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THE WEIL REPRESENTATION AND GAUSS SUMS

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## THE WEIL REPRESENTATION AND GAUSS SUMS

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We use the Weil representation to evaluate certain Gauss sums over a local field, up to  $\pm 1$ . Also we construct a cocycle on  $\mathrm{Sp}(2m, \mathbb{R})$  with a simple formula on the maximal compact torus and we show how to lift homomorphisms  $j : \mathrm{Sp}(2n, \mathbb{R}) \rightarrow \mathrm{Sp}(2m, \mathbb{R})$  to the double covers of these groups.

### 1. Introduction.

Let  $F$  be a self-dual local field of char  $\neq 2$ , for instance  $F = \mathbb{R}, \mathbb{C}$ , or a finite extension of  $\mathbb{Q}_p$ . For most of the paper we will assume  $F \neq \mathbb{C}$ . Let  $\chi$  be a nontrivial additive character of  $F$ . Then all additive characters of  $F$  have the form  $\lambda\chi$  for some  $\lambda \in F$ , where  $\lambda\chi(t) = \chi(\lambda t)$ . We consider the following unitary operators on  $L^2(F^m)$ :

$$(1.1) \quad (\mathbf{a}(A)\Phi)(X) = |\det A|_F^{1/2} \Phi(XA) \quad \text{for } A \in \mathrm{GL}_m(F),$$

$$(1.2) \quad \mathbf{n}(B)\Phi(X) = \chi(XBX^T/2) \Phi(X) \quad \text{for } B = B^T \in M_m(F),$$

$$(1.3) \quad (\mathcal{F}_j\Phi)(X) = \int_{F^j} \Phi(Y_1, \dots, Y_j, X_{j+1}, \dots, X_m) \chi(X_1Y_1 + \dots + X_jY_j) dY,$$

$$(1.4) \quad (\iota(t)\Phi)(X) = t\Phi(X), \quad t \in \mathbf{T} = \{z \in \mathbb{C} \mid z\bar{z} = 1\}.$$

Here  $\Phi$  is a nice function in  $L^2(F^m)$  (to be precise,  $\Phi$  belongs to the Schwartz space  $\mathcal{S}(F^m)$ ),  $dY$  is an additive Haar measure on  $F^j$  normalized so that  $\mathcal{F}_j^2 = \mathbf{a}(\mathrm{diag}\{-I_j, I_{m-j}\})$  for  $0 \leq j \leq m$ , and  $|a|_F$  for  $a \in F$  is the modulus function, determined by  $d(ya) = |a|_F dy$  for a Haar measure  $dy$  on  $(F, +)$ . All our vectors are row vectors. We will usually suppress the symbol  $\iota$ ; that is, identify  $t$  with  $\iota(t)$  for  $t \in \mathbf{T}$ . Let  $\mathrm{Mp} = \mathrm{Mp}(F^m)$  be the topological group generated by all the above operators. We call this the metaplectic group. This group is independent of  $\chi$  since  $\lambda\chi(XBX^T/2) = \chi(X(\lambda B)X^T/2)$  and  $\mathbf{a}\left(\begin{smallmatrix} \lambda I_j & \\ & I_{m-j} \end{smallmatrix}\right)\mathcal{F}_{j,\lambda\chi} = \mathcal{F}_{j,\lambda\chi}$ , where we have added a subscript to the Fourier

transform  $\mathcal{F}_j$  for clarification. Let  $\mathrm{Sp} = \mathrm{Sp}(2m, F)$  denote the symplectic group consisting of all  $2m \times 2m$  matrices  $\tau$  such that  $\tau^T w_m \tau = w_m$ , where  $w_m = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ ,  $I = I_m$ . This group is generated by the matrices  $\{\underline{a}(A) \mid A \in \mathrm{GL}_m(F)\}$ ,  $\{\underline{n}(B) \mid B = B^T \in M_m(F)\}$ , and  $w_m$ , where

$$\underline{a}(A) = \begin{pmatrix} A & \\ & A^{-T} \end{pmatrix} \quad \underline{n}(B) = \begin{pmatrix} I & B \\ & I \end{pmatrix} \quad (A \in \mathrm{GL}_m(F), B = B^T \in M_m(F)).$$

**Theorem** (Segal, Shale, Weil). *There is a homomorphism  $\pi = \pi_\chi : \mathrm{Mp} \rightarrow \mathrm{Sp}$  such that*

$$\pi(\mathbf{a}(A)) = \underline{a}(A), \quad \pi(\mathbf{n}(B)) = \underline{n}(B), \quad \pi(\iota(t)) = I_{2m}, \quad \pi(\mathcal{F}_j) = w_j,$$

where

$$(1.5) \quad w_j = \begin{pmatrix} 0_j & & I_j \\ & I_{m-j} & 0_{m-j} \\ -I_j & & 0_j \\ & 0_{m-j} & I_{m-j} \end{pmatrix}.$$

Moreover, there is an exact sequence of topological groups

$$(1.6) \quad 1 \rightarrow \mathbf{T} \xrightarrow{\iota} \mathrm{Mp} \xrightarrow{\pi} \mathrm{Sp} \rightarrow 1.$$

If  $F = \mathbb{C}$ , this sequence splits. If  $F \neq \mathbb{C}$  then  $\mathrm{Mp}$  contains a subgroup  $\widetilde{\mathrm{Sp}} = \widetilde{\mathrm{Sp}}(2m, F)$  such that  $\pi|_{\widetilde{\mathrm{Sp}}} : \widetilde{\mathrm{Sp}} \rightarrow \mathrm{Sp}$  is a nontrivial two-fold cover of  $\mathrm{Sp}$ .

From now on assume  $F \neq \mathbb{C}$ . The realization  $\widetilde{\mathrm{Sp}} \subset U(L^2(F^m))$  is known as the Weil, oscillator, or Segal-Weil-Shale representation. The group  $\mathrm{Mp}$  and the projection  $\pi$  have much better definitions in terms of a certain centralizing property these operators have with respect to a unitary representation of the Heisenberg group; see [R1, §3.2]. Note that

$$(1.7) \quad \pi_{\lambda\chi}(\sigma) = \begin{pmatrix} I & \\ & \lambda I \end{pmatrix} \pi_\chi(\sigma) \begin{pmatrix} I & \\ & \lambda^{-1} I \end{pmatrix} \quad \text{for } \sigma \in \mathrm{Mp}.$$

The main result of this paper is that a certain Gauss sum is computed up to  $\pm 1$ . The idea of the proof is to compare two sections of the homomorphism  $\pi$ . If  $F = \mathbb{R}$  or if  $F$  is a nonarchimedean field whose residue characteristic is  $\neq 2$  then there is a splitting homomorphism  $\mathbf{k} : K \rightarrow \mathrm{Mp}$ , where  $K$  is a certain maximal compact subgroup of  $\mathrm{Sp}$ . This section can be compared to the standard section  $r_0 : \mathrm{Sp} \rightarrow \mathrm{Mp}$  (see (4.1)) which was defined by Rao. Define  $x_0 : K \rightarrow \mathbf{T}$  by the formula  $r_0(k) = x_0(k) \mathbf{k}(k)$ . We will find

an expression for  $x_0(k)$  as a Gauss sum. The operators in  $\widetilde{\text{Sp}}$  are known explicitly, thus  $y_0(g)$  and  $y(k)$  may be found (Prop. 2 and Lemma 7) such that  $y_0(g)r_0(g) \in \widetilde{\text{Sp}}$  and  $y(k)k(k) \in \widetilde{\text{Sp}}$  for each  $g \in \text{Sp}$ ,  $k \in K$ . Then  $x_0(k)y_0(k)y(k)^{-1} \in \widetilde{\text{Sp}} \cap \mathbf{T} = \{\pm 1\}$ . In this way the exact value of the Gauss sum  $x_0(k)$  may be computed, up to  $\pm 1$ . A technical difficulty is that one must find an explicit formula for the Bruhat decomposition on  $K$  in order to compare the sections  $r_0$  and  $k$ ; this is done in Lemma 9.

Now we state our result explicitly. Let  $F = \mathbb{R}$  or  $F =$  a self-dual nonarchimedean local field such that 2 is a unit in the ring  $\mathfrak{o}_F$  of algebraic integers of  $F$ . Let  $\chi(t) = e^{2\pi it}$  if  $F = \mathbb{R}$ , and let  $\chi$  be any additive character such that  $\alpha_\chi = \mathfrak{o}_F$  if  $F$  is nonarchimedean, where

$$(1.8) \quad \alpha_\chi = \{ x \in F \mid \chi(xy) = 1 \quad \text{for all } y \in \mathfrak{o}_F \}.$$

For example, if  $F = \mathbb{Q}_p$ , one could take  $\chi = \chi_p$  to be the unique additive character such that  $\chi_p(a/p^n) = e^{-2\pi ia/p^n}$  when  $a, n \in \mathbb{Z}$ , and if  $F$  is a finite extension of  $\mathbb{Q}_p$  one could take  $\chi = \lambda\chi_p \circ \text{tr}_{F/\mathbb{Q}_p}$ , where  $\lambda$  is a generator for the inverse different of  $F$ . Let  $K = \text{Sp}(2m, \mathfrak{o}_F)$  if  $F$  is nonarchimedean and

$$K = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \mid (\alpha + i\beta)(\alpha - i\beta)^T = I_m \right\} \quad \text{if } F = \mathbb{R}.$$

Define  $\Phi_0 \in L^2(F^m)$  to be the characteristic function of  $\mathfrak{o}_F^m$  if  $F$  is nonarchimedean and  $\Phi_0(X) = e^{-\pi X \cdot X}$  if  $F = \mathbb{R}$ . Let  $k = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in K$  and  $j = \text{rank}(C)$ . It is not hard to see (Lemma 8) that there exist  $\pi_1, \pi_2 \in \text{SO}(m, F)$  such that all entries of  $\pi_i$  are 0, 1, or  $-1$  and such that the top left  $j \times j$  minor of  $\pi_1 C \pi_2$  is invertible. Let

$$(1.9) \quad \pi_1 C \pi_2 = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}, \quad \pi_1 D \pi_2 = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix},$$

where  $C_1, D_1$  are  $j \times j$  matrices and  $C_4, D_4$  are  $(m - j) \times (m - j)$  matrices. For  $a, b \in F^\times$  let  $(a, b)_F \in \{\pm 1\}$  denote the Hilbert symbol:  $(a, b)_F = 1$  or  $-1$  according as  $a$  is or is not a norm in  $F(\sqrt{b})$ . In particular, if  $F = \mathbb{R}$  then  $(a, b)_\mathbb{R} = -1$  iff  $a$  and  $b$  are both negative.

**Main Theorem.** *Let  $k \in K - P$  and  $\pi_1, \pi_2, C_1, C_2, \dots, D_4$  be as above. Put  $\eta = \det \begin{pmatrix} -C_1 & D_2 \\ -C_3 & D_4 \end{pmatrix}$ . Then  $\eta \neq 0$ , and the quantity*

$$(1.10) \quad x_0(k) = |\det C_1| |\eta|^{-1/2} \int_{F^j} \chi(y(D_1 C_1^T + D_2 C_2^T)y^T/2) \Phi_0(yC_1, yC_2) dy$$

*is independent of the choice of  $\pi_1$  and  $\pi_2$ . Furthermore,  $x_0(k)^2 = (\eta, -1)_F$  if  $F$  is nonarchimedean and  $x_0(k)^2 = \text{sign}(\eta) (-i)^j \det(D - iC)$  if  $F = \mathbb{R}$ .*

In §2 we discuss the Weil index, which is fundamental to the study of the Weil representation, and we give a new method to compute the operators belonging to  $\widetilde{\text{Sp}}$ . In §3 we discuss splittings of the maximal compact subgroup of  $\text{Sp}$  into  $\text{Mp}$ . The main theorem is proved in §4, and as an application we construct in Proposition 10 a section  $r_+ : \text{Sp} \rightarrow \widetilde{\text{Sp}}$  such that the associated cocycle  $c_+$  has the following nice properties: a)  $c_+(pg_1, g_2) = c_+(g_1, g_2) = c_+(g_1, g_2p)$  for all  $p = \begin{pmatrix} A & * \\ 0 & * \end{pmatrix} \in \text{Sp}$  with  $\det A > 0$  and all  $g_1, g_2 \in \text{Sp}$ ; b)  $c_+$  has a simple formula on the standard maximal compact torus of  $\text{Sp}$ ; and c)  $c_+$  coincides with the Kubota cocycle when  $m = 1$ . In §5 we show how to lift homomorphisms  $j : \text{Sp}(2n, \mathbb{R}) \rightarrow \text{Sp}(2m, \mathbb{R})$  to the double covers of these groups.

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### 2. The Weil index.

We first show that  $\widetilde{\text{Sp}}$  is equal to the commutator subgroup of  $\text{Mp}$  (this fact is known). By computing some commutators explicitly, we are led to consider some products of Gauss sums which Rao calls the Weil index. We will give a new proof that the Weil index defines a homomorphism of the Witt group and a new way to compute the operators in  $\widetilde{\text{Sp}}$ . If  $H$  is any group, let  $(H, H)$  denote its commutator.

**Lemma 1.**  *$\widetilde{\text{Sp}}$  is equal to the commutator subgroup of  $\text{Mp}$ . If  $G_1$  is a subgroup of  $\text{Mp}$  and  $\pi(G_1) = \text{Sp}$  then  $G_1 = H\widetilde{\text{Sp}} = H(\text{Mp}, \text{Mp})$ , where  $H = G_1 \cap \mathbf{T}$ . Also,  $\widetilde{\text{Sp}} = (\widetilde{\text{Sp}}, \widetilde{\text{Sp}})$ .*

*Proof.* Let  $G_0 = (\text{Mp}, \text{Mp})$ . Then  $\pi|_{G_0}$  is surjective, because  $\text{Sp} = (\text{Sp}, \text{Sp})$ . Now let  $G_1$  be any subgroup of  $\text{Mp}$  such that the restriction of  $\pi$  to  $G_1$  is surjective and let  $H = G_1 \cap \mathbf{T}$ . Then  $(G_1, G_1)$  contains  $G_0$  because given any  $A, B$  in  $\text{Mp}$  there are constants  $t_1$  and  $t_2$  in  $\mathbf{T}$  such that  $t_1A$  and  $t_2B$  belong to  $G_1$ ; thus  $ABA^{-1}B^{-1} = (t_1A)(t_2B)(t_1A)^{-1}(t_2B)^{-1} \in (G_1, G_1)$ . In particular,  $G_0 \subset (\widetilde{\text{Sp}}, \widetilde{\text{Sp}}) \subset \widetilde{\text{Sp}}$ . If the inclusion of  $G_0$  in  $\widetilde{\text{Sp}}$  were proper, then the exact sequence (1.6) would be split. So  $\widetilde{\text{Sp}} = (\widetilde{\text{Sp}}, \widetilde{\text{Sp}}) = G_0 \subset G_1 \subset \text{Mp}$ . Given any  $g \in G_1$  there exists  $a \in \mathbf{T}$  such that  $ag \in \widetilde{\text{Sp}}$ . Since  $\widetilde{\text{Sp}} \subset G_1$ ,  $a = (ag)g^{-1} \in G_1 \cap \mathbf{T} = H$ . Since  $g = a^{-1}(ag)$ , we see  $G_1 \subset H\widetilde{\text{Sp}} \subset G_1$ . □

Let  $P$  be the subgroup of matrices  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}$  such that  $C = 0$ . Then  $P$  is generated by  $\underline{a}(A)\underline{n}(B)$  such that  $A \in \text{GL}_m(F)$  and  $B = B^T \in M_m(F)$ , and the section  $r_P : P \rightarrow \text{Mp}$  given by

$$(2.1) \quad r_P(\underline{a}(A)\underline{n}(B)) = \mathbf{a}(A)\mathbf{n}(B)$$

is easily seen to be a homomorphism. Since  $P$  and  $\{w_j \mid j = 0, \dots, m\}$  generate  $\text{Sp}$  ([R1, Lemma 2.14]), their lifts  $\{r_P(p) \mid p \in P\}$  and  $\mathcal{F}_j$ , together with  $\mathbf{T}$ , generate  $\text{Mp}$ . We will use this information to compute  $\widetilde{\text{Sp}} = (\text{Mp}, \text{Mp})$  and see how Gauss sums arise in the process. In the following let  $\mathcal{F} = \mathcal{F}_m$ .

Since  $(\text{GL}(m, F), \text{GL}(m, F)) = \text{SL}(m, F)$  for any field  $F$  (Milnor, J., *Introduction to Algebraic K-Theory*, Ann. of Math. Studies No. 72, p. 25 and 28), it is not hard to see that  $(P, P)$  is generated by  $\underline{a}(A)$  such that  $A \in \text{SL}_m(F)$  and  $\underline{n}(B)$  such that  $B = B^T \in M_m(F)$ , where  $I = I_m$ . Since  $\mathbf{a}$  and  $\mathbf{n}$  are homomorphisms,

$$(2.2) \quad \mathbf{a}(A) \in \widetilde{\text{Sp}} \quad \text{and} \quad \mathbf{n}(B) \in \widetilde{\text{Sp}} \quad \text{for all } A \in \text{SL}_m(F) \quad \text{and} \quad B = B^T \in M_m(F).$$

Since  $\widetilde{\text{Sp}} = (\text{Mp}, \text{Mp})$  is normal in  $\text{Mp}$  and  $\begin{pmatrix} I & \\ & B \end{pmatrix} = w_m \underline{n}(-B) w_m^{-1}$ ,

$$(2.3) \quad \mathcal{F} \mathbf{n}(-B) \mathcal{F}^{-1} \in \widetilde{\text{Sp}} \cap \pi^{-1} \left( \begin{pmatrix} I & \\ & B \end{pmatrix} \right).$$

Now we compute the fiber in  $\widetilde{\text{Sp}}$  over  $\underline{a}(B^{-1})w_m$  when  $B$  is symmetric and invertible. The Bruhat decomposition on the big cell ( $\det C \neq 0$ ) has the form

$$(2.4) \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \underline{n}(AC^{-1}) \underline{a}(-C^{-T}) w_m \underline{n}(C^{-1}D).$$

Note that  $AC^{-1}$  and  $C^{-1}D$  are symmetric. Taking  $\begin{pmatrix} I & \\ & -B \end{pmatrix}$  in place of  $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$  when  $B$  is invertible gives

$$\underline{n}(B^{-1}) \begin{pmatrix} I & \\ & -B \end{pmatrix} \underline{n}(B^{-1}) = \underline{a}(B^{-1})w_m.$$

An element of  $\widetilde{\text{Sp}}$  which lies in the fiber over the left side of the equation is

$$\mathbf{n}(B^{-1}) \mathcal{F} \mathbf{n}(B) \mathcal{F}^{-1} \mathbf{n}(B^{-1}) \in \widetilde{\text{Sp}}.$$

On the other hand, the fiber in  $\text{Mp}$  over the right hand side contains  $\mathbf{a}(B^{-1}) \mathcal{F}$ . Hence there is a constant  $\gamma(B) \in \mathbf{T}$  (which depends on  $\chi$ ) such that

$$(2.5) \quad \gamma(B) \mathbf{a}(B^{-1}) \mathcal{F} \\ = \mathbf{n}(B^{-1}) \mathcal{F} \mathbf{n}(B) \mathcal{F}^{-1} \mathbf{n}(B^{-1}) \in \widetilde{\text{Sp}} \cap \pi^{-1} \left( \begin{pmatrix} & B^{-1} \\ -B & \end{pmatrix} \right).$$

This constant is the same as  $\gamma(f)$  of [W, No. 14] or the “Weil index”  $\gamma_F(f)$  of [R1, Appendix], where  $f$  is the character of second degree on  $F^m$  given by  $f(X) = \chi(XBX^T/2)$ .

Weil has shown that  $\gamma$  is a homomorphism of the Witt group ([W, No. 25]); that is,

$$(2.6) \quad \gamma(R^TBR) = \gamma(B), \quad \gamma \left( \begin{matrix} B_1 \\ B_2 \end{matrix} \right) = \gamma(B_1)\gamma(B_2)$$

for all  $R \in GL_m(F)$  and all square matrices  $B_1, B_2$ . Here is a different proof of these facts. Take any  $R \in GL_m(F)$  such that  $R^TBR = \text{diag}\{b_1, \dots, b_m\}$  with  $b_i \in F^\times$ . It will suffice to show  $\gamma(B) = \prod \gamma(b_i)$ . Let  $\varphi$  be any nonzero element of  $L^2(F)$ , and define  $\varphi^{\otimes m} \in L^2(F^m)$  by  $\varphi^{\otimes m}(X) = \prod \varphi(X_i)$ . Let  $\Phi(X) = \varphi^{\otimes m}(XR)$ . The right side of (2.5) evaluated at  $\Phi$  is the function

$$\begin{aligned} \Phi_1(X) &= \chi(XB^{-1}X^T/2) \int_{F^m} \chi(YBY^T/2) (\mathcal{F}^{-1} \mathbf{n}(B^{-1}) \Phi)(Y) \chi(XY^T) dY \\ &= \chi(XB^{-1}X^T/2) \int_{F^m} \chi(YBY^T/2) \int_{F^m} \chi(ZB^{-1}Z^T/2) \Phi(Z) \chi((X-Z)Y^T) dZ dY. \end{aligned}$$

Let  $X' = XR = (X'_1, \dots, X'_m)$ . Now change variables  $Y \mapsto YR^{-T}$ ,  $Z \mapsto ZR$ . Then

$$\begin{aligned} \Phi_1(X) &= \prod_{i=1}^m \chi(b_i^{-1}X_i'^2/2) \int_F \chi(b_i y^2/2) \int_F \chi(b_i^{-1}z^2/2) \varphi(z) \chi((X'_i - z)y) dz dy. \end{aligned}$$

On the other hand, the left side of (2.5) evaluated at  $\Phi$  is

$$\begin{aligned} \Phi_1(X) &= \gamma(B) |\det B|_F^{-1/2} \int_{F^m} \Phi(Y) \chi(XB^{-1}Y^T) dY \\ &= \gamma(B) \prod_{i=1}^m |b_i|_F^{-1/2} \int_F \varphi(y) \chi(b_i^{-1}X'_i y) dy \\ &= \gamma(B) \prod |b_i|_F^{-1/2} \hat{\varphi}(b_i^{-1}X'_i), \end{aligned}$$

where  $\hat{\varphi}$  is the Fourier transform of  $\varphi$ . Comparing the two expressions for  $\Phi_1(X)$ , we find  $\gamma(B) = \prod \gamma(b_i)$ , where  $\gamma(b)$  for  $b \in F^\times$  is the constant in **T** such that

$$(2.7) \quad \begin{aligned} \gamma(b) |b|^{-1/2} \hat{\varphi}(b^{-1}x) &= \chi(b^{-1}x^2/2) \int_F \chi(by^2/2) \int_F \chi(b^{-1}z^2/2) \varphi(z) \chi((x-z)y) dz dy. \end{aligned}$$

Note that our notation is consistent: if  $m = 1$  and  $B = b$  then  $\gamma(B) = \prod_{i=1}^1 \gamma(b) = \gamma(b)$ . This proves (2.6).

If  $F = \mathbb{R}$  and  $\chi(t) = e^{2\pi i \lambda t}$ ,  $\gamma(b)$  may be evaluated by taking  $\phi(x) = e^{-\pi x^2}$  in formula (2.7) and applying [Ig, Ch. I, §2, Lemma 1]. The result is

$$(2.8) \quad \gamma(b) = \begin{cases} e^{2\pi i/8} & \text{if } \lambda b > 0 \\ e^{-2\pi i/8} & \text{if } \lambda b < 0. \end{cases}$$

If  $F$  is nonarchimedean with a discrete valuation  $v$ , then by [W, No. 27],

$$(2.9) \quad \gamma(b) = \tilde{\gamma}(b)/|\tilde{\gamma}(b)|, \quad \tilde{\gamma}(b) = \sum_{\nu=-\infty}^{+\infty} \int_{v(y)=\nu} \chi(by^2/2) dy,$$

where  $dy$  is an additive Haar measure. This is recognizable as a Gauss sum.

In ([W, No. 28]) it is shown that for all  $a, b \in F^\times$ ,

$$(2.10) \quad \gamma(-b) = \bar{\gamma}(b), \quad \gamma(1) \gamma(-a) \gamma(-b) \gamma(ab) = (a, b)_F,$$

where  $(a, b)_F = 1$  or  $-1$  according as  $a$  is or is not a norm in  $F(b^{1/2})$ . From this it is easy to deduce that  $\gamma(b)^8 = 1$ , and even  $\gamma(a)^4 \gamma(b)^4 = 1$ , for all  $a, b \in F^\times$ . Other formulas are gathered in the appendix of [R1].

**Proposition 2.** For all  $A \in \text{GL}_m(F)$  and  $B = B^T \in \text{M}_m(F)$ ,

$$(\det A, -1)_F^{1/2} \mathbf{a}(A) \in \widetilde{\text{Sp}}, \quad \mathbf{n}(B) \in \widetilde{\text{Sp}}, \quad \gamma(1)^j \mathcal{F}_j \in \widetilde{\text{Sp}}.$$

*Proof.* These formulas can be deduced from [R1, Def. 5.2 and Cor. A.5] or from [R2, Th. 4.1]; here we give a different proof. The first assertion when  $\det A = 1$ , the second assertion in general, and the third assertion when  $j = m$  have already been shown (equations (2.2) and (2.5)). For arbitrary  $A \in \text{GL}_m(F)$ , write  $A = BA_1$  with  $B = \text{diag}\{\det A, I_{m-1}\}$  and  $A_1 \in \text{SL}_m(F)$ . Then

$$\frac{\gamma(\det A)}{\gamma(1)} \mathbf{a}(A) = \{\gamma(B) \mathbf{a}(B) \mathcal{F}\} \{\gamma(I) \mathcal{F}\}^{-1} \{\mathbf{a}(A_1)\}.$$

Here we have used that  $\gamma(B)/\gamma(I) = \gamma(\det A)/\gamma(1)$  by (2.6). Each term in brackets belongs to  $\widetilde{\text{Sp}}$  by (2.5) and (2.2). Now

$$\{\gamma(1)/\gamma(a)\}^2 = \gamma(1)^2 \gamma(-a)^2 = (a, -1)_F$$

by (2.10), so  $\gamma(\det A)/\gamma(1) = \pm(\det A, -1)_F^{1/2}$ . This proves the first assertion. One can deduce the last assertion by embedding  $\text{Sp}(2j, F)$  into  $\text{Sp}(2m, F)$  ( $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ , where  $A = \begin{pmatrix} a & \\ & I_{m-j} \end{pmatrix}$ ,  $B = \begin{pmatrix} b & \\ & 0_{m-j} \end{pmatrix}$ ,  $C = \begin{pmatrix} c & \\ & 0_{m-j} \end{pmatrix}$ ,  $D = \begin{pmatrix} d & \\ & I_{m-j} \end{pmatrix}$ ) and applying the argument of (2.4-5) to the image of  $\text{Sp}(2j, F)$ . □

### 3. Splitting of $K$ .

In this section we recall how to obtain the splitting of a maximal compact subgroup  $K$  of  $\text{Sp}$ . First let us discuss the case where  $F = \mathbb{R}$ . Let  $U_m = \{ Z \in M_m(\mathbb{C}) \mid Z\bar{Z}^T = I_m \}$ , and define  $u : U_m \rightarrow \text{Sp}$  by the formula

$$(3.1) \quad u(A + iB) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

It is well known that  $\text{Sp}$  acts on the Siegel upper half space, and by an easy calculation  $u(U_m)$  is the stabilizer of  $iI_m$ , hence it is a maximal compact subgroup of  $\text{Sp}$ . The nontrivial additive character  $\chi$  which appears in (1.2) and (1.3) has the form  $\chi(t) = e^{2\pi i \lambda t}$ , where  $\lambda \in \mathbb{R}^\times$ . Define  $\Phi_0 \in L^2(\mathbb{R}^m)$  by

$$\Phi_0(X_1, \dots, X_m) = e^{-\pi|\lambda| \sum X_j^2}.$$

**Lemma 3.** *Let  $F = \mathbb{R}$  and  $K = u(U_m)$ . There is a splitting homomorphism  $\mathbf{k} : K \rightarrow \text{Mp}$  such that  $\mathbf{k}(k)(\Phi_0) = \Phi_0$  for all  $k \in K$ .*

*Proof.* It is well known that  $\pi^{-1}(K)$  stabilizes  $\mathbb{C}^\times \Phi_0$ ; see [Ig, Ch. I, §9] or [B, Prop. 3.2(a)]. Let  $\tilde{\tau} \in \text{Mp}$  and  $\pi(\tilde{\tau}) = k \in K$ . Then  $\tilde{\tau}\Phi_0 = c\Phi_0$ , and  $c \in \mathbb{T}$  since  $\tilde{\tau}$  is a unitary operator. Thus one takes  $\mathbf{k}(k) = c^{-1}\tilde{\tau}$ .  $\square$

There is a similar result in the nonarchimedean case ([W, No. 19] or [Kz, Lemma 2]). Let  $F$  be a nonarchimedean field,  $\mathfrak{o}_F$  its ring of integers,  $\Lambda$  a lattice in  $F^m$ , and  $\Lambda'$  the dual lattice:

$$\Lambda' = \{ x \in F^m \mid \chi(x \cdot y) = 1 \quad \text{for all } y \in \Lambda \}.$$

Let  $K \subset \text{Sp}$  be the stabilizer of  $\Lambda \oplus \Lambda'$ :

$$K = \{ \sigma \in \text{Sp} \mid (x, y)\sigma \in \Lambda \oplus \Lambda' \quad \text{for all } x \in \Lambda, y \in \Lambda' \}.$$

Then  $K$  is a maximal compact subgroup of  $\text{Sp}$ . Since  $\Lambda = \mathfrak{o}_F^m \alpha$  for some  $\alpha \in \text{GL}_m(F)$ , the various  $K$  for different choices of  $\Lambda$  are conjugate to one another. If  $\Lambda = \mathfrak{o}_F^m$  and  $\mathfrak{a}_\chi = \mathfrak{o}_F$  (see (1.8)) then  $\Lambda' = \mathfrak{o}_F^m$  and  $K = \text{Sp}(2m, \mathfrak{o}_F)$ .

**Lemma 4.** *Let  $F$  be a nonarchimedean field such that  $2 \in \mathfrak{o}_F^\times$ . Let  $\Phi_0$  be the characteristic function of  $\Lambda$ . There is a splitting homomorphism  $\mathbf{k} : K \rightarrow \widetilde{\text{Sp}}$  such that  $\mathbf{k}(k)\Phi_0 = \Phi_0$  for all  $k \in K$ .*

*Proof.* This is proved in [Kz, Lemma 2]. (N.B. Kazhdan's article contains an error in the group law for the Heisenberg group, but the reasoning is correct in the case where  $2 \in \mathfrak{o}_F^\times$ .)  $\square$

**Lemma 5.** *Let  $F = \mathbb{R}$  or  $F$  as in the previous lemma. If  $p \in P \cap K$  then  $r_P(p) = \mathbf{k}(p)$ . Let  $\hat{P} = w_m^{-1} P w_m = \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp} \mid B = 0 \}$  and define  $r_{\hat{P}} : \hat{P} \rightarrow \text{Mp}$  by  $r_{\hat{P}}(w_m^{-1} p w_m) = \mathcal{F}^{-1} r_P(p) \mathcal{F}$ . If  $\hat{p} \in \hat{P} \cap K$  then  $r_{\hat{P}}(\hat{p}) = \mathbf{k}(\hat{p})$ .*

*Proof.* Suppose  $p \in P \cap K$ . We can write  $p = \underline{n}(B) \underline{a}(A)$ . From the definition of  $K$  it is clear that  $\det A$  is a unit. Write  $r_P(p) = c \mathbf{k}(k)$ , with  $c \in \mathbf{T}$ . Since  $\Phi_0(0) = 1$  and  $\mathbf{k}(k) \Phi_0 = \Phi_0$ , we know  $c = r_P(p) \Phi_0(0) = \mathbf{n}(B) \mathbf{a}(A) \Phi_0(0) = |\det A|^{1/2} = 1$ . Next suppose  $\hat{p} = w_m^{-1} p w_m \in \hat{P} \cap K$ . Let  $\Phi'_0 = \mathcal{F} \Phi_0$  and  $K' = w_m K w_m^{-1}$ . If  $F = \mathbb{R}$  then  $\Phi'_0 = \Phi_0$  and  $K' = K$ . If  $F$  is nonarchimedean then  $\Phi'_0$  is a constant multiple of the characteristic function of  $\Lambda'$  and  $K'$  is the stabilizer of  $\Lambda' \oplus \Lambda$ ; that is,  $\Phi'_0(0)^{-1} \Phi'_0, K'$  are defined like  $\Phi_0, K$  with the roles of  $\Lambda$  and  $\Lambda'$  reversed. Moreover  $p \in P \cap K'$ , hence  $r_P(p) \Phi'_0 = \Phi'_0$ . Thus  $r_{\hat{P}}(\hat{p}) \Phi_0 = \mathcal{F}^{-1} r_P(p) \Phi'_0 = \mathcal{F}^{-1} \Phi'_0 = \Phi_0$ , so that  $r_{\hat{P}}(\hat{p}) = \mathbf{k}(\hat{p})$ . □

**Lemma 6.** *If  $F$  is nonarchimedean and  $2 \in \mathfrak{o}_F^\times$  then  $(a, b)_F = 1$  for all  $a, b \in \mathfrak{o}_F^\times$ .*

*Proof.* This proof may be found in [W, Theorem 5]. Choose  $\chi$  such that  $\mathfrak{a}_\chi = \mathfrak{o}_F$  (see (1.8)). Let  $\Lambda = \mathfrak{o}_F^m$ ; thus  $K = \text{Sp}(2m, \mathfrak{o}_F)$ . Note that  $\mathbf{k}(w_m) = \mathcal{F}$ , since  $\mathcal{F} \Phi_0 = \Phi_0$ . Consider equation (2.5) with  $m = 1$  and  $B = b$ , where  $b \in \mathfrak{o}_F^\times$ . By Lemma 5, (2.5) says

$$\gamma(b) \mathbf{k}(\underline{a}(b^{-1})) \mathbf{k}(w_m) = \mathbf{k}(\underline{n}(b^{-1})) \mathbf{k}(w_m) \mathbf{k}(\underline{n}(b)) \mathbf{k}(w_m)^{-1} \mathbf{k}(\underline{n}(b^{-1})).$$

Since  $\mathbf{k}$  is a homomorphism,  $\gamma(b) = 1$ . Then  $(a, b)_F = 1$  by (2.10). □

**Lemma 7.** *If  $F$  is nonarchimedean and  $2 \in \mathfrak{o}_F^\times$  then  $\mathbf{k}(k) \in \widetilde{\text{Sp}}$  for all  $k \in K$ . If  $F = \mathbb{R}$  then  $\pm(\det Z)^{1/2} \mathbf{k}(u(Z_\lambda)) \in \widetilde{\text{Sp}}$  for all  $k \in K$ , where  $Z_\lambda = Z$  or  $\bar{Z}$  according as  $\lambda > 0$  or  $\lambda < 0$ .*

*Proof.* First assume  $F$  is nonarchimedean. It is well known that  $K$  is generated by  $K \cap P$  and  $K \cap \hat{P}$ , where  $\hat{P} = w_m^{-1} P w_m$ . Suppose  $p = \underline{n}(B) \underline{a}(A) \in P \cap K$ . Then  $\mathbf{k}(p) = r_P(p)$  by Lemma 5. Now  $\det A \in \mathfrak{o}_F^\times$ , so  $r_P(p) \in \text{Sp}$  by Lemma 6 and Prop. 2. Next suppose  $\hat{p} = w_m^{-1} p w_m \in \hat{P} \cap K$ . By Lemma 5,  $\mathbf{k}(\hat{p}) = r_{\hat{P}}(\hat{p})$ . Now  $r_P(p) \in \widetilde{\text{Sp}}$  by the same argument as above. (Note that  $p \in P \cap K'$  in the proof of Lemma 5.) Since  $\widetilde{\text{Sp}}$  is normal in  $\text{Mp}$ ,  $r_{\hat{P}}(\hat{p}) \in \widetilde{\text{Sp}}$  also.

Next assume  $F = \mathbb{R}$ . In [R2, Th. 4.1] a homomorphism  $\tilde{\sigma} \mapsto r(\tilde{\sigma})$  from the universal cover of  $\text{Sp}$  into  $\text{Mp}$  is constructed. From the construction it is easy to see that for each  $\sigma \in \text{Sp}$ , the set  $\{ r(\tilde{\sigma}) \mid \pi(\tilde{\sigma}) = \sigma \}$  has cardinality two. By Lemma 1 it follows that the image of  $r$  is  $\widetilde{\text{Sp}}$ . When  $\lambda = 1$ , the result now follows from [R2, Prop. 4.2]. Let us add a subscript  $\chi$  to remind

ourselves that the definition of  $\mathbf{k}$  depends on  $\chi$ . From (1.7) and the definition of  $\Phi_0$  it is not hard to deduce that  $\mathbf{k}_\chi(u(Z)) = \mathbf{k}_{\lambda\chi}(u(Z_\lambda))$  for all  $\lambda \in \mathbb{R}^\times$ ,  $Z \in U_m$ . The result follows.  $\square$

#### 4. Proof of the Main Theorem.

In this section we will prove the Main Theorem, which was stated in the introduction, and give an application. Let  $F = \mathbb{R}$  or  $F$  = a self-dual nonarchimedean field such that 2 is a unit in  $\mathfrak{o}_F$ . Let  $\chi(t) = e^{2\pi it}$  if  $F = \mathbb{R}$ , and let  $\chi(t)$  be an additive character of  $F$  such that  $\mathfrak{a}_\chi = \mathfrak{o}_F$  (see (1.8)) if  $F$  is nonarchimedean. Let  $K = u(U_m)$  if  $F = \mathbb{R}$  (see (3.1)) and  $K = \text{Sp}(2m, \mathfrak{o}_F)$  if  $F$  is nonarchimedean. For each  $k \in K$  we will define a number  $x_0(k) \in \mathbf{T}$ , which will turn out to be the same as the Gauss sum  $x_0(k)$  that appears in the statement of the Main Theorem. Moreover  $x_0(k)^2$  can easily be evaluated using results of §§2-3.

Let us explain the definition of  $x_0(k)$ . By the Bruhat decomposition ([R1, Lemma 2.14]), every element  $g \in \text{Sp}$  can be written in the form  $g = p_1 w_j p_2$  for some  $p_1, p_2 \in P$  and  $0 \leq j \leq m$ . Consider the *standard section*  $r_0 : \text{Sp} \rightarrow \text{Mp}$  given by

$$(4.1) \quad r_0(p_1 w_j p_2) = r_p(p_1) \mathcal{F}_j r_P(p_2).$$

This is well-defined by virtue of [R1, Th. 3.5(3)]. Define  $x_0(k) \in \mathbf{T}$  for  $k \in K$  by the formula

$$(4.2) \quad r_0(k) = x_0(k) \mathbf{k}(k).$$

Suppose  $y(k), y_0(g)$  are elements of  $\mathbf{T}$  such that  $y(k) \mathbf{k}(k) \in \widetilde{\text{Sp}}$  and  $y_0(g) r_0(g) \in \widetilde{\text{Sp}}$ . Then  $x_0(k) y_0(k) y(k)^{-1} \in \widetilde{\text{Sp}} \cap \mathbf{T} = \{ \pm 1 \}$ , thus

$$(4.3) \quad x_0(k)^2 = y(k)^2 y_0(k)^{-2}.$$

By virtue of Lemma 7, Prop. 2, and (2.8), we can take (for  $g = \begin{pmatrix} A_1 & * \\ 0 & * \end{pmatrix} w_j \begin{pmatrix} A_2 & * \\ 0 & * \end{pmatrix} \in \text{Sp}$  and  $k \in K$ )

$$(4.4) \quad y(k) = \det(u^{-1}(k))^{1/2} \quad \text{and} \quad y_0(g) = e^{2\pi i j/8} (\text{sign}(\det A_1 A_2))^{1/2} \quad \text{if } F = \mathbb{R},$$

$$(4.5) \quad y(k) = 1 \quad \text{and} \quad y_0(g) = (\det A_1 A_2, -1)_F^{1/2} \quad \text{if } F \text{ is nonarchimedean.}$$

Now we begin the computation of  $x_0(k)$ . Suppose  $k = p_1 w_j p_2$  is the Bruhat decomposition of  $k$  and  $p_1 = \underline{n}(B_1) \underline{a}(A_1)$ . Note that  $j$  is the rank of the bottom left  $m \times m$  minor of  $k$ . Then

$$(4.6) \quad x_0(k) = (r_0(k)\Phi_0)(0) = |\det A_1|^{1/2} \int_{F^j} (r_P(p_2)\Phi_0)(y, 0) dy.$$

Thus we need to calculate  $A_1$  and  $p_2$ . This is accomplished in the next two lemmas.

**Lemma 8.** *Given  $C \in M_m(F)$  of rank  $j$ , there exist  $\pi_1, \pi_2 \in \text{SO}(m, F)$  with all entries 0, 1, or  $-1$  such that the top left  $j \times j$  minor of  $\pi_1 C \pi_2$  is invertible.*

*Proof.* One can permute columns to make the first  $j$  columns of  $C$  linearly independent, then one can exchange rows to make the top left  $j \times j$  minor of  $C$  invertible. The resulting matrix is  $\pi'_1 C \pi'_2$ , where  $\pi'_1$  and  $\pi'_2$  are permutation matrices. One can always arrange for an even number of row exchanges (to make  $\det \pi'_i = 1$ ) unless  $m = 2, j = 1$ . When  $m = 2$ , one could choose  $\pi_i = I_2$  or  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . □

**Lemma 9.** *Suppose  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2m, F)$ , the rank of  $C$  is  $j$ , and the top left  $j \times j$  minor of  $C$  is invertible. Write*

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} \quad D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$$

*in block form, where  $C_1, D_1$  are  $j \times j$  matrices and  $C_4, D_4$  are  $(m-j) \times (m-j)$  matrices. Then  $g = p_1 w_j p_2$ , where  $w_j = \pi(\mathcal{F}_j)$  is the matrix of (1.5) and  $p_1, p_2 \in P$  have the form*

$$p_1 = \begin{pmatrix} (\alpha'^T)^{-1} & * \\ & \alpha' \end{pmatrix}, \quad \alpha' = \begin{pmatrix} -C_1 & D_2 \\ -C_3 & D_4 \end{pmatrix},$$

$$p_2 = \begin{pmatrix} I_m & B' \\ & I_m \end{pmatrix} \begin{pmatrix} \alpha & \\ & (\alpha^T)^{-1} \end{pmatrix}, \quad \alpha = \begin{pmatrix} I_j & C_1^{-1} C_2 \\ & I_{m-j} \end{pmatrix},$$

$$B' = \begin{pmatrix} \beta & \\ & 0_{m-j} \end{pmatrix}, \quad \beta = C_1^{-1}(D_1 + D_2 C_2^T C_1^{-T}) \in M_j(F).$$

*(In particular, we are asserting that  $\alpha'$  is invertible and  $\beta$  is symmetric, so that  $p_1, p_2$  really belong to  $P$ .)*

*Proof.* First,  $DC^T$  is symmetric since  $g \in \text{Sp}$ , thus  $D_1 C_1^T + D_2 C_2^T$  is symmetric. This implies  $\beta = \beta^T$ , so  $p_2 \in P$ . Now  $C\alpha^{-1} = \begin{pmatrix} C_1 & 0 \\ C_3 & * \end{pmatrix}$ . Since

$\text{rank}(C\alpha^{-1}) = \text{rank}(C) = \text{rank}(C_1)$ , the \* is really 0. Thus

$$gp_2^{-1} = \begin{pmatrix} A\alpha^{-1} & * \\ \begin{pmatrix} C_1 & 0 \\ C_3 & 0 \end{pmatrix} & D' \end{pmatrix}, \quad D' = \begin{pmatrix} -C_1\beta & 0 \\ -C_3\beta & 0 \end{pmatrix} + D\alpha^T.$$

By direct calculation,  $D' = \begin{pmatrix} 0 & D_2 \\ D'_3 & D_4 \end{pmatrix}$  for some matrix  $D'_3$ . Since  $gp_2^{-1} \in \text{Sp}$ , we know  $\begin{pmatrix} C_1 & 0 \\ C_3 & 0 \end{pmatrix} D'^T = \begin{pmatrix} 0 & C_1 D_3^T \\ 0 & * \end{pmatrix}$  is symmetric; since  $C_1$  is invertible, this forces  $D'_3 = 0$ . Thus the bottom  $m$  rows of  $gp_2^{-1}$  are  $\begin{pmatrix} C_1 & 0 & 0 & D_2 \\ C_3 & 0 & 0 & D_4 \end{pmatrix}$ . These rows must be linearly independent, so  $\alpha'$  is invertible. Moreover, the bottom  $m$  rows of  $\text{diag}\{\alpha'^T, \alpha'^{-1}\}gp_2^{-1}$  are the same as the bottom  $m$  rows of  $w_j$ . Since this matrix is symplectic, it is easy to check from the relations  $A^T C = C^T A$ ,  $B^T D = D^T B$ ,  $A^T D - C^T B = I$  for all  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}$  that the top  $m$  rows of  $\text{diag}\{\alpha'^T, \alpha'^{-1}\}gp_2^{-1}$  must have the form

$$\begin{pmatrix} A'_1 & 0 & I_j & -A_3'^T \\ A'_3 & I_{m-j} & 0 & B'_4 \end{pmatrix}$$

with  $A'_1$  and  $B'_4$  symmetric. (Here the matrices in the top row have height  $j$ , the matrices in the first and third columns have width  $j$ .) Thus

$$\begin{pmatrix} I & E \\ 0 & I \end{pmatrix} \text{diag}\{\alpha'^T, \alpha'^{-1}\}gp_2^{-1} = w_j, \quad \text{where } E = \begin{pmatrix} A'_1 & A_3'^T \\ A'_3 & -B'_4 \end{pmatrix}.$$

This completes the proof. □

*Proof of Main Theorem.* Define  $x_0(k)$  as in (4.2). Let  $k = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in K - P$  and choose  $\pi_1, \pi_2$  as in Lemma 8. Let  $\alpha_i = \underline{a}(\pi_i)$  for  $i = 1, 2$  and  $k' = \alpha_1 k \alpha_2$ . Then  $x_0(k) = x_0(k')$  because

$$(4.7) \quad r_0(k') = \mathbf{a}(\pi_1) r_0(k) \mathbf{a}(\pi_2) = \mathbf{k}(\alpha_1) x_0(k) \mathbf{k}(k) \mathbf{k}(\alpha_2) = x_0(k) \mathbf{k}(k').$$

Define  $C_1, \dots, C_4$  and  $D_1, \dots, D_4$  by (1.9). Now apply Lemma 9 to the matrix  $k'$ . By (4.6) and Lemma 9, the matrix  $\alpha' = \begin{pmatrix} -C_1 & D_2 \\ -C_3 & D_4 \end{pmatrix}$  is invertible, and in the notation of Lemma 9,

$$\begin{aligned} x_0(k') &= |\det(\alpha')|^{-1/2} \int_{F^j} (\mathbf{n}(B') \mathbf{a}(\alpha) \Phi_0)(y, 0) dy \\ &= |\det(\alpha')|^{-1/2} \int_{F^j} \chi(y\beta y^T/2) \Phi_0(y, yC_1^{-1}C_2) dy. \end{aligned}$$

By changing variables  $y \mapsto yC_1$  we obtain (1.10). Finally,  $x_0(k)^2$  can be computed from (4.3), (4.4), and (4.5). □

**Example.** For  $F = \mathbb{Q}_p$  ( $p \geq 3$ ) and  $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}(2, \mathbb{Z}_p)$ , we have  $x_0(k) = 1$  if  $c = 0$ , and otherwise

$$x_0(k) = |c|^{1/2} \int_{c^{-1}\mathbb{Z}_p} \chi(dcy^2/2) dy = |c|^{-1/2} \int_{\mathbb{Z}_p} \chi(dc^{-1}y^2/2) dy.$$

If  $c$  is a unit, this integral is one. If  $c$  is not a unit, then  $\int_{p^{-\nu}\mathbb{Z}_p} \chi(dc^{-1}y^2/2) dy = 0$  for all  $\nu > 0$  by a standard argument in the theory of Gauss sums, so  $x_0(k) = \gamma(d/c)$  by (2.9). In summary, for  $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}(2, \mathbb{Z}_p)$ ,

$$x_0(k) = \begin{cases} \gamma(c/d) & \text{if } c \neq 0 \text{ and } p|c \\ 1 & \text{if } c = 0 \text{ or } c \in \mathbb{Z}_p^\times. \end{cases}$$

**Proposition 10.** Let  $F = \mathbb{R}$ ,  $\chi(t) = e^{2\pi it}$ . There is a section  $r_+ : \text{Sp}(2m, \mathbb{R}) \rightarrow \widetilde{\text{Sp}}(2m, \mathbb{R})$  such that

- a)  $r_+(p_1gp_2) = r_P(p_1)r_+(g)r_P(p_2)$  whenever  $p_1, p_2 \in P_+ = \{ \begin{pmatrix} A & * \\ 0 & * \end{pmatrix} \in P \mid \det(A) > 0 \}$ ; and
- b) If  $k = u(\text{diag}\{e^{i\theta_1}, \dots, e^{i\theta_m}\})$  ( $-\pi < \theta_j \leq \pi$ ) then

$$(4.8) \quad r_+(k) = e^{-\delta\pi i/4} (e^{\delta\pi i/2})^{1/2} e^{i\theta/2} \mathbf{k}(k),$$

where  $\theta = \sum \theta_j$ ,  $\delta = \sum \delta_j$ , and  $\delta_j = -1, 0, 1, 2$  according as  $\theta_j \in (-\pi, 0)$ ,  $\theta_j = 0$ ,  $\theta_j \in (0, \pi)$ , or  $\theta_j = \pi$ . Define a cocycle  $c_+ : \text{Sp} \times \text{Sp} \rightarrow \{\pm 1\}$  by the formula  $r_+(g_1)r_+(g_2) = c_+(g_1, g_2)r_+(g_1g_2)$  for  $g_1, g_2 \in \text{Sp}$ . If  $m = 1$  then  $c_+$  coincides with the Kubota cocycle.

*Proof.* Let  $W_+ = \{w_j, w_-w_j \mid j = 0, \dots, m\}$ , where  $w_- = \underline{a} \begin{pmatrix} -1 & \\ & I_{m-1} \end{pmatrix}$ . It can easily be seen from the Bruhat decomposition that  $\text{Sp} = \sqcup_{w \in W_+} P_+ w P_+$ . Let  $\varepsilon : W_+ \rightarrow \text{Mp}$  be the restriction to  $W_+$  of  $k \mapsto \det(u^{-1}(k))^{1/2} \mathbf{k}(k)$ , where we choose the branch of the square root function which has argument in  $(-\pi/2, \pi/2]$ , and define  $r_+ : \text{Sp} \rightarrow \widetilde{\text{Sp}}$  by

$$r_+(p_1wp_2) = r_P(p_1)\varepsilon(w)r_P(p_2) \quad \text{for } w \in W_+ \text{ and } p_1, p_2 \in P_+.$$

This section is well-defined by [R1, Theorem 3.5(3)], and it takes values in  $\widetilde{\text{Sp}}$  by Prop. 2 and Lemma 7. Clearly (a) is satisfied. Now we prove (b). Define  $\nu : \text{Sp} \rightarrow \mathbf{T}$  by  $r_+(g) = \nu(g)r_0(g)$ , where  $r_0(g)$  is the standard section (4.1). Note that  $\nu(w) = (\det(u^{-1}(w)))^{1/2}$  for  $w \in W_+$  because  $r_0(w)\Phi_0 = \Phi_0$ . By (4.2),

$$r_+(k) = \nu(k)x_0(k)\mathbf{k}(k).$$

Thus we need to show that for  $k$  as in part (b),

$$(4.9) \quad \nu(k)x_0(k) = e^{-\delta\pi i/4} (e^{\delta\pi i/2})^{1/2} e^{i\theta/2}.$$

If  $A_1, A_2 \in \text{SO}_m(\mathbb{R})$  and  $k' = \underline{a}(A_1) k \underline{a}(A_2)$  then  $\nu(k') = \nu(k)$  because  $r_0$  and  $r_+$  are constant on double cosets  $P_+ w P_+$ , and  $x_0(k') = x_0(k)$  by (4.7). Thus the left side of (4.9) does not change if the  $\theta_j$  are permuted or if an even number of the  $\theta_j$  are shifted by  $\pi$ . The right side is also invariant under such manipulations of the  $\theta_j$ , because  $\theta - \delta\pi/2$  and  $e^{\delta\pi i/2}$  are unchanged. Thus we are reduced to the case where  $\sin\theta_\ell \neq 0$  for  $1 \leq \ell \leq j$ ,  $\sin\theta_\ell = 0$  for  $\ell > j$ , and  $\theta_\ell \in [0, \pi)$  for all  $\ell \geq 2$ .

If  $j = 0$  then  $k = I$  or  $k = w_-$  according as  $\theta_1 = 0$  or  $\theta_1 = \pi$ . If  $k = I$ , both sides of (4.9) are 1. If  $k = w_-$  then  $\delta = 2$ ,  $\theta = \pi$ ,  $x_0(k) = 1$ ,  $\nu(k) = i$ , and again both sides are equal. Now consider the case  $j \geq 1$ . Then  $\sin\theta_1 \neq 0$ , so  $\delta_1 = \pm 1$ ,  $\delta_2 = \dots = \delta_j = 1$ , and  $\delta_\ell = 0$  for all  $\ell > j$ . Let  $w = w_j$  or  $w_- w_j$  according as  $\delta_1 = 1$  or  $\delta_1 = -1$ . Then  $k \in P_+ w P_+$ , so  $\nu(k) = (\delta_1 i^j)^{1/2} = (e^{\delta\pi i/2})^{1/2}$ . It only remains to show  $x_0(k) = e^{-\delta\pi i/4} e^{i\theta/2}$ . By (1.10),

$$x_0(k) = \prod_{\ell=1}^j |\sin\theta_\ell|^{-1/2} \int_{\mathbb{R}} e^{-\pi i y^2 \sin\theta_\ell \cos\theta_\ell} e^{-\pi y^2 \sin^2\theta_\ell} dy.$$

By [Ig, Ch. I, Lemma 1], it can easily be verified that each term in the product has positive real part, and that up to a positive constant coming from the self-dual Haar measure, its square is  $|\sin\theta_\ell|^{-1} (1 + i \cot\theta_\ell)^{-1} = e^{i\theta_\ell - (\pi i \delta_\ell/2)}$ . Since  $x_0(k) \in \mathbf{T}$  a priori, the positive constant is one. Note that  $\theta_\ell - (\pi \delta_\ell/2) \in (-\pi/2, \pi/2)$  for all  $\ell \leq j$ , therefore  $\text{Re}(e^{i\theta_\ell/2} e^{-\delta_\ell \pi i/4}) > 0$ . This completes the proof of (b).

Finally we want to show that if  $m = 1$  then  $c_+$  coincides with the Kubota cocycle  $c_K$  which is defined in [K]. We recall the definition of  $c_K$ : if  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$  let  $x(g) = c$  or  $d$  according as  $c \neq 0$  or  $c = 0$ . In terms of the Hilbert symbol which was defined after (1.9),

$$c_K(g_1, g_2) = (x(g_1), x(g_2))_{\mathbb{R}} (-x(g_1) x(g_2), x(g_1 g_2))_{\mathbb{R}}.$$

We can reduce the proof of (c) to the case where  $g_i \in K$  by the following argument, which was suggested by the referee. It is clear that  $x(pg)$ ,  $x(gp)$ , and  $x(g)$  have the same sign if  $p \in P_+$ , thus  $c_K(pg_1, g_2) = c_K(g_1, g_2 p) = c_K(g_1, g_2)$ . Moreover  $c_K(g, p) = 1 = c_K(p, g)$  because  $(x(g), 1)_{\mathbb{R}} = (-x(g), x(g))_{\mathbb{R}} = 1$ . Obviously  $c_+$  satisfies the same relations because of the property (a) of  $r_+$  and because  $r_+(I_{2m}) = 1$ . Also we know the cocycle relation

$$c(x, y) c(xy, z) = c(y, z) c(x, yz)$$

for  $c = c_+$  or  $c = c_K$ , where  $x, y, z \in \text{SL}_2(\mathbb{R})$ . For arbitrary  $g_1$  and  $g_2$  write  $g_1 = p_1 k_1$ ,  $g_2 = p_2 k_2$ ,  $g_3 = g_1 g_2 = p_1 p_3 k_3$  with  $p_i \in P_+$  and  $k_i \in K$ . Take

$x = k_1, y = p_2, z = k_2$  in the cocycle relation. Then  $xy = k_1 p_2 = p_3 k_3 k_2^{-1}$ , hence

$$(4.10) \quad c(g_1, g_2) = c(k_1, p_2 k_2) = c(p_3 k_3 k_2^{-1}, k_2) c(k_1, p_2) c(p_2, k_2)^{-1} = c(k_3 k_2^{-1}, k_2).$$

Thus it suffices to show  $c_+$  and  $c_K$  coincide on  $K \times K$ . Let  $g_j = u(e^{i\theta_j})$  for  $j = 1, 2, 3$ , where  $\theta_j \in (-\pi, \pi]$  and  $e^{i(\theta_1+\theta_2)} = e^{i\theta_3}$ , and let  $x_j = 1$  or  $-1$  according as  $\theta_j \in (-\pi, 0]$  or  $\theta_j \in (0, \pi]$ . Then  $c_+(k_1, k_2) = e^{i(\theta_1+\theta_2)/2} / e^{i\theta_3/2}$ , and this is equal to  $-1$  iff  $x_1 = x_2 = -x_3$ . Since  $x_j \equiv x(g_j) \pmod{\mathbb{R}_+}$ , it can easily be seen that the two cocycles agree.  $\square$

### 5. Embedding $\widetilde{\text{Sp}}(2n, \mathbb{R})$ into $\widetilde{\text{Sp}}(2m, \mathbb{R})$ .

In this section let  $F = \mathbb{R}$ ,  $\chi(t) = e^{2\pi it}$ . Let  $P_+ = \{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \text{Sp} \mid \det A > 0 \}$ . We will prove the following proposition.

**Proposition 11.** *Consider a continuous homomorphism  $j : \text{Sp}(2n, \mathbb{R}) \rightarrow \text{Sp}(2m, \mathbb{R})$  such that  $j(P_+^{(n)}) \subset P_+^{(m)}$  and  $j(K^{(n)}) \subset K^{(m)}$ , where for clarity we have superscripted our symbols with the degree of the underlying symplectic space. Let  $\psi = \det \circ u^{-1} : K \rightarrow \mathbf{T}$ . There is an integer  $N$  such that*

$$(5.1) \quad (\psi^{(m)} \circ j)(k) = (\psi^{(n)}(k))^N \quad \text{for } k \in K^{(n)}.$$

Define  $\tilde{j} : \text{Mp}(\mathbb{R}^n) \rightarrow \text{Mp}(\mathbb{R}^m)$  by

$$(5.2) \quad \tilde{j}(\zeta r_P(p) \mathbf{k}(k)) = \zeta^N r_P(j(p)) \mathbf{k}(j(k)) \quad \text{for } \zeta \in \mathbf{T}, p \in P_+^{(n)}, k \in K^{(n)}.$$

Then  $\tilde{j}$  is a continuous homomorphism,  $\pi \circ \tilde{j} = j \circ \pi$ , and  $\tilde{j}|_{\widetilde{\text{Sp}}(2n, \mathbb{R})}$  takes values in  $\widetilde{\text{Sp}}(2m, \mathbb{R})$ . Moreover,  $\tilde{j}|_{\widetilde{\text{Sp}}}$  is the only continuous function from  $\widetilde{\text{Sp}}(2n, \mathbb{R})$  into  $\widetilde{\text{Sp}}(2m, \mathbb{R})$  such that  $\pi \circ \tilde{j} = j \circ \pi$  and  $\tilde{j}(1) = 1$ .

If one takes  $n = 1$  and  $j \left( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) = \begin{pmatrix} \alpha I & \beta T \\ \gamma T^{-1} & \delta I \end{pmatrix}$ , where  $T$  is a symmetric invertible  $m \times m$  real matrix, and if one identifies  $\widetilde{\text{Sp}}(2, \mathbb{R})$  with  $\text{Sp} \times \{\pm 1\}$  via the bijection  $\delta r_+(g) \mapsto (g, \delta)$  for  $r_+$  as given in Prop. 10 and  $\delta \in \{\pm 1\}$ , then the formula (6.2) is a concise statement of [RS, Th. 1.2].

*Proof.* Let  $\tilde{j}$  be as in (5.2),  $\widetilde{K} = \widetilde{\text{Sp}} \cap \pi^{-1}(K) = \{ \pm \psi(k)^{1/2} \mathbf{k}(k) \mid k \in K \}$  (by Lemma 7). If  $\zeta \mathbf{k}(k) \in \widetilde{K}^{(n)}$  then  $\tilde{j}(\zeta \mathbf{k}(k)) = \zeta^N \mathbf{k}(j(k)) \in \widetilde{\text{Sp}}(2m, F)$  by (5.1). (An integer  $N$  exists as in (5.1) because any element of  $\text{Hom}(U_n, \mathbf{T})$

is a power of determinant.) It is clear that  $\tilde{j}$  is well-defined, continuous, and  $\pi \circ \tilde{j} = j \circ \pi$ , thus we have a commutative diagram

$$\begin{CD} \widetilde{\mathrm{Sp}}(2n, \mathbb{R}) @>\tilde{j}>> \widetilde{\mathrm{Sp}}(2m, \mathbb{R}) \\ @V\pi VV @VV\pi V \\ \mathrm{Sp}(2n, \mathbb{R}) @>j>> \mathrm{Sp}(2m, \mathbb{R}). \end{CD}$$

We still need to show  $\tilde{j}$  is a homomorphism. Let

$$U = \{pk \in \mathrm{Sp} \mid p \in P_+, \arg \psi(k) \in [-\pi/2, \pi/2]\}$$

and  $\phi : U \rightarrow \widetilde{\mathrm{Sp}}(2n, \mathbb{R})$  the homeomorphism:  $\phi(pk) = \psi(k)^{1/2} r_P(p) \mathbf{k}(k)$  (where  $\arg \psi(k)^{1/2} \in [-\pi/4, \pi/4]$ ). Since  $\phi(U)$  generates  $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ , it will suffice to show

$$\tilde{j}(\tilde{u}\tilde{g}) = \tilde{j}(\tilde{u})\tilde{j}(\tilde{g})$$

for all  $\tilde{u} = \phi(u) \in \phi(U)$ ,  $\tilde{g} \in \widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ . Both sides lie in the fiber of  $\widetilde{\mathrm{Sp}}(2m, \mathbb{R})$  over the point  $j(u)j(g) \in \mathrm{Sp}(2m, \mathbb{R})$  ( $g = \pi(\tilde{g})$ ), so the equality holds up to  $\pm 1$ . Let  $F : U \rightarrow \{\pm 1\}$ ,  $F(u) = \tilde{j}(\tilde{u}\tilde{g})\tilde{j}(\tilde{g})^{-1}\tilde{j}(\tilde{u})^{-1}$ , where  $\tilde{g} \in \widetilde{\mathrm{Sp}}$  is fixed. Then  $F$  is continuous and  $F(I_{2m}) = 1$ . Since  $U$  is connected,  $F$  is identically one. This proves  $\tilde{j}$  is a homomorphism. The characterization of  $\tilde{j}$  as the only continuous lift of  $j$  taking 1 to 1 follows from an elementary fact about covering projections ([Sp, Ch. 2, Sec. 2, Th. 2]).

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