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Up to polynomial coordinate substitutions, we find the list of all rational primitive polynomials in two complex variables whose zero fiber is isomorphic to $\mathbf{C}^*$.

1. Introduction.

Let $p(x,y)$ and $q(x,y)$ be polynomials in two complex variables. We shall say that these polynomials are equivalent if there exists a polynomial automorphism α of $\mathbf{C}^2$ and an affine automorphism β of $\mathbf{C}$ for which $p = \beta \circ q \circ \alpha$. Consider the set of polynomials which have a fiber isomorphic to a given algebraic curve R. It is natural to look for a list of non-equivalent polynomials such that every polynomial from this set is equivalent to one of the polynomials from the list. If such a list exists we shall say that there is a classification of polynomials with this fiber R. This problem is equivalent to the problem of classification of all smooth polynomial embeddings of R into $\mathbf{C}^2$ up to a polynomial automorphism. The remarkable Abhyankar-Moh-Suzuki theorem [**AM**], [**Su1**] says that all smooth polynomial embeddings of the complex line into $\mathbf{C}^2$ are equivalent to linear embeddings. Moreover, V. Lin and M. Zaidenberg [**LZ**] obtained the classification of polynomial injections of $\mathbf{C}$ into $\mathbf{C}^2$ (i.e. they found a description of all polynomials whose zero fiber is homeomorphic to $\mathbf{C}$). Later W. Neumann and L. Rudolph [**NR**] reproved these theorems and W. Neumann obtained the classification for all polynomials whose zero fiber is diffeomorphic to a once punctured Riemann surface of genus ≤ 2.

The papers [**AM**], [**NR**], and [**N**] use the following theorem [**AS**]: if the zero fiber of a polynomial is a once punctured Riemann surface, then every other fiber of this polynomial is once punctured. The situation is drastically changed when the zero fiber R has two or more punctures. The behavior of punctures on the other fibers becomes more complicated and there is no analogue of the above theorem.

The Lin-Zaidenberg theorem is based on the following elegant fact. If a polynomial has at most one degenerate fiber (and it is so in the case of a contractible fiber) then the polynomial is isotrivial, i.e. its generic fibers are pairwise isomorphic. Isotrivial polynomials form a narrow class and its classification was obtained later in [**K1**], [**Z1**].

In this paper we shall study the case when R is isomorphic to $\mathbf{C}^*$ (the simplest case of a twice punctured surface). None of the above approaches works. The number of punctures on the generic fiber of the corresponding polynomial may be arbitrarily large and the polynomial may have a second degenerate fiber. This makes the problem difficult and we can obtain a classification for polynomials with a $\mathbf{C}^*$-fiber only under some additional conditions on the generic fibers of polynomials. Namely, we assume that these polynomials are rational, i.e. their generic fibers are m times punctured Riemann spheres. Even under this assumption the problem is complicated and only the cases when $m = 1$ or 2 were considered earlier [**Sa1**, **Sa2**, **Z1**, **Z3**]. The final classification for $m = 2$ was obtained in [**Z1**, **Z3**] by Zaidenberg. "Deformations" of Zaidenberg's polynomials were used later [**ACL**] to obtain examples of polynomials which are not equivalent to linear ones and have all fibers smooth and irreducible. These examples are important in connection with the Jacobian conjecture. P. Cassou-Noguès also noted that the coordinate functions in the recent counterexample of Pinchuk [**P**] to the real Jacobian Conjecture are deformations of Zaidenberg's polynomials. This shows that the study of rational polynomials with a $\mathbf{C}^*$-fiber may lead to interesting consequences. In this paper we shall prove the following fact.

Main theorem. *Let* $p\colon \mathbf{C}^2 \to \mathbf{C}$ *be a primitive rational polynomial whose zero fiber* Γ_0 *is isomorphic to* $\mathbf{C}^*$. *Suppose that* Γ_0 *is degenerate. Then there is a polynomial coordinate system* (x, y) *in* $\mathbf{C}^2$ *for which the polynomial* $p(x, y)$ *coincides with one of the following forms*

$$a(\psi^{nm+1} + (\psi^n + x)^m)/x^m \tag{1}$$

$$a(\psi^{nm-1} + (\psi^n + x)^m)/x^m \tag{2}$$

where $a \in \mathbf{C}^*, n$ *and* m *are natural,* $m \geq 2, n \geq 1$, *in formula* (2) $n \geq 2$ *in the case of* $m = 2$, $\psi(x, y) = x^m y + a_{m-1}x^{m-1} + \cdots + a_1 x - 1$, *and all coefficients* $a_{m-1}, \ldots, a_1$ *are determined uniquely by the condition that each of the above forms must be a polynomial.*

Let us describe briefly the scheme of the proof. The technique from [**Z1**], [**Z2**], and [**Sa1**] in combination with the Ramanujam-Morrow Theorem [**R**], [**M**] enables us to show that there is some "symmetry" between the fibers over 0 and ∞ for an extension $\bar{p} : \bar{X} \to \mathbf{CP}^1$ of p. The proof of this fact is long and computational, and, therefore, we place it in the Appendix. Using this symmetry, we find the dual graph of the curve $\hat{D} = \hat{X} - \mathbf{C}^2$ where $\hat{p} : \hat{X} \to \mathbf{CP}^1$ is another extension of p such that $\hat{D}$ is of simple normal crossing type (which will be abbreviated by SNC-type in what follows). The form of this graph implies that the second degenerate fiber of p contains a $\mathbf{C}^*$-component which does not meet some line. After this step the Main Theorem can be obtained from the following result which is interesting by itself.

Proposition. *Let Γ_0 and C be closed disjoint affine algebraic curves in $\mathbf{C}^2$. Suppose that Γ_0 is isomorphic to $\mathbf{C}^*$ and C is isomorphic to $\mathbf{C}$. Then there exists a coordinate system (x, y) in $\mathbf{C}^2$ for which C is the y-axis and the curve Γ_0 is given by one of the following equations*

(i) $x^n + \sigma^k(x, y) = 0$;

(ii) $x^n\sigma^k(x, y) + 1 = 0$;

where n, k are relatively prime natural numbers, $\sigma(x, y) = x^m y + g(x)$ with $g \in \mathbf{C}[x]$, $\deg g < m$, and $g(0) \neq 0$ for $m > 0$.

Note that the polynomials given by (i) correspond to non-rational polynomials. It is worth mentioning that there exist non-rational polynomials with a $\mathbf{C}^*$-fiber which are not equivalent to polynomials of this type. Examples of such polynomials were constructed recently by P. Russell and by P. Cassou-Noguès.

2. Preliminaries.

In this section we introduce notation, terminology, recall some known theorems, and prove several simple facts. The ground field is always $\mathbf{C}$ in this paper.

2.1. Let $p : X \to B$ be a morphism from a smooth algebraic surface X into a smooth algebraic curve B. (For instance, $X = \mathbf{C}^2$, $B = \mathbf{C}$, and p is a polynomial.) Put $\Gamma_b = p^{-1}(b)$ for every $b \in B$.

Definition. We shall say that a fiber Γ_b is generic if for a certain neighborhood U of b in B the following commutative diagram holds

$$\begin{tikzcd}
p^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & \Gamma_b \times U \arrow[dl, "\rho"] \\
& U &
\end{tikzcd}$$

where φ is a C^∞-diffeomorphism and ρ is the natural projection. If a fiber is not generic we shall call it degenerate.

2.2. Definition. A polynomial p is primitive if its generic fibers are connected, otherwise it is nonprimitive (for example, $p(x, y) = x^2$ is nonprimitive).

The study of nonprimitive polynomials can be reduced to the primitive case due to the following fact which is actually the Stein factorization.

Theorem ([**F**], [**LZ**]). *For every non-primitive polynomial* $q(x,y)$ *there exist a primitive polynomial* $p(x,y)$ *and a polynomial in one variable* $h(z)$ *so that* $q(x,y) = h(p(x,y))$.

Therefore, from now on we shall restrict ourselves to primitive polynomials only.

2.3. Let p be a primitive polynomial, let Γ be the generic fiber of p, and let $\chi(\Gamma_b)$ be the Euler characteristics of Γ_b. Suppose that the set $S \subset \mathbf{C}$ is such that Γ_b is degenerate iff $b \in S$. We shall call S *the degeneration set* of p. It is well-known that S is finite [**T**].

Theorem ([**Su1**], [**Su2**], see also [**Z2**]). *For every primitive polynomial* p *in two variables the following formula holds*

$$\sum_{b \in S} (\chi(\Gamma_b) - \chi(\Gamma)) = 1 - \chi(\Gamma)$$

where S *is the degeneration set of* p. *Moreover,* $\chi(\Gamma_b) \geq \chi(\Gamma)$, *and this inequality becomes the equality if and only if* Γ_b *is generic.*

Remark. When p is not primitive the first statement of the theorem is still true, but the second statement holds only when generic fibers do not contain components isomorphic to $\mathbf{C}$ or $\mathbf{C}^*$. (We do not use this remark further.)

Corollary. *Let the zero fiber* Γ_0 *of a primitive polynomial* p *be isomorphic to* $\mathbf{C}^*$. *Then either* Γ_0 *is generic or there is only one degenerate fiber other than* Γ_0.

Proof. Suppose that Γ_0 is degenerate. Since $\chi(\Gamma_0) = 0$, we have, by Theorem 2.3,

$$\sum_{b \in S-0} (\chi(\Gamma_b) - \chi(\Gamma)) - \chi(\Gamma) = 1 - \chi(\Gamma).$$

Hence

$$\sum_{b \in S-0} (\chi(\Gamma_b) - \chi(\Gamma)) = 1.$$

Since every term in the above sum is a positive integer, by Theorem 2.3, there is only one term. $\square$

Notation. Multiplying the polynomial by a constant, if necessary, we shall always suppose that under the assumption of Corollary the second degenerate fiber is $\Gamma_1 = p^{-1}(1)$.

2.4. Let $p : X \to B$ be as in 2.1. Standard results of the theory of resolution of singularities yield the existence of smooth compactifications $\bar{X}$ of X and $\bar{B}$ of B so that the mapping $p : X \to B$ can be extended to a regular mapping $\bar{p} : \bar{X} \to \bar{B}$. (When $B = \mathbf{C}$ then $\bar{B}$ coincides, of course, with $\mathbf{CP}^1$.)

Definition. We shall call the mapping $\bar{p}$ an extension of p. An irreducible component E of the curve $\bar{D} = \bar{X} - X$ is called *horizontal* if the restriction of $\bar{p}$ to E is not a constant mapping (which implies automatically that this restriction is surjective). Otherwise, it is called *vertical.*

A degenerate fiber of a polynomial p can be reducible even when p is primitive, in other words this fiber can consist of more than one irreducible algebraic curve (component). We shall need information about the number of irreducible components of the degenerate fibers of a polynomial p, and we can define this number in terms of extensions of the polynomial p. Since p is primitive, the generic fiber of $\bar{p}$ is connected, i.e. it is a smooth compact Riemann surface. Recall that the polynomial p is *rational* if the generic fiber of $\bar{p}$ is isomorphic to the Riemann sphere. The following theorem was proved in [**Sa1**] for rational polynomials and in [**K2**] for the general case.

Theorem. *Let $\bar{p} : \bar{X} \to \mathbf{CP}^1$ be an extension of a primitive polynomial p, and let S be the degeneration set of p. Suppose that γ_b is the number of irreducible components in the fiber Γ_b of p, and n is the number of horizontal components in the curve $\bar{D} = \bar{X} - \mathbf{C}^2$. Then*

$$\sum_{b \in S} (\gamma_b - 1) \leq n - 1.$$

Moreover, if p is rational, then $n - 1 = \sum_{b \in S} (\gamma_b - 1)$.

2.5. Let $\bar{p} : \bar{X} \to \bar{B}$ be an extension of a morphism $p : X \to B$ from a smooth algebraic surface X into a smooth curve B.

Definition. This extension is called pseudominimal if there are no (-1)-curves among the vertical components of $\bar{D} = \bar{X} - X$. (Recall that a (-1)-curve in a compact smooth algebraic surface is a rational curve whose selfintersection number is -1. The surface remains smooth after contracting this curve to a point.)

Proposition [**Z2**, Lemma 3.5]. *Let $\bar{p}$ be a pseudominimal extension of p. Suppose that the generic fiber of $\bar{p}$ is connected and that g is its genus. Let $\overline{\Gamma}_o$ be the closure of the fiber $\Gamma_o = p^{-1}(o)$ in $\bar{X}$ where $o \in B$. Then the arithmetic genus of $\overline{\Gamma}_o$ is $\leq g$ and the equality holds if and only if the divisors $\overline{\Gamma}_o$ and $p^*(o)$ coincide, i.e the fiber $\bar{p}^{-1}(o)$ contains no vertical components of $\bar{D}$.*

Since the arithmetical genus of a smooth non-multiple rational curve is zero we have

Corollary. *Suppose that $\bar{p}$ is pseudominimal. Let $g = 0$ and $\overline{\Gamma}_o$ be a smooth rational curve. Suppose that Γ_o is not a multiple fiber of the mapping p. Then the fiber $\bar{p}^{-1}(o)$ contains no vertical components of $\bar{D}$.*

2.6. Let p be a rational polynomial and let $\hat{p} : \hat{X} \to \mathbf{CP}^1$ be an extension (may be non-pseudominimal). Let C be a non-multiple component of Γ_o where $o \in \mathbf{C}$ and let $\hat{C}$ be its closure in $\hat{p}^{-1}(o)$. By Corollary 2.5, one may reduce the fiber $\hat{p}^{-1}(o)$ to this component $\hat{C}$ by blowing $\hat{X}$ down. The following fact shows that every fiber of $\hat{p}$ can be reduced to one component without any extra assumption since $\hat{X}$ is a rational ruled surface.

Theorem ([**GH**, Chap. 4, Sec. 3]). *There exists a commutative diagram*

$$\begin{tikzcd}
\hat{X} \arrow[rr, "\delta"] \arrow[dr, "\hat{p}"'] & & Q \arrow[dl, "q"] \\
& \mathbf{CP}^1 &
\end{tikzcd}$$

where Q is a Hirzebruch surface, q is the natural projection, and δ is a composition of blowing-ups.

2.7. If $\bar{p} : \bar{X} \to \bar{B}$ is a pseudominimal extension of $p : X \to B$ then $\bar{X}$ is not necessarily an NC-completion of X, i.e. the divisor $\bar{D} = \bar{X} - X$ may be not of normal crossing type.

Definition. An extension $\hat{p} : \hat{X} \to \bar{B}$ of a morphism $p : X \to B$ is called quasiminimal if $\hat{X}$ is an NC-completion of X and it is minimal, i.e. the completion stops being an NC-completion after contracting any vertical (-1)-curve in the divisor $\hat{D} = \hat{X} - X$.

It is clear that for every pseudominimal extension $\bar{p} : \bar{X} \to \bar{B}$ of $p : X \to B$ there exists a composition of blowing-ups $\sigma : \hat{X} \to \bar{X}$ such that the extension $\hat{p} = \bar{p} \circ \sigma$ is quasi-minimal and the restriction of σ is an isomorphism between $\bar{X} - \bar{D}_v$ and $\hat{X} - \hat{D}_v$ where $\bar{D}_v$ and $\hat{D}_v$ are the unions of the vertical components of the divisors $\bar{D}$ and $\hat{D}$ respectively. Vice versa, for every quasi-minimal extension $\hat{p}$ one can find a pseudominimal extension $\bar{p}$ such that the above properties hold. By construction, the curve $\hat{D}$ is simply connected if the curve $\bar{D}$ is simply connected. When $\hat{D}$ is simply connected (and this is the case we shall deal with) it has no non-smooth components (i.e. there is no component which has ordinary double points). In this case $\hat{X}$ is called an SNC-completion of X and the divisor $\hat{D}$ is of SNC-type (simple normal crossing type).

2.8. Definition. We shall say that a fiber Γ_b of p is generic relative to the extension $\bar{p}$, if the fiber $\bar{p}^{-1}(b)$ is not a degenerate fiber of $\bar{p}$ and the horizontal components of the curve $\bar{D}$ meet the fiber $\bar{p}^{-1}(b)$ normally.

Since we permanently work with polynomial extensions we shall need to know the connection between the generic fibers of the polynomial p and its generic fibers relative to the extension $\bar{p}$. It is not difficult to check the following fact (e.g., see [**Z2**, Proposition 3.6]).

Proposition. *Let $\bar{p}$ be a pseudominimal extension of a polynomial p. Then Γ_b (where $b \neq \infty$) is a generic fiber of p iff $\bar{p}^{-1}(b)$ is generic relative to $\bar{p}$.*

Corollary. *Let $\hat{p}$ be a quasi-minimal extension of a polynomial p. Then Γ_b (where $b \neq \infty$) is a generic fiber of p iff $\hat{p}^{-1}(b)$ is generic relative to $\hat{p}$.*

2.9. Let $\hat{D}$ be a complete algebraic curve of SNC-type in a compact algebraic surface $\hat{X}$. The *dual graph* $G(\hat{D})$ of $\hat{D}$ is a weighted graph whose vertices are the irreducible components of $\hat{D}$, edges between vertices are the ordinary double points that belong to the corresponding components, and the weights over vertices are the selfintersection numbers of the corresponding components. The *valency* of a vertex in the graph is the number of the incident edges. A vertex is called *an endpoint, a linear point*, or *a branch point* of the graph if its valency is 1,2, or > 2 respectively. Two vertices in the graph are neighbors if they are joined by an edge (i.e. the corresponding components in $\hat{D}$ have a common point). The dual graph is *linear* if it has no branch points.

Let E be a vertex of $G(\hat{D})$. By $G(\hat{D}) - E$ we denote the graph obtained from $G(\hat{D})$ by removing E and deleting the edges at E. Each connected component of the graph $G(\hat{D}) - E$ is called *a branch* at E.

It is well known that for every SNC-completion $\hat{X}$ of $\mathbf{C}^2$ the graph of the curve $\hat{D}$ is a tree of rational curves. In particular, $\hat{D}$ is connected and simply connected. (In fact the curve $\bar{D} = \bar{X} - \mathbf{C}^2$ is connected and simply connected for every completion $\bar{X}$ of $\mathbf{C}^2$.) Note that if $\hat{D}$ contains a (-1)-component which corresponds to a linear point or an endpoint E of $G(\hat{D})$ then by contracting this component we obtain a new curve $\tilde{D}$ which is still of SNC-type and whose graph is a tree. When E is an endpoint then the graph $G(\tilde{D})$ coincides with $G(\hat{D}) - E$, except for the weight of the former neighbor of E which is increased by 1. If E is a linear point then $G(\tilde{D})$ can be obtained from $G(\hat{D}) - E$ by joining the former neighbors of E with an edge and increasing their weights by 1. The graph $G(\tilde{D})$ may contain a linear or end point of weight -1, and one can contract the corresponding component again.

Definition. By an *RM-procedure*, we understand a sequence of successive contractions of (-1)-components which correspond to linear points and endpoints in the graph $G(\hat{D})$ and in subsequent images of $G(\hat{D})$ during these contractions. This procedure keeps going until we obtain a graph which has no linear points and endpoints of weight -1.

The remarkable Ramanujam-Morrow theorem shows that the final graph is linear and gives its complete description. Here is the part of this theorem which will be used later in this paper.

Theorem (Ramanujam-Morrow [**R**], [**M**]). *Let $\hat{X}$ be a smooth algebraic compact surface and let $\hat{D}$ be a divisor of SNC-type in $\hat{X}$. Suppose $\hat{X} - \hat{D}$ is isomorphic to $\mathbf{C}^2$. Then every RM-procedure reduce $\hat{D}$ to a curve whose dual graph has one of the following representations in Fig.* 1

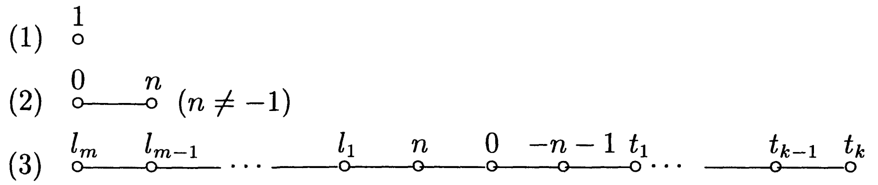

Figure 1. Ramanujam-Morrow graphs.

where $l_i \leq -2$, $t_j \leq -2$, $n > 0$, and k and m are nonnegative integers. Moreover, l_1 and t_1 cannot be simultaneously -2.

2.10. Lemma. *Let p be a primitive polynomial and let $\bar{p} : \bar{X} \to \mathbf{CP}^1$ be an extension. For each $b \in \mathbf{C}$ and every connected component A of the set $\bar{p}^{-1}(b) - p^{-1}(b)$ there exists exactly one horizontal irreducible component $\bar{H}$ of the curve $\bar{D} = \bar{X} - \mathbf{C}^2$ for which $A \cap \bar{H} \neq \emptyset$. Moreover, the set $\bar{H} \cap A$ consists of one point, and each horizontal component of $\bar{D}$ meets the fiber $\bar{p}^{-1}(\infty)$ at one point as well.*

Proof. Since p is primitive, the generic fiber of p, and, therefore, the generic fiber of $\bar{p}$ are connected. Since $\bar{X}$ is compact this implies that every fiber of $\bar{p}$ is connected. After a sequence of blowing-ups one may suppose that $\bar{D}$ is of SNC-type. (These blowing-ups do not change the number of connected components in $\bar{p}^{-1}(b) - p^{-1}(b)$ and the number of horizontal irreducible component in $\bar{D}$.) Each horizontal component meets $\bar{p}^{-1}(\infty) \subset \bar{D}$. Since $\bar{p}^{-1}(\infty)$ is connected, the statement of this lemma follows from the fact that $\bar{D}$ is connected simply connected. □

2.11. Definition. Let $\bar{D}$ be as in the previous lemma. A horizontal component E of $\bar{D}$ is called *a section* if the restriction of $\bar{p}$ to E is a one-to-one mapping.

Suppose that p is a primitive polynomial whose zero fiber Γ_0 is isomorphic to $\mathbf{C}^*$. Recall that if Γ_0 is degenerate, then p has one more degenerate fiber Γ_1, by Corollary 2.3.

Lemma. *Let p, Γ_0 be as above. Suppose that Γ_0 is degenerate and Γ_1 is the second degenerate fiber. Let $\bar{p} : \bar{X} \to \mathbf{CP}^1$ be an extension of p. Then both the number of horizontal components of $\bar{D}$ and the number of irreducible components in Γ_1 do not exceed 2. In the case of a rational polynomial p*

both these numbers are 2, and at least one of the horizontal components is not a section.

Proof. Let $\bar{\Gamma}_0$ be the closure of Γ_0 in $\bar{X}$. After some blowing-ups (if necessary) one may suppose that the curve $\bar{\Gamma}_0$ is smooth in $\bar{X}$. Since Γ_0 has two punctures, the set $\bar{\Gamma}_0 - \Gamma_0$ consists of two points. The fiber $\bar{p}^{-1}(0)$ is connected and, hence, the number of connected components in $\bar{p}^{-1}(0) - \Gamma_0$ is ≤ 2. By Lemma 2.10, there are at most two horizontal components in $\bar{D}$. By Theorem 2.4, the number of irreducible components in the second degenerate fiber does not exceed two. If p is rational and $\bar{D}$ has only one horizontal component then Γ_1 is irreducible. Therefore, p must be equivalent to a linear polynomial ([**Sa1**, Theorem A]), i.e. p cannot have a $\mathbf{C}^*$-fiber. This shows that in the case of rational p there are two horizontal components in $\bar{D}$. By Theorem 2.4, there are two irreducible components in the second degenerate fiber. If both horizontal components are sections then the generic fiber of p is $\mathbf{C}^*$ and Γ_0 must be generic, by Theorem 2.3. This contradicts the assumption that the zero fiber is not generic. □

It is worth mentioning that there was a wrong claim in [**K**] that at most one horizontal component in an extension of any rational polynomial may be different from a section. An example of a rational polynomial whose extension has more than one horizontal component different from a section was constructed in [**AC**].

2.12. Lemma. *Let the assumption be as in* 2.11. *Suppose that $\bar{H}_1$ and $\bar{H}_2$ are horizontal components of $\bar{D}$. Then for each $k = 1, 2$ and each $b \neq 0, \infty$ the component $\bar{H}_k$ meets the fiber $\bar{p}^{-1}(b)$ normally, the set $\bar{H}_k \cap \bar{p}^{-1}(b)$ contains only smooth points of $\bar{p}^{-1}(b)$ which belong to non-multiple components of $\bar{p}^*(b)$. (In other words the local intersection index of $\bar{H}_k$ and $\bar{p}^{-1}(b)$ is 1.) Moreover, if the horizontal component is a section, the same is true for $b = 0, \infty$.*

Proof. Let $\bar{H}$ be one of the horizontal components. Let the mapping $\bar{p}|_{\bar{H}} : \bar{H} \to \mathbf{CP}^1$ be m-sheeted. The set $\bar{p}^{-1}(0) - \Gamma_0$ consists of two connected components. Since $\bar{D}$ has two horizontal components and each of them intersects $\bar{p}^{-1}(0)$, the set $\bar{p}^{-1}(0) \cap \bar{H}$ consists of one point, by Lemma 2.10. The same is true for the set $\bar{p}^{-1}(\infty) \cap \bar{H}$. If $m > 1$ these points are branch points of index m for the projection $\bar{p}|_H : H \to \mathbf{CP}^1$. By the Riemann-Hurwitz formula, there is no other branch point. In the case of $m = 1$ there is no branch point at all. It remains to note that if the local intersection index of $\bar{H}$ and $\bar{p}^{-1}(b)$ at a point $x \in \bar{H} \cap \bar{p}^{-1}(b)$ is ≥ 2 then x must be a branch point of the mapping $\bar{p}|_{\bar{H}}$. □

Remark. The fact that for $b \neq 0, 1, \infty$ the fiber $\bar{p}^{-1}(b)$ meets $\bar{H}$ normally can be easily obtained from Proposition 2.8. The only new information, which we get from Lemma 2.12, is that $\bar{p}^{-1}(1)$ meets $\bar{H}$ normally as well.

2.13. The next proposition enables us to describe polynomials with a $\mathbf{C}^*$-fiber in many cases.

Proposition. *Let Γ_0 and C be disjoint closed affine algebraic curves in $\mathbf{C}^2$. Suppose that Γ_0 is isomorphic to $\mathbf{C}^*$ and C is isomorphic to $\mathbf{C}$. Then there exists a coordinate system (x, y) in $\mathbf{C}^2$ for which C is the y-axis and the curve Γ_0 is given by one of the following equations*
(i) $x^n + \sigma^k(x, y) = 0$;
(ii) $x^n \sigma^k(x, y) + 1 = 0$;
where n, k are relatively prime natural numbers, $\sigma(x, y) = x^m y + g(x)$ with $g \in \mathbf{C}[x]$, $\deg\, g < m$, and $g(0) \neq 0$ for $m > 0$.

Proof. According to the Abhyankar-Moh-Suzuki Theorem [**AM**], [**Su1**] one may suppose that C coincides with the axis $x = 0$. Let Γ_0 be the zero fiber of a primitive polynomial $p(x, y) = \Sigma a_{ij} x^i y^j$. Note that there exists $j_0 > 0$ such that $a_{ij_0} \neq 0$ for some i since otherwise Γ_0 is a line. Choose natural $s > 0$ so that $sj > i$ for every pair (i, j) such that $j > 0$ and $a_{i,j} \neq 0$. Then one can represent $p(x, x^{-s}y)$ as $x^e h(x, y)$, where e is an integer, x does not divide the polynomial $h(x, y)$, and $h(0, 0) = 0$.

It is clear that the curve $\Gamma_0' = \{(x, y) | h(x, y) = 0\}$ is homeomorphic to $\mathbf{C}$. (It is so since the birational mapping $(x, y) \to (x, x^{-s}y)$ establishes an isomorphism between Γ_0 and $\Gamma_0' - (0, 0)$. More precisely: Γ_0' is the proper transform of Γ_0 under this mapping.) By the Lin-Zaidenberg Theorem [**LZ**], one may suppose that the curve $\Gamma_0' \cup C$ is given by the zero fiber of a quasi-homogeneous polynomial $u^r(u^l + v^k)$ in a certain coordinate system (u, v) ($u = f_1(x, y), v = f_2(x, y)$, where f_1 and f_2 are polynomials giving an automorphism). In this system $C = \{u = 0\}$. Thus we may suppose $f_1(x, y) = x$ and, therefore, $f_2(x, y) = y + \varphi(x)$. In particular, $h(x, y) = x^l + (y + \varphi(x))^k$. Passing to $p(x, y)$, we obtain the desired conclusion. $\square$

Remark. In the above proposition one may assume that C is only homeomorphic to $\mathbf{C}$. In order to show that C is actually smooth one may use the following argument. If C is not smooth then it follows from the Lin-Zaidenberg Theorem that $\mathbf{C}^2 - C$ admits a natural $\mathbf{C}^*$-action. It is not difficult to check that Γ must be an orbit of this action. But these orbits are not closed which is a contradiction. We do not need this stronger version of Proposition later. It is also worth mentioning that this Proposition is a generalization of Saito's Theorem on **C***-polynomials [**Sa2**] and Zaidenberg's Theorem on **C***-actions [**Z4**].

Corollary. *Let* Γ_0 *and* C *be as in the above proposition. Suppose that* Γ_0 *is the zero fiber of a primitive polynomial. Then either* Γ_0 *is generic or* p *is non-rational.*

Proof. Suppose that p is equivalent to one of the polynomials (ii) from Proposition 2.13. Then $p^{-1}(c)$ is given by $y = x^{-m}[(c-1)x^{-n/k} - g(x)]$ which implies that Γ_0 is generic. If p is equivalent to one of the polynomials (i) then the generic fiber of p is isomorphic to the curve $x^n + y^k = 1$ with extra punctures. (In order to see this it suffices to note that $(x,y) \to (x, \sigma(x,y))$ is a birational morphism.) When neither n nor k is 1 then the curve $x^n + y^k = 1$ has a positive genus, i.e. p is non-rational. Consider $n = 1$. Then $p^{-1}(c)$ is given by $y = x^{-m}[(c-x)^{1/k} - g(x)]$ which implies that Γ_0 is generic. The case when $k = 1$ is similar. ☐

2.14. Notation and Terminology. We conclude this section with citing notation we shall use in the remainder of this article. We always denote by p a primitive rational polynomial with fibers $\Gamma_b = p^{-1}(b)$ for $b \in \mathbf{C}$. The zero fiber Γ_0 is degenerate and is isomorphic to $\mathbf{C}^*$. By $\bar{p} : \bar{X} \to \mathbf{CP}^1$ and $\hat{p} : \hat{X} \to \mathbf{CP}^1$ we denote extensions of p. The complement of $\mathbf{C}^2$ in $\bar{X}$ (respectively $\hat{X}$) is denoted by $\bar{D}$ (respectively $\hat{D}$). Recall that these curves are always simply connected. The extension $\hat{p}$ is always quasi-minimal and, therefore, the curve $\hat{D}$ is of SNC-type. For every SNC-curve $\hat{D}$ its dual graph is denoted by $G(\hat{D})$. By Lemma 2.11, we know that $\hat{D}$ (resp. $\bar{D}$) has only two horizontal components $\hat{H}_1$ and $\hat{H}_2$ (resp. $\bar{H}_1, \bar{H}_2$). At least one of them is not a section, by Lemma 2.11. We always suppose that $\hat{H}_2$ (resp. $\bar{H}_2$) is not a section. Due to Corollary 2.3 we know that there is one more degenerate fiber of p, which is always $\Gamma_1 = p^{-1}(1)$. It contains two irreducible components C_1 and C_2, by Lemma 2.11. The closures of these components in $\hat{X}$ are $\hat{C}_1, \hat{C}_2$ respectively. Later we shall see that either C_1 or C_2 is a non-multiple component of $p^{-1}(1)$. After proving this we shall always suppose that C_2 is not multiple.

Since we shall work a lot with graphs we have to introduce some terminology. Let G_1, G_2 be subgraphs of the graph $G = G(\hat{D})$. The subgraph G_1 is contractible if the curve that consists of components corresponding to its vertices is contractible. (Recall that an algebraic curve C in a smooth closed algebraic surface Y is called contractible if there exist another smooth closed algebraic surface Z, a point $z \in Z$, and a morphism $\varphi : Y \to Z$ which is a composition of blowing-ups of Z at z and infinitely near points to z such that $\varphi^{-1}(z) = C$.) By $G_1 \cup G_2$ we denote the subgraph of G that contains all vertices of G_1 and G_2 and all edges between these vertices that belong to G. The graph $G - G_1$ is obtained form G by removing all vertices of G_1 from G and deleting all edges incident to these vertices. Let E be a component in $\hat{D}$. We denote the corresponding vertex of $G = G(\hat{D})$ by the same letter E. We say that E is a (-1)-vertex if its weight is -1, i.e. E is a (-1)-curve. Let $\tilde{D}$ be the curve obtained from $\hat{D}$ after several contractions in an RM-procedure. Suppose that a component F is not contracted after these steps. Then, by abusing notation, we denote the image of the vertex F in $\tilde{D}$ and in $G(\tilde{D})$ by the same letter F unless it may cause misunderstanding. Some subgraphs are denoted by rectangles in the figures of graphs. A rectangle may correspond to an empty subgraph unless the opposite is stated.

We shall consider later linear graphs with n vertices, each of which has weight -2. We call such a graph *standard* and denote it by $S(n)$.

3. The first description of $G(\hat{D})$.

The central result of this section is Proposition 3.6 which gives some essential features of graph $G(\hat{D})$ (see Fig. 2). In particular, this first description of $G(\hat{D})$ implies that the fiber $\hat{p}^{-1}(0)$ is irreducible (Proposition 3.7) which is a key for obtaining the graph of the fiber $\hat{p}^{-1}(\infty)$ in Section 4.

3.1. By Theorem 2.6, for every $b \in \mathbf{CP}^1$ the fiber $\hat{p}^{-1}(b)$ can be contracted to a smooth rational irreducible curve (since the fibers of morphism q from Theorem 2.6 are irreducible). In other words there exists a morphism $\delta : \hat{X} \to \dot{X}$ which is a composition of blowing-ups of a smooth closed algebraic surface $\dot{X}$ so that $\delta^{-1}(\dot{E}) = \hat{p}^{-1}(b)$ where $\dot{E}$ is a smooth irreducible rational curve in $\dot{X}$ and the restriction of δ to $\hat{X} - \hat{p}^{-1}(b)$ is an isomorphism between $\hat{X} - \hat{p}^{-1}(b)$ and $\dot{X} - \dot{E}$. By the universal property of blowing-ups, there exists a morphism $\dot{p} : \dot{X} \to \mathbf{CP}^1$ such that $\hat{p} = \dot{p} \circ \delta$ and $\dot{E} = \dot{p}^{-1}(b)$. Suppose we have compositions of blowing-ups $\delta_1 : \hat{X} \to \tilde{X}$ and $\delta_2 : \tilde{X} \to \dot{X}$ for which $\delta = \delta_2 \circ \delta_1$. Put $\tilde{p} = \dot{p} \circ \delta_2 : \tilde{X} \to \mathbf{CP}^1$. Since the preimage of every SNC-curve under blowing up remains an SNC-curve we may speak about the graphs of $\hat{p}^{-1}(b)$ and $\tilde{p}^{-1}(b)$.

Lemma. *Let G be the graph of a fiber $\tilde{p}^{-1}(b)$. Suppose that this fiber contains at least two irreducible components. Then*

(1) *all weights of G are negative and G contains a (-1)-vertex;*

(2) *if E is a (-1)-vertex in G then E is a linear point or an endpoint;*

(3) *two (-1)-vertices in G cannot be neighbors when $\tilde{p}^{-1}(b)$ consists of more than two components;*

(4) *if E is a linear point of weight -1 then it is a multiple component of the divisor $\tilde{p}^*(b)$, and, therefore, all components of the curve $\delta_1^{-1}(E)$ are multiple in the divisor $\hat{p}^*(b)$.*

Proof. In order to obtain the fiber $\tilde{p}^{-1}(b)$ from $\dot{E}$ one has to blow $\dot{X}$ up at a point from $\dot{E}$ and, perhaps, to repeat blowing up the resulting surfaces at points from the fibers over b several times (we need at least one blowing-up since $\tilde{p}^{-1}(b)$ is not irreducible). After each blowing-up we obtain a fiber over b whose dual graph is a tree of rational curves and which contain a (-1)-curve as a result of the last blowing-up. Since $\dot{E}$ is a fiber of $\dot{p}$ its self-intersection number $\dot{E} \cdot \dot{E} = 0$. Hence the weights of the dual graph of the fiber over b in the first blowing-up of $\dot{X}$ are already negative which implies (1). Assume now that a (-1)-vertex E is a branch point of G. In order to reduce the fiber over b to an irreducible curve one has to contract a branch of G at E. After this the weight of E becomes non-negative, i.e. this component cannot be shrunk further. Thus one need to contract all other branches at E. This makes the weight of E positive in contradiction with the fact that the selfintersection of the fiber must be 0. Thus (2) holds. The same reason implies (3).

If E is a linear point of G it appears in the blowing-up procedure after blowing up an ordinary double point of the fiber over b. Hence the multiplicity of E in $\tilde{p}^*(b)$ is at least 2. □

3.2. Proposition. *Let E be a branch point of $G = G(\hat{D})$ of weight -1. Then*

(i) *the irreducible component E of the curve $\hat{D}$ cannot be contracted in any Ramanujam-Morrow procedure, and after this procedure the weight of E becomes non-negative;*

(ii) *at most two branches of G at E are non-contractible.*

Proof. One cannot contract E at once in an RM-procedure since it is a branch point. Thus in order to contract E one must contract a branch at E first. We have to contract a neighbor of E at some step while contracting this branch. But the weight of E becomes non-negative after this step. Hence E cannot be contracted. This implies that if more than two branches are non-contractible at E then the graph G cannot be reduced to a linear graph via an RM-procedure which is a contradiction. □

Corollary. (i) *Let E and F be branch points of G. Suppose that E is a (-1)-vertex. Consider all branches at F that do not contain E. Then all of them except possibly for one are contractible.*

(ii) *Let E be a branch point of G of weight -1 and valency ≥ 4 (we do not assume the existence of another branch point now), and let G^1, G^2 be branches of at E. Then $G^1 \cup G^2$ contains either a non-branch (-1)-vertex or a vertex of zero weight.*

Proof. Note that the branch at F that contains E is non-contractible, by

Proposition 3.2 (i). If there exist two other non-contractible branches at F, then F remains a branch point after any RM-procedure. Contradiction.

Assume that G^1 and G^2 do not contain (-1)-vertices which are not branch points of G. Hence none of the vertices in these subgraphs can be contracted. Moreover, since E is non-contractible these vertices have no contractible neighbors in an RM-procedure, i.e. all of these vertices preserve their weights during this procedure. By Proposition 3.2 (ii), all other branches are contractible and after contracting them we obtain a positive weight of E, since the number of these contractible branches is ≥ 2. Hence one of the neighbors of E from G^1 or G^2 must have a zero weight, by Theorem 2.9. ☐

3.3. Lemma. *There is no linear point or endpoint of weight* -1 *in* $G(\hat{D})$ *except for, possibly,* $\hat{H}_1$ *and* $\hat{H}_2$.

Proof. Let E be a linear point or an endpoint in $G(\hat{D})$ of weight -1. If it is different from $\hat{H}_1$ and $\hat{H}_2$ it corresponds to a vertical component of $\hat{D}$. After contracting E we obtain a new extension $\bar{p} : \bar{X} \to \mathbf{CP}^1$ such that the curve $\bar{D} = \bar{X} - \mathbf{C}^2$ is of SNC-type. This contradicts quasi-minimality of $\hat{p}$. ☐

3.4. By quasi-minimality of the extension $\hat{p}$, horizontal components $\hat{H}_1$ and $\hat{H}_2$ meets the fiber $\hat{p}^{-1}(\infty)$ normally. Denote by G_∞ the subgraph of $G(\hat{D})$ that corresponds to the fiber $\hat{p}^{-1}(\infty)$.

Lemma. *The curves* $\hat{H}_1$ *and* $\hat{H}_2$ *meet* $\hat{p}^{-1}(\infty)$ *at different components denoted by* E_1 *and* E_2 *respectively. All weights of the graph* $G_\infty - (E_1 \cup E_2)$ *are* ≤ -2. *The weights of* E_1 *and* E_2 *are also negative and at least one of them is* -1.

Proof. By Theorem 2.6, the fiber $\hat{p}^{-1}(\infty)$ can be contracted to an irreducible curve in the way we did in the proof of Lemma 3.1. After this contraction we obtain a new extension $\bar{p} : \bar{X} \to \mathbf{CP}^1$ with the following properties: the fiber $\bar{E} = \bar{p}^{-1}(\infty)$ is irreducible and non-multiple (since the same is true for the fibers of the morphism q from Theorem 2.6), and $\bar{X} - \bar{E}$ is isomorphic to $\hat{X} - \hat{p}^{-1}(\infty)$. Then the curve $\bar{D}$ is simply connected and its horizontal components $\bar{H}_1, \bar{H}_2$ meet $\bar{E}$ at points a_1, a_2 respectively, by Lemma 2.10. (May be $a_1 = a_2$.) Since $\bar{H}_2$ is not a section its intersection index with $\bar{E}$ is not 1. Since $\bar{E}$ is not a multiple fiber of $\bar{p}$ the curve $\bar{H}_2$ cannot meet $\bar{E}$ normally. This means that in order to obtain the quasi-minimal extension $\hat{p} : \hat{X} \to \mathbf{CP}^1$ we have to blow $\bar{X}$ up at a_2. In particular, $\hat{p}^{-1}(\infty)$ is not irreducible. By Lemma 3.1, all the weight of G_∞ are negative and it contains a (-1)-vertex E. This vertex must be either a linear point or an endpoint of G_∞. Note that E must be a branch point of $G(\hat{D})$, by Lemma 3.3, i.e. E is a neighbor of at least one of the vertices $\hat{H}_1, \hat{H}_2$, by Lemma 3.1 (2). Assume that $\hat{H}_1$ and $\hat{H}_2$ are neighbors of E simultaneously. In particular, there is no other (-1)-vertex in G_∞. Assume that E is a linear point of G_∞. Then the valency of E in $G(\hat{D})$ is 4. Consider the two branches at E whose union is $G_\infty - E$. By Corollary 3.2 (ii), one of them has a vertex of zero weight which is a contradiction. Assume E is an endpoint of G_∞. Since the other vertices of G_∞ have weights ≤ -2 it cannot be a linear graph, otherwise induction by the number of vertices shows that the fiber $\hat{p}^{-1}(\infty)$ cannot be contracted to the irreducible component $\bar{E}$ with selfintersection 0. Thus G_∞ has a branch point F. The branches of G_∞ at F that do not contain E are not contractible. This contradicts Corollary 3.2 (i). Thus $\hat{H}_1$ and $\hat{H}_2$ meets $\hat{p}^{-1}(\infty)$ at different components E_1 and E_2. As we mentioned before each (-1)-vetex from G_∞ must be a neighbor of either $\hat{H}_1$ or $\hat{H}_2$ in $G(\hat{D})$. Hence Lemma 3.1 (1) concludes the proof. □

3.5 Lemma. *Under the assumption of Lemma* 3.4 *one of the weights of* E_1 *and* E_2 *must be* ≤ -2. *When* $\hat{H}_1$ *is a section the weight of* E_1 *is* ≤ -2 *and, therefore, the weight of* E_2 *is* -1.

Proof. By Lemma 3.4, these weights are negative. Assume that both E_1 and E_2 are (-1)-vertices. By Lemma 3.1 (3), there are no more vertices in G_∞ and, by Lemma 3.3, E_1 and E_2 are branch points of $G(\hat{D})$. By Proposition 3.2 (i), the weights of E_1 and E_2 become non-negative after an RM-procedure. By Theorem 2.9, E_1 and E_2 must become neighbors after this procedure. Note that the weights of the vertices in the connected component of $G(\hat{D}) - (E_1 \cup E_2)$ that is between E_1 and E_2 are ≤ -2, by Lemma 3.4, i.e. none of these vertices can be contracted in an RM-procedure. Thus there is no vertices between E_1 and E_2 in $G(\hat{D})$, i.e. they are neighbors in $G(\hat{D})$ and in G_∞. This contradicts Lemma 3.1 (3). Therefore, one of the weights is ≤ -2.

Suppose that $\hat{H}_1$ is a section and assume that the weight of E_1 is -1. By Lemma 3.3, E_1 cannot be an end point of G_∞. (Otherwise it is a linear point of $G(\hat{D})$.) Hence E_1 is a linear point in G_∞ and, therefore, a multiple component of the divisor $\hat{p}^*(\infty)$, by Lemma 3.1 (4). In particular, the intersection number of E_1 and $\hat{H}_1$ is ≥ 2 which contradicts Lemma 2.12. Thus the weight of E_1 must be ≤ -2 when $\hat{H}_1$ is a section. □

Convention. From now on we always suppose now that the weight of E_2 is -1.

3.6. Proposition. *The graph* $G(\hat{D})$ *looks like the graph in Fig.* 2. *More precisely:*

(i) *The subgraph* G_∞ *coincide with* $G^1_\infty \cup E_1 \cup G^0_\infty \cup E_2 \cup G^2_\infty$.

(ii) *The subgraph* G^2_∞ *is non-empty.*

(iii) *The subgraphs* $E_2 \cup G^2_\infty, \hat{H}_1 \cup G^1_1$, $\hat{H}_2 \cup G^2_1$, *and* G^1_∞ *are linear.*

(iv) *One of the branches at* E_2 *which is different from* G^2_∞ *must be contractible.*

(v) *The weight of* $\hat{H}_2$ *is* ≥ -1 *and* $\hat{H}_1$ *is a* (-1)*-vertex.*

Proof. The first two statements follow from Lemmas 3.3, 3.4, and 3.5, and from Convention 3.5. Assume that the graph $E_2 \cup G^2_\infty$ contains a branch point F which should be different from E_2, by Lemma 3.1 (2). The branches at F which do not contain E_2 are non-contractible, by Lemma 3.4. But this contradicts Corollary 3.2 (i) (in order to see this put $E = E_2$). Thus the subgraph $E_2 \cup G^2_\infty$ is linear. Exactly the same argument implies the rest of the statement (iii).

Since G^2_∞ does not contain (-1)-vertices it is non-contractible. By Proposition 3.2, one of the branches at E_2 which is different from G^2_∞ must be contractible, i.e (iv) is proven.

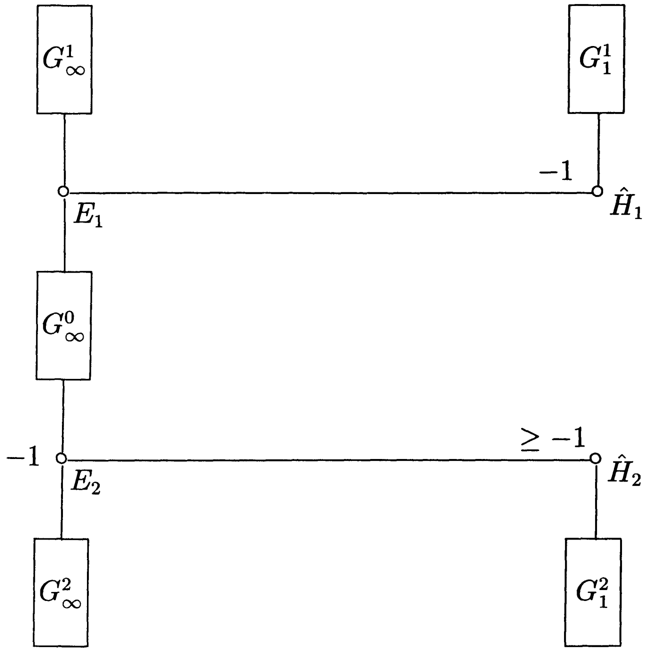

Figure 2. The first description of $G(\hat{D})$.

First consider the case when the branch $\hat{H}_2 \cup G^2_1$ is contractible. It follows from Lemma 3.3 that we cannot contract vertices from G^2_1 at the first step of an RM-procedure. Hence $\hat{H}_2$ should be a (-1)-vertex in this case. After contracting $\hat{H}_2 \cup G_1^2$ the weight of E_2 becomes non-negative (see Proposition 3.2). By Theorem 2.9, the neighbor of E_2 after an RM-procedure must have a non-negative weight as well. Hence, since the weights of $G_\infty - E_2$ are ≤ -2, by Lemmas 3.4 and 3.5, we have to contract some vertices in $\hat{H}_1 \cup G_1^1$. Lemma 3.3 implies that $\hat{H}_1$ should be contracted first, i.e. it is a (-1)-vertex in this case.

In the second case we can contract the branch at E_2 that contains $\hat{H}_1$. Same argument as above shows that $\hat{H}_1$ must be a (-1)-vertex and the weight of $\hat{H}_2$ is ≥ -1. □

Corollary. *The subgraphs* G_1^1 *and* G_1^2 *from Fig. 2 do not contain linear points and endpoints of weight* -1.

Proof. Assume the contrary and let F be such (-1)-vertex in, say, G_1^1. Since $\hat{H}_1 \cup G_1^1$ is linear one can see that F must be a linear point or an end point of $G(\hat{D})$ which contradicts Lemma 3.3. □

3.7. Lemma. *The vertices of the subgraphs* G_1^1 *and* G_1^2 *from Fig. 2 correspond to components of the fiber* $\hat{p}^{-1}(1)$.

Proof. By Corollary 2.8, the vertices from $G_1^1 \cup G_1^2$ correspond to components from either $\hat{p}^{-1}(1)$ or $\hat{p}^{-1}(0)$ since all other fibers are generic. Assume that one of subgraphs, say G_1^1, corresponds to components from $\hat{p}^{-1}(0)$. By Corollary 2.5, $\hat{p}^{-1}(0)$ can be contracted to the component that is the closure of Γ_0 in $\hat{X}$. Hence the subgraph G_1^1 is contractible, i.e. it contains a (-1)-vertex F. This contradicts Lemma 3.3. By an analogous argument, the vertices of G_1^2 cannot correspond to components from $\hat{p}^{-1}(0)$. □

This implies the following fact.

Proposition. *The fiber* $\hat{p}^{-1}(0)$ *consists of one irreducible component. Moreover, suppose that* m_k *is the intersection number of* $\hat{H}_k$ *and the fiber of* $\hat{p}$ *where* $k = 1, 2$. *Then* $\hat{H}_1$ *and* $\hat{H}_2$ *meet* $\hat{p}^{-1}(0)$ *at different points* a_1 *and* a_2 *respectively, and the contact order between* $\hat{H}_k$ *and* $\hat{p}^{-1}(0)$ *at* a_k *is* m_k.

4. The fiber over ∞.

4.1. The aim of this section is to describe the graph G_∞ of the fiber $\hat{p}^{-1}(\infty)$. First we introduce some *notation which will be used in the rest of this paper.*

Let Q, q, δ be the same as in Theorem 2.6. We consider the following subvarieties of Q: $Q^1 = q^{-1}(\mathbf{C})$, $Q^2 = q^{-1}(\mathbf{C}^*)$, and $Q^3 = q^{-1}(\mathbf{C} - \{0, 1\})$.

We put also $H_k = \delta(\hat{H}_k)$ $(k = 1, 2)$. Since the fibers $\hat{p}^{-1}(b)$ are irreducible for $b \in \mathbf{C} - \{0, 1\}$, by Corollary 2.8, the restriction of δ to $\hat{p}^{-1}(\mathbf{C} - \{0, 1\})$ is an isomorphism between $\hat{p}^{-1}(\mathbf{C}-\{0,1\})$ and Q^3. Moreover, since the fiber $\hat{p}^{-1}(0)$ is irreducible, by Proposition 3.7, the restriction of δ to $\hat{p}^{-1}(\mathbf{C}-\{1\})$ is also isomorphism between $\hat{p}^{-1}(\mathbf{C}-\{1\})$ and $Q^1-q^{-1}(1)$. Hence H_1 and H_2 meets $q^{-1}(0)$ at different points c_1 and c_2 respectively, H_k is smooth at c_k, and the contact order between H_k and $q^{-1}(0)$ and H_k is m_k where m_k is the same as in Proposition 3.7.

Introduce a coordinate system $(x,(y_1:y_2))$ in $Q^1=\mathbf{C}\times\mathbf{CP}^1$ so that $q(x,(y_1:y_2))=x$ and the coordinates of c_1 and c_2 in $q^{-1}(0)$ are $(0:1)$ and $(1:0)$ respectively. Consider the antiholomorphic mapping $'\varphi: Q^1\to Q^1$ given by

$$'\varphi(x,(y_1:y_2))=(\bar{x},(\bar{y}_1:\bar{y}_2))$$

(where $\bar{a}$ means the complex conjugate of number a) and consider the isomorphism $''\varphi: Q^2\to Q^2$ given by

$$''\varphi(x,(y_1:y_2))=(1/x,(y_1:y_2)).$$

Let $'H_k$ be the closure of $'\varphi(H_k)$ in Q and $''H_k$ be the closure of $''\varphi(H_k)$ in Q.

Convention. For every curve F in Q (or in Q^l with $l<k$) we denote by F^k the curve $F\cap Q^k$. Similarly, if ψ is a morphism from Q (or Q^l) then ψ_k is the restriction of ψ to Q^k. For instance, $'H_k^3={}'H_k\cap Q^3$ and $q_3=q\,|_{Q^3}$.

Lemma. *There exists an isomorphism* $\xi\colon Q^3\to Q^3$ *such that* $\xi('H_k^3)={}''H_k^3$ *and* $q_3=q_3\circ\xi$.

4.2. The proof of Lemma 4.1 is very computational and, therefore, we prefer to hide it in the Appendix. In this section we extract a consequence from it. In order to do this we need an intermediate step.

Let $\tilde{X}_1,\tilde{X}_2,X_1,X_2$ be smooth algebraic surfaces such that $X_k\subset\tilde{X}_k$, and let $\tilde{p}_k:\tilde{X}_k\to\mathbf{CP}^1$ be nonconstant morphisms such that every non-empty fiber $\tilde{p}_k^{-1}(c)$ is compact. Put $p_k=\tilde{p}_k|_{X_k}$ and suppose that $\kappa: X_1\to X_2$ is an isomorphism so that $\alpha\circ p_1=p_2\circ\kappa$ where α is an automorphism of $\mathbf{CP}^1$. Suppose also that $\tilde{p}_2(\tilde{X}_2)$ does not contain $\alpha(b)$ for some point $b\in\mathbf{CP}^1$. Let $\tilde{F}_{1k},\dots,\tilde{F}_{lk}$ be irreducible curves in $\tilde{X}_k$ such that $\tilde{p}_k$ is not constant on any of them. Put $F_{ik}=\tilde{F}_{ik}\cap X_k$ and suppose that $\kappa(F_{j1})=F_{j2}$. Denote by $\bar{p}_k:\bar{X}_k\to\mathbf{CP}^1$ an extension of $\tilde{p}_k$ and by $\bar{F}_{jk}$ the closure of $\tilde{F}_{jk}$ in $\bar{X}_k$.

Lemma. *Suppose that* $\bar{F}_{11},\dots,\bar{F}_{l1}$ *meet* $\bar{p}_1^{-1}(b)$ *at different points* $a_{11},\dots,a_{l1}$, *that* $\bar{F}_{j1}$ *is smooth at* a_{j1}, *and that the contact order between* $\bar{F}_{j1}$ *and* $\bar{p}_1^{-1}(b)$ *is* n_j. *Then one may choose an extension* $\bar{p}_2$ *so that* $\bar{F}_{12},\dots,\bar{F}_{l2}$ *meet the fiber* $\bar{p}_2^{-1}(\alpha(b))$ *at different points* $a_{12},\dots,a_{l2}$, *that* $\bar{F}_{j2}$ *is smooth at* a_{j2} , *and that the contact order of* F_{j2} *and* $\bar{p}_2^{-1}(\alpha(b))$ *is* n_j.

Proof. Let $S = \alpha^{-1}(\mathbf{CP}^1 - \tilde{p}_2(\tilde{X}_2))$. Put $X_1' = \tilde{p}_1^{-1}(S) \cup X_1$. Glue X_1' and $\tilde{X}_2$ along $X_1 \approx X_2$ via κ and we obtain the desired compactification of $\tilde{X}_2$. □

4.3. Now we are ready to extract a consequence from Lemma 4.1.
Proposition. *Let $\tilde{p}$ be the restriction of $\hat{p}$ to $\hat{X} - \hat{p}^{-1}(\infty)(= \delta^{-1}(Q^1))$. Then there exists an extension $\bar{p} : \bar{X} \to \mathbf{CP}^1$ of $\tilde{p}$ such that*

(i) *the fiber $\bar{p}^{-1}(\infty)$ is irreducible;*

(ii) *$\bar{H}_1$ and $\bar{H}_2$ meet $\bar{p}^{-1}(\infty)$ at different points a_1 and a_2 respectively;*

(iii) *for each $k = 1, 2$ the curve $\bar{H}_k$ is smooth at a_k and the contact order between $\bar{H}_k$ and $\bar{p}^{-1}(\infty)$ is m_k where m_k is the same as in Proposition 3.7.*

Proof. Recall that the contact order of H_1 and $q^{-1}(0)$ at c_1 is m_1 and H_1 is smooth at c_1. Hence H_1 is given by $x = y^{m_1} f(y)$ in the local coordinate system (x, y) with origin at c_1 where $y = y_1/y_2$ and f is a holomorphic function such that $f(0) \neq 0$. The definitions of $'H_1$ and $'\varphi$ imply that the local equation for $'H_1$ is $x = y^{m_1}\overline{f(\bar{y})}$ (where "bar" means the complex conjugate). Hence $'H_1$ is smooth at c_1 and has the contact order m_1 with $q^{-1}(0)$. Similar fact holds, of course, for $'H_2$. Application of Lemma 4.2 to the isomorphism ξ implies the existence of an extension of q_3 such that the closures of $''H_1^3$ and $''H_2^3$ meet the fiber over 0 at different points with multiplicities m_1 and m_2 respectively and, moreover, these points are smooth points of the closures of $''H_1^3$ and $''H_2^3$ in Q^1 respectively. Application of Lemma 4.2 to the isomorphism $''\varphi$ implies the existence of an extension of q_3 with similar properties of the curves H_1^3 and H_2^3 over ∞. The last application of Lemma 4.2 to the isomorphism $\delta \mid_{\delta^{-1}(Q^3)}$ yields the desired conclusion. □

4.4. Recall that by $S(m)$ (where $m \geq 0$) we denote a linear graph with m vertices each of which has weight -2. Such graphs will be referred as standard in the sequel.

Lemma. *There exists a quasi-minimal extension $\hat{p} : \hat{X} \to \mathbf{CP}^1$ of p such that the graph G_∞ of the fiber $\hat{p}^{-1}(\infty)$ is linear and looks like in Fig.* 3a. *One of the horizontal components of $\hat{D}$ is a section.*

Proof. Let $\bar{p} : \bar{X} \to \mathbf{CP}^1$ be as in Lemma 4.3. In particular $\bar{H}_1$ and $\bar{H}_2$ meet $\bar{p}^{-1}(\infty)$ at different points with multiplicities m_1 and m_2 respectively. Consider two cases: (1) m_1 and $m_2 > 1$ and (2) $m_1 = 1$. Note that $\bar{D} - \bar{p}^{-1}(\infty)$ consists of two connected components each of which is an SNC-type curve (since $\bar{D} - \bar{p}^{-1}(\infty)$ is isomorphic to $\hat{D} - \hat{p}^{-1}(\infty)$, by construction and

Lemma 3.6). Hence in case (1) in order to obtain a quasi-minimal extension from $\bar{p}$ we have to keep blowing $\bar{X}$ up at a_1, a_2 and infinitely near points until the horizontal components meets the fiber over ∞ normally.

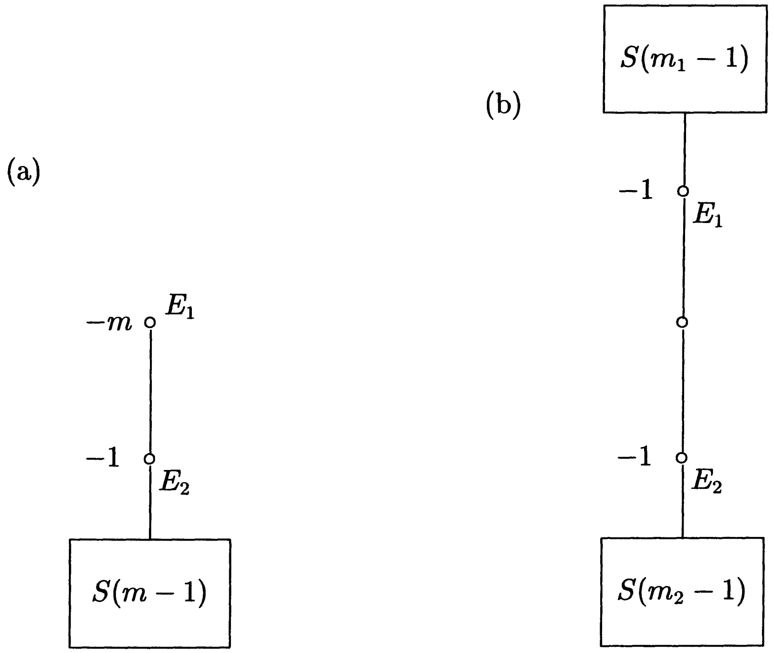

Figure 3. The graph G_∞.

It happens when the graph of the fiber over ∞ looks like in Fig. 3b. Note this graph contains two (-1)-vertices which contradicts Lemma 3.5. Thus this case does not hold. In (2) m_2 must be ≥ 2 since otherwise both horizontal components are sections which contradicts Lemma 2.11. Replace further m_2 by m. In order to obtain a quasi-minimal extension from $\bar{p}$ we have to blow $\bar{X}$ up at a_2 and $m-1$ infinitely near points to a_2. This leads to the graph G_∞ looking as in Fig. 3a. □

5. The fiber Γ_1.

From now on we suppose that G_1 *is the graph of* $\hat{p}^{-1}(1)$. Let the notation be as in Section 2.14. Note that due to Corollary 2.13 neither C_1 nor C_2 is isomorphic to **C**. Recall that $\hat{C}_k$ is the closure of C_k in $\hat{X}$.

5.1. Lemma. *Either* $\hat{C}_1$ *or* $\hat{C}_2$ *is a non-multiple component of* $\hat{p}^*(1)$.

Proof. Let m_k be the intersection number $\hat{H}_k \cdot \hat{p}^{-1}(0)$. (We know already that $m_1 = 1$ but it is not essential here.) Note that $m_1 + m_2 \geq 3$ since otherwise the generic fiber of p is $\mathbf{C}^*$ which contradicts our assumption about p. Thus $\hat{H}_1 \cup \hat{H}_2$ meets $\hat{p}^{-1}(1)$ at m_1+m_2 different points which belong to non-multiple components of $\hat{p}^{-1}(1)$, by Lemma 2.12. Note that $\hat{p}^{-1}(1) \cap \hat{D}$ consists of at most two connected components, by Lemma 3.7. The curve $\hat{H}_1 \cup \hat{H}_2$ meets each of these components at one point, by Lemma 2.10. Thus $\hat{H}_1 \cup \hat{H}_2$ must meet either $\hat{C}_1$ or $\hat{C}_2$ which concludes the proof. ☐

Convention. From now on we suppose that C_2 is not a multiple component of p.

5.2. Recall that we denote the closure of C_k in $\hat{X}$ by $\hat{C}_k$.

Lemma.
(i) *The subgraph* $G_1 - \hat{C}_2$ *is contractible,*
(ii) $\hat{C}_2$ *is an endpoint,*
(iii) $\hat{C}_1$ *is a linear point or an end point in this graph with weight* -1*,*
(iv) *the subgraph* $G_1 - (\hat{C}_1 \cup \hat{C}_2)$ *coincides with* $G_1^1 \cup G_1^2$ *and all its weights are* ≤ -2*,*
(v) *the graph* G_1 *is linear.*

Proof. Let G' be a connected component of the subgraph $G_1 - \hat{C}_2$. Since C_2 is not a multiple component of the fiber Γ_1, all components of the curve corresponding to the subgraph G' can be shrunk one after another, by Corollary 2.5, which implies (i). Thus G' contains a (-1)-vertex F. Assume it is different from $\hat{C}_1$. Note that $G' \subset G_1^1 \cup G_1^2 \cup \hat{C}_1$, by Lemma 3.7, i.e. F belongs to $G_1^1 \cup G_1^2$. This contradicts Corollary 3.6. Thus the only way to contract G' is to require that it contains $\hat{C}_1$ which is a linear point or an endpoint of weight -1. In particular, $G_1 - \hat{C}_2$ consist of one connected component only (if there are two components one of them does not contain $\hat{C}_1$ and, therefore, cannot be shrunk). Thus $\hat{C}_2$ is an endpoint, i.e. (ii) and (iii) hold. By Lemma 3.1 (1) and Corollary 3.6, the weights of $G_1 - (\hat{C}_1 \cup \hat{C}_2)$ are ≤ -2, i.e. (iv) holds.

Assume that G_1 is not linear and F is a branch point. Let $\dot{G}$ be the branch of G_1 at F that contains $\hat{C}_1$. Assume that $\dot{G}$ contains $\hat{C}_2$. Then the other branches of G_1 at F are non-contractible, by (iv), and one cannot contract $\hat{p}^{-1}(1)$ to $\hat{C}_2$ in contradiction with Corollary 2.5. Hence $\dot{G}$ does not contain $\hat{C}_2$. While contracting $G_1 - \hat{C}_2$ one must contract $\dot{G}$ first due to (iv). After this we obtain a new graph in which F must be a linear (-1)-vertex otherwise this graph cannot be contracted further. By Lemma 3.1 (4), all vertices of $\dot{G}$ correspond to multiple components of $\hat{p}^*(1)$. By Lemma 2.12, $\hat{H}_k$ cannot meet any vertex of $\dot{G}$. Assume that $\dot{G} - \hat{C}_1$ contains a non-empty connected component which does not contain any neighbor of F. Then this component must be either G_1^1 or G_1^2. But Fig. 2 implies $\hat{H}_k$ meets G_1^k when this subgraph is non-empty. Hence this connected component does not exist and $\hat{C}_1$ is an endpoint of $\dot{G}$ and G_1. This implies that $\hat{D} \cap \hat{C}_1$ consists of one point a and $C_1 = \hat{C}_1 - a$ is isomorphic to $\mathbf{C}$ in contradiction with the remark in the beginning of 5.1. Hence (v) is true. ☐

5.3. Lemma. *Suppose that G_1 does not coincide with $\hat{C}_1 \cup \hat{C}_2$. Then $\hat{C}_1$ and $\hat{C}_2$ are not neighbors in G_1.*

Proof. Assume that $\hat{C}_1$ and $\hat{C}_2$ are neighbors. Since G_1 is linear and $\hat{C}_2$ is an endpoint, by Lemma 5.2, only one vertex of $G_1 - \left(\hat{C}_1 \cup \hat{C}_2\right)$ is a neighbor of $\hat{C}_1$, and let us say that the corresponding irreducible components meet at a point a. Note that $C_1 = \hat{C}_1 - \left(a \cup \left(\hat{C}_1 \cap \left(\hat{H}_1 \cup \hat{H}_2\right)\right)\right)$. Recall that C_1 is not isomorphic to $\mathbf{C}$, by Corollary 2.13. Hence $\hat{C}_1 \cap (\hat{H}_1 \cup \hat{H}_2)$ is not empty and $\hat{C}_1$ is a non-multiple component of $\hat{p}^*(1)$, by Lemma 2.12. Hence, by Lemma 5.2, $\hat{C}_1$ is an endpoint of G_1 which means that $G_1 = \hat{C}_1 \cup \hat{C}_2$. Contradiction. ☐

5.4. The following fact can be proven easily by induction.

Proposition. *If G is a linear contractible graph with no (-1)-vertex, except for possibly an endpoint, then this endpoint is indeed a (-1)-vertex and the rest of weights is -2.*

Corollary. *If the graph $G_1 - (\hat{C}_1 \cup \hat{C}_2)$ consists of one connected component then it is standard. Moreover, $\hat{C}_2$ is a (-1)-vertex in this case.*

Proof. The first statement follows immediately from Proposition 5.4, Lemmas 5.3 and 5.2 (i), (iv), and (v). The second statement follows from the fact that the selfintersection of the fiber $\hat{p}^{-1}(1)$ is 0. ☐

We shall need the description of G_1 under some additional assumption which will be used in the next section.

Lemma. *Let the notation be as in Lemma* 5.2. *Suppose that neither G_1^1 nor G_1^2 is empty. Let m and n be natural, and $m \geq 2$, $n \geq 2$.*

(a) *If G_1^2 (resp. G_1^1) is a standard graph $S(n-1)$, then the subgraph G_1^1 (resp. G_1^2) is the union of a standard graph $S(m-1)$ and the neighbor V_1 of $\hat{C}_1$ whose weight $-n-1$.*

(b) *If G_1^1 is a linear graph such that it consists of standard graphs $S(m-2)$, $S(n-2)$, a vertex F of weight -3 between these two standard graphs, and if an endpoint of $S(n-2)$ is a neighbor of $\hat{C}_1$, then the neighbor V_1 of $\hat{C}_1$ in G_1^2 has weight $-n$ and* (b′) *either the subgraph $G_1^2 - V_1$ is empty,*

(b″) *or it consists of a standard graph and the neighbor V_2 of V_1 whose weight is $-m-1$.*

Therefore in all these cases the graph $G_1 - \hat{C}_2$ coincides with one of the graphs in Fig. 4.

Proof. Consider (a). Recall that $G_1^1 \cup \hat{C}_1 \cup G_1^2$ is contractible and $\hat{C}_1$ is the only (-1)-vertex in this subgraph, by Lemma 5.2. Assumption (a) implies that we can contract to $\hat{C}_1 \cup G_1^2$ first. After this contraction we obtain a new graph such that all vertices except for V_1 have the same weight as in G_1 (since we have not contracted their neighbors, by construction). In particular, all weights in this new graph except for the weight of V_1 are different from -1, by Lemma 5.2. The weight of V_1 in this new graph is -1 and the rest of the weights must be -2, by Proposition 5.4. Note that while contracting $\hat{C}_1 \cup G_1^2$ we shrink n neighbors of V_1. Hence the weight of V_1 in G_1 is $-n-1$ which implies (a).

Consider (b). One may contract $S(n-2) \cup \hat{C}_1$. After this we obtain a new graph in which all vertices except for F and V_1 have the same weights as in $G_1 - \hat{C}_2$, i.e they are ≤ -2, by Lemma 5.2. The weight of F in this new graph is -2, by construction. Thus the weight of V_1 in this new graph is -1. Note that while contracting $\hat{C}_1 \cup S(n-2)$ we shrink $n-1$ neighbors of V_1. Hence the weight of V_1 in G_1 is $-n$. Note we may contract $G_1^1 \cup \hat{C}_1 \cup V_1$ now. Indeed, since after contracting $\hat{C}_1 \cup S(n-2)$ the weight of V_1 becomes -1 and the weight of F becomes -2, one can contract the vertices from $V_1 \cup F \cup S(m-2)$ as well. If $G_1^2 \neq V_1$ then after this contraction the weight of V_2 must be -1 and the rest of the weights are -2, by Proposition 5.4. This implies that the weight of V_2 in G_1 was $-m-1$ and that the graph $G_1^2 - (V_1 \cup V_2)$ is standard. □

5.5. Suppose that $G_1 - \hat{C}_2$ looks like one of the graphs in Fig. 4. There are two ways for $\hat{C}_2$ to be connected with this graph. Namely, $\hat{C}_2$ is either the upper endpoint or the lower endpoint of G_1.

Lemma. *Let G_1^1 and G_1^2 be non-empty.*

(a) *Suppose that $G_1 - \hat{C}_2$ looks like in Fig. 4a. If $\hat{C}_2$ is the upper endpoint of G_1 then V_1 and all vertices of $S(n-1)$ are multiple components of the divisor $\hat{p}^*(1)$. If $\hat{C}_2$ is the lower endpoint of G_1 then all vertices of $S(n-1)$ except for the upper endpoint of G_1 are multiple components of $\hat{p}^*(1)$.*

(b′) *Suppose that $G_1 - \hat{C}_2$ looks like in Fig* 4b′ *and $\hat{C}_2$ is the upper endpoint of G_1. Then V_1 is a multiple component of the divisor $\hat{p}^*(1)$.*

(b″) *Suppose that $G_1 - \hat{C}_2$ looks like in Fig* 4b″. *Then V_1 is a multiple component of the divisor $\hat{p}^*(1)$. If $\hat{C}_2$ is the lower endpoint then all vertices* *below V_1 are also multiple components of $\hat{p}^*(1)$.*

Proof. The proof of the statements (a), (b'), and (b") is based on the same idea. We contract some components in $G_1 - \hat{C}_2$ so that V_1 becomes a linear (-1)-vertex in the image of G_1. This contraction generates a morphism $\sigma : \hat{X} \to \tilde{X}$ which in its turn generates $\tilde{p} : \tilde{X} \to \mathbf{CP}^1$ so that $\hat{p} = \tilde{p} \circ \sigma$. By Lemma 3.1 (4), $\sigma(V_1)$ is a multiple component of $\tilde{p}$ and, therefore, V_1 is a multiple component of $\hat{p}$. In order to make V_1 a linear (-1)-vertex one must contract $\hat{C}_1 \cup S(n-1)$ in the case of the first statement from (a), and $\hat{C}_1 \cup S(n-2) \cup F$ in cases (b') and (b"). The rest of statement (a) can be checked in the same manner. □

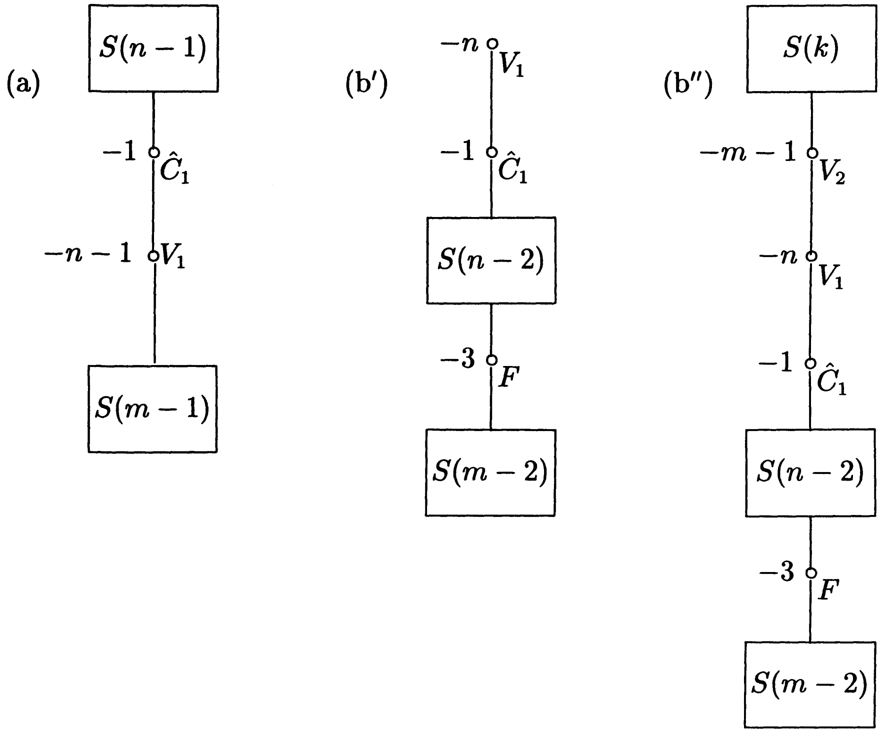

Figure 4. The graph $G_1 - \hat{C}_2$.

6. The graph $G(\hat{D})$.

In this section we still denote the graph of $\hat{p}^{-1}(1)$ by G_1. We also use notation from Fig. 2 and Lemma 3.6. By Lemma 4.4, the graph G_∞ looks like in Fig. 3a. In particular, G^0_∞ and G^1_∞ are empty and the weight of E_1 is $-m$. As we mentioned in 3.1 $\hat{p}^{-1}(1)$ is an SNC-curve and it meets $\hat{D}$ normally, by Lemma 2.12. Hence $\hat{D} \cup \hat{p}^{-1}(1) = \hat{D} \cup \hat{C}_1 \cup \hat{C}_2$ is an SNC-curve and we may speak about its graph. The aim of this section is the following

Theorem. *The graph* $G(\hat{D} \cup \hat{C}_1 \cup \hat{C}_2)$ *looks like one of the graphs in Fig.* 5.

It is worth mentioning that the righ-hand side vertical parts of these graphs correspond to the subgraph G_1 and in each of these graphs the number of edges between vertices $\hat{C}_2$ and $\hat{H}_2$ is $m - 1$.

6.1. We prove this Theorem in several steps using the fact that either $\hat{H}_2 \cup G_1^2$ or $E_1 \cup \hat{H}_1 \cup G_1^1$ is contractible, by Lemma 3.6 (iv).

6.1.1. Lemma. *Suppose that* $\hat{H}_2 \cup G_1^2$ *is contractible and that* G_1^1 *and* G_1^2 *are not empty. Then the graph* $G(\hat{D} \cup \hat{C}_1 \cup \hat{C}_2)$ *looks like in Fig.* 5a.

Proof. By Proposition 5.4, the weight of $\hat{H}_2$ is -1, and G_1^2 is a standard graph, say $S(n-1)$ where $n > 1$. By Lemma 5.4, G_1^1 is a linear graph consisting of a standard graph $S(k)$ and a vertex V_1 of weight $-n-1$. After contracting $\hat{H}_2 \cup G_1^2$ the weight of the image of E_2 becomes $n-1$ and one can see that this new graph can be reduced further to a graph from Theorem 2.9 via an RM-procedure only if $k = m - 1$ and $\hat{H}_1$ and V_1 are not neighbors in $G(\hat{D})$, i.e. $G(\hat{D})$ looks like in Fig. 5a.

It remains to check the position of $\hat{C}_1$ and $\hat{C}_2$. Note that, since G_1^1 and G_1^2 are not empty, $\hat{C}_1$ is a multiple component of the divisor $\hat{p}^*(1)$, by Lemma 3.1 (4) and Lemma 5.4. Therefore, $\hat{H}_2$ does not meet $\hat{C}_1$, by Lemma 2.12. Since $G_1 - \hat{C}_2$ looks like in Fig. 4a, $\hat{C}_2$ cannot be the upper endpoint of G_1. Otherwise, all vertices of G_1^2 are multiple components of $\hat{p}^*(1)$, by Lemma 5.5, i.e. $\hat{H}_2$ meets a multiple component which contradicts again Lemma 2.12. Since $\hat{H}_1$ is a section and since it meets G_1^1 it does not meet $\hat{C}_1$ or $\hat{C}_2$. According to Proposition 3.6 it meets G_1^1 at an endpoint, and, as we mentioned above, this endpoint is not V_1. This yields Fig. 5a. Note also that the intersection number of $\hat{H}_2$ and each fiber of p is m since m is the same as m_2 in Proposition 3.7. (Recall that we replaced m_2 by m in 4.4.) By Lemma 2.12, $\hat{H}_2$ meets $\hat{p}^{-1}(1)$ at m different points. It follows from Fig. 5a that only one of these points does not belong to C_2. Hence the number of edges between $\hat{C}_2$ and $\hat{H}_2$ is $m - 1$. □

Remark. The argument at the end of the proof about the number of edges between $\hat{C}_2$ and $\hat{H}_2$ will be valid for all graphs in Fig. 5 and 6.

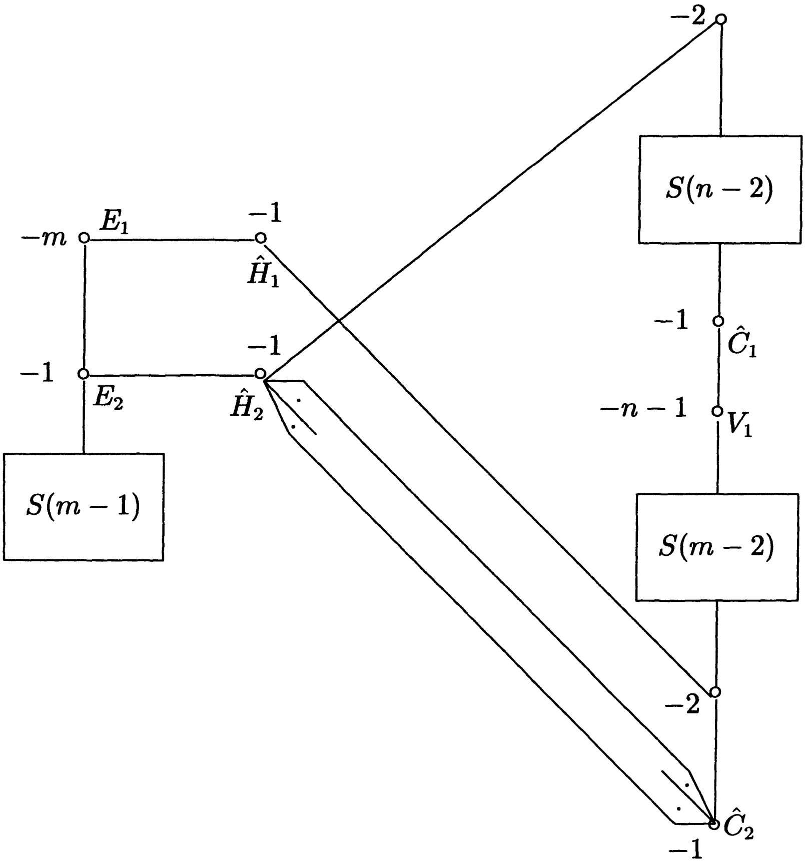

Figure 5a. The graph $G(\hat{D} \cup (\hat{C}_1 \cup \hat{C}_2))$.

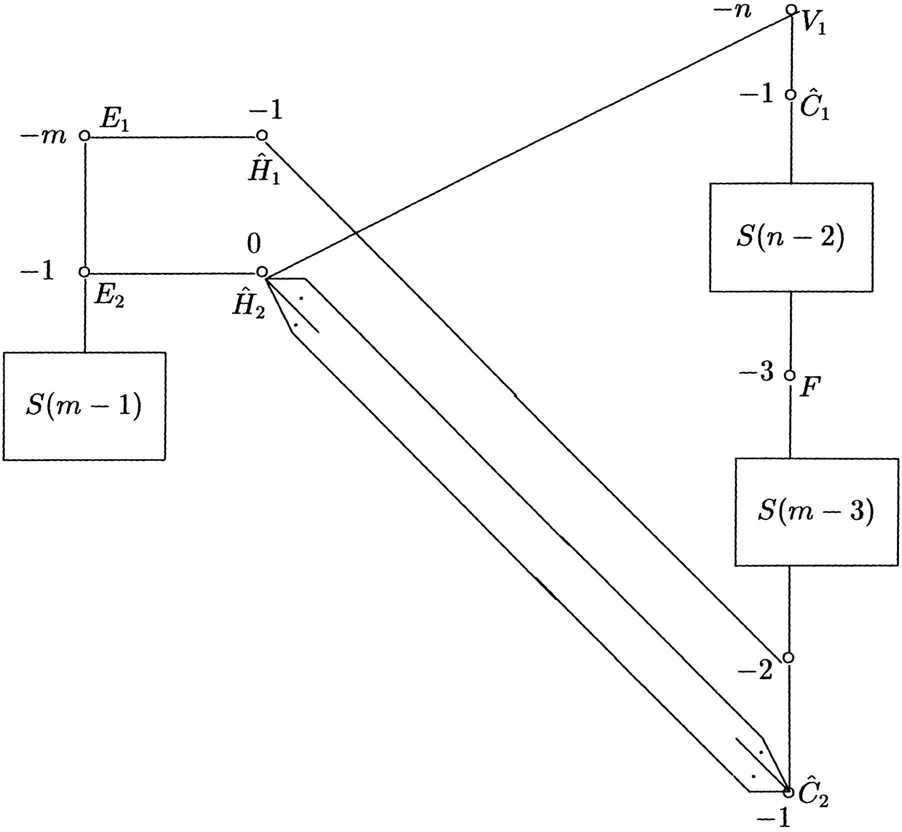

Figure 5b. The graph $G(\hat{D} \cup (\hat{C}_1 \cup \hat{C}_2))$.

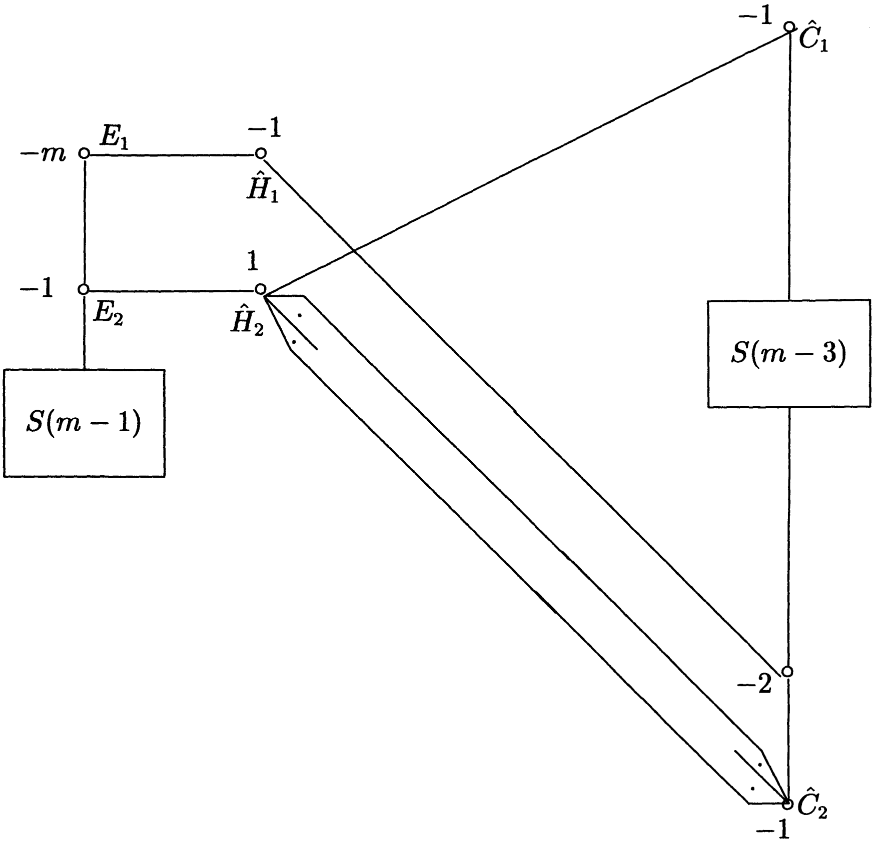

Figure 5c. The graph $G(\hat{D} \cup (\hat{C}_1 \cup \hat{C}_2))$. (In this graph $m > 2$).

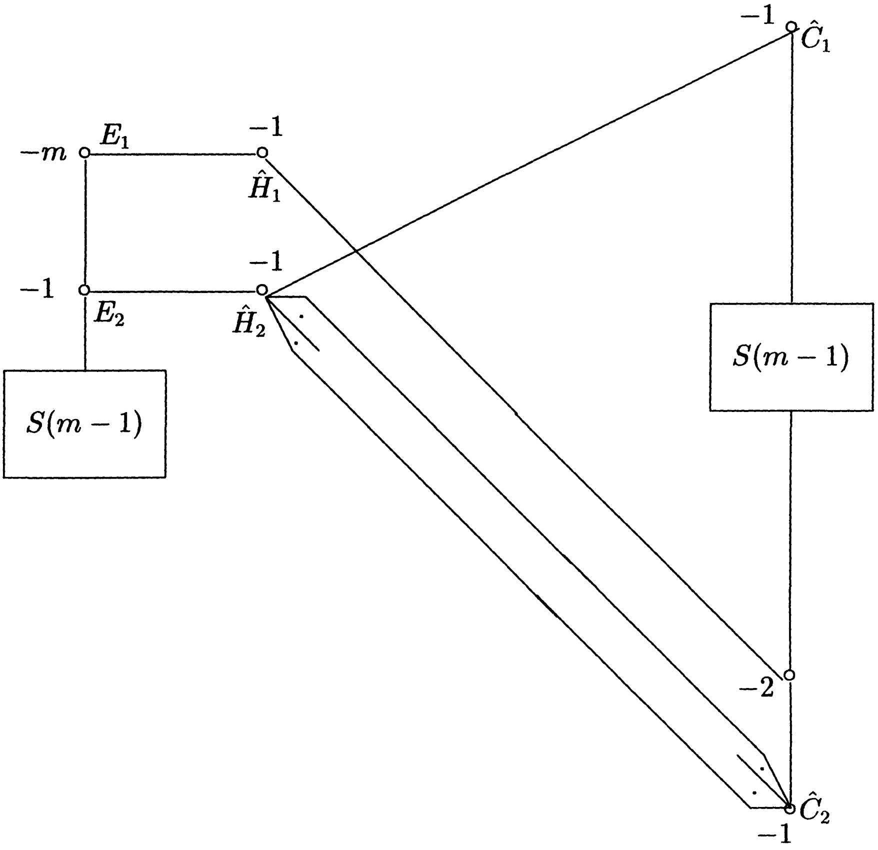

Figure 5d. The graph $G(\hat{D} \cup (\hat{C}_1 \cup \hat{C}_2))$.

6.1.2. Lemma. *Suppose that* $E_1 \cup \hat{H}_1 \cup G_1^1$ *is contractible, but* $\hat{H}_1 \cup G_1^1$ *is non-contractible. Let* G_1^1 *and* G_1^2 *be non-empty. Then* $G(\hat{D} \cup \hat{C}_1 \cup \hat{C}_2)$ *looks like in Fig.* 5b *or like in Fig.* 6a *and* 6b.

Proof. By the assumption of Lemma, in some step of an RM-procedure we have to contract the image of E_1 while the image of $\hat{H}_1 \cup G_1^1$ is not empty yet. Therefore, $G_1^1 = S(m-2) \cup G'$. After contracting $E_1 \cup \hat{H}_1 \cup S(m-2)$, the image of G' must be contractible. If F is the vertex in G' which is the neighbor of $S(m-2)$ then one can see that the weights of $G' - F$ in this last image are the same as in the original graph $G(\hat{D})$, i.e. none of them is -1, by Lemma 5.2. By Proposition 5.4, this means that the weight of F in this image is -1 and all other weights are -2, i.e. $G' - F = S(n-2)$. By construction, only two neighbors of F are shrunk before F while contracting $E_1 \cup \hat{H}_1 \cup S(m-2)$. This means that the weight of F in $G(\hat{D})$ is -3. Note also that after contracting of $E_1 \cup \hat{H}_1 \cup G_1^1$ the weight of E_2 becomes $n-1$. Since the weights of G_1^2 are ≤ -2 (Lemma 5.2) the weight of $\hat{H}_2$ must be 0, by Theorem 2.9. There are two possible forms of the subgraph G_1^2 described in Lemma 5.4 (b')-(b"). Form (b') and Theorem 2.9 yield the same $G(\hat{D})$ as in Fig. 5b. The same argument, which was used at the end of the proof of Lemma 6.1.1, shows that in Fig. 5b $\hat{H}_2$ does not meet $\hat{C}_1$ and that $\hat{C}_2$ is the lower endpoint of G_1 which concludes the description of Fig. 5b.

Assume that G_1^2 has form (b"). This graph has two endpoints one of which is V_1. Assume that the weight of the other endpoint is different from $-n$. By Theorem 2.9, V_1 must be a neighbor of $\hat{H}_2$. On the other hand V_1 is a multiple component of $\hat{p}^*(1)$, by Lemma 5.5, and it cannot meet $\hat{H}_2$, by Lemma 2.12. Hence case (b") does not hold unless the other endpoint of G_1^2 is a neighbor of $\hat{H}_2$ and, therefore, has weight $-n$. The last condition holds only when $n = 2$ and $k \geq 1$ or when $n = m+1$ and $k = 0$. When $n = 2$ the last statement from Theorem 2.9 implies also that $k = 1$. This yields $G(\hat{D})$ as in Fig. 6a and 6b.

The same argument as in 6.1.1 shows that in Fig. 6a and 6b $\hat{C}_2$ must be the lower endpoint of G_1 and $\hat{H}_2$ does not meet $\hat{C}_1$ which concludes the description of Fig. 6a and 6b. □

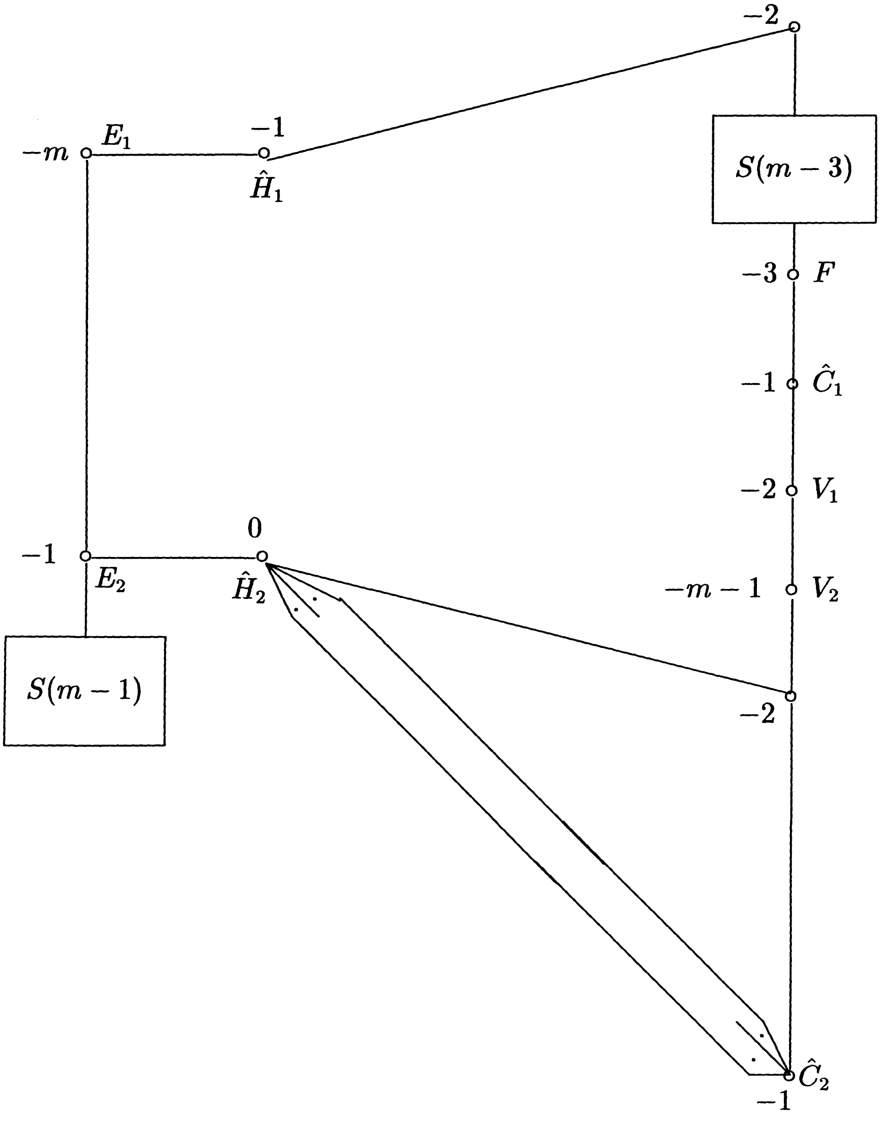

Figure 6a. The graph $G(\hat{D} \cup (\hat{C}_1 \cup \hat{C}_2))$. (When $m = 2$ the vertices above F are absent and $\hat{H}_1$ is a neighbor of F.)

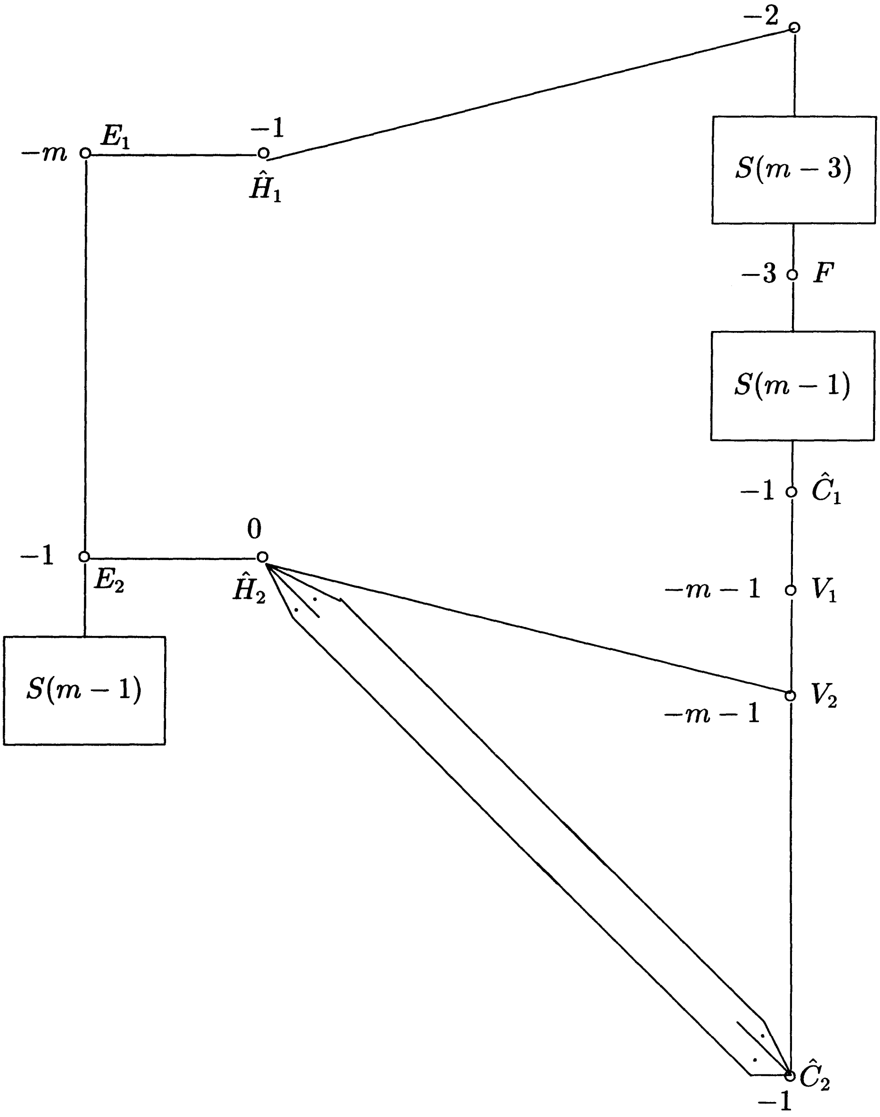

Figure 6b. The graph $G(\hat{D} \cup (\hat{C}_1 \cup \hat{C}_2))$. (When $m = 2$ the vertices above F are absent and $\hat{H}_1$ is a neighbor of F.)

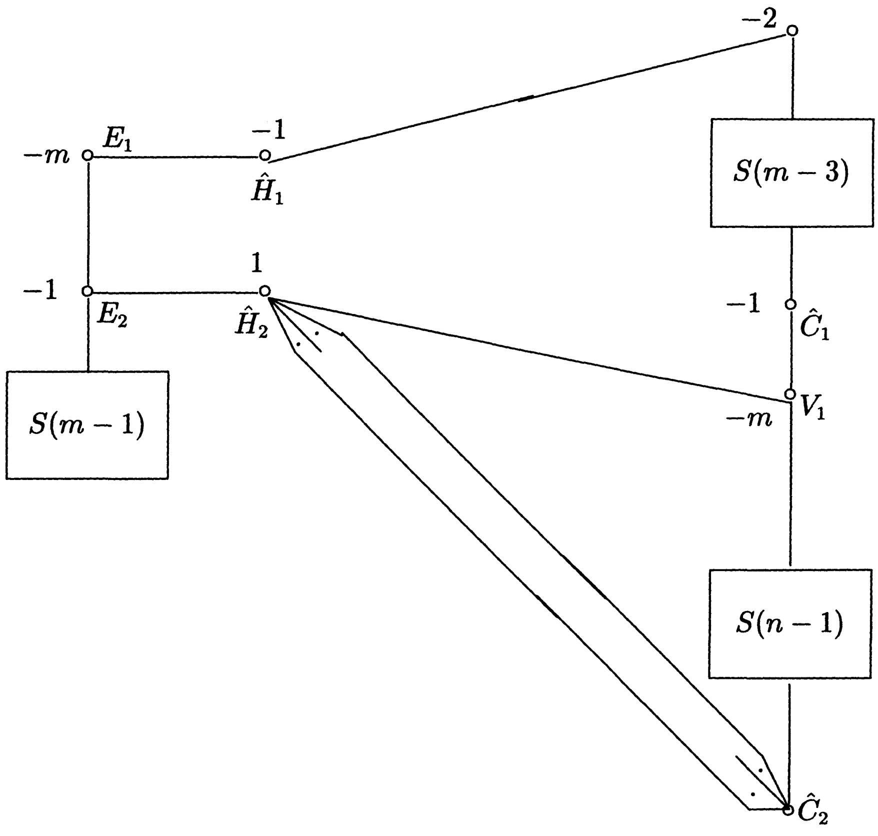

Figure 6c. The graph $G(\hat{D} \cup (\hat{C}_1 \cup \hat{C}_2))$. (In this graph $m > 2$.)

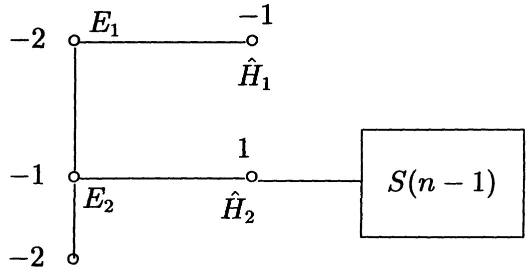

Figure 6d. The graph $G(\hat{D})$.

6.1.3. Lemma. *Let* G_1^1 *and* G_1^2 *be non-empty. Suppose that* $E_1 \cup \hat{H}_1 \cup G_1^1$ *is contractible and* $\hat{H}_1 \cup G_1^1$ *is contractible. Then* $G(\hat{D} \cup \hat{C}_1 \cup \hat{C}_2)$ *looks like in Fig.* 6c.

Proof. By Proposition 5.4, G_1^1 must be a standard graph $S(k)$. After contracting $\hat{H}_1 \cup G_1^1$ in an RM-procedure we have to contract the image of E_1, i.e. its weight must be -1. This implies that $k = m - 2$. In particular, since G_1^1 is non-empty $m > 2$. After contracting $E_1 \cup \hat{H}_1 \cup G_1^1$ the weight of E_2 becomes 0. Hence E_2 survives an RM-procedure and it must have a neighbor of a non-negative weight after this procedure, by Theorem 2.9. Since G_1^2 has weights ≤ -2, by Lemma 5.2, the weight of $\hat{H}_2$ is 1, by Theorem 2.9. By Lemma 5.4, G_1^2 is a linear graph consisting of a standard graph $S(n-1)$ and a vertex V_1 of weight $-m < -2$. The last statement of Theorem 2.9 implies that $\hat{H}_2$ is a neighbor of V_1. This leads to $G(\hat{D})$ as in Fig. 6c. The position of $\hat{C}_1$ and $\hat{C}_2$ may checked in a manner similar to 6.1.1 ($\hat{H}_1$ meets the upper endpoint of $S(m-2)$ since it is the only non-multiple component, by Lemma 5.5). □

6.1.4. Lemma. *Let either* G_1^2 *or* G_1^1 *be empty. Then* $G(\hat{D})$ *looks like one of the graphs in Fig* 5c, *Fig.* 5d (*without vertices* $\hat{C}_1$ *and* $\hat{C}_2$), *and Fig.* 6d.

Proof. Recall that $G_1 - (\hat{C}_1 \cup \hat{C}_2)$ is standard, by Corollary 5.4. Thus $G_1^1 = S(k), G_1^2 = S(l)$ where $k, l \geq 0$ and $kl = 0$. We need to consider several possibilities.

Case 1: the graph $\hat{H}_2 \cup G_1^2$ is contractible, i.e. $\hat{H}_2$ is a (-1)-vertex, by Proposition 5.4. After contracting this subgraph and the subgraph $\hat{H}_1 \cup G_1^1$ we obtain the linear graph $E_1 \cup E_2 \cup S(m-1)$ where the weights of E_1 and E_2 become $k - m + 1$ and l respectively. Theorem 2.9 implies that $k = m, l = 0$, (i.e. G_1^2 is empty), and $G(\hat{D})$ looks like in Fig 5d.

Case 2: the graph $\hat{H}_2 \cup G_1^2$ is non-contractible, the weight of $\hat{H}_2$ is ≥ 0 (by Lemma 3.6), and $E_1 \cup \hat{H}_1 \cup G_1^1$ is contractible.

Subcase 2a: $k = 0, l \geq 0$. One can contract $E_1 \cup \hat{H}_1$. This means that $m = 2$. After this contraction the weight of E_2 becomes 0 and Theorem 2.9 implies that the weight of $\hat{H}_2$ is 1. Hence $G(\hat{D})$ looks as in Fig. 6d with $n = l + 1$.

Subcase 2b: $k > 0, l = 0$. One can see that the only way to contract $E_1 \cup \hat{H}_1 \cup S(k)$ is to require that $k = m - 2$, i.e. $m > 2$. After this contraction the weight of E_2 becomes 0. Thus the weight of $\hat{H}_2$ is 1, by Theorem 2.9, and we deal with Fig. 5c. □

6.2. We shall need the following procedure. Contract all components of $\hat{p}^{-1}(1)$ except for $\hat{C}_2$ (we can do this, by Lemma 5.2) and contract all components of $\hat{p}^{-1}(\infty)$ except for one. We obtain a morphism $\delta : \hat{X} \to Q$ where δ and Q are the same as in Theorem 2.6. Put $H_k = \delta(\hat{H}_k)$ and let q be the same as in 2.6. Then $E = q^{-1}(\infty)$ and H_1 generate a basis in the second homology group of Q. (Recall H_1 is a section, i.e. $H_1 \cdot E = 1$.) This implies that $H_2 \cong mH_1 + sE$ since the intersection $H_2 \cdot E$ is m. This also implies that a basis of the second homology group in $\hat{X}$ consists of $\hat{C}_1, \hat{H}_1$, and the components of the curve B which is the union of all components of $\hat{D}$ except for $\hat{H}_1$ and $\hat{H}_2$.

Lemma. *Let $\hat{H}_2$ be homology equivalent to $k\hat{C}_1 + l\hat{H}_1 + U$ where U is a linear combination of components of B and $\hat{H}_1$. Then $k = \pm 1$.*

Proof. We have another basis of the second homology group of $\hat{X}$ generated by the components of $\hat{D}$ [**R**]. Note that in order to obtain the second basis from the first one it suffices to replace $\hat{C}_1$ by $\hat{H}_2$. Hence the determinant of the transition matrix coincides with k. This transition matrix must be invertible and, therefore, the determinant must be ± 1. □

Convention. From now on we suppose that $q^{-1}(\infty) = \delta(E_1)$ where E_1 is from Fig. 2, i.e. in the description of δ we have to contract all components of $\hat{p}^{-1}(\infty)$ except for E_1 (we can do this since the graph of the fiber $\hat{p}^{-1}(\infty)$ looks like in Fig. 3a).

6.3. Lemma. *Let the notation be as in* 6.2.

(a) *Suppose that the subgraph G_1 of $G(\hat{D} \cup \hat{C}_1 \cup \hat{C}_2)$ looks like in Fig.* 6a. *Then*

$$\delta^*(H_1) \cong \hat{H}_1 + 4m\hat{C}_1 + U_1$$

and

$$\delta^*(H_2) \cong \hat{H}_2 + (2m - 1)\hat{C}_1 + U_2$$

where U_1 and U_2 are linear combinations of components of B.

(b) *Suppose that the subgraph* G_1 *of* $G(\hat{D} \cup \hat{C}_1 \cup \hat{C}_2)$ *looks like in Fig.* 6b. *Then*

$$\delta^*(H_1) \cong \hat{H}_1 + m(m+2)\hat{C}_1 + U_1$$

and

$$\delta^*(H_2) \cong \hat{H}_2 + (m^2+m-1)\hat{C}_1 + U_2$$

where U_1 *and* U_2 *are linear combinations of components of* B.

(c) *Suppose that the subgraph* G_1 *of* $G(\hat{D} \cup \hat{C}_1 \cup \hat{C}_2)$ *looks like in Fig.* 6c. *Then*

$$\delta^*(H_1) \cong \hat{H}_1 + ((m-1)n+1)\hat{C}_1 + U_1$$

and

$$\delta^*(H_2) \cong \hat{H}_2 + (m-1)n\hat{C}_1 + U_2$$

where U_1 *and* U_2 *are linear combinations of components of* B.

(d) *Suppose that* $\hat{C}_1$ *and* $\hat{C}_2$ *are endpoints of* G_1, *i.e.* $G_1 - (\hat{C}_1 \cup \hat{C}_2)$ *is a standard graph* $S(n-1)$, *by Corollary* 5.4. *Let* $n > 1$ *and let* V_j *be the endpoint of* $S(n-1)$ *which is the neighbor of* $\hat{C}_j$ *in* G_1 *(may be* $V_1 = V_2$*). Suppose that* $\hat{H}_2$ *meets* $\hat{C}_1$ *at* l *points.*

(d′) *If the section* $\hat{H}_1$ *meets* $\hat{C}_1$ *and* $\hat{H}_2$ *meets* V_2 *then*

$$\delta^*(H_1) \cong \hat{H}_1 + n\hat{C}_1 + U_1$$

$$\delta^*(H_2) \cong \hat{H}_2 + (nl+1)\hat{C}_1 + U_2$$

where U_1 *and* U_2 *are linear combinations of components of* B.

(d″) *If the section* $\hat{H}_1$ *meets* V_2 *(and, therefore,* $\hat{H}_2$ *does not meet* $S(n-1)$, *by Lemma* 2.10*) then*

$$\delta^*(H_1) \cong \hat{H}_1 + \hat{C}_1 + U_1$$

$$\delta^*(H_2) \cong \hat{H}_2 + nl\hat{C}_1 + U_2$$

where U_1 *and* U_2 *are linear combinations of components of* B.

Proof. All cases are similar and we consider (a) only. Recall that morphism δ is a composition of blowing-ups $\delta_s \circ \cdot \circ \delta_1$. Put $\sigma_j = \delta_j \circ \cdot \circ \delta_1 : X_j \to Q$ and let E_j be the exceptional divisor of δ_j. Suppose that D is an SNC-divisor in Q and that the blowing-up δ_j takes place at the common point of components E' and E'' of the divisor $\sigma_{j-1}^*(D)$. Then the multiplicity of E_j in $\sigma_j^*(D)$ is the sum of multiplicities of E' and E'' in $\sigma_{j-1}^*(D)$. Hence the multiplicity of $\hat{C}_1$ in the divisor $\delta^*(H_k)$ (which is the coefficient before $\hat{C}_1$ in the formula (a) for $\delta^*(H_k)$ $(k = 1,2)$ in the statement of the Lemma) must be the sum of multiplicities of its neighbors F and V_1 in the graph from Fig.

6a. Let H_1', H_2', F' and V_j' be the images of $\delta^*(H_1), \delta^*(H_2), F$, and V_j after contracting $\hat{C}_1$.

When $m = 2$ an easy computation shows that the multiplicities of F' in H_1' and H_2' are 3 and 1 respectively, the multiplicities of V_1' in H_1' and H_2' are 5 and 2 respectively, and the multiplicities of V_2' in H_1' and H_2' are 2 and 1 respectively.

Note that in general case after contracting $\hat{C}_1$ the graph of the fiber over 1 is the union of $S(m-1)$, V_1' which is a (-1)-vertex, V_2' whose weight is $-m-1$, a vertex of weight -2, and the image of $\hat{C}_2$. The vertex F' is the endpoint of $S(m-1)$ that is a neighbor of V_1'. One may contract V_1' and obtain a similar linear graph but with m replaced by $m-1$. Therefore, we may apply induction which shows that the multiplicities of F', V_1', and V_2' in H_1' are $2m-1, 2m+1$, and 2 respectively, and in H_2' they are $m-1, m$, and 1 respectively. Hence the multiplicities of $\hat{C}_2$ in $\hat{H}_1$ and $\hat{H}_2$ are $4m$ and $2m-1$ respectively. ☐

6.4. *The Proof of Theorem* 6.1.

We need to check the position of $\hat{C}_1$ and $\hat{C}_2$ in Fig. 5c and 5d (in particular, the fact that there is only one edge between $\hat{C}_1$ and $\hat{H}_2$) and we have to show that none of graphs from Fig. 6 can hold.

Case of Fig. 5c. Recall that in this case $G_1^1 = S(m-2)$ with $m > 2$ and G_1^2 is empty. Since $\hat{H}_1$ meets $S(m-2)$ and since $\hat{H}_1$ is a section it does not meet $\hat{C}_1$ and $\hat{C}_2$. The second horizontal component $\hat{H}_2$ meets $\hat{p}^{-1}(1)$ only at points from $\hat{C}_1$ or $\hat{C}_2$, by Lemma 2.10. Let it meet $\hat{C}_1$ at l points and, thus, $\hat{C}_2$ at $m-l$ points. Let V_1 and V_2 be the endpoints of $S(m-2)$. Suppose that V_j is the neighbor of $\hat{C}_j$ in G_1. One may always suppose that $\hat{H}_1$ meets V_2 (otherwise just switch indices of $\hat{C}_1$ and $\hat{C}_2$). Let δ, Q, q, H_j, E be as in 6.2. Since $H_2 \cong mH_1 + sE$, Lemma 6.3 (d") implies

$$\hat{H}_2 \cong m\hat{H}_1 + (m-(m-1)l)\hat{C}_1 + U$$

where U is again a combination of components of B. The coefficient before $\hat{C}_1$ is ± 1, by Lemma 6.2. Hence either $l = 1$ and we deal with Fig. 5c or $m = 3$ and $l = 2$. But in this case $V_1 = V_2$ and switching the indices of $\hat{C}_1$ and $\hat{C}_2$ we obtain again Fig. 5c.

Case of Fig. 5d. Similar argument implies that

$$\hat{H}_2 \cong m\hat{H}_1 + (m-(m+1)l)\hat{C}_1 + U$$

where U is a linear combination of components of B. Hence $l = 1$ which shows that the position of $\hat{C}_1$ and $\hat{C}_2$ in Fig. 5d is correct.

Case of Fig 6a. Since $H_2 \cong mH_1 + sE$ Lemma 6.3 (a) implies

$$\hat{H}_2 \cong m\hat{H}_1 + (4m^2 - 2m + 1)\hat{C}_1 + U$$

where U is again a combination of components of B. Hence the coefficient before $\hat{C}_1$ is not ±1 and we have to disregard this case, by Lemma 6.2.

Case of Fig 6b. Since $H_2 \cong mH_1 + sE$ Lemma 6.3 (b) implies

$$\hat{H}_2 \cong m\hat{H}_1 + (m^3 + m^2 - m + 1)\hat{C}_1 + U$$

where U is again a combination of components of B. Hence the coefficient before $\hat{C}_1$ is not ±1 and we have to disregard this case, by Lemma 6.2.

Case of Fig 6c. Since $H_2 \cong mH_1 + sE$ Lemma 6.3 (c) implies

$$\hat{H}_2 \cong m\hat{H}_1 + [(m-1)^2 n + m]\hat{C}_1 + U$$

where U is again a combination of components of B. Hence the coefficient before $\hat{C}_1$ is not ±1 and we have to disregard this case, by Lemma 6.2.

Case of Fig. 6d. (We owe the argument in this case to the referee.) First consider $n > 1$. Since G_1^1 is empty and since $\hat{H}_2$ meets $S(n-1)$ the section $\hat{H}_1$ meets $\hat{p}^{-1}(1)$ only at one point of $\hat{C}_1 \cup \hat{C}_2$. One may suppose that it meets $\hat{C}_1$ since the components $\hat{C}_1$ and $\hat{C}_2$ are symmetric in this case. Note that $\hat{H}_2$ cannot meet $\hat{C}_1$. Otherwise, since $m = 2$, it does not meet $\hat{C}_2$. Hence C_2 is obtained from $\hat{C}_2$ by deleting one point, i.e. it is isomorphic to $\mathbf{C}$ in contradiction with Corollary 2.13. Let V_1 and V_2 be the endpoints of $S(n-1)$. Suppose that V_j is the neighbor of $\hat{C}_j$ in G_1. First consider the case when $\hat{H}_2$ meets V_1. Again δ, Q, q, H_j are the same as in 6.2. Recall that the morphism δ is obtained by contracting all components in the fiber $\hat{p}^{-1}(\infty)$ but E_1 and all components in the fiber $\hat{p}^{-1}(1)$ but $\hat{C}_2$. Hence one may check that H_2 is smooth and meets H_1 at one point with contact order $n-1$, i.e $H_1 \cdot H_2 = n-1$. The description of δ easily implies that $H_1 \cdot H_1 = n-1$ and $H_2 \cdot H_2 = n+2$. Recall that $H_2 \cong mH_1 + sE$ and $m = 2$. Since $H_1 \cdot E = 1$ in order to get $H_1 \cdot H_2 = n-1$ we must require that $s = -(n-1)$, i.e $H_2 = 2H_1 - (n-1)E$. Since $E \cdot E = 0$ we have $H_2 \cdot H_2 = 0$ in contradiction with the result of our previous computation.

Thus $\hat{H}_2$ meets V_2. Since $H_2 \cong mH_1 + sE$, $m = 2$, and since $\hat{H}_2$ does not meet $\hat{C}_1$ Lemma 6.3 (d') implies that

$$\hat{H}_2 \cong 2\hat{H}_1 + (2n-1)\hat{C}_1 + U$$

where U is again a combination of components of B. Hence the coefficient before $\hat{C}_1$ is not ±1 and we have to disregard this case.

Now consider Fig 6d with $n = 1$. Hence $\hat{p}^{-1}(1) = \hat{C}_1 \cup \hat{C}_2$. Since $m = 2$ the fiber $\hat{p}^{-1}(1)$ meets $\hat{D}$ at three points none of which is $\hat{C}_1 \cap \hat{C}_2$, by Lemma 2.12. Thus $\hat{D}$ meets either $\hat{C}_1$ or $\hat{C}_2$ at one point, i.e. either C_1 or C_2 is isomorphic to **C**. This contradicts Corollary 2.13. ☐

The graphs $G(\hat{D} \cup \hat{C}_1 \cup \hat{C}_2)$ from Fig. 5 imply that $\hat{H}_2$ meets $\hat{p}^{-1}(1) - C_2$ at one point $\hat{b}_2$ and that $\hat{C}_1 \cap \hat{D}$ consists of two points. Hence we have

6.4.1. Corollary. *The curve C_1 is isomorphic to $\mathbf{C}^*$ and $\hat{H}_2$ meets $\hat{p}^{-1}(1) - C_2$ at one point $\hat{b}_2$.*

Let the notation be as in 6.2. Recall that morphism δ from 6.2 implies the contraction of all components of $\hat{p}^{-1}(1)$ but $\hat{C}_2$ and all components of $\hat{p}^{-1}(\infty)$ but E_1.

6.4.2. Corollary. *The surface Q is a quadric $\mathbf{CP}^1 \times \mathbf{CP}^1$ such that q is the projection to the second factor and H_1 is a section for this projection.*

Proof. One can see from Fig. 5 that $H_1 \cdot H_1 = 0$. The statement of Lemma follows from the fact that the only Hirzebruch surface which admits a zero section is the quadric. ☐

6.4.3. Put $Q^1 = Q - q^{-1}(\infty), H_k^1 = H_k \cap Q^1$, and $b = H_1^1 \cap H_2^1$.

Corollary. *In the above notation there exists a coordinate system (u, v) in $\mathbf{C}^2 = Q^1 - H_1$ so that $q(u, v) = u$ and the curve $H_2^1 - b$ is given by the parametric equations $u = t^m$ and $v = (t-1)^{-1}$ with $t \in \mathbf{C} - \{1\}$.*

Proof. It follows also from Corollary 6.4.1 and the description of the morphism δ that H_2 meets the fiber $q^{-1}(0)$ at one point a, the fiber $q^{-1}(\infty)$ at one point c, and the curve H_1 at one point b with contact order 1, i.e $H_1 \cdot H_2 = 1$. Therefore, every section of the projection q which is homologically equivalent to H_1 meets H_2 at one point. Thus one may consider the morphism $r : Q \dashrightarrow H_2 \cong \mathbf{CP}^1$ that assigns to each point in Q the intersection of H_2 with the section through this point which is homologically equivalent to H_1. (The existence of such a section follows from Corollary 6.4.2.) Choose a coordinate on H_2 so that a corresponds to -1, c corresponds to 0, and b corresponds to ∞. Then the restriction of functions q and r to $\mathbf{C}^2 = Q - (H_1 \cup q^{-1}(\infty))$ produces the desired coordinate system. ☐

7. Main Theorem.

7.1. Let K be the curve that consists of all components of $\hat{D}$ but $\hat{H}_2$ and let Q^1, q, H_1 be as in 6.4.3. Note that $\hat{X} - (K \cup \hat{C}_1)$ is naturally isomorphic to $Q^1 - H_1^1$ and, therefore, it is isomorphic to $\mathbf{C}^2$. Under this isomorphism the curve $H^* := \hat{H}_2 - (\{\hat{b}_2\} \cup (\hat{H}_2 \cap K))$ (where $\hat{b}_2$ is from 6.4.1) is mapped onto $H_2^1 - b$ from 6.4.3. Our construction of the polynomial forms is based on the following simple observation.

Lemma. *Let $X_1 = \hat{X} - (K \cup \hat{C}_1)$ and $X_2 = \hat{X} - \hat{D}$. Let φ be a primitive polynomial on X_1 such that $\hat{H}_2 \cap X_1 = \varphi^{-1}(0)$ and let $\hat{\varphi}$ be the rational function on $\hat{X}$ which extends φ. Let $\hat{L}$ be an irreducible curve in $\hat{X}$ such that $L := \hat{L} \cap X_1$ is isomorphic to $\mathbf{C}$ and disjoint from $\hat{H}_2 \cap X_1$. Let f be a primitive polynomial on X_1 such that $L = f^{-1}(0)$ and let $\hat{f}$ be the rational function on $\hat{X}$ which extends f. Then*

(1) *the curve $\hat{C}_1 \cap X_2$ is the zero fiber of a polynomial that coincides with the restriction of either $\hat{\varphi}$ to X_2 or $\hat{\varphi}^{-1}$ to X_2;*

(2) *the curve $\hat{L} \cap X_2$ is the zero fiber of a polynomial that is the restriction of a rational function $\hat{f}\hat{\varphi}^m$ where $m \in Z$.*

Proof. We denote by U_k (where k is natural) a divisor which is an integer combination of irreducible components of K. Suppose that the zero fiber of a primitive polynomial ψ on X_2 is the curve $\hat{C}_1 \cap X_2$ and $\hat{\psi}$ is the rational function on $\hat{X}$ which extends ψ. Then the divisor of $\hat{\psi}$ is $\hat{C}_1 + l\hat{H}_2 + U_1$ and the divisor of $\hat{\varphi}$ is $n\hat{C}_1 + \hat{H}_2 + U_2$ where $n, l \in \mathbf{Z}$. Hence the divisor of $\hat{\varphi}\hat{\psi}^{-n}$ is $(1 - nl)\hat{H}_2 + U_3$. Since the divisor of a rational function is homologically trivial we see that $(nl - 1)\hat{H}_2$ is homologically equivalent to U_3. But the components of $\hat{D}$ form a basis of the second homology group of $\hat{X}$ [**R**]. Thus U_3 is the zero divisor, $nl = 1$, i.e. $n = \pm 1$, and $\hat{\psi} = c\hat{\varphi}^{\pm 1}$ where c is a nonzero constant.

Suppose that $h(x) = 0$ is a polynomial equation of the curve $\hat{L} \cap X_2$ in X_2 and $\hat{h}$ is the rational function on $\hat{X}$ which extends h. The divisor of $\hat{f}$ is $\hat{L} + s\hat{C}_1 + U_4$ and the divisor of $\hat{h}$ is $\hat{L} + m\hat{H}_2 + U_5$ where $s, m \in \mathbf{Z}$. Using again the fact that irreducible components of $K \cup \hat{C}_1$ are linearly independent as elements of the second homology group of $\hat{X}$, one can see that $\hat{h}$ coincides with $\hat{f}\hat{\varphi}^m$ up to a nonzero constant factor. □

7.2. Since X_1 is isomorphic to the surface $Q^1 - H_1^1$ from Corollary 6.4.3 there exists a coordinate system (u, v) on X_1 such that $p(u, v) = u$ and the curve H^* is given by the parametric equations $u = t^m$ and $v = (t - 1)^{-1}$. Thus H^* is given by the zero fiber of the polynomial $\varphi(u, v) = v^m u - (v + 1)^m$. Note that the line $L = \{v = 0\}$ does not meet H^* and matches with the hypothesis of 7.1. We would like to emphasize that the existence of this line L is a key of the proof of Main Theorem.

Lemma. *Let φ, u, v, L be as above, let X_1, X_2 be as in 7.1, and let $\hat{\varphi}$ and $\hat{v}$ be the rational functions on $\hat{X}$ that extend φ and v respectively.*

(1) *If* $G(\hat{D})$ *looks like in Fig.* 5a, *then the primitive polynomial on* X_2 *whose zero fiber is* C_1 *coincides with the restriction of* $\hat{\varphi}$ *to* X_2, *and the primitive polynomial on* X_2 *whose zero fiber is* L *coincides with the restriction of* $\hat{v}\hat{\varphi}^n$ *to* X_2.

(2) *If* $G(\hat{D})$ *looks like in Fig.* 5b, *then the restriction of* $\hat{\varphi}^{-1}$ *to* X_2 *is a primitive polynomial whose zero fiber is* C_1, *and the zero fiber of the polynomial that is the restriction of* $\hat{v}\hat{\varphi}^{-n}$ *to* X_2 *is* L.

(3) *If* $G(\hat{D})$ *looks like in Fig.* 5c, *then a primitive polynomial whose zero fiber is* C_1 *coincides with the restriction of* $\hat{\varphi}^{-1}$ *to* X_2, *and a primitive polynomial on* X_2 *whose zero fiber is* L *coincides with the restriction* $\hat{v}\hat{\varphi}^{-1}$ *to* X_2.

(4) *If* $G(\hat{D})$ *looks like in Fig.* 5d, *then a primitive polynomial whose zero fiber is* C_1 *coincides with the restriction of* $\hat{\varphi}$ *to* X_2, *and a primitive polynomial on* X_2 *whose zero fiber is* L *coincides with the restriction* $\hat{v}\hat{\varphi}$ *to* X_2.

Proof. Embed X_1 into the surface $Q^1 \approx \mathbf{C} \times \mathbf{CP}^1$ so that v can be extended to a regular mapping $Q^1 \to \mathbf{CP}^1$ and the natural projection $q_1 : Q^1 \to \mathbf{C}$ is the extension of the function u. Put $H_1^1 = Q^1 - X_1$ and H_2^1 equal to the closure of H^* in Q^1. The divisor of the extension of φ to Q^1 is $H_2^1 - mH_1^1$. Note that the point $b = H_2^1 \cap H_1^1$ corresponds to $u = 1, v = \infty$. Consider the local coordinate system $(\tilde{u}, \tilde{v}) = (u - 1, v^{-1})$ at b. The function

$$\varphi(u,v) = uv^m - (v+1)^m = v^m \left[u - 1 - \frac{(v+1)^m - v^m}{v^m} \right]$$

can be rewritten in this new coordinate system as $\tilde{v}^{-m}(\tilde{u} - g(\tilde{v}))$ where g is a polynomial. Hence H_2^1 meets H_1^1 normally at b which is a point of indeterminacy of type x/y^m for the extension of φ. In order to obtain the surface $\hat{X} - \hat{p}^{-1}(\infty)$ we need to blow Q^1 up at b and infinitely near points in such a way that after this blowing-up the graph of the fiber over 1 looks like a subgraph G_1 in Fig. 5. Let n and m be as in Fig. 5. Then induction in n and m shows that the divisor of $\hat{\varphi}$ contains the component $\hat{C}_1$ with coefficient 1 in cases (a) and (d), and with coefficient -1 in cases (b) and (c). Hence $\hat{\varphi} \mid_{X_2}$ is a polynomial on X_2 in cases (a) and (d), and $\hat{\varphi}^{-1} \mid_{X_2}$ is a polynomial in cases (b) and (c). It is also easy to check using induction in n and m that the divisor of the extension $\hat{v}$ of v to $\hat{X}$ contains $\hat{C}_1$ with coefficient $-n$ in cases (a) and (b), and with coefficient -1 in cases (c) and (d). Hence in case (a) $\hat{v}\hat{\varphi}^n$ is a polynomial on X_2 which does not equal zero on C_1. In case (d) we have the same with $n = 1$. In cases (b) and (c) such a polynomial on X_2 is given by $\hat{v}\hat{\varphi}^{-n}$ with $n \geq 1$. Note that these polynomials on X_2 have zero fiber equal to L. ☐

Remark. Note that $L = \hat{L} \cap X_2$ is isomorphic to $\mathbf{C}$.

7.3. Recall that C_1 is isomorphic to $\mathbf{C}^*$, by Corollary 6.4.1. Consider the case when the subgraph of G^1 looks like in Fig. 5a. By the Abhyankar-Moh-Suzuki Theorem [**AM**], [**Su**] and Proposition 2.13, there is a coordinate system (x, y) in X_2 such that $\hat{v}\hat{\varphi}^n|X_2 = x$ and

$$(a) \qquad \hat{\varphi}|X_2 = \begin{cases} \sigma^k + x^l \\ \text{or} \\ x^l\sigma^k - 1 \end{cases}$$

where $\sigma(x, y) = x^s y + g(x), \deg g < s$ and $g(0) = -1$. (When $s = 0$ we suppose that $\sigma(x, y) = y$.) For Fig 5d we have the same formulas but with $n = 1$.

For Fig. 5b we have $\hat{v}\hat{\varphi}^{-n}|X_2 = x$ and

$$(b) \qquad \hat{\varphi}^{-1}|X_2 = \begin{cases} \sigma^k + x^l \\ \text{or} \\ x^l\sigma^k - 1. \end{cases}$$

For Fig 5c we have the same formulas but with $n = 1$.

Lemma. *The number k equals 1 in formulas (a) and (b) above, i.e. one can suppose that in case (a) $\hat{\varphi} = x^s y + a_{s-1}x^{s-1} + \cdots + a_1 x - 1$ and in case (b) $\hat{\varphi}^{-1} = x^s y + a_{s-1}x^{s-1} + \cdots + a_1 x - 1$.*

Proof. Consider the first expression for $\hat{\varphi}$ in case (a). Note that $k = 1$ if the system $\sigma^k(x, y) + x^l - d = x - c = 0$ has one root for every generic complex numbers c and d. Since $\sigma^k + x^l = \varphi = d$ and $v\varphi^n = x = c$, one has $v = c/d^n$. Putting this value of v in the equation $\varphi(u, v) = v^m u - (1 + v)^m - d = 0$, we can see that this equation has only one root. Thus $k = 1$. If $l \geq s$ we replace y by $y + x^{l-s+1}$ and obtain the desired form of $\hat{\varphi}$. Same argument enables us to obtain the desired conclusion in the other case. ☐

7.4. Main theorem. *Let $p\colon \mathbf{C}^2 \to \mathbf{C}$ be a primitive rational polynomial whose zero fiber Γ_0 is isomorphic to $\mathbf{C}^*$. Suppose that Γ_0 is degenerate. Then there is a polynomial coordinate system (x, y) in $\mathbf{C}^2$ for which the polynomial $p(x, y)$ coincides with one of the following forms*

$$(1) \qquad a(\psi^{nm+1} + (\psi^n + x)^m)/x^m$$

$$(2) \qquad a(\psi^{nm-1} + (\psi^n + x)^m)/x^m,$$

where $a \in \mathbf{C}^*$, n and m are natural, $m \geq 2, n \geq 1$, in formula (2) $n \geq 2$ in the case of $m = 2$, $\psi(x, y) = x^m y + a_{m-1}x^{m-1} + \cdots + a_1 x - 1$, and all *coefficients* $a_{m-1}, \dots, a_1$ *are determined uniquely by the condition that each of the above forms must be a polynomial.*

Proof. Multiplying p by a nonzero number we may suppose that Γ_1 is the second degenerate fiber of p. Let (u, v) be the coordinate system that we used in Lemma 7.2. Recall that $p(x, y) = u$, by construction, and $\varphi(u, v) = v^m u - (v+1)^m$. Hence $p(x, y) = (\varphi + (1+v)^m)/v^m$. According to the argument in 7.3 $v = x\varphi^{-n}$ in cases (a) and (d). Thus $p(x, y) = (\varphi^{1+nm} + (x + \varphi^n)^m)/x^m$. In cases (b) and (c) $v = x\varphi^n$ and $p(x, y) = (\varphi^{1-m(n+1)} + (\varphi^{-(n+1)} + x)^m)/x^m$. Putting $\psi = \varphi$ in cases (a), (d) and $\psi = \varphi^{-1}$ in cases (b), (c) we obtain the formulas (1) and (2). The polynomial $\psi(x, y)$ coincides with $x^s y + a_{s-1}x^{s-1} + \dots + a_1 x - 1$. If $s < m$ then one can see that the numerator in forms (1) and (2) contains the monomial $x^s y$ with a nonzero coefficient, i.e., p is not a polynomial. Hence $s \geq m$. If $s > m$ and the numerator does not contain the monomial x^m then it is easy to check that Γ_0 contains the line $x = 0$, but it is not so. If this monomial belongs to the numerator with a nonzero coefficient then Γ_0 does not meet the line $x = 0$. Hence either Γ_0 is not degenerate or p is not rational, by Corollary 2.13. Contradiction. Hence $m = s$. When $n = 1$ in formula (2) we deal with Fig. 5c and, therefore, m must be > 2. Note also that the coefficient before x^j in the numerator for $0 < j < m$ is of form $ka_j + g_j(a_1, \dots, a_{j-1})$ where k is a nonzero integer and g_j is a polynomial (and g_1 is constant). If we want p to be a polynomial we have to require that these coefficients are zero which yields the claim about $a_1, \dots, a_{m-1}$. □

7.5. Let f, g be polynomials given by forms (1) or (2) in Main Theorem. If these forms have different discrete parameters then there is no automorphism β of $\mathbf{C}^2$ for which $f \circ \beta = g$. We shall follow **[Z1]** in the proof of this fact. Let a, n, m be the same as in Main Theorem. We say that $f \in A_1(a, n, m)$ if f is given by form (1) with the corresponding parameters a, n, m. If f is given by form (2) with given a, n, m we say that $f \in A_2(a, n, m)$.

Theorem. *Let* $f \in A_k(a, n, m)$ *and* $h \in A_l(a', n', m')$. *If* f *is equivalent to* h *up to a polynomial automorphism of* $\mathbf{C}^2$ *then* $k = l, a = a', n = n', m = m'$.

Proof. Note that $f^{-1}(a)$ is the second degenerate fiber for f and $h^{-1}(a')$ is the second degenerate fiber for h. Since any automorphism preserves degenerate fibers, $a = a'$. By construction, the generic fiber of f is the $m+1$ times punctured Riemann sphere. Hence we must have the same for h and $m = m'$. One can see that the fiber $f^{-1}(a)$ has a component of multiplicity n. Therefore $n = n'$.

Assume, to reach a contradiction that $f \in A_1(a, n, m)$ and $h \in A_2(a, n, m)$, and there is a polynomial automorphism $\beta(x, y) = (\beta_1(x, y), \beta_2(x, y))$ for which $f \circ \beta = h$. By Lemma 7.3, the multiple component of $f^{-1}(a)$ is given by $r(x,y) = x^m y + g(x) = 0$ and the multiple component of $h^{-1}(a)$ is given by $\tau(x,y) = x^m y + \tilde{g}(x) = 0$. By Nullstellensatz, $r \circ \beta = c\tau$ ($c \in \mathbf{C}^*$). Hence it is easy to show that $\deg_x \beta_1 = 1, \deg_y \beta_1 = 0$, $\deg_x \beta_2 = 0$, $\deg_y \beta_2 = 1$. Moreover, $\beta_1(x,y) = c'x$ and $\beta_2(x,y) = c''y$ ($c', c'' \in \mathbf{C}^*$) . (Indeed, if $\beta_1(x,y) = c'x + d'$ with $d' \neq 0$, then $r \circ \beta$ contains the monomial $x^{m-1}y$ with a nonzero coefficient, but this is not so.) Let $f = (\varphi^{1+nm} + (\varphi^n + x)^m)/x^m$ and $h = (\psi^{nm-1} + (\psi^n + x)^m)/x^m$. Since $\varphi = 0$ is the multiple component of $f^{-1}(a)$ and $\psi = 0$ is the multiple component of $h^{-1}(a)$, we have $\tilde{c}\varphi \circ \beta = \psi$. Put $z = \varphi \circ \beta$. Then the mapping $(x,y) \to (x,z)$ is birational. Note that $f \circ \beta$ has the form $(z^{1+nm} + (z^n + x)^m)/x^m$ in the coordinate system (x,z) and h has the form $((\tilde{c}z)^{nm-1} + ((\tilde{c}z)^n + x)^m)/x^m$. These two expressions are not equal, i.e., $f \circ \beta \neq h$. □

A. Appendix: The proof of Lemma 4.1.

The proof of the existence of the isomorphism ξ from Lemma 4.1 consists of two steps. First, we reduce the problem to a question about some Laurent polynomials. Second, we establish some symmetry of the coefficients of these polynomials which enables us to solve this question.

A.1. Reduction.

We revive notation from Section 4.1. We introduce also $Q^4 = Q^2 - q^{-1}(\omega_1)$ where ω_k ($k = 1, 2$) is the group of m_k-roots of unity.

A.1.1. Lemma. *The numbers m_1 and m_2 are relatively prime.*

Proof. The mapping δ generates a homomorphism δ_* of the second homology groups. Recall that a basis of the second homology group of Q consists of two elements E and F where E may be viewed as a fiber of q. The irreducible components of $\hat{D}$ generate a basis in the second homology group of $\hat{X}$ [**R**]. Obviously, the image of every vertical component of $\hat{D}$ under δ_* is a multiple of E and $\delta_*(\hat{H}_k) = m_k F + n_k E$. Since δ_* is surjective its image contains F. This is possible only if m_1 and m_2 are relatively prime. □

Remark. Note that either $m_2 > 1$ or $m_1 > 1$ since otherwise the generic fiber of p is isomorphic to $\mathbf{C}^*$ in contradiction with our assumption about this polynomial. We suppose in this section that $m_2 \geq 2$. (If this condition does not hold we can switch the numbers m_1 and m_2.) Using the fact that m_1 and m_2 are relatively prime, we suppose also that m_2 is even if and only if $m_1 = 1$. (If the last condition does not hold we can again switch m_1 and m_2.)

A.1.2. Recall that in the notation of 4.1 for every curve F in Q (or in Q^l with $l < k$) the curve F^k is $F \cap Q^k$. Similarly, if ψ is a morphism from Q

(or Q^l) then ψ_k is the restriction of ψ to Q^k. Consider the action μ of ω_1 on Q^1 given by $\mu_\varepsilon(x,(y_1:y_2)) = (\varepsilon x,(y_1:y_2))$ for every $\varepsilon \in \omega_1$. It generates a natural morphism $\tau : Q^1 \to Q^1/\omega_1 = Q^1$. Note that $\tau_2 : Q^2 \to Q^2$ is an unramified covering of Q^2 and that $Q^4 = \tau^{-1}(Q^3)$. Let $H^1_{\tau,k} = \tau^{-1}(H^1_k)$. Denote by $'H^l_{\tau,k}$ (resp. $''H^l_{\tau,k}$) the image of $H^l_{\tau,k}$ under $'\varphi$ (resp. $''\varphi$) where $'\varphi$ and $''\varphi$ are defined in 4.1. It is easy to see that $'H^l_{\tau,k} = \tau^{-1}('H^l_k)$ and $''H^l_{\tau,k} = \tau^{-1}(''H^l_k)$. The proof of the next lemma uses some properties of the curve $H^1_{\tau,2}$ which will be checked in A.1.3.

Lemma. *Suppose that there exists an automorphism $\zeta : Q^4 \to Q^4$ such that*

(i) $\zeta('H^4_{\tau,k}) = ''H^4_{\tau,k}$ *for* $k = 1,2$;

(ii) $q_4 \circ \zeta = q_4$.

Then Lemma 4.1 *is true.*

Proof. Let $\mu^0_\varepsilon = \zeta^{-1} \circ \mu_\varepsilon \circ \zeta$. We need to show that $\mu^0_\varepsilon = \mu_\varepsilon$ for every $\varepsilon \in \omega_1$. Then one can see from definitions that ζ can be pushed down to an automorphism ξ of Q^3 with the desired properties.

By construction, μ_ε and μ^0_ε preserve $'H^4_{\tau,2}$ and we consider the restriction of both actions to this curve. In Lemma A.1.3 (iii) below we shall show that there exists a normalization $\nu : \mathbf{C} \to H^1_{\tau,2}$ such that $q \circ \nu(s) = s^{m_2}$ where s is a coordinate on $\mathbf{C}$. This implies the existence of normalization $'\nu : \mathbf{C} \to 'H^1_{\tau,2}$ so that $q \circ '\nu(s) = s^{m_2}$. Since $q \circ \mu_\varepsilon = \varepsilon q$ and $q \circ \mu^0_\varepsilon = \varepsilon q$ the restrictions of μ_ε and μ^0_ε to $'H^1_{\tau,2}$ generate automorphisms of $\mathbf{C}$ which preserve the origin $s = 0$. Hence these automorphisms are homothetic transformations and, therefore, they are commutative. Thus the restrictions of μ_ε and μ^0_ε to $'H^1_{\tau,2}$ are commutative and we may view the restriction of the mappings $'\mu_\varepsilon = \mu^{-1}_\varepsilon \circ \mu^0_\varepsilon$ to this curve as an ω_1-action.

Note that $q_4 \circ '\mu_\varepsilon = q_4$. Hence it suffices to show that the restriction of this mapping to the generic fiber $E = \mathbf{CP}^1$ of q_4 is identical. Consider the set $S = E \cap 'H^4_{\tau,2}$. By construction, $'\mu_\varepsilon$ preserves S. Since $S \subset 'H^4_{\tau,2}$ the restriction of the mappings $'\mu_\varepsilon$ to S may be viewed as an ω_1-action on S. Recall that $'H^4_{\tau,2}$ is irreducible, by Lemma A.1.3 below. Hence every orbit of $'\mu_\varepsilon$ in S is of the same size l and, of course, l is a divisor of m_1. But S consists of m_2 points. Since m_1 and m_2 are relatively prime this implies that $l = 1$, i.e. the restriction of $'\mu_\varepsilon$ to S is identity. If $m_2 \geq 3$ we are done since the restriction of $'\mu_\varepsilon$ to E is a linear fractional transformation and thus it is identity as well. When $m_2 = 2$ then $m_1 = 1$, by Remark A.1.1. Hence the group ω_1 is trivial which implies again the desired conclusion. $\square$

A.1.3. We need to consider the curves $H^1_{\tau,1}$ and $H^1_{\tau,2}$ from A.1.2 more closely.

Lemma. (i) *The curves* $H^4_{\tau,k}$ $(k = 1,2)$ *are smooth and do not meet each other;*

(ii) *the curve* $H^1_{\tau,1}$ *consists of* m_1 *irreducible components each of which is a section, i.e. the i-th component* $(i = 1,\dots,m_1)$ *has a normalization given in the coordinate system* $(x,(y_1 : y_2))$ *on* $Q^1 \cong \mathbf{C}\times\mathbf{CP}^1$ *by formulas* $x = t, y_2/y_1 = e_{1,i}(t)$ *where t runs over* $\mathbf{C}$ *and* $e_{1,i}$ *is a rational function of t (which may be identically* ∞*);*
(iii) *there exists a normalization* $\mathbf{C}\to H^1_{\tau,2} \subset Q^1 \cong \mathbf{C}\times\mathbf{CP}^1$ *of* $H^1_{\tau,2}$ *given by* $x = s^{m_2}, y_2/y_1 = e_2(s^{m_1})$ *where s is a coordinate on* $\mathbf{C}$ *and* e_2 *is a rational function (in particular,* $H^1_{\tau,2}$ *is irreducible);*
(iv) *the function* $e_2(t)$ *has a simple zero at* $t = 0$ *and the function* $e_{1,i}(t)$ *has a pole at* $t = 0$ *for every* $i = 1,\dots,m_1$.

Proof. The curve $\hat{H}^1_k = \hat{H}_k - \hat{p}^{-1}(\{\infty\})$ is isomorphic to $\mathbf{C}$ since $\hat{p}^{-1}(\infty)\cap\hat{H}_k$ is a point, by Lemma 2.12. Since the restriction of δ to $\hat{X} - \hat{p}^{-1}(\{1,\infty\})$ is an isomorphism the mapping $\delta\mid_{\hat{H}^1_k}$ may be viewed as a normalization of the curve $H^1_k = H_k \cap Q^1$. Moreover, $H^1_k - q^{-1}(1)$ is smooth and H^1_1 does not meet H^1_2 outside the fiber $q^{-1}(1)$. By construction, $\tau_2 : Q^2 \to Q^2$ is an unramified covering and the restriction of τ to each fiber of q_2 generates an isomorphism of fibers of q_2. This implies that the curves $H^4_{\tau,k}$ $(k = 1,2)$ are smooth and and do not meet each other which yields (i).

The restriction of $\hat{p}$ to $\hat{H}^2_k$ is an m_k-sheeted cyclic covering of $\mathbf{C}^*$. In particular, one may introduce a coordinate t on $\hat{H}^1_k$ so that $p(t) = t^{m_k}$. Hence the curve $H^1_k \subset Q^1 \cong \mathbf{C}\times\mathbf{CP}^1$ has the following parametric representation $x = t^{m_1}, y_2/y_1 = e_k(t)$ where e_k is a rational function. Since the mapping τ in the coordinate system $(x,(y_1 : y_2))$ has the following form $(x,(y_1,y_2)) \to (x^{m_1},(y_1 : y_2))$ the curve $H^1_{\tau,k} = \tau^{-1}(H^1_k)$ $(k = 1,2)$ is given by the equations $x^{m_1} = t^{m_k}, y_2/y_1 = e_k(t)$.

For $k = 1$ this implies that $H^1_{\tau,1}$ consists of m_1 components and a normalization of the i-th component may be chosen in the form $x = t, y_2/y_1 = e_1(\varepsilon t)$ where $\varepsilon \in \omega_1$. This yields (ii).

For $k = 2$ the curve $H^1_{\tau,2}$ is irreducible since m_1 and m_2 are relatively prime and, by putting $t = s^{m_1}$, we obtain the normalization of this curve given in (iii).

Recall that H^1_1 and H^1_2 meet the fiber $q^{-1}(0)$ at different points c_1 and c_2 which coincide with the points $(0 : 1)$ and $(1 : 0)$ respectively in the coordinate system $(y_1 : y_2)$ on $q^{-1}(0) \cong \mathbf{CP}^1$ (see 4.1). Hence $e_{1,i}(t)$ has a pole at $t = 0$ for every i. As we mentioned in the beginning of the proof the curve H^1_2 is smooth at c_2. Hence $e_2(t)$ must have a simple zero at $t = 0$ unless $m_2 = 1$. But m_2 cannot be 1 due to Remark A.1.1 which concludes the proof. ☐

A.1.4. Let F be a component of $H^1_{\tau,1}$ and let A be the union of $H^1_{\tau,2}$ and the other components of $H^1_{\tau,1}$. We want to modify these curves using birational mappings described in the following

Lemma. *Let F be a section in Q^1, i.e it meets each fiber of q_1 at one point. Suppose that A is another closed curve in Q^1 such that q_1 is non-constant on each component of A. Let $a \in A \cap F$ and $b = q_1(a)$. Then there exists a birational mapping of Q^1 into itself such that*

(i) *its restriction to $Q^1 - q_1^{-1}(b)$ is an automorphism which preserves the function $q\,|_{Q^1-q^{-1}(b)}$;*

(ii) *the proper transforms of A and F do not meet in the fiber over b.*

Moreover, suppose that the mapping $q_1\,|_A$ is m-sheeted, $m > 1$, and $\nu_A^{-1}(a)$ consists of m points where $\nu_A : A^{\mathrm{norm}} \to A$ is a normalization of A. Then

(iii) *the proper transform of A meets the fiber over b at more than one point.*

Proof. Our main tool will be Nagata's elementary operations between ruled surfaces. Let $E = q^{-1}(b)$. Since F is a section F meets E at one point a which belongs to A, by assumption. Choose a local coordinate system (z, t) with origin at a so that $q(z, t) = t$. Since F is a section one may suppose that its local equation is $z = 0$. The local equation of A is $z^k = t^l g(t)$ where g is holomorphic and $g(0) \neq 0$. Consider the following birational mapping. First we blow Q^1 up at a. After this the curve E is replaced by two (-1)-curves E_1 and E_2 where E_1 is the proper transform of E_2. Contract E_1. As a result we obtain a new sample of Q^1 in which the fiber E is replaced by E' and the curves F and A are replaced by their proper transforms F' and A'. One may choose a local coordinate (z', t') system with origin at $a' = E' \cap F'$ so that $z' = z/t$ and $t' = t$. In this system the local equation of F' is $z' = 0$. When $l \leq k$ one can check that A' does not contain a' and, therefore, does not meet F'. When $l > k$ the local equation of A' is $z'^k = t'^{l-k} g(t')$. We see that the contact order between A' and F' at a' is less than the contact order between A and F at a. Thus repeating this procedure we finally obtain proper transforms F'' and A'' of F and A which do not meet each other in the fiber over b. Suppose that A'' meets the fiber over b at one point a''. Assumption on normalization implies that A consists of m branches in a neighborhood of a'' such that their local equations are $z'' = g_j(t'')$ $(j = 1, \dots, m)$. Repetition of blowing-ups and blowing-downs in the fiber over b makes some of these branches disjoint eventually. □

A.1.5. Recall that F is a component of $H^1_{\tau,1}$ and A is the union of $H^1_{\tau,2}$ and the other components of $H^1_{\tau,1}$. By Lemma A.1.4, we may find a birational mapping θ of Q^1 into itself so that $\theta\,|_{Q^1-q^{-1}(\omega_1)}$ is an automorphism which preserves $q\mid_{Q^1-q^{-1}(\omega_1)}$, the proper transforms of F and A do not meet, and the proper transform of A meets $q^{-1}(b)$ at least at two points for every $b \in \omega_1$. Suppose that the proper transform of $H^1_{\tau,1}$ consists of components $F_1, \dots, F_{m_1}$ where F_{m_1} is the proper transform of F, and the proper transform of $H^1_{\tau,2}$ is $\bar{H}$. In order to make notation shorter denote by H the curve $\bar{H}^2 = \bar{H} \cap Q^2$. The advantage of the long trip from H_1 and H_2 to these curves is that we can represent $H, F_1, \dots, F_{m_1-1}$ as affine curves in $Q^2 - F_{m_1} \cong \mathbf{C}^* \times \mathbf{C}$. Introduce a coordinate system (x,y) in $Q^1 - F_{m_1}$ so that the restriction of q to $Q^1 - F_{m_1}$ is the projection to the x-axis. It follows from Lemma A.1.3 (iii) that $H^1_{\tau,2}$ meets the fiber $q^{-1}(0)$ at one point only. Hence $\bar{H}$ meets $q^{-1}(0)$ at one point only.

Lemma. *There exists a coordinate system* (x,y) *in* $Q^2 - F_{m_1} \cong \mathbf{C}^* \times \mathbf{C}$ *such that the* y*-coordinate of the point* $\bar{H} \cap q^{-1}(0)$ *is 0 and the curves* $H, F_1, \dots, F_{m_1-1}$ *have the following properties:*

(i) *the curves* F_i^4 $(i = 1, \dots, m_1 - 1)$ *do not meet each other, and* H^4 *is smooth;*

(ii) $H \cup \bigcup_{i=1}^{m_1-1} F_i$ *meets* $q^{-1}(b)$ *at least at two points for every* $b \in \omega_1$*;*

(iii) *for each* $i = 1, \dots, m_1 - 1$ *there exists a normalization* $\nu_i : \mathbf{C}^* \to F_i \subset \mathbf{C}^* \times \mathbf{C}$ *of* F_i *such that* $\nu_i(t) = (t, f_i(t))$ *where* t *is a coordinate on* $\mathbf{C}^*$ *and* $f_i(t) = a_{i,n_i}t^{n_i} + a_{i,n_i-1}t^{n_i-1} + \dots + a_{i,k_i}t^{k_i}$ *is a Laurent polynomial;*

(iv) *there exists a normalization* $\nu : \mathbf{C}^* \to H \subset \mathbf{C}^* \times \mathbf{C}$ *of* H *so that* $\nu(t) = (t^{m_2}, h(t))$ *where* $h(t) = d_n t^n + d_{n-1}t^{n-1} + \dots + d_k t^k$ *is a Laurent polynomial;*

(v) $k = m_1$ *and, in particular,* m_2 *and* k *are relatively prime.*

Proof. Properties (i)-(iv) follow immediately from Lemma A.1.3 and the description of θ. For (v) we need to consider the birational mapping θ more accurately. It is more convenient to denote now our usual coordinate system (which was used in A.1.1 and A.1.3) on the first sample of Q^1 by $(x', (y'_1 : y'_2))$. Put $y' = y'_2/y'_1$. Recall that q_1 is the projection to the x'-axis in the first sample of Q^1. Since θ is an isomorphism outside the set $q_1^{-1}(\omega_1)$ which preserves q_1 the restriction of θ to $Q^1 - (q_1^{-1}(\omega_1) \cup F_{m_1})$ has form $(x', (y'_1 : y'_2)) \to (x, y)$ such that $x = x'$ and $y = L(x', y')$ where for every $x' \in \mathbf{C} - \omega_1$ the mapping $L(x', y')$ is a linear fractional transformation $(r_1(x')y' + r_2(x'))/(r_3(x')y' + r_4(x'))$ and r_1, r_2, r_3, r_4 are polynomials for which the roots of the polynomial $r_0 = r_1r_4 - r_2r_3$ are contained in ω_1. Hence $\bar{H}$ which is the proper transform of $H^1_{\tau,2}$ is given by $x = t^{m_2}, y = h(t) = (r_1(t^{m_2})e_2(t^{m_1}) + r_2(t^{m_2}))/(r_3(t^{m_2})e_2(t^{m_1}) + r_4(t^{m_2}))$ where e_2 is from Lemma A.1.3 (iii). Recall that $e_2(t)$ has a simple zero at $t = 0$. Hence, since the y-coordinate of the point $\bar{H} \cap q^{-1}(0)$ is zero we have $h(0) = 0$. This implies $r_2(0) = 0$. The assumption on r_0 implies that $r_1(0)r_4(0) \neq 0$. If $m_1 < m_2$ then, using again the fact that e_2 has a simple zero at the origin, one can show that the first nonzero term of the Taylor series of $h(t)$ at $t = 0$ is dt^{m_1} which yields (v). If $m_2 < m_1$ then the same argument implies that this Taylor series contains a nonzero term dt^{m_1}. It may also contain terms of form $d_i t^i$ where $i < m_1$, but i must be divisible by m_2. Due to the remark after this theorem the coordinate system (x, y) can be changed so that all terms whose exponents are multiples of m_2 have zero coefficients which yields the desired conclusion. □

Remark. We have some freedom in the choice of the coordinate system (x, y) from Lemma A.1.5 since we can always use a substitution $(x, y) \to (x, cx^l y + g(x))$ where c is a nonzero constant, $l \in \mathbf{Z}$, and g is a Laurent polynomial. Using this freedom we can suppose further that $d_i = 0$ for i divisible by m_2.

A.1.6. Lemma. *Suppose that*

(i) $n = -k$;

(ii) $n_i = -k_i$ *for every* $i = 1, \dots, m_1 - 1$;

(iii) $d_{-j} = \bar{d}_j$ *and* $a_{i,-j} = \bar{a}_{i,j}$ *for every* j *and every* $i = 1, \dots, m_1 - 1$.

Then Lemma 4.1 *is true.*

Proof. As usual, put $'F_i = '\varphi_2(F_i), ''F_i = ''\varphi_2(F_i)$, $'H = '\varphi_2(H)$, and $''H = \varphi_2(H)$. Assumptions (i)-(iii) and the description of H and F_i given in A.1.5 immediately imply that $'F_i = ''F_i$ and $'H = ''H$. We are going to show that this implies the existence of ζ from Lemma A.1.2 and, therefore, the existence of ξ form Lemma 4.1. Put $\zeta = ''\varphi_4 \circ \theta_4^{-1} \circ ''\varphi_4 \circ '\varphi_4 \circ \theta_4 \circ '\varphi_4$ where θ is from A.1.5. By construction, this mapping is a diffeomorphism which preserves the function q_4, and $\zeta('H^4_{r,k}) = ''H^4_{r,k}$. We need to check also that this mapping is an automorphism which is equivalent to the fact that $'\varphi_4 \circ \theta_4 \circ '\varphi_4$ is an automorphism. This is obvious. Indeed, in the local coordinate system (x, y) from A.1.5 the mapping θ is given by $(x, y) \to (x, L(x, y))$ where L is a rational function. Hence $'\varphi_4 \circ \theta_4 \circ '\varphi_4$ is given by $(x, y) \to (x, \overline{L(\bar{x}, \bar{y})})$ which is a regular mapping and, therefore, an automorphism. □

A.2. Symmetry of the coefficients.

We put $\varepsilon = \exp(2\pi\sqrt{-1}/m_2)$ *and suppose that "bar" means complex conjugate for the rest of the paper.* Most of the computation in this section is based on the following observation.

A.2.1 Lemma. *Let* $g(t) = b_{l_2} t^{l_2} + b_{l_2 - 1} t^{l_2 - 1} + \cdots + b_{l_1} t^{l_1} \in \mathbf{C}[t, t^{-1}]$ *be a Laurent polynomial where* $b_{l_1} b_{l_2} \neq 0$. *Suppose that all roots of* g *have absolute* *value one. Then for every i between 0 and* $l_2 - l_1$ *we have* $b_{l_2}\bar{b}_{l_1+i} = b_{l_2-i}\bar{b}_{l_1}$. *In particular, if* $b_{l_2-j} = \bar{b}_{l_1+j}$ *for some j such that* $0 \leq j \leq l_2 - l_1$ *then* $b_{l_2-i} = \bar{b}_{l_1+i}$ *for every i.*

Proof. Consider the Laurent polynomials $g(t^{-1})$ and $\overline{g(\bar{t})}$. Clearly, if λ is a root of $g(t)$ then λ^{-1} is a root of the above two polynomials, i.e. they have common roots. Hence $g(t^{-1}) = ct^l\overline{g(\bar{t})}$ where c is a nonzero constant and $l = -l_1 - l_2$. This implies the desired conclusion. □

A.2.2. Lemma. *Let the notation be as in* A.1.5. *Then* $n \equiv \pm k (\text{mod } m_2)$, *and n and* m_2 *is relatively prime.*

Proof. Since the x-coordinates of the singular points of H belong to ω_1, the roots of the Laurent polynomial $h_s(t) = h(\varepsilon^s t) - h(t)$ have absolute value 1 for every s which is not a multiple of m_2. Note that $h_s(t) = b_n^s t^n + \cdots + b_k^s t^k$ where $b_i^s = (\varepsilon^{si} - 1)d_i$. Suppose that $\varepsilon^{ns} \neq 1$ and $\varepsilon^{ks} \neq 1$. Then $b_k^s, b_n^s \neq 0$. By the Vieta Theorem, $|b_n^s/b_k^s| = 1$. Suppose first that n and $k \not\equiv (m_2/2)(\text{mod } m_2)$. Then s can be chosen 2, and $|b_n^2/b_k^2| = |b_n^1/b_k^1| \cdot |(1+\varepsilon^n)/(1+\varepsilon^k)|$. Hence $|1+\varepsilon^n| = |1+\varepsilon^k|$ which is possible only when $n = \pm k(\text{mod } m_2)$.

Now let either n or $k = (m_2/2)(\text{mod } m_2)$. In particular, m_2 is even and $m_1 = 1$, by Remark A.1.1. Hence $k = 1$. The case $m_2 = 2$ is trivial since $d_i = 0$ for i divisible by m_2 (see Remark A.1.5). We want to show that m_2 cannot be greater than 2 when $n = (m_2/2)(\text{mod } m_2)$, and we need to consider two cases.

Case 1: assume that $m_2 \geq 6$. By comparing $|b_n^3/b_k^3|$ and $|b_n^1/b_k^1|$, one can see that $|1 + \varepsilon^n + \varepsilon^{2n}| = |1 + \varepsilon^k + \varepsilon^{2k}|$. Since $\varepsilon^n = -1$ the left-hand side of this equality is 1. Since $\varepsilon^k = \varepsilon$ the right-hand side is $|1 + \varepsilon + \varepsilon^2|$ which is not 1 when $m_2 \geq 6$. Contradiction.

Case 2: assume that $m_2 = 4$. Since $m_1 = 1$ we have $\omega_1 = \{1\}$. Hence the x-coordinate of every singular point of H is 1. This means that the only root of each Laurent polynomial h_s is 1. Consider $h_1(t) = h(\sqrt{-1}t) - h(t)$ and $h_3(t) = h(-\sqrt{-1}t) - h(t)$. Due to the remark about the roots of these polynomials both of them coincide with $t^k(t-1)^{n-k}$ up to constant factors. On the other hand $h_3(t) = -h_1(-\sqrt{-1}t)$ which is a contradiction. (The original argument in this last case was very complicated. The proof above belongs to the referee.)

Since $n = \pm k(\text{mod } m_2)$ and $k = m_1$, by Lemma A.1.5 (v), the numbers n and m_2 must be relatively prime, by Lemma A.1.1. □

A.2.3. Lemma. *In the notation of Lemma* A.1.5 $|d_k| = |d_n|$.

Proof. Let b_i^s be as in the proof of Lemma A.2.2. Since $|b_n^1| = |b_k^1|$, by the Vieta Theorem, we have $|d_n(1-\varepsilon^n)| = |d_k(1-\varepsilon^k)|$. Hence $|d_n| = |d_k|$ since $|1-\varepsilon^n| = |1-\varepsilon^k|$ in the virtue of Lemma A.2.2. □

Convention. From now on we suppose that $d_n = \bar{d}_k$. Due to the above Corollary we can always achieve this by a coordinate substitution from Remark A.1.5.

A.2.4. Lemma. *Let the notation be as in Lemma* A.1.5. *Then* $d_{n-i} = \bar{d}_{k+i}$ *for every* i *between 0 and* $n-k$. *If* $n = k(\text{mod } m_2)$ *then* $d_i \neq 0$ *only if* $i - k = 0(\text{mod } m_2)$.

Proof. Suppose first that $n = -k(\text{mod } m_2)$. As in Lemma A.2.2 introduce the Laurent polynomial $h_s(t) = h(\varepsilon^s t) - h(t) = \sum_{i=k}^{n} b_i^s t^i$ where $s \neq 0(\text{mod } m_2)$ and $b_i^s = (\varepsilon^{si} - 1)d_i$. Recall that the absolute value of every root of h_s is 1. Since $d_n = \bar{d}_k$, by Convention A.2.3, and $\varepsilon^{sn} = \bar{\varepsilon}^{ks}$ we have $b_n^s = \bar{b}_k^s$. Lemma A.2.1 implies that $b_{n-i}^s = \bar{b}_{k+i}^s$ for every s. Hence $d_{n-i}(\varepsilon^{n-i} - 1) = \bar{d}_{k+i}(\bar{\varepsilon}^{k+i} - 1)$, i.e. $d_{n-i} = \bar{d}_{k+i}$. (We use the fact that $d_i = 0$ when i is divisible by m_2.)

Consider the case when $n = k(\text{mod } m_2)$. By Lemma A.2.1, $b_n^s \bar{b}_{k+i}^s = b_{n-i}^s \bar{b}_k^s$, but now $\varepsilon^{sn} = \varepsilon^{ks}$. Suppose that $2n \neq 0(\text{mod } m_2)$. Then s can be chosen 2 and $b_i^2 = b_i^1(\varepsilon^i + 1)$. Hence $b_n^1 \bar{b}_{k+i}^1(\varepsilon^n + 1)(\varepsilon^{-k-i} + 1) = b_{n-i}^1 \bar{b}_k^1(\varepsilon^{n-i} + 1)(\varepsilon^{-k} + 1)$ and for nonzero b's we have $\varepsilon^{-n} + \varepsilon^{n-i} - \varepsilon^n - \varepsilon^{-n-i} = (1 - \varepsilon^{-i})(\varepsilon^{-n} - \varepsilon^n) = 0$. The last equality holds only if $i = 0(\text{mod } m_2)$. Thus $b_{k+i}^1 = 0$ when $i \neq 0(\text{mod } m_2)$ and $b_j^1 = (\varepsilon^k - 1)d_j$. Hence $d_{n-i} = \bar{d}_{k+i}$.

Let $2n = 0(\text{mod } m_2)$. Then, by Lemma A.2.2 and Remark A.1.1, $n = \pm 1(\text{mod } m_2)$, i.e. $m_2 = 2$. Hence $d_i = 0$ for even i, by Remark A.1.5. The equality $d_{n-i} = \bar{d}_{k+i}$ holds since $n = -k(\text{mod } 2)$. □

A.2.5. Note that if $n = -k(\text{mod } m_2)$ then, using automorphism $(x, y) \to (x, x^l y)$ (where (x, y) is a coordinate system from A.1.5), we may suppose that $n = -k$.

Lemma. *Let* $f_i(t)$ *be as in* A.1.5. *Suppose that* $n = -k$. *Then for every* $i = 1, \dots, m_1 - 1$

(i) $n_i = -k_i$ *and*

(ii) *for every* j *we have* $a_{i,-j} = \bar{a}_{i,j}$.

Proof. First note that since the x-coordinates of the intersection points of F_i and H has absolute value 1 the Laurent polynomial $f(t) = h(t) - f_i(t^{m_2})$ has only roots with absolute value 1. Let $f(t) = \sum_{j=r}^{s} c_j t^j$ with $c_r c_s \neq 0$. We have to consider several cases

(1) $s = n_i m_2 > n > -n > k_i m_2 = r$;

(2) $s = n > n_i m_2 > k_i m_2 > -n = r$;

(3) $s = n_i m_2 > n > k_i m_2 > -n = r$; and

(4) $s = n > n_i m_2 > -n > k_i m_2 = r$.

Consider (1). Assume that $j_0 = s - n < -n - r$. Then, by definition of f, we have $c_{s-j_0} \neq 0$ and $c_{r+j_0} = 0$ which contradicts Lemma A.2.1. Similarly, one cannot have $s - n > -n - r$, i.e. $s - n = -n - r$ and, therefore, $s = -r$ and $n_i = -k_i$. By construction and by Convention A.2.3, $c_{s-j_0} = d_n = \bar{d}_{-n} = \bar{c}_{-s+j_0}$. Hence $c_{s-j} = \bar{c}_{-s+j}$ for every j, by Lemma A.2.1. Since $d_{jm_2} = 0$, by Remark A.1.5, we have $c_{jm_2} = a_{i,j}$ which implies (ii) in this case.

Exactly the same argument works in (2) and we consider (3). One may suppose that $j_0 = n_i m_2 - n \neq k_i m_2 - r = n + k_i m_2$. Indeed, otherwise $2n = 0 (\operatorname{mod} m_2)$, i.e. m_2 is even and, by Remark A.1.1, $m_1 = 1$. The statement of Lemma is true since $m_1 - 1 = 0$. Assume $j_0 < k_i m_2 - r$. Then, by definition of f, we have $c_{s-j_0} \neq 0$ and $c_{r+j_0} = 0$ which contradicts Lemma A.2.1. Similarly one cannot have $j_0 > k_i m_2 - r$ and we have to disregard (3) unless $m_2 = 2$. Exactly the same argument shows that (4) does not hold, except for the case $m_2 = 2$ which is obvious. □

A.2.6. Lemma. *Under the assumption of Lemma* A.1.5 $n \neq k (\operatorname{mod} m_2)$ *unless* $m_2 = 2$.

Proof. Assume the contrary. The second statement of Lemma A.2.4 implies that $d_j \neq 0$ only if $j - k = 0 (\operatorname{mod} m_2)$. We are going to show that this fact contradicts Lemma A.1.5 (ii). Let $f(t) = \sum_{j=r}^{s} c_j t^j$ has the same meaning as in the proof of Lemma A.2.5. We have again cases

(1) $s = n_i m_2 > n > k > k_i m_2 = r$;

(2) $s = n > n_i m_2 > k_i m_2 > k = r$;

(3) $s = n_i m_2 > n > k_i m_2 > k = r$; and

(4) $s = n > n_i m_2 > k > k_i m_2 = r$.

Consider (1). Note that $j_0 = s - n \neq k - r$. Indeed, otherwise $2n = 0 (\operatorname{mod} m_2)$. Since $m_2 \neq 2$ this implies that n and m_2 are not relatively prime which contradicts Lemmas A.1.5 (v) and A.2.2. Assume $j_0 < k - r$. Then, by definition of f, we have $c_{s-j_0} \neq 0$ and $c_{r+j_0} = 0$ which contradicts Lemma A.2.1. Similarly, one cannot have $s - n > k - r$. Therefore, we have to disregard (1) and, similarly, (2).

The second statement of Lemma A.2.4 and the construction of f imply that $c_j = d_j$ when $j - k = 0 (\operatorname{mod} m_2)$, $c_j = a_{i,l}$ when $j = m_2 l$, and $c_j = 0$

in all other cases. Consider (3). Note that $c_s = a_{i,n_i}$ and $c_r = d_k$. Put $\lambda_i = c_s/\bar{c}_r$. By Lemma A.2.1, $c_{s-j} = \lambda_i \bar{c}_{r+j}$. Put $j = m_2 l$ then $a_{i,n_i-l} = \lambda_i \bar{d}_{k+m_2 l}$. Since $d_n = \bar{d}_k$, by Convention A.2.3, and, therefore, $d_{n-j} = \bar{d}_{k+j}$ we have $a_{i,n_i-l} = \lambda_i d_{n-m_2 l}$. Hence

$$\lambda_i f_i(t^{m_2}) = t^{n_i m_2 - n} h(t).$$

Same argument in case (4) gives similar formula. Suppose that v is a root of h. Then in notation from A.1.5 the above formula implies then that the point $b = (v, 0)$ (in coordinate system (x, y) from A.1.5) is a selfintersection point of H and the multiplicity of H at this point is $\geq m_2$. Moreover, for every i the curve F_i must meet this point as well. Hence the curve $H \cup \bigcup_{i=1}^{m_1-1} F_i$ from A.1.5 meets the fiber $q^{-1}(b)$ at this point only which is a contradiction. Thus this case does not hold. □

Combination of the above Lemma and Lemmas A.2.2, A.2.5 gives

A.2.7. Lemma. *Applying an automorphism of* $(x, y) \to (x, x^l y)$ (*where* (x, y) *is the coordinate system from* A.1.5 *and* $l \in \mathbf{Z}$) *one may suppose that conditions* (i)-(iii) *from Lemma* A.1.6 *hold, and, therefore, Lemma* 4.1 *is true.*

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