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INTERPOLATING BLASCHKE PRODUCTS GENERATE H^∞

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The algebra of bounded analytic functions on the open unit disc is generated by the set of Blaschke products having simple zeros which form an interpolating sequence.

Let H^∞ be the algebra of bounded analytic functions in the unit disc \mathbb{D} and set

$$\|f\| = \sup_{z \in \mathbb{D}} |f(z)|,$$

for $f \in H^\infty$. A *Blaschke product* is an H^∞ function of the form

$$B(z) = \prod_{\nu=1}^{\infty} \frac{-\bar{z}_\nu}{|z_\nu|} \frac{z - z_\nu}{1 - \bar{z}_\nu z}$$

with $\sum(1 - |z_\nu|) < \infty$. In [5] D.E. Marshall proved that H^∞ is the closed linear span of the Blaschke products: given $f \in H^\infty$ and $\varepsilon > 0$, there are constants c_1, \dots, c_n and Blaschke products B_1, \dots, B_n such that

$$(1) \quad \|f + c_1 B_1 + \dots + c_n B_n\|_\infty < \varepsilon.$$

In fact, Marshall proved that the unit ball of H^∞ is the uniformly closed convex hull of the set of Blaschke products (including $B \equiv 1$).

A Blaschke product $B(z)$ is called an *interpolating Blaschke product* if

$$(2) \quad \inf_{\nu} (1 - |z_\nu|^2) |B'(z_\nu)| = \delta_B > 0,$$

because of the Carleson theorem that (2) holds if and only if every interpolation problem

$$f(z_\nu) = w_\nu, \quad \nu = 1, 2, \dots,$$

for $\{w_\nu\} \in l^\infty$, has a solution $f \in H^\infty$. Although the interpolating Blaschke products comprise a small subset of the set of all Blaschke products, they play a central role in the theory of H^∞ . See the last three chapters of [3]. The theorem in this paper helps explain why interpolating Blaschke products are so important in that theory.

Theorem. *H^∞ is the closed linear span of the interpolating Blaschke products.*

In other words, (1) is true with the additional proviso that each of B_1, \dots, B_n is an interpolating Blaschke product.

The theorem solves a problem posed in [3] and [4]. It is not known if the set of interpolating Blaschke products is norm dense in the set of all Blaschke products. It is also not known if the unit ball of H^∞ is the closed convex hull of the set of all interpolating Blaschke products.

Recently, Marshall and A. Stray [6] proved the theorem in the special case that f extends continuously to almost every point of $\partial\mathbb{D}$, and our proof closely follows their reasoning. In particular, the idea of comparing (11) and (12) and the argument deriving the theorem from Lemma 3 below are both due to them. We thank Violant Marti for making the drawings.

The *hyperbolic distance* between $z \in \mathbb{D}$ and $w \in \mathbb{D}$ is

$$\rho(z, w) = \log \left(\frac{1 + \left| \frac{z-w}{1-\bar{w}z} \right|}{1 - \left| \frac{z-w}{1-\bar{w}z} \right|} \right),$$

and the *hyperbolic derivative* of an analytic function f is

$$(1 - |z|^2) |f'(z)|.$$

The hyperbolic derivative is invariant under conformal changes in $z \in \mathbb{D}$.

The Blaschke product with zeros $\{z_\nu\}$ is an interpolating Blaschke product if and only if the following conditions both hold:

$$(3) \quad \inf_{\nu \neq \mu} \rho(z_\mu, z_\nu) > 0$$

and

$$(4) \quad \sum_{z_\nu \in Q} (1 - |z_\nu|) < Cl(Q)$$

for all $Q = \{re^{i\theta} : \theta_0 < \theta < \theta_0 + \ell(Q), 1 - \ell(Q) < r < 1\}$. See [1] or Chapter VII of [3].

Lemma 1. *Let B be a Blaschke product and let $\{z_\nu\}$ be its zeros, counted with their multiplicities. Then the following are equivalent:*

- (a) $B = B_1 \dots B_N$, with each B_j an interpolating Blaschke product.
- (b) Condition (4) holds.
- (c) There exist positive constants ρ_0, δ_0 such that for each z_ν there is w_ν with

$$(5) \quad \rho(z_\nu, w_\nu) \leq \rho_0$$

and

$$(6) \quad (1 - |w_\nu|^2) |B'(w_\nu)| \geq \delta_0.$$

In [6] it is shown that if B satisfies one of these conditions, then B is the uniform limit of a sequence of interpolating Blaschke products.

Proof of Lemma 1. The equivalence between (a) and (b) is in [7]. Assume (c) holds, let

$$Q = \{re^{i\theta} : \theta_0 < \theta < \theta_0 + \ell(Q), 1 - \ell(Q) < r < 1\},$$

and set

$$T(Q) = \{re^{i\theta} \in Q : 1 - \ell(Q) < r < 1 - 2^{-1}\ell(Q)\}.$$

To prove (4), we may assume there exists $z_\nu \in T(Q)$. Let w_ν satisfy (5) and (6). Then there exists a_ν such that $\rho(a_\nu, z_\nu) < \rho_0$ and $|B(a_\nu)| \geq m = m(\rho_0, \delta_0) > 0$. Then the inequalities

$$\begin{aligned} \log m^{-2} &\geq \log |B(a_\nu)|^{-2} \geq \sum_{z_\mu \in Q} \frac{(1 - |z_\mu|^2)(1 - |a_\nu|^2)}{|1 - \overline{a_\nu}z_\mu|^2} \\ &\geq \frac{A(\rho_0, \delta_0)}{\ell(Q)} \sum_{z_\mu \in Q} (1 - |z_\mu|) \end{aligned}$$

show that (4) holds.

If (a) holds, there exists $C > 0$ such that

$$|B(z)| \geq C \prod_{j=1}^N \inf_{\{B_j(z_\nu)=0\}} \left| \frac{z - z_\nu}{1 - \overline{z_\nu}z} \right|.$$

Fix $\delta_0 > 0$. Given z_ν , there exists ζ_ν such that $\rho(z_\nu, \zeta_\nu) \leq \delta_0$ and $|B(\zeta_\nu)| \geq m = m(\delta_0) > 0$, and then the geodesic arc from z_ν to ζ_ν contains a point w_ν at which (6) holds. □

We write \mathcal{F} for the set of finite products of interpolating Blaschke products. By the remark following Lemma 1, it is enough to prove (1) with each $B_j \in \mathcal{F}$, and by Marshall’s theorem it is also enough to prove (1) when $f = B$ is a Blaschke product.

Fix a Blaschke product B and let $0 < \alpha < \beta < 1$, $M = 2^K > 1$, and $\delta < 1$ be constants which will be determined later. We may assume $|B(0)| > \beta$. Consider “squares” of the form

$$Q_{n,j} = \{re^{i\theta} : 2\pi j2^{-n} \leq \theta < 2\pi(j+1)2^{-n}; 1 - 2^{-n} \leq r < 1\}$$

and their top halves

$$T(Q_{n,j}) = Q_{n,j} \cap \{re^{i\theta} : 1 - 2^{-n} \leq r < 1 - 2^{-n-1}\}.$$

Let $G_1 = \{Q_1^{(1)}, Q_2^{(1)}, \dots\}$ be the set of maximal $Q_{n,j}$ for which

$$\inf_{T(Q_{n,j})} |B(z)| < \alpha.$$

The squares in G_1 have disjoint interiors. Write $S_{p,j}^{(1)}$, $1 \leq p \leq M = 2^K$, for 2^K different $Q_{n+K,j} \subset Q_k^{(1)} = Q_{n,j}$. If M is fixed and $1 - \beta$ is small, then by Harnack's inequality

$$(7) \quad \sup_{T(S_{p,j}^{(1)})} |B(z)| < \beta.$$

Now let $H_1 = \{V_1^{(1)}, V_2^{(1)}, \dots\}$ be the set of maximal $Q_{n,j}$ such that

$$V_k^{(1)} \subset Q_p^{(1)}$$

for some $Q_p^{(1)}$ and

$$\inf_{T(V_k^{(1)})} |B(z)| > \beta.$$

Since $|B|$ has nontangential limit 1 almost everywhere,

$$\sum_{V_k^{(1)} \subset Q_p^{(1)}} \ell(V_k^{(1)}) = \ell(Q_p^{(1)}).$$

If $(1 - \beta)/(1 - \alpha)$ is small, then

$$l(V_k^{(1)}) < \frac{1}{M} l(Q_p^{(1)})$$

when $V_k^{(1)} \subset Q_p^{(1)}$, again by Harnack's inequality. Hence $V_k^{(1)} \subset S_{p,j}^{(1)}$, for some p, j , because of (7).

Next let $G_2 = \{Q_1^{(2)}, Q_2^{(2)}, \dots\}$ be the set of maximal $Q_{n,j}$ such that

$$Q_{n,j} \subset V_k^{(1)} \in H_1$$

and

$$\inf_{T(Q_{n,j})} |B(z)| < \alpha.$$

If $(1 - \beta)/(1 - \alpha)$ is small, then

$$(8) \quad \sum_{Q_j^{(2)} \subset V_k^{(1)}} \ell(Q_j^{(2)}) < \varepsilon \ell(V_k^{(1)})$$

(see [3, p. 334]).

We form the $S_{p,k}^{(2)}$ as before and continue, obtaining $Q_j^{(m)}, S_{p,j}^{(m)}$ and $V_k^{(m+1)}$ with

$$Q_j^{(m)} \supset S_{p,j}^{(m)} \supset V_k^{(m+1)}.$$

See Figure 1. Then $B(z)$ has zeros only in

$$\bigcup_{m,j} \left(Q_j^{(m)} \setminus \bigcup_{V_k^{(m+1)} \subset Q_j^{(m)}} V_k^{(m+1)} \right).$$

In fact, if $1 - \alpha$ is small enough, all zeros from

$$Q_j^{(m)} \setminus \bigcup_{V_k^{(m+1)} \subset Q_j^{(m)}} V_k^{(m+1)}$$

fall into

$$\bigcup_{p=1}^M R_{p,j}^{(m)} = \bigcup_{p=1}^M \left(S_{p,j}^{(m)} \setminus \bigcup_{V_k^{(m+1)} \subset S_{p,j}^{(m)}} V_k^{(m+1)} \right),$$

and we require $1 - \alpha$ to be that small.

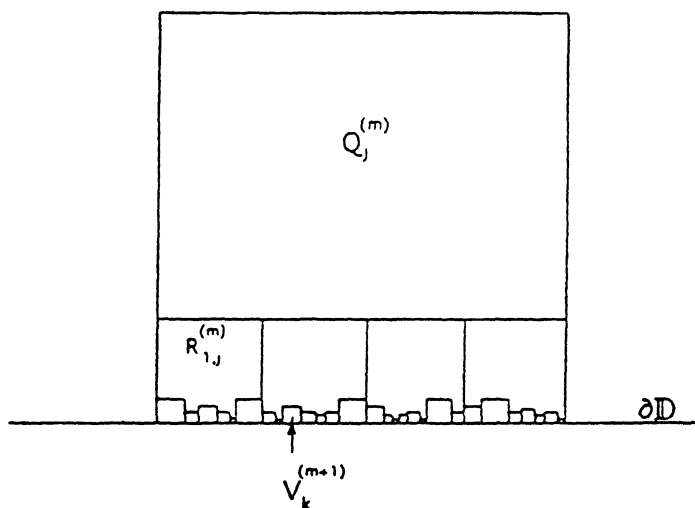


Figure 1.

Now factor

$$B = B_1 B_2 \cdots B_M,$$

where for fixed p, B_p has zeros only in $\bigcup_{m,j} R_{p,j}^{(m)}$. Fix p , set

$$\Gamma_{p,j}^{(m)} = \partial R_{p,j}^{(m)} \setminus \partial S_{p,j}^{(m)},$$

and mark points $z_\nu^* = z_\nu^*(m, p, j)$ on $\Gamma_{p,j}^{(m)}$ with

$$(9) \quad \rho(z_\nu^*, z_{\nu+1}^*) = \delta.$$

Let B_p^* be the Blaschke product with zeros $\bigcup_{m,j} z_\nu^*(m, p, j)$. Then by (3), (4), (8) and (9), B_p^* is an interpolating Blaschke product.

Lemma 2. $|B_p^*| \leq \delta^{1/4}$ on $\bigcup_{m,j} R_{p,j}^{(m)}$.

Proof. Clearly $|B_p^*| < \delta$ on $\bigcup_{m,j} \Gamma_{p,j}^{(m)}$. Fix one $R_{p,j}^{(m)}$. Then for any $\varepsilon > 0$, the harmonic measure

$$\omega \left(z, \Gamma_{p,j}^{(m)}, \mathbb{D} \setminus \bigcup \overline{\{V_k^{(m+1)} \subset S_{p,j}^{(m)}\}} \right) > \frac{1}{4} - \varepsilon$$

for all $z \in R_{p,j}^{(m)}$, provided $(1 - \beta)/(1 - \alpha)$ is small. Since $\log |B_p^*(z)|$ is harmonic, that shows $|B_p^*| \leq \delta^{1/4}$ on $R_{p,j}^{(m)}$. \square

Lemma 3. *There exist $A = A(\alpha, \beta, \delta, M)$ and $\eta = \eta(\alpha, \beta, \delta, M) > 0$ so that if*

$$(10) \quad \inf_{\xi \in \bigcup_{m,j} R_{p,j}^{(m)}} \rho(z, \xi) > A$$

and if

$$|B_p B_p^*(z)| = \delta^{1/8},$$

then

$$(1 - |z|^2) \left| (B_p B_p^*)'(z) \right| \geq \eta.$$

Proof. We have

$$(11) \quad \frac{1}{4} \log \frac{1}{\delta} = \log |B_p B_p^*(z)|^{-2} \sim \sum_\nu \frac{(1 - |z|^2)(1 - |z_\nu|^2)}{|1 - \bar{z}_\nu z|^2},$$

where $\{z_\nu\}$ is the zero set of $B_p B_p^*$. On the other hand,

$$(12) \quad (1 - |z|^2) \frac{(B_p B_p^*)'(z)}{B_p B_p^*(z)} = \bar{z} \sum_\nu \frac{(1 - |z|^2)(1 - |z_\nu|^2)}{|1 - \bar{z}_\nu z|^2} \left(\frac{\frac{1}{z} - z_\nu}{z - z_\nu} \right).$$

By (10) there is A' so that if $|z - z_\nu| < A'(1 - |z|)$, then $z_\nu \in R_{p,j}^{(m)}$ where $\ell(S_{p,j}^{(m)}) < 1 - |z|$. See Figure 2.

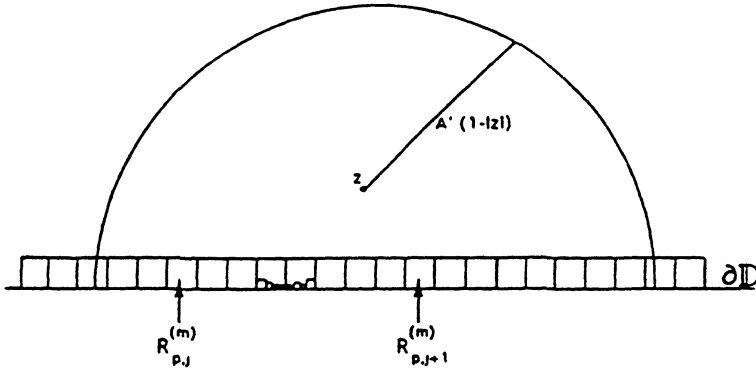


Figure 2.

If $(1 - \alpha)$ is small compared to $1/M$, then $\inf_{T(S_{p,j}^{(m)})} |B(z)| \geq C(\alpha) > 0$ and

$$\sum_{\{z_n \in R_{p,j}^{(m)}; B(z_n)=0\}} (1 - |z|^2) \leq C_1(\alpha)\ell(S_{p,j}^{(m)}),$$

where $C_1(\alpha)$ tends to 0 if α tends to 1. Therefore

$$\begin{aligned} \sum_{|z_\nu - z| < A'(1 - |z|)} \frac{(1 - |z_\nu|^2)(1 - |z|^2)}{|1 - \bar{z}_\nu z|^2} &\leq \frac{1}{1 - |z|^2} \sum_{|z_\nu - z| < A'(1 - |z|)} (1 - |z_\nu|^2) \\ &\leq \frac{1}{\delta M} (1 + \varepsilon + \varepsilon^2 + \dots) \\ &\quad + \frac{C_1(\alpha)}{M} (1 + \varepsilon + \varepsilon^2 + \dots). \end{aligned}$$

Take M so large (and consequently $1 - \alpha$ so small) that

$$\sum_{|z_\nu - z| < A'(1 - |z|)} \frac{(1 - |z_\nu|^2)(1 - |z|^2)}{|1 - \bar{z}_\nu z|^2} < \frac{1}{16} \log \frac{1}{\delta}.$$

If $|z - z_\nu| > A'(1 - |z|)$ then

$$\left| \arg \left(\frac{\frac{1}{z} - z_\nu}{z - z_\nu} \right) \right| < c(A')$$

where $c(A') \rightarrow 0$ as $A' \rightarrow \infty$. Hence

$$\begin{aligned} & \left| \sum_{|z-z_\nu| \geq A'(1-|z|)} \frac{\bar{z}(1-|z|^2)(1-|z_\nu|^2)}{|1-\bar{z}_\nu z|^2} \left(\frac{\frac{1}{z} - z_\nu}{z - z_\nu} \right) \right| \\ & \geq \cos^{-1}(c(A')) \sum_{|z-z_\nu| \geq A'(1-|z|)} \left| \frac{\bar{z}(1-|z|^2)(1-|z_\nu|^2)}{|1-\bar{z}_\nu z|^2} \left(\frac{\frac{1}{z} - z_\nu}{z - z_\nu} \right) \right|. \end{aligned}$$

Consequently,

$$\begin{aligned} & (1-|z|^2) \left| (B_p B_p^*)'(z) \right| \\ & \geq \delta^{1/8} \left(\left| \sum_{|z-z_\nu| \geq A'(1-|z|)} \frac{\bar{z}(1-|z|^2)(1-|z_\nu|^2)}{|1-\bar{z}_\nu z|^2} \left(\frac{\frac{1}{z} - z_\nu}{z - z_\nu} \right) \right| \right. \\ & \quad \left. - \sum_{|z-z_\nu| < A'(1-|z|)} \left| \frac{\bar{z}(1-|z|^2)(1-|z_\nu|^2)}{|1-\bar{z}_\nu z|^2} \left(\frac{\frac{1}{z} - z_\nu}{z - z_\nu} \right) \right| \right) \\ & \geq \delta^{1/8} \left(\cos^{-1}(c(A')) \frac{11}{16} \log(1/\delta) - \frac{1}{16} \log(1/\delta) \right), \end{aligned}$$

and if A' is large, that proves the lemma. □

With Lemma 3, the remainder of the proof is just like in the Marshall-Stray paper [6]. There is $\gamma, |\gamma| = \delta^{1/8}$, so that

$$\frac{B_p B_p^* - \gamma}{1 - \bar{\gamma} B_p B_p^*} = C_p$$

is a Blaschke product, by a theorem of Frostman [2]. Suppose $C_p(z) = 0$. Then

$$|B_p B_p^*(z)| = \delta^{1/8}$$

and

$$(1-|z|^2) \left| C_p'(z) \right| = \frac{(1-|z|^2)}{1-|\gamma|^2} \left| (B_p B_p^*)'(z) \right|.$$

Thus by Lemma 3

$$(1-|z|^2) \left| C_p'(z) \right| \geq \frac{\eta}{1-\delta^{1/4}}$$

if (10) holds. But if (10) fails, then there is $\xi \in \bigcup_{m,j} R_{p,j}^{(m)}$ with $\rho(z, \xi) < A$. By Lemma 2, $|B_p B_p^*(\xi)| \leq \delta^{1/4}$. Somewhere along the hyperbolic geodesic from z to ξ there is a point w with

$$(1-|w|^2) \left| (B_p B_p^*)'(w) \right| > \eta' > 0$$

and $\rho(z, w) < A$. So by Lemma 1, C_p is a finite product of interpolating Blaschke products and $B_p B_p^* \in \mathcal{F}$.

For σ very small, replace B_p^* by

$$\tilde{B}_p^* = \frac{B_p^* - \sigma}{1 - \bar{\sigma} B_p^*},$$

which is again an interpolating Blaschke product by [3, p. 404]. Repeating the above argument with \tilde{B}_p^* , we see that

$$\tilde{C}_p = \frac{B_p \tilde{B}_p^* - \tilde{\gamma}}{1 - \tilde{\gamma} B_p \tilde{B}_p^*}$$

is also a finite product of interpolating Blaschke products for some $\tilde{\gamma}$. Thus also $B_p \tilde{B}_p^* \in \mathcal{F}$. But then since

$$B_p \tilde{B}_p^* = -\sigma B_p + (1 - |\sigma|^2) B_p B_p^* + \dots,$$

we conclude that $B_p \in \mathcal{F}$. □

References

- [1] L. Carleson, *An interpolation problem for bounded analytic functions*, Amer. J. Math., **80** (1958), 921-930.
- [2] O. Frostman, *Potential d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions*, Medd. Lunds. Univ. Mat. Sem., **3** (1935), 1-118.
- [3] J. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.
- [4] P. Jones, *Ratios of interpolating Blaschke products*, Pacific J. Math., **95**(2) (1981), 311-321.
- [5] D. Marshall, *Blaschke products generate H^∞* , Bull. Amer. Math. Soc., **82** (1976), 494-496.
- [6] D. Marshall and A. Stray, *Interpolating Blaschke products*, Pacific J. Math., **173** (1996), 491-499.
- [7] G. Mc.Donald and C. Sundberg, *Toeplitz operators on the disc*, Indiana U. Math. J., **28** (1979), 595-611.

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