 0 \) when \( u \) has the form of (6). As in the proof of (2) of Proposition 5, we only prove that \( \varepsilon_r(\mu) \to 0(r \to \infty) \) when \( u = p_1^{\alpha(1)} \cdot p_2^{\alpha(2)} \), where \( p_1(z) = |z - b_1| \), \( p_2(z) = |z - b_2| \), \( 0 < \alpha(1), \alpha(2) < 2 \), and \( b_1 \) is in \( D \), \( b_2 \) is in \( \partial D \). We suppose that \( B_j, M_1 \), and \( \varepsilon \) are as in the proof of (2) of Proposition 5. By the definition of \( \varepsilon_r(\mu) \), we have that

\[ \varepsilon_r(\mu) = \sup(u \chi_{D_r(\alpha)\varepsilon})(a) \times \tilde{u}(a)^{-1}. \]
Moreover,

\[(u \chi_{D_r(a)^c})^\sim(a) \times \tilde{u}(a)^{-1} \leq (u \chi_{D_r(a)^c})^\sim(a) \times (u^{-1})^\sim(a) \leq (u \chi_{D_r(a)^c})^\sim(a) \times \varepsilon^{-\alpha(1)-\alpha(2)} \int_{B_0} |k_\alpha|^2 \, dm + (u \chi_{D_r(a)^c})^\sim(a) \times \varepsilon^{-\alpha(2)} \cdot (p_{1-\alpha(1)})^\sim(a) + M_1 \times \varepsilon^{-\alpha(1)} \times C \int_{D \setminus D_r(0)} |1 - \bar{a}z|^{-\alpha(2)} \, dm,\]

where

\[C = \|\phi_a(b_2) - z\|_{\infty}^{\alpha(2)} \times \|1 - \bar{a}z\|_{\infty}^{\alpha(2)} \times \int_{2D} |w|^{-\alpha(2)} \, dm.\]

Since \(u\) is bounded, therefore \(\{u \circ \phi_a; a \in D\}\) is uniformly integrable (cf. [1, p. 120]), moreover applying the same argument in the proof of this proposition when \(u\) has the form of (5), Rudin’s lemma implies that a set of functions \(\{|1 - \bar{a}z|^{-\alpha(2)}; a \in D\}\) is also uniformly integrable, hence we conclude that \(\varepsilon_r(\mu) \to 0 (r \to \infty)\). The proof of the latter half of (6) of this proposition is similar that in the above.

If \(u\) has the form of (7), then by the similar arguments in the proof of (3) of Proposition 5, set \(j(+) = \{j; \alpha(j) \geq 0\}, \quad j(-) = \{j; \alpha(i) < 0\}\). And put \(u_1 = \prod_{j(+)} p_j^{\alpha(j)}, \quad u_2 = \prod_{j(-)} p_j^{\alpha(j)}\), then

\[(u \chi_{D_r(a)^c})^\sim(a) \times \tilde{u}(a)^{-1} \leq (u \chi_{D_r(a)^c})^\sim(a) \times (u^{-1})^\sim(a)\]

\[= (u_1 u_2 \chi_{D_r(a)^c})^\sim(a) \times (u_1^{-1} u_2^{-1})^\sim(a).\]

Therefore, the desired result follows from the Cauchy-Schwarz’s inequality and (6) of this proposition. \(\boxdot\)

**Corollary 3.** Suppose that \(d \nu = \nu d m\) and there is a constant \(C > 0\) such that

\[\int_D |f|^2 \, \nu \leq C \int_D |f|^2 \, d \mu\]

for all \(a\) in \(D\), then the following are true.

(1) If \(v(z) = (1 - |z|^2)^\alpha\) for some \(-1 < \alpha \leq 1\), and there exist \(l > 0\) and \(\gamma' = \gamma'_l > 0\) such that

\[\hat{\mu}_l(a) \leq \gamma'(1 - |a|^2)^\alpha\]

for all \(a\) in \(D\), then there exist \(r > 0\) and \(\gamma = \gamma_r > 0\) such that

\[(1 - |a|^2)^\alpha \leq \gamma \hat{\mu}_r(a)\]

for all \(a\) in \(D\).
(2) If \( v = \prod p_j^{\alpha(j)} \), where \( p_j(z) = |z - b_j| \), \( b_i \neq b_j (i \neq j) \), and \( 0 < \alpha(j) < 2 \) for \( b_j \) in \( D \cup \partial D \) or \( -2 < \alpha(j) < 0 \) for \( b_j \) in \( D \cup \partial D \), and if there exist \( l > 0 \) and \( \gamma' = \gamma'_l > 0 \) such that

\[
\hat{\mu}_l(a) \leq \gamma' \prod_{j \in \Lambda} |a - b_j|^\alpha(j)
\]

for all \( a \) in \( D \), then there exist \( r > 0 \) and \( \gamma = \gamma_r > 0 \) such that

\[
\prod_{j \in \Lambda} |a - b_j|^\alpha(j) \leq \gamma \hat{\mu}_r(a)
\]

for all \( a \) in \( D \), where \( \Lambda = \{ j ; b_j \text{ is in } \partial D \} \).

(3) If \( v = \prod p_j^{\alpha(j)} \) where \( p_j(z) = |z - b_j| \), \( b_i \neq b_j (i \neq j) \), and \( -1 < \alpha(j) < 1 \) for \( b_j \) in \( D \cup \partial D \), and if there exist \( l > 0 \) and \( \gamma = \gamma'_l > 0 \) such that

\[
\hat{\mu}_l(a) \leq \gamma' \prod_{j \in \Lambda} |a - b_j|^\alpha(j)
\]

for all \( a \) in \( D \), then there exist \( r > 0 \) and \( \gamma = \gamma_r > 0 \) such that

\[
\prod_{j \in \Lambda} |a - b_j|^\alpha(j) \leq \gamma \hat{\mu}_r(a)
\]

for all \( a \) in \( D \), where \( \Lambda = \{ j ; b_j \text{ is in } \partial D \} \).

Proof. We show that (1) is true. By the fact in the proof of Corollary 1, and the fact that \( u(z) = (1 - |z|^2)^\alpha \) satisfies the \((A_2)\)-condition for all \( \alpha > -1 \) (see [6]), the hypothesis in (1) of the Corollary and Proposition 1 imply the \((\mu, \nu)\)-Carleson inequality. Hence, Theorem 7 and Proposition 9 show that the assertion is true.

Similarly, (2) and (3) follow from Proposition 1, Lemma 2, (5) of Proposition 4, Theorem 6, Theorem 7, and Proposition 9. \( \square \)

References


Received July 1, 1993 and revised June 1, 1994. For the first author, this research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education.

HOKKAIDO UNIVERSITY
JAPAN
Isometric immersions of $H^n_1$ into $H^{n+1}_1$

Kinetsu Abe

Rotationally symmetric hypersurfaces with prescribed mean curvature

Marie-Françoise Bidaut-Véron

The covers of a Noetherian module

Jian-Jun Chuai

On the odd primary cohomology of higher projective planes

Mark Foskey and Michael David Slack

Unit indices of some imaginary composite quadratic fields. II

Mikihito Hirabayashi

Mixed automorphic vector bundles on Shimura varieties

Min Ho Lee

Trace ideal criteria for Toeplitz and Hankel operators on the weighted Bergman spaces with exponential type weights

Peng Lin and Richard Rochberg

On quadratic reciprocity over function fields

Kathy Donovan Merrill and Lynne Walling

$(A_2)$-conditions and Carleson inequalities in Bergman spaces

Takahiko Nakazi and Masahiro Yamada

A note on a paper of E. Boasso and A. Larotonda: “A spectral theory for solvable Lie algebras of operators”

C. Ott

Tensor products with anisotropic principal series representations of free groups

Carlo Pensavalle and Tim Steger

On Ricci deformation of a Riemannian metric on manifold with boundary

Ying Shen

The Weyl quantization of Poisson SU(2)

Albert Jeu-Liang Sheu

Weyl’s law for $SL(3, \mathbb{Z}) \backslash SL(3, \mathbb{R}) / SO(3, \mathbb{R})$

Eric George Stade and Dorothy Irene Wallace (Andreoli)

Minimal hyperspheres in two-point homogeneous spaces

Per Tomter

Subalgebras of little Lipschitz algebras

Nikolai Isaac Weaver