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**FROM THE L^1 NORMS OF THE COMPLEX HEAT KERNELS
TO A HÖRMANDER MULTIPLIER THEOREM FOR
SUB-LAPLACIANS ON NILPOTENT LIE GROUPS**

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This paper aims to prove a Hörmander multiplier theorem for sub-Laplacians on nilpotent Lie groups. We investigate the holomorphic functional calculus of the sub-Laplacians, then we link the L^1 norm of the complex time heat kernels with the order of differentiability needed in the Hörmander multiplier theorem. As applications, we show that order $d/2 + 1$ suffices for homogeneous nilpotent groups of homogeneous dimension d , while for generalised Heisenberg groups with underlying space \mathbf{R}^{2n+k} and homogeneous dimension $2n + 2k$, we show that order $n + (k + 5)/2$ for k odd and $n + 3 + k/2$ for k even is enough; this is strictly less than half of the homogeneous dimension when k is sufficiently large.

1. Introduction.

We begin with the classical Laplacian $(-\Delta)$ on the Euclidean space \mathbf{R}^d . The multiplier theorem of L. Hörmander [Ho] gives a sufficient condition on a function $m : \mathbf{R}^+ \rightarrow \mathbf{C}$ for the operator $m(-\Delta)$ to be bounded on $L^p(\mathbf{R}^d)$ whenever $1 < p < \infty$, namely, when m satisfies the condition that

$$(1) \quad \lambda^k \left| m^{(k)}(\lambda) \right| \leq c \quad \forall \lambda \in \mathbf{R}^+$$

for $0 \leq k \leq s = [d/2] + 1$ where c is a constant and $[d/2]$ is the integral part of $d/2$.

By using fractional differentiation, the value of s in condition (1) can be improved slightly but it is known that for $(-\Delta)$ on \mathbf{R}^d , the value cannot be improved beyond $s = d/2$. We call s the order of the Hörmander multiplier theorem.

A lot of work has been done to obtain results of this type for other operators. E.M. Stein [St] proved a general result for a large class of operators, but only when the function m is of Laplace transform type, a rather restrictive condition. This was later improved by M. Cowling [Co], using the transference method and interpolation. For the sub-Laplacian L on a homogeneous nilpotent Lie group G of homogeneous dimension d , the following

results are known. A. Hulanicki and Stein proved a Hörmander multiplier theorem for L with order $3d/2 + 2$ when G is a stratified group; this was reported by G.B. Folland and Stein [FS]. L. De Michele and G. Mauceri [DM] improved Hulanicki and Stein's results and obtained order $d/2 + 1$. Recently, M. Christ [Ch] investigated the problem carefully, and proved a Hörmander multiplier theorem with order $d/2$ when G is a homogeneous nilpotent group. His principal result was then reproved and extended by Mauceri and S. Meda [MM].

All the above results rely on certain estimates on the heat kernels, L^2 information derived from the spectral theorem, and the Calderón–Zygmund operator theory. However, the factors controlling the order s were to some extent hidden by the complexity of the proofs.

One open question is whether the condition $s \geq d/2$ is necessary as in the Euclidean case [Ch]. Another natural question is to decide what factors control the order s . It seemed that $s = d/2$ is the optimal value [Ch], but recently D. Müller and Stein (conference announcement) showed that for the Heisenberg group of homogeneous dimension $2n + 2$, the order can be lowered to $n + 1/2$.

In this paper, we show that the order s is controlled by the behaviour of the $L^1(G)$ norm of the heat kernels for complex time (Theorem 2). As a corollary, we obtain $s = d/2 + 1$ for homogeneous nilpotent groups (Theorem 3). Although this order is not optimal, our proof is different from and much easier than the previous proofs. Further, if G is the generalised Heisenberg group of homogeneous dimension $2n + 2k$, with underlying manifold \mathbf{R}^{2n+2k} , then we obtain a Hörmander multiplier theorem with order $s = n + k/2 + \beta$ where $\beta = 5/2$ for k odd and $\beta = 3$ for k even (Theorem 4). This order is strictly less than half the homogeneous dimension when k is sufficiently large.

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2. H_∞ functional calculus.

The references for this section are the papers of A. McIntosh [Mc] and Cowling, I. Doust, McIntosh, and A. Yagi [CDMY].

Definition. A closed operator L in a Banach space X is said to be of type ω , $0 \leq \omega < \pi$, if its spectrum is a subset of the closed sector $S_\omega = \{z \in C \mid |\arg z| \leq \omega\} \cup \{0\}$, and the resolvents $(L - \lambda I)^{-1}$ satisfy the inequality

$$\|(L - \lambda I)^{-1}\| \leq c_\mu |\lambda|^{-1}$$

when $|\arg \lambda| \geq \mu > \omega$.

For $\mu > \omega$, let $H_\infty(S_\mu^0)$ be the usual space of bounded holomorphic functions in the open sector S_μ^0 , which is just the interior of S_μ . Further, define

$$\Psi(S_\mu^0) = \left\{ m \in H_\infty(S_\mu^0) \mid \exists s > 0, c > 0 \text{ such that } |m(z)| \leq \frac{c|z|^s}{1 + |z|^{2s}} \right\}.$$

Suppose that $\omega < \theta < \mu$. Let γ be the contour defined by the function

$$\gamma(t) = \begin{cases} te^{i\theta} & \text{if } 0 \leq t < \infty \\ -te^{-i\theta} & \text{if } -\infty < t \leq 0. \end{cases}$$

We adopt the definitions of H_∞ functional calculus of [Mc], as follows. For $m \in \Psi(S_\mu^0)$, then

$$m(L) = \frac{1}{2\pi i} \int_\gamma (L - \lambda I)^{-1} m(\lambda) d\lambda.$$

The above integral is absolutely convergent in the norm topology and $m(L)$ is a bounded linear operator which is independent of the choice of θ . For $m \in H_\infty(S_\mu^0)$, we define

$$m(L) = \frac{1}{2\pi i} (I + L)^2 L^{-1} \int_\gamma (L - \lambda I)^{-1} \frac{\lambda m(\lambda)}{(1 + \lambda)^2} d\lambda$$

when L is a one to one operator of type ω with dense domain and dense range. This definition is consistent with the previous one when $m \in \Psi(S_\mu^0)$.

We now define $\Lambda_{\infty,1}^\alpha(\mathbf{R}^+)$ to be the class of all bounded measurable functions $m : \mathbf{R}^+ \rightarrow \mathbf{C}$ such that $\|m\|_{\Lambda_{\infty,1}^\alpha} < \infty$, where

$$\|m\|_{\Lambda_{\infty,1}^\alpha} = \|m\|_\infty + \sum_{n \in \mathbf{Z}} 2^{|n|\alpha} \|(m \circ \exp) * \phi_n^\vee\|_\infty;$$

in this definition, for all ξ in \mathbf{R} ,

$$\begin{aligned} \phi_0(\xi) &= (2 - 2|\xi|)_+ - (1 - 2|\xi|)_+, \\ \phi_1(\xi) &= (1 - 2|\xi - 1|)_+ + \left(\frac{1}{2} - \left|\xi - \frac{3}{2}\right|\right)_+, \end{aligned}$$

and

$$\phi_{n\epsilon}(\xi) = \phi_1(2^{-n}\epsilon\xi),$$

when $n = 1, 2, 3, \dots$ and $\epsilon = \pm 1$; here ϕ^\vee denotes the inverse Fourier transform of ϕ . It is not hard to check, using Fourier analysis, that if condition

(1) holds when $k = 0, 1, 2, \dots, s$, then $m \in \Lambda_{\infty,1}^\alpha(\mathbf{R}^+)$ when $\alpha < s$. It was observed by Coifman that there were similarities between having functional calculus for bounded analytic functions in all sectors and Hörmander-type theorems. The following theorem is proved in [CDMY] (Theorem 4.10).

Theorem 1. *Suppose that L is a one-one operator of type 0 in $L^p(X)$, $1 < p < \infty$. Then the following conditions are equivalent:*

(i) *L admits a bounded $H_\infty(S_\mu^0)$ functional calculus for all positive μ and there exist positive constants C and α such that*

$$(2) \quad \|m(L)\| \leq C\mu^{-\alpha} \|m\|_{H_\infty(S_\mu^0)} \quad \forall m \in H_\infty(S_\mu) \quad \forall \mu > 0;$$

(ii) *L admits a bounded $\Lambda_{\infty,1}^\alpha(\mathbf{R}^+)$ functional calculus.*

In this paper, we prove that the H_∞ functional calculus of the sub-Laplacian on a homogeneous nilpotent group satisfies (2), hence there is a Hörmander type functional calculus. Note that to establish the existence of the H_∞ functional calculus, we just need to prove (2) for m in $\Psi(S_\mu)$, for the extension to m in $H_\infty(S_\mu)$ then follows from the Convergence Lemma in [CDMY] (Lemma 2.1).

In the rest of this paper, the constants C and c may vary from line to line.

3. The L^1 norms of the heat kernels and the Hörmander multiplier theorem.

Let \mathfrak{g} be a finite dimensional nilpotent Lie algebra. Assume that

$$\mathfrak{g} = \bigoplus_{i=1}^m \mathfrak{g}_i$$

as a vector space, where $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ for all i, j , and \mathfrak{g}_1 generates \mathfrak{g} as a Lie algebra.

Let G be the associated connected, simply connected Lie group. Then G has homogeneous dimension d given by the formula

$$d = \sum_{j=1}^m j \dim(\mathfrak{g}_j),$$

where $\dim(\mathfrak{g}_j)$ denotes the dimension of \mathfrak{g}_j .

Consider any finite subset $\{X_k\}$ of \mathfrak{g}_1 which spans \mathfrak{g}_1 . Each X_k can be identified with a unique left invariant vector field on G . Define

$$L = - \sum_k X_k^2;$$

then L is a left invariant second order differential operator. We define $L^p(G)$ with respect to Haar measure (and denote the corresponding norms by $\|\cdot\|_p$), then L is non-negative self-adjoint on $L^2(G)$ and it admits a spectral resolution

$$L = \int_0^\infty \lambda dP_\lambda.$$

For any bounded Borel function on $[0, \infty)$, we can define

$$m(L) = \int_0^\infty m(\lambda) dP_\lambda$$

which is bounded on $L^2(G)$, and the corresponding operator norm, which we denote by $\|m(L)\|_{2 \rightarrow 2}$, satisfies $\|m(L)\|_{2 \rightarrow 2} = \|m\|_\infty$.

Note that the operators $m(L)$ given by the spectral theorem and in Section 2 are identical when both definitions are applicable.

We need the following lemma which gives the upper bounds on the heat kernel and its derivatives.

Lemma . *Let h_z be the kernel of e^{-zL} , $\operatorname{Re} z > 0$, and $\arg z = \theta$. Then the following estimates hold:*

$$(3) \quad |h_z(x)| \leq C (|z| \cos \theta)^{-\frac{d}{2}} \exp \left\{ -c \cos \theta \frac{\|x\|^2}{|z|} \right\}$$

$$(4) \quad |X_i h_z(x)| \leq C (|z| \cos \theta)^{-\frac{d+1}{2}} \exp \left\{ -c \cos \theta \frac{\|x\|^2}{|z|} \right\}.$$

Proof. The following estimates on the heat kernel $h_t(x)$ and its derivatives for $t > 0$ are well known (e.g. see Saloff-Coste [Sa] and its references):

$$|h_t(x)| \leq C t^{-\frac{d}{2}} \exp \left\{ -c \frac{\|x\|^2}{t} \right\}$$

$$|X_i h_t(x)| \leq C t^{-\frac{d+1}{2}} \exp \left\{ -c \frac{\|x\|^2}{t} \right\}.$$

The required estimates then follow by interpolation as in Theorem 3.4.8 of Davies [Da]. \square

We now represent the operator $m(L)$, using the semigroup e^{-zL} .

As in Section 2, for $m \in \Psi(S_\delta)$, we choose the contour $\gamma = \gamma_- + \gamma_+$, where

$$\gamma_+(t) = te^{i\mu} \quad \text{if } 0 \leq t < \infty$$

$$\gamma_-(t) = -te^{-i\mu} \quad \text{if } -\infty < t \leq 0$$

with $\delta > \mu$, and write

$$m(L) = \frac{1}{2\pi i} \int_{\gamma} (L - \lambda I)^{-1} m(\lambda) d\lambda.$$

Assume $\lambda \in \gamma_+$; then we have

$$(L - \lambda I)^{-1} = \int_{\Gamma_+} e^{\lambda z} e^{-zL} dz$$

where the curve Γ_+ is defined by $\Gamma_+(t) = te^{i\theta}$ for $t \geq 0$ and $\theta = (\pi - \mu)/2$. Therefore

$$\begin{aligned} m_+(L) &= \frac{1}{2\pi i} \int_{\gamma_+} \left[\int_{\Gamma_+} e^{\lambda z} e^{-zL} dz \right] m(\lambda) d\lambda \\ &= \int_{\Gamma_+} \left[\frac{1}{2\pi i} \int_{\gamma_+} e^{\lambda z} m(\lambda) d\lambda \right] e^{-zL} dz, \end{aligned}$$

by a change in the order of integration. Define Γ_- similarly: $\Gamma_-(t) = te^{-i\theta}$ for $t \geq 0$. A similar argument shows that

$$\begin{aligned} m_-(L) &= \frac{1}{2\pi i} \int_{\gamma_-} \left[\int_{\Gamma_-} e^{\lambda z} e^{-zL} dz \right] m(\lambda) d\lambda \\ &= \int_{\Gamma_-} \left[\frac{1}{2\pi i} \int_{\gamma_-} e^{\lambda z} m(\lambda) d\lambda \right] e^{-zL} dz, \end{aligned}$$

and therefore

$$m(L) = \int_{\Gamma_+} e^{-zL} n_+(z) dz + \int_{\Gamma_-} e^{-zL} n_-(z) dz,$$

where

$$n_{\pm}(z) = \frac{1}{2\pi i} \int_{\gamma_{\pm}} e^{\lambda z} m(\lambda) d\lambda,$$

which implies the bound

$$(5) \quad |n_{\pm}(z)| \leq \frac{1}{2\pi} \|m\|_{\infty} (\cos \theta)^{-1} |z|^{-1}.$$

Consequently, the kernel of $K_m(x)$ of $m(L)$ is given by

$$(6) \quad K_m(x) = \int_{\Gamma_+} h_z(x) n_+(z) dz + \int_{\Gamma_-} h_z(x) n_-(z) dz.$$

We now state our main theorem.

Theorem 2. *Let h_z be the kernel of e^{-zL} , $\operatorname{Re} z > 0$, $\arg z = \theta$. Assume that for some $\ell > 0$ the $L^1(G)$ norm of the complex time heat kernel h_z satisfies*

$$\|h_z\|_1 \leq C (\cos \theta)^{-\ell}.$$

Then the operator $m(L)$ can be extended to a bounded operator on $L^p(G)$ for all $p \in (1, \infty)$ if the function m satisfies the Hörmander condition (1) of order $s = \ell + 1$.

Proof. We denote the Haar measure by dx and the control distance associated to the sub-Laplacian L by d . We write $d(e, x) = \|x\|$, where e is the identity element of G .

Our plan of proof is to prove that L has a bounded holomorphic functional calculus as in (i) of Theorem 1 with $\alpha = \ell + 1$. Then Theorem 2 follows from Theorem 1.

Let $m \in \Psi(S_\mu^0)$. To apply Calderón–Zygmund operator theory, we first prove the following estimate

$$(7) \quad I = \int_{\|x\| \geq 2\|y\|} |K_m(x) - K_m(y^{-1}x)| dx \leq C \|m\|_\infty (\cos \theta)^{-(\ell+1+\epsilon)}.$$

Using (5) and (6), and changing the order of integration, we have

$$(8) \quad I \leq C \|m\|_\infty (\cos \theta)^{-1} \int_\Gamma \int_{\|x\| \geq 2\|y\|} |h_z(x) - h_z(y^{-1}x)| dx |z|^{-1} d|z|,$$

where \int_Γ is short for $\int_{\Gamma_+} + \int_{\Gamma_-}$. We write

$$(9) \quad \int_{\|x\| \geq 2\|y\|} |h_z(x) - h_z(y^{-1}x)| dx \\ = \left(\int_{\|x\| \geq 2\|y\|} |h_z(x) - h_z(y^{-1}x)| dx \right)^\alpha \left(\int_{\|x\| \geq 2\|y\|} |h_z(x) - h_z(y^{-1}x)| dx \right)^{1-\alpha},$$

where α will be specified later. We estimate the second factor:

$$(10) \quad \left(\int_{\|x\| \geq 2\|y\|} |h_z(x) - h_z(y^{-1}x)| dx \right)^{1-\alpha} \leq (2\|h_z\|_1)^{1-\alpha} \leq C (\cos \theta)^{-\ell(1-\alpha)}.$$

To estimate the first factor, we use the upper bound on $X_i h_z$ in the lemma to obtain

$$(11) \quad \int_{\|x\| \geq 2\|y\|} |h_z(x) - h_z(y^{-1}x)| dx \\ \leq C \|y\| \int_{\|x\| \geq \|y\|} (|z| \cos \theta)^{-\frac{d+1}{2}} e^{-c \cos \theta \|x\|^2 / |z|} dx,$$

where the constant c in the right hand side of (11) is half the constant c in the right hand side of (4). We now use polar coordinates in G , and deduce that

$$\begin{aligned} & \int_{\|x\| \geq 2\|y\|} |h_z(x) - h_z(y^{-1}x)| dx \\ & \leq C \|y\| (|z| \cos \theta)^{-\frac{d+1}{2}} \int_{\|y\|}^{\infty} e^{-c \cos \theta r^2/|z|} r^{d-1} dr \\ & = C \|y\| (|z| \cos \theta)^{-\frac{d+1}{2}} \left(\frac{|z|}{c \cos \theta} \right)^{\frac{d}{2}} \int_{c\|y\|^2 \cos \theta/|z|}^{\infty} e^{-s} s^{\frac{d}{2}-1} ds \\ & \leq C \left(\frac{\|y\|^2 \cos \theta}{|z|} \right)^{\frac{1}{2}} (\cos \theta)^{-d-1} e^{-c\|y\|^2 \cos \theta/|z|} \left[1 + \left(\frac{\|y\|^2 \cos \theta}{|z|} \right)^{\frac{d}{2}-1} \right]. \end{aligned}$$

Consequently,

$$\begin{aligned} (12) \quad & \int_{\Gamma} \left(\int_{\|x\| \geq 2\|y\|} |h_z(x) - h_z(y^{-1}x)| dx \right)^{\alpha} |z|^{-1} d|z| \\ & \leq C \int_{\Gamma} \left(\left(\frac{\|y\|^2 \cos \theta}{|z|} \right)^{\frac{1}{2}} (\cos \theta)^{-d-1} e^{-c\|y\|^2 \cos \theta/|z|} \right. \\ & \quad \left. \cdot \left[1 + \left(\frac{\|y\|^2 \cos \theta}{|z|} \right)^{\frac{d}{2}-1} \right] \right)^{\alpha} |z|^{-1} d|z| \\ & \leq C \int_0^{\infty} \left(t^{\frac{1}{2}} (\cos \theta)^{-d-1} e^{-ct} [1 + t^{\frac{d}{2}-1}] \right)^{\alpha} t^{-1} dt \\ & \leq c_{\alpha} (\cos \theta)^{-\alpha(d+1)}, \end{aligned}$$

where c_{α} becomes large as $\alpha \rightarrow 0$. We combine the inequalities (8) to (10) and (12), to get

$$(13) \quad I = \int_{\|x\| \geq 2\|y\|} |K_m(x) - K_m(y^{-1}x)| dx \leq c_{\alpha} \|m\|_{\infty} (\cos \theta)^{-1-\ell(1-\alpha)-\alpha(d+1)}.$$

By choosing α in (13) sufficiently small, interpolation shows that for any p , $1 < p \leq 2$, there exists a constant $c_{\epsilon,p}$ for any $\epsilon > 0$ such that

$$\|m(L)\|_{L^p(G)} \leq c_{\epsilon,p} \|m\|_{\infty} (\cos \theta)^{-\ell-1-\epsilon}.$$

We now fix p , $1 < p \leq 2$. To get rid of ϵ , we choose $p_1 = \frac{p+1}{2}$ and ϵ sufficiently small in the estimate of $\|m(L)\|_{L^{p_1}(G)}$, then interpolation between p_1 and 2 gives us the desired estimate.

The case $p > 2$ follows from duality. □

4. Hörmander multiplier theorems for sub-Laplacians on Lie groups.

4.1. Nilpotent Lie groups. Theorem 2 reduces the difficult task of controlling the kernel K_m of the operator $m(L)$ as in (7) to finding the $L^1(G)$ norms of the complex heat kernels h_z . The obvious next question is how large the norms $\|h_z\|_1$ are.

To obtain a sharp estimate on $\|h_z\|_1$ in the general setting of nilpotent Lie groups might be difficult but we can get a useful upper bound on $\|h_z\|_1$ without much difficulty. That result is the content of the following theorem.

Theorem 3. *Let L be a sub-Laplacian on a homogeneous nilpotent Lie group G of homogeneous dimension d , as in Section 3. Then for each $\epsilon > 0$, there exists $c_\epsilon > 0$ such that the $L^1(G)$ norms of the complex heat kernels satisfy*

$$\|h_z\|_1 \leq c_\epsilon (\cos \arg z)^{-\frac{d}{2}-\epsilon}.$$

Consequently, the operator $m(L)$ can be extended to a bounded operator on $L^p(G)$ for all $p \in (1, \infty)$ if m satisfies the Hörmander condition (1) up to order $s = \frac{d}{2} + 1$.

Proof. We first estimate the $L^2(G)$ norms of the complex heat kernels as follows. Let $z = t + iv$ and denote the norm of the operator e^{-zL} from $L^2(G)$ to $L^\infty(G)$ by $\|e^{-zL}\|_{2 \rightarrow \infty}$. We then have

$$\|h_z\|_2 = \|e^{-zL}\|_{2 \rightarrow \infty}.$$

By spectral theory, e^{-ivL} is an isometry on $L^2(G)$, so

$$\|e^{-zL}\|_{2 \rightarrow \infty} = \|e^{-tL}\|_{2 \rightarrow \infty}.$$

We conclude that

$$(14) \quad \|h_z\|_2 = \|h_t\|_2 = Ct^{-\frac{d}{4}} = C(\operatorname{Re} z)^{-\frac{d}{4}}.$$

The middle equality holds by homogeneity.

We observe that by homogeneity, $\|h_z\|_1 = \|h_{z/|z|}\|_1$, hence we can assume $|z| = 1$.

To estimate $\|h_z\|_1$, we denote $\cos \arg z$ by σ , choose $\beta = \frac{1}{2} + v$ and break G into two parts:

$$\begin{aligned} G_1 &= \{x \in G \mid \|x\| < \sigma^{-\beta}\} \\ G_2 &= \{x \in G \mid \|x\| \geq \sigma^{-\beta}\}. \end{aligned}$$

We then have

$$\begin{aligned}
 (15) \quad \int_{G_1} |h_z(x)| dx &\leq (\text{vol } G_1)^{\frac{1}{2}} \left(\int_{G_1} |h_z(x)|^2 dx \right)^{\frac{1}{2}} \\
 &\leq (\text{vol } G_1)^{\frac{1}{2}} \left(\int_G |h_z(x)|^2 dx \right)^{\frac{1}{2}} \\
 &\leq C \sigma^{-\frac{d}{2} - \frac{dv}{2}}.
 \end{aligned}$$

To estimate $\int_{G_2} |h_z(x)| dx$, we use the estimate (3) of the lemma, and then integrate in polar coordinates. It turns out that

$$\begin{aligned}
 (16) \quad \int_{G_2} |h_z(x)| dx &\leq C \int_{\sigma^{-\beta}}^{\infty} \sigma^{-\frac{d}{2}} \exp\{-c\sigma r^2\} r^{d-1} dr \\
 &= C \sigma^{-d} \int_{\sigma^{1-2\beta}}^{\infty} \exp\{-cs\} s^{\frac{d}{2}-1} ds \\
 &\leq c_{d,v}
 \end{aligned}$$

where $c_{d,v}$ depends only on d and v . It follows from (15) and (16) that by choosing $v = \frac{2\epsilon}{d}$, there exists c_ϵ such that

$$\|h_z\| \leq c_\epsilon \sigma^{-\frac{d}{2} - \epsilon}.$$

To complete the proof, we apply Theorem 2, and then interpolate to get rid of ϵ (as in the proof of Theorem 2). □

4.2. Generalised Heisenberg groups. In the proof of Theorem 3, estimate (14) shows that the $L^2(G)$ norm of the complex heat kernels is a multiple of $(\cos \arg z)^{-d/4}$. If we use this estimate to obtain an upper bound for the $L^1(G)$ norm of the complex heat kernels, we have the power $d/2$. This is the reason why our Theorem 3 as well as previously known proofs which utilise the $L^2(G)$ estimate only obtain order $s \geq d/2$.

To improve the order beyond half the homogeneous dimension, we need a sharper estimate on the $L^1(G)$ norm of the complex heat kernels. This can be done for the generalised Heisenberg groups (or H-type groups).

We now give a brief definition of generalised Heisenberg groups. For more details, see the thesis of J. Randall [Ra1] and its references.

Let \mathfrak{g} be a 2-step nilpotent Lie algebra with an inner product. Let ζ be the centre of \mathfrak{g} and ϑ the orthogonal complement of ζ in \mathfrak{g} . For $v \in \vartheta$, let $f_\vartheta = (\ker \text{ad } v) \cap \vartheta$, and denote by ϑ_v the orthogonal complement of f_ϑ in ϑ . Then \mathfrak{g} is called an H-type algebra or a generalised Heisenberg algebra if $\text{ad } v : \vartheta_v \rightarrow \zeta$ is a surjective isometry for every unit vector $v \in \vartheta$.

The connected simply connected Lie group G , associated with $\underline{\mathfrak{g}}$ is called an H-type or generalised Heisenberg group.

For the generalised Heisenberg algebra $\underline{\mathfrak{g}} = \vartheta \oplus \zeta$, let $\dim(\vartheta) = 2n$ and $\dim(\zeta) = k$; then G is a stratified group with dilations $\gamma_r(v, \xi) = (rv, r^2\xi)$, for $(v, \xi) \in \vartheta \oplus \zeta$, and homogeneous dimension $d = 2n + 2k$.

We can also define the sub-Laplacian L on G . The heat kernel $h_z(x)$ has an explicit representation which can be used to estimate its $L^1(G)$ norm, (see [Ra1]). Our next theorem is

Theorem 4. *The $L^1(G)$ norm for h_z satisfies the following estimate:*

$$\|h_z\|_1 \leq \frac{c}{(\cos \arg z)^{n+\ell}} \text{ where } \ell = \begin{cases} \frac{k+3}{2} & \text{for } k \text{ odd} \\ \frac{k}{2} + 2 & \text{for } k \text{ even.} \end{cases}$$

Hence the operator $m(L)$ can be extended to a bounded operator on $L^p(G)$ for all $p \in (1, \infty)$, if m satisfies the Hörmander condition (1) up to order

$$s = \begin{cases} n + \frac{k+5}{2} & \text{for } k \text{ odd} \\ n + \frac{k}{2} + 3 & \text{for } k \text{ even.} \end{cases}$$

Proof. The estimate on the $L^1(G)$ norm of the complex time heat kernels, which uses the explicit representation of the heat kernels, is the main result of [Ra2].

The second part of this theorem is a consequence of Theorem 2. \square

NOTE:

(a) The order s obtained in this theorem is strictly less than half of the homogeneous dimension when k is sufficiently large.

(b) The Hörmander multiplier result in Theorem 4 can be obtained by direct estimate on the kernel K_m of the operator $m(L)$, using the explicit representation of the complex heat kernels [D2].

(c) After this paper was written up, it came to the author's knowledge that, by using the real variable method, W. Hebisch was successful in proving that on a product of generalised Heisenberg groups Hörmander type multiplier theorem for the sub-Laplacian is true with the order $s = \frac{D}{2} + \epsilon$, $\epsilon > 0$, where D is the euclidean dimension of the group [He].

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