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Let X be a compact metric space of dimension d. In previous work, we have shown that for all sufficiently large n, every element of the identity component $U_0(C(X) \otimes M_n)$ of the unitary group $U(C(X) \otimes M_n)$ is a product of at most 4 exponentials of skewadjoint elements. On the other hand, if X is a manifold then some elements of $U_0(C(X) \otimes M_n)$ require at least about d/n^2 exponentials. Similar qualitative behavior (with different bounds: 5 and $d/(2n^2)$) holds for the problem of factoring elements of the identity component $\mathrm{inv}_0(C(X) \otimes M_n)$ of the invertible group as products of exponentials of arbitrary elements of the algebra. In this paper, we identify the sets of finite products of 10 other types of elements of $\mathrm{inv}_0(C(X) \otimes M_n)$, and we show that the minimum lengths of factorizations have the same qualitative behavior as the two exponential factorization problems above (after a suitable minor modification in 3 of the 10 cases). We obtain upper bounds for large n that range from 5 to 22, and lower bounds approximately of the form rd/n^2 with r ranging from 1/16 to 2. The classes of elements we consider all make sense in general unital C*-algebras. They are: unipotents, positive invertibles, selfadjoint invertibles, symmetries, *-symmetries, commutators of elements of $\mathrm{inv}_0(A)$ and $U_0(A)$, accretive elements, accretive unitaries, and positive-stable elements (real part of spectrum positive). The last three classes are the ones requiring the slight modification; without it, lengths of factorization behave like exponential length rather than exponential rank.

Introduction.

If A is a unital C*-algebra, then the C* exponential rank of A, denoted $\mathrm{cer}(A)$, is the smallest $n \in \{1, 1+\varepsilon, 2, 2+\varepsilon, \ldots, \infty\}$ such that every element of the identity component of the unitary group of A is a product of at most n exponentials of skewadjoint elements. (We say u is a product of $k + \varepsilon$ exponentials if it is a limit of products of k exponentials.) In [**13**] it is proved that for $n \geq 2$ and X a compact manifold, $\mathrm{cer}(C(X) \otimes M_n)$ is at least as large as about $\dim(X)/n^2$, but on the other hand is bounded above by a finite function of n and $\dim(X)$ which is at most 4 for n sufficiently large (depending on d).

In this paper, we consider the problem of factoring suitable invertible elements of $C(X) \otimes M_n$ as products of a number of other kinds of factors, such as positive invertible elements, unipotent elements, symmetries, and commutators of homotopically trivial invertibles and unitaries. Including exponential rank and several other problems that have been studied before, we consider a total of 12 factorization problems. (A detailed list is given at the beginning of Section 1.) We prove in this paper that 9 of them have the same qualitative behavior on $C(X) \otimes M_n$ as described above for C* exponential rank (although with different constants). The remaining 3 problems behave like C* exponential length [**24**], but, after a suitable small modification, they too behave as above. Including the already known cases for completeness, we thus give 12 theorems which say essentially the same thing about different factorization problems in $C(X) \otimes M_n$.

We find it striking that 9 different factorization problems, and slight modifications of 3 others, all have the same qualitative behavior on algebras of the form $C(X) \otimes M_n$. In particular, the same topological obstruction seems to prevent short factorizations when $\dim(X) >> n \geq 2$ (even when X is contractible), and then seems to disappear for large n.

In Section 1, we describe 13 different factorization problems. (We only get 12 theorems because for $C(X) \otimes M_n$, although not in general C*-algebras, two of the problems turn out to be identical.) We introduce notation, prove several general lemmas, and discuss known results on the factorization problems for M_n and $L(H)$. The remaining three sections contain the proofs of our theorems; the arrangement is described near the end of Section 1.

Unless otherwise specified, we consider only compact metric spaces X; in some of our problems, this saves some technicalities involving dimensions of compact spaces that are not second countable. If $a : X \to M_n$ is a function, then $\det(a)$ is the function $x \mapsto \det(a(x))$. We let SU_n and SL_n denote the unitaries and invertibles in M_n with determinant 1. If Z is any metric space (for example, SU_n), then $C(X, Z)$ denotes the space of continuous functions from X to Z with the topology of uniform convergence.

I would like to thank Terrance Quinn for valuable e-mail correspondence, and Ian Putnam for calling my attention to Quinn's work on factorizations into positive elements. It was questions about such factorizations that led to this paper.

1. Generalities on Factorization.

Let A be a unital C*-algebra, let $U(A)$ and $\mathrm{inv}(A)$ be its unitary and invertible groups, and let $U_0(A)$ and $\mathrm{inv}_0(A)$ be their identity components. In this paper, we consider the lengths of factorizations of elements of $\mathrm{inv}_0(A)$, when the factorization is possible at all, into selfadjoint invertible elements, positive invertible elements, commutators of elements of $U_0(A)$ or $\mathrm{inv}_0(A)$, exponentials of skewadjoint or arbitrary elements of A, and the classes of operators given in the following definition. (The terminology in the definition specializes for M_n, except as noted, to that used in the survey article [**29**] on factorization problems in M_n.)

1.1 Definition. Let A be a unital C*-algebra. Then an element $a \in A$ is called:

(1) a *symmetry* ("involution" in [**29**]) if $a^2 = 1$;

(2) a **-symmetry* ("symmetry" in [**29**]) if a is unitary and $a^2 = 1$;

(3) *unipotent* if $a - 1$ is nilpotent;

(4) *quasiunipotent* (not in [**29**]) if $a - 1$ is quasinilpotent;

(5) *positive-stable* if $\mathrm{sp}(a) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$;

(6) *accretive* if $(a + a^*)/2$ is positive and invertible;

(7) *unitary accretive* (not in [**29**]) if a is accretive and unitary.

We point out that a positive-stable (written without the hyphen in [**29**]) element is not necessarily positive. Our definition of accretive agrees with that in [**29**] but conflicts with terminology used elsewhere. (See Section 4.) To clarify the significance of accretive and positive-stable elements, we prove in Section 4 that an accretive element is positive-stable (and hence invertible), and that $a \in A$ is accretive if and only if $\mathrm{Re}(\varphi(a)) > 0$ for every state φ.

Including the ones mentioned before the definition, there are altogether 13 classes. Somewhat over half of them are discussed in [**29**] for M_n and (sometimes) for $L(H)$. Unlike [**29**], we insist that positive and selfadjoint elements be invertible, and, rather than using the most obvious generalization of [**29**], we use commutators of elements of $\mathrm{inv}_0(A)$. Of the remaining classes, products of exponentials are trivial in M_n. Unitary commutators seem to have been overlooked in [**29**]. Quasiunipotent elements seem to be a more natural class than unipotent elements in a C*-algebra, but in M_n (the primary focus of [**29**]) and in $C(X) \otimes M_n$ (the primary focus here), both classes are the same. Finally, we consider accretive unitaries because of their close connection with exponential length. (See Section 4.)

The paper [**29**] discusses a large number of other factorization problems.

Some, such as products of symmetric matrices and pseudoinvolutions, do not make sense in abstract C*-algebras without additional structure. Others, such as products of projections, partial isometries, and nilpotent elements, make sense in an abstract C*-algebra but do not take place in $\mathrm{inv}_0(A)$, even with simple modifications. The problem of factorization into normal elements, also considered in [**29**], can be easily modified to take place in $\mathrm{inv}_0(A)$, but it then has a trivial solution, given by the polar decomposition.

Only a few factorization problems have been considered in more general C*-algebras. The two exponential factorizations were formally introduced in [**11**] and [**18**], and there are now quite a few results; see the survey article [**16**]. Commutators have also been considered by a number of authors; references are also given in [**16**]. Factorizations into positive elements have recently been studied by Quinn [**19**], [**20**]. Unipotent triangular matrices are used in [**5**] in a purely algebraic setting, which is nevertheless very useful to us, but we do not include this class with the 13 classes above because it is not intrinsic.

1.2 Definition. Let $\mathcal{C}$ be one of the 13 classes of invertible elements of a C*-algebra mentioned before or in Definition 1.1. Let A be a unital C*-algebra. We define the following sets and numbers:

(1) $P_{\mathcal{C}}(A)$ is the set of all finite products of elements of A of the class $\mathcal{C}$.

(2) $\overline{P}_{\mathcal{C}}(A)$ is the closure of $P_{\mathcal{C}}(A)$ in $\mathrm{inv}(A)$.

(3) $\mathrm{rk}_{\mathcal{C}}(A)$ is the smallest number n such that every element of $P_{\mathcal{C}}(A)$ is a product of at most n elements in the class $\mathcal{C}$; it is ∞ if no such n exists. We call it the $\mathcal{C}$-rank of A.

(4) $\overline{\mathrm{rk}}_{\mathcal{C}}(A)$ is the smallest integer n such that products of n elements of the class $\mathcal{C}$ are dense in $\overline{P}_{\mathcal{C}}(A)$; it is ∞ if no such n exists. We call it the $\overline{\mathcal{C}}$-rank of A.

Unlike our definition of exponential rank [**11**], [**18**], we do not allow the values $n + \varepsilon$ here. We exclude them partly for simplicity and partly because for some of the classes we consider, a limit of products of n members of the class seems unlikely to be a product of $n+1$ members of the class. For example, if $\mathcal{C}$ is the class of positive invertible elements, then Theorem 3 of [**10**] shows that $\overline{\mathrm{rk}}_{\mathcal{C}}(L(H)) \leq 5$. However, until very recently the best known upper bound for $\mathrm{rk}_{\mathcal{C}}(L(H))$ was 17. (See [**30**]. This has just been improved to 7 in [**17**]. But it is still not clear whether $\mathrm{rk}_{\mathcal{C}}(L(H)) = \overline{\mathrm{rk}}_{\mathcal{C}}(L(H))$.)

If $\mathcal{C}$ is the class of exponentials of skewadjoint elements, then the exponential rank can be recovered as follows:

$$\mathrm{cer}(A) = n \quad \text{if and only if} \quad \mathrm{rk}_{\mathcal{C}}(A) = \overline{\mathrm{rk}}_{\mathcal{C}}(A) = n.$$
$$\mathrm{cer}(A) = n + \varepsilon \quad \text{if and only if} \quad \mathrm{rk}_{\mathcal{C}}(A) = n + 1 \text{ and } \overline{\mathrm{rk}}_{\mathcal{C}}(A) = n.$$

The same thing can be done whenever the elements of a general class C include a neighborhood of 1 in $P_C(A)$.

1.3 Proposition. *Let C be any of the 13 classes mentiond in or before Definition* 1.1. *Then $P_C(A) \subset \overline{P}_C(A) \subset \mathrm{inv}_0(A)$ for every unital* C*-*algebra A.*

Proof. We show that all the elements of class C are in $\mathrm{inv}_0(A)$. The proposition will then follow from the fact that $\mathrm{inv}_0(A)$ is a closed subgroup of $\mathrm{inv}(A)$.

The commutators are in $\mathrm{inv}_0(A)$ by definition. It is a standard fact that the exponentials are in $\mathrm{inv}_0(A)$. All the other classes satisfy conditions ensuring that the spectrums of their elements do not separate 0 from ∞; thus, their elements are all exponentials. (For the classes of accretive elements and accretive unitaries, use Corollary 4.3 below to see this.) □

If H is a separable infinite dimensional Hilbert space, then in almost all cases $P_C(L(H))$ is known to be either $U(L(H))$ or $\mathrm{inv}(L(H))$, depending only on whether all the elements of the class C are unitaries or not. (See [**29**]. For the classes of positive-stable elements and accretive elements, this follows by comparison with the class of positive invertible elements. For the class of accretive unitaries, see Section 4.) In particular, $\overline{P_C}(L(H)) = P_C L(H))$. Furthermore, in these cases $\mathrm{rk}_C(L(H)) < \infty$, with known upper bounds varying from 1 (for exponentials of skewadjoint elements) to 7 (for positive invertible elements); again, see [**29**] and Section 10 of [**16**].

For $A = M_n$, things are a little more complicated. It turns out that $P_C(A)$ is always closed in $\mathrm{inv}(M_n)$, and always has the form $\{a \in G : \det(a) \in H\}$, where G is either $U(M_n)$ or $\mathrm{inv}(M_n)$, and H is a closed subgroup of the multiplicative group $\mathbb{C} - \{0\}$, not depending on n. Generally $\mathrm{rk}_C(M_n)$ is bounded above by a small constant independent of n (between 1 and 5), and often this constant is known to be the exact value, except for the trivial case $n = 1$.

For commutative C*-algebras these problems are almost all trivial, but an important distinction arises.

1.4 Proposition. *Let X be compact Hausdorff (not necessarily metric), and let C be one of the 13 classes considered in and before Definition* 1.1.

(1) *If X is not totally disconnected, and C is the class of positive-stable, accretive, or unitary accretive elements, then $\mathrm{rk}_C(C(X)) = \overline{\mathrm{rk}}_C(C(X)) = \infty$.*

(2) *If C is any of the remaining 10 classes, then $\mathrm{rk}_C(C(X)) = \overline{\mathrm{rk}}_C(C(X)) = 1$ for any X.*

Part (2) is obvious and its proof is omitted. Part (1) will be proved in Section 4, where results for totally disconnected X will also be given.

We now turn to the case $C(X) \otimes M_n$, with $n \geq 2$. Some more notation is required to state our results:

1.5 Definition. For each of our 13 classes $\mathcal{C}$, we define

$$N_{\mathcal{C}}(n, d) = \sup_{\dim(X) \leq d} \mathrm{rk}_{\mathcal{C}}(C(X) \otimes M_n),$$

where X runs through all compact metric spaces whose covering dimension [**6**] $\dim(X)$ is at most d.

We note that, for compact metric spaces, all three of the usual dimensions agree. See Theorem 1.7.7 of [**6**].

For each of the 10 classes of Proposition 1.4 (2), we will prove a result of the following form:

(1) For any compact metric space X, we have

$$P_{\mathcal{C}}(C(X) \otimes M_n) = \overline{P}_{\mathcal{C}}(C(X) \otimes M_n),$$

and both are equal to

$$\{a \in \mathrm{inv}_0(C(X) \otimes M_n) : a(x) \in P_{\mathcal{C}}(M_n) \text{ for all } x \in X\}.$$

(2) $N_{\mathcal{C}}(n, d) < \infty$ for all $n \geq 1$ and $0 \leq d < \infty$.

(3) For each fixed $d < \infty$, we have $\lim_{n \to \infty} \sup N_{\mathcal{C}}(n, d) \leq N_{\mathcal{C}}$, for some explicitly given finite number $N_{\mathcal{C}}$ (betwen 4 and 22, depending on $\mathcal{C}$).

(4) If X is a compact manifold with boundary, and $n \geq 2$, then

$$\overline{\mathrm{rk}}_{\mathcal{C}}(C(X) \otimes M_n) \geq f_{\mathcal{C}}(n, \dim(X)),$$

for some explicitly given function $f_{\mathcal{C}}$ with $f_{\mathcal{C}}(n, d) \to \infty$ linearly as $d \to \infty$ for each fixed n.

Note that $\overline{\mathrm{rk}}_{\mathcal{C}}(A) \leq \mathrm{rk}_{\mathcal{C}}(A)$ for any $\mathcal{C}$ and A.

For the remaining three classes, (1) still holds, but $\mathrm{rk}_{\mathcal{C}}(C(X) \otimes M_n)$ is generally infinite. We will, however, still recover analogs of (2), (3), and (4) by restricting to appropriate commutators, or, in this context, elements with determinant 1.

The proofs of these results (except for several that have already been proved elsewhere) occupy the remaining three sections. The three slightly exceptional classes are treated together in Section 4, where their connection with exponential length [**24**], [**18**] is demonstrated. The other classes of nonunitary elements are treated in the next section, where the lower bounds are shown to follow immediately from results on Banach exponential rank, while most of the upper bounds can be obtained from a theorem of Dennis and Vaserstein on factorization into unipotent triangular matrices. The remaining unitary cases are treated in Section 3. Again, lower bounds follow easily from exponential rank results, but upper bounds require more work.

The bounds we present, both upper and lower, are the best we can easily obtain given the results already known. We have not seriously attempted to find the best possible bounds in any of the problems we consider. We have also not investigated more general C*-algebras. Thus, we can state three problems (out of many possible):

1.6 Problem. Improve the upper and lower bounds given in this paper for $\mathrm{rk}_{\mathcal{C}}(C(X) \otimes M_n)$ for the various classes $\mathcal{C}$ considered.

1.7 Problem. For those classes $\mathcal{C}$ for which it is not obvious, characterize $P_{\mathcal{C}}(A)$ and $\overline{P}_{\mathcal{C}}(A)$ for an arbitrary unital C*-algebra A.

Of course, these two problems are not equally interesting for all classes $\mathcal{C}$.

1.8 Problem. Does there exist a C*-algebra A such that $\mathrm{rk}_{\mathcal{C}}(A)$ is finite for one of the 10 classes in Proposition 1.4(2), but infinite for another one?

2. Factorization of invertible elements.

We start this section by stating the factorization theorem for exponentials, in effect proved in [**13**]. We then prove the factorization theorems for unipotent elements, positive invertible elements, selfadjoint invertible elements, symmetries, and commutators of elements of $\mathrm{inv}_0(A)$.

The lower bounds in these results are obtained from the lower bound for exponentials, and the upper bounds are obtained from a factorization theorem for upper and lower triangular unipotent matrices due to Dennis and Vaserstein [**5**]. (The upper bounds for commutators are already in [**5**].)

2.1 Theorem. *Let $\mathcal{C}$ be the class of exponentials. Then:*

(1) $P_{\mathcal{C}}(C(X) \otimes M_n) = \overline{P}_{\mathcal{C}}(C(X) \otimes M_n) = \mathrm{inv}_0(C(X) \otimes M_n)$.

(2) $N_{\mathcal{C}}(n, d) < \infty$ *for* $n \geq 1$ *and* $0 \leq d < \infty$.

(3) *For fixed* $d < \infty$, *we have* $\lim_{n\to\infty} \sup N_{\mathcal{C}}(n, d) \leq 5$.

(4) *If X is a compact manifold with boundary and $n \geq 2$, then*

$$\overline{\mathrm{rk}}_{\mathcal{C}}(C(X) \otimes M_n) \geq \frac{\dim(X) - 2}{2(n^2 - 1)}.$$

Proof. (1) is well known (see [**2**, Proposition 3.4.3]), (2) follows from Corollary 3.2 of [**13**] and Proposition 4.6 of [**11**], (3) is Theorem 3.4 of [**13**], and (4) is Theorem 2.3 of [**13**]. □

For later use, it is important to note that the example which proves part (4) of this theorem is actually a homotopically trivial element of $C(X, SU_n)$.

2.2 Proposition. *Let X be a compact manifold with boundary, and let $n \geq 2$. Let l be the least integer such that $l \geq [\dim(X) - 2]/[2(n^2 - 1)]$. Then there is a homotopically trivial $a \in C(X, SU_n)$ which is not a limit of products of fewer than l exponentials of elements in $C(X) \otimes M_n$.*

Proof. Let $m = 2(n^2 - 1)(l - 1) + 2$. Then $m \leq \dim(X)$. The proof of Theorem 2.3 of [**13**] exhibits a contractible compact subset X_0 of $\mathbb{R}^m$, and $u \in C(X_0, SU_n)$, such that u is not a limit of products of fewer than l exponentials. On the other hand, $u \in U_0(C(X_0) \otimes M_n)$ since X_0 is contractible. Therefore $u = \exp(ih_1)\exp(ih_2)\dots\exp(ih_N)$ for some N and selfadjoint $h_1, \dots, h_N \in C(X_0) \otimes M_n$. The proof of Lemma 2.3 of [**11**] (see the claim (*) there) shows that $h_1, \dots, h_N$ may be chosen to have values in

$$L = \{a \in M_n : a \text{ is selfadjoint and } \operatorname{tr}(a) \in 2\pi\mathbf{Z}\}.$$

Since X is a manifold with $\dim(X) \geq m$, we can identify X_0 with some homeomorphic subset of X. Since L is topologically the disjoint union of vector spaces and X_0 is connected, the Tietze extension theorem provides $k_1, \dots, k_N : X \to L$ such that $k_j|_{X_0} = h_j$. Then $v = \exp(ik_1)\cdots\exp(ik_N) \in C(X, SU_n)$ is homotopically trivial, since L can be retracted onto $\{(2\pi r/n) \cdot 1 : r \in \mathbf{Z}\}$. By restriction to X_0, we see that v is not a limit of products of fewer than l exponentials. □

The following theorem is not of the form we are concentrating on in this paper, because triangularity is not an intrinsic property of elements of $C(X) \otimes M_n$. However, it has in effect been proved elsewhere, and parts (1)-(3) will be used to prove the corresponding parts of several later theorems. (Part (4) is included merely for completeness.) Note that the factors are triangular (presumably alternating upper and lower triangular), not merely pointwise triangularizable.

2.3 Theorem. *Let $\mathcal{C}$ be the class of unipotent triangular matrices, that is, elements of $A \otimes M_n$ which are either upper or lower triangular and whose diagonal entries are all 1. Then:*

(1) $P_{\mathcal{C}}(C(X) \otimes M_n) = \overline{P}_{\mathcal{C}}(C(X) \otimes M_n) = \{a \in \operatorname{inv}_0(C(X) \otimes M_n) : \det(a) = 1\}$.

(2) $N_C(n,d) < \infty$ *for* $n \geq 1$ *and* $0 \leq d < \infty$.

(3) *For fixed* $d < \infty$, *we have* $\lim_{n\to\infty} \sup N_C(n,d) \leq 6$.

(4) *If* X *is a compact manifold with boundary and* $n \geq 2$, *then*

$$\overline{\mathrm{rk}}_C(C(X) \otimes M_n) \geq \frac{\dim(X) - 2}{2(n^2 - 1)}.$$

Proof. We first note that it suffices to prove parts (1) and (2) for elementary matrices (unipotent triangular matrices with at most one nonzero off-diagonal entry), since every elementary matrix is unipotent triangular and every unipotent triangular matrix is a product of at most $(n-1)(n-2)/2$ elementary matrices.

(1) We observe, by factoring out the determinant in a homotopy, that $\{a \in \mathrm{inv}_0(C(X) \otimes M_n) : \det(a) = 1\}$ is the set of homotopically trivial elements of $C(X, SL_n)$. Indeed, if $a \in \mathrm{inv}_0(C(X) \otimes M_n)$ and $\det(a) = 1$, we can choose a homotopy $t \mapsto a_t$ from a at $t = 0$ to 1 at $t = 1$. Let $(t, x) \mapsto \xi_t(x)$ be the continuous branch of $\det(a_t(x))^{-1/n}$ which is 1 when $t = 1$. Then $t \mapsto \xi_t a_t$ is a homotopy in $C(X, SL_n)$ from a to $\xi_1 a_1$, which is locally constant with values of the form $\lambda \cdot 1$ with $\lambda^n = 1$. Since SL_n is connected, it is easy to connect $\xi_1 a_1$ to 1 in $C(X, SL_n)$.

By Lemma 9 of [**28**], the set of finite products of elementary matrices contains a neighborhood of 1 in $C(X, SL_n)$. It is clearly a connected subgroup, and it is therefore open. So it is the identity component. Therefore the previous paragraph shows that

$$P_C(C(X) \otimes M_n) = \{a \in \mathrm{inv}_0(C(X) \otimes M_n) : \det(a) = 1\}.$$

Since this subset is closed in $\mathrm{inv}(C(X) \otimes M_n)$, it is also equal to $\overline{P}_C(C(X) \otimes M_n)$.

(2) This follows from Theorem 4 of [**28**].

(3) The stable rank $\mathrm{sr}(C(X))$ is the greatest integer not exceeding $1 + \dim(X)/2$, by [**27**]. Using (2), the result now follows from Theorem 20 of [**5**].

(4) This follows from Proposition 2.2, since every unipotent triangular matrix is an exponential. □

As an immediate corollary, we get the theorem on unipotent factorizations.

2.4 Theorem. *Let* C *be the class of unipotent elements. Then:*

(1) $P_C(C(X) \otimes M_n) = \overline{P}_C(C(X) \otimes M_n) = \{a \in \mathrm{inv}_0(C(x) \otimes M_n) : \det(a) = 1\}$.

(2) $N_C(n,d) < \infty$ *for* $n \geq 1$ *and* $0 \leq d < \infty$.

(3) *For fixed* $d < \infty$, *we have* $\lim_{n\to\infty} \sup N_C(n,d) \leq 6$.

(4) *If X is a compact manifold with boundary and $n \geq 2$, then*

$$\overline{\mathrm{rk}}_C(C(X) \otimes M_n) \geq \frac{\dim(X) - 2}{2(n^2 - 1)}.$$

Proof. Part(4) follows from Proposition 2.2, one inclusion in (1) is immediate, and all the rest follows from the previous theorem. □

The following two lemmas enable us to apply Theorem 2.3 to other factorization problems.

2.5 Lemma. *Let $a \in C(X) \otimes M_n$ be triangular, with diagonal elements $\alpha_1(x), \ldots, \alpha_n(x)$ at $x \in X$. Assume that, for each $x \in X$, the numbers $\alpha_j(x)$ are all distinct. Then there is $s \in \mathrm{inv}_0(C(X) \otimes M_n)$ such that $sas^{-1} = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$.*

Proof. The proof is the same for upper and lower triangular matrices; we do only the upper triangular case. The proof is by induction on n. The case $n = 1$ is trivial, so suppose the result has been proved for n, and let $a \in C(X) \otimes M_{n+1}$ be upper triangular, with diagonal elements $(\alpha_1, \ldots, \alpha_{n+1})$ with distinct values at each $x \in X$. We write

$$a(x) = \begin{pmatrix} a_0(x) & \xi(x) \\ 0 & \alpha_{n+1}(x) \end{pmatrix},$$

with $a_0 \in C(X) \otimes M_n$ upper triangular, and with $\xi : X \to \mathbb{C}^n$ continuous (thinking of elements of $\mathbb{C}^n$ as column vectors). By the induction hypothesis, there is $s_0 \in \mathrm{inv}_0(C(X) \otimes M_n)$ such that $s_0 a_0 s_0^{-1} = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$. Furthermore, $\alpha_{n+1}(x)$ is not an eigenvalue of $a_0(x)$, since the $\alpha_j(x)$ are distinct. Therefore we can define

$$s(x) = \begin{pmatrix} s_0(x) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -(\alpha_{n+1}(x) - a_0(x))^{-1}\xi(x) \\ 0 & 1 \end{pmatrix}$$

for $x \in X$, and one checks that $sas^{-1} = \mathrm{diag}(\alpha_1, \ldots, \alpha_{n+1})$. □

2.6 Lemma. *Let $a \in C(X) \otimes M_n$ be the product of $k \geq 2$ unipotent triangular matrices. Then a is a product of $2k$ positive invertible elements, and also a product of $2k$ symmetries.*

Proof. Write $a = a_1 a_2 \cdots a_k$ with a_k triangular and unipotent. Let c be any invertible diagonal matrix. Write

$$a = (a_1 c)(c^{-1} a_2 c^2)(c^{-2} a_3 c^3) \cdot \cdots \cdot (c^{-(k-2)} a_{k-1} c^{k-1})(c^{-(k-1)} a_k).$$

Let b_j be the j-th factor in this expression. Then each b_j is again triangular, and the diagonal entries of b_j are the same as those of c (for $j < k$) or those of $c^{-(k-1)}$ (for $j = k$). If the diagonal entries of $c(x)$ and $c(x)^{-(k-1)}$ are all distinct for each $x \in X$, then Lemma 2.5 provides invertible elements s_j such that $s_j b_j s_j^{-1} = c$ for $j < k$ and $s_j b_j s_j^{-1} = c^{-(k-1)}$ for $j = k$.

To write a as a product of $2k$ positive invertible elements, we now follow the proof of Theorem 2 of [**25**]. Take $c = \mathrm{diag}(\alpha_, \dots, \alpha_n)$ with $(\alpha_, \dots, \alpha_n)$ distinct positive real numbers. For $j < k$, we then have $b_j = (s_j^* s_j)^{-1}(s_j^* c s_j)$, a product of two positive invertible elements. For $j = k$, use $c^{-(k-1)}$ in place of c to get the same thing.

To write a as a product of $2k$ symmetries, we follow the proof of Theorem 5 of [**25**]. Suppose first that n is even, so $n = 2m$. Let $\lambda_1, \dots \lambda_m$ be complex numbers of the form $\lambda_j = \exp(2\pi i \theta_j)$, with $\theta_1, \dots, \theta_m$ real, irrational, and algebraically independent over $\mathbf{Q}$. Take $c = \mathrm{diag}(\lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1}, \dots, \lambda_m, \lambda_m^{-1})$. Then c and $c^{-(k-1)}$ have all their diagonal entries distinct. As in [**25**], the equation

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \lambda^{-1} \\ \lambda & 0 \end{pmatrix}$$

can be used to show that c and $c^{-(k-1)}$ are each products of two symmetries. So b_j, being similar to c or $c^{-(k-1)}$, is also a product of two symmetries.

If n is odd, let $n = 2m + 1$, choose $\lambda_1, \dots, \lambda_m$ as before, and take $c = \mathrm{diag}(\lambda_1, \lambda_1^{-1}, \dots, \lambda_m, \lambda_m^{-1}, 1)$. Then proceed as before. □

2.7 Theorem. *Let be the class of positive invertible elements. Then:*

(1) $P_C(C(X) \otimes M_n) = \overline{P}_C(C(X) \otimes M_n) = \{a \in \mathrm{inv}_0(C(X) \otimes M_n) : \det(a) > 0\}$.

(2) $N_C(n, d) < \infty$ *for* $n \geq 1$ *and* $0 \leq d < \infty$.

(3) *For fixed* $d < \infty$, *we have* $\lim_{n \to \infty} \sup N_C(n, d) \leq 12$.

(4) *If* X *is a compact manifold with boundary and* $n \geq 2$, *then*

$$\overline{\mathrm{rk}}_C(C(X) \otimes M_n) \geq \frac{\dim(X) - 2}{n^2 - 1} - 1.$$

Proof. (1) Since

$$\{a \in \mathrm{inv}_0(C(X) \otimes M_n) : \det(a) > 0\}$$

is closed in $\mathrm{inv}(C(X) \otimes M_n)$ and obviously contains $P_C(C(X) \otimes M_n)$, it is enough to show that it is contained in $P_C(C(X) \otimes M_n)$. So let $a \in \mathrm{inv}_0(C(X) \otimes M_n)$ and suppose $\det(a) > 0$. Then

$$\det(a)^{-1/n}\, a \in \mathrm{inv}_0(C(X) \otimes M_n)$$

and has determinant 1. By Theorem 2.3(1) and Lemma 2.6, it is a product of positive invertible elements. Therefore so is

$$a = (\det(a)^{1/n} \cdot 1)(\det(a)^{-1/n} a).$$

(2) This follows from the corresponding part of Theorem 2.3 by using Lemma 2.6 just as above.

(3) Fix $d < \infty$. Combine Lemma 2.6 and Theorem 2.3(3) as in the argument for part (1) to obtain the following statement: For all sufficiently large n, all X with $\dim(X) = d$, and all $a \in \mathrm{inv}_0(C(X) \otimes M_n)$, there are 12 positive invertible elements $a_1, \ldots, a_{12} \in C(X) \otimes M_n$ such that $a = (\det(a)^{1/n} \cdot 1)a_1 a_2 \cdots a_{12}$. Now $\det(a)^{1/n} a_1$ is again a positive invertible element. Thus a is a product of 12 positive invertible elements.

(4) Theorem 2.8 of [**29**] implies that a product of two positive invertible elements of M_n is similar to a positive invertible element of M_n, and so has spectrum in the right half plane. It follows that the product of two positive invertible elements of $C(X) \otimes M_n$ is an exponential. Now use Proposition 2.2. □

2.8 Theorem. *Let C be the class of selfadjoint invertible elements. Then:*

(1) $P_C(C(X) \otimes M_n) = \overline{P}_C(C(X) \otimes M_n) = \{a \in \mathrm{inv}_0(C(X) \otimes M_n) :$ $\det(a)$ *is real*$\}$.

(2) $N_C(n, d) < \infty$ *for* $n \geq 1$ *and* $0 \leq d < \infty$.

(3) *For fixed* $d < \infty$, *we have* $\lim_{n \to \infty} \sup N_C(n, d) \leq 13$.

(4) *If* X *is a compact manifold with boundary and* $n \geq 2$, *then*

$$\overline{\mathrm{rk}}_C(C(X) \otimes M_n) \geq \frac{\dim(X) - 2}{2(n^2 - 1)}.$$

Proof. Parts (1), (2), and (3) follow from the corresponding parts of Theorem 2.7 by multiplying by the selfadjoint invertible element

$$w(x) = \begin{cases} 1 & \det(a(x)) > 0 \\ \mathrm{diag}(-1, 1, \ldots, 1) & \det(a(x)) < 0. \end{cases}$$

Part (4) follows from Proposition 2.2, since every selfadjoint invertible element is an exponential. □

2.9 Theorem. *Let C be the class of symmetries* (*called involutions in* [**25**] *and* [**29**]). *Then:*

(1) $P_C(C(X) \otimes M_n) = \overline{P}_C(C(X) \otimes M_n) = \{a \in \mathrm{inv}_0(C(X) \otimes M_n) : \det(a) =$ $\{\pm 1\}$ *for all* $x\}$.

(2) $N_C(n,d) < \infty$ *for* $n \geq 1$ *and* $0 \leq d < \infty$.

(3) *For fixed* $d < \infty$, *we have* $\lim_{n\to\infty} \sup N_C(n,d) \leq 13$.

(4) *If* X *is a compact manifold with boundary and* $n \geq 2$, *then*

$$\overline{\mathrm{rk}}_C(C(X) \otimes M_n) \geq \frac{\dim(X) - 2}{2(n^2 - 1)}.$$

Proof. Parts (1), (2), and (3) are proved in the same way as in the proof of Theorems 2.7 and 2.8. (Note that the element w from the proof of Theorem 2.8 is a symmetry. Also, one uses the part of Lemma 2.6 that refers to symmetries.)

Part (4) follows from Proposition 2.2, since every symmetry is an exponential. □

2.10 Theorem. *Let* C *be the class of multiplicative commutators of elements of* $\mathrm{inv}_0(A)$. *Then:*

(1) $P_C(C(X)\otimes M_n) = \overline{P}_C(C(X)\otimes M_n) = \{a \in \mathrm{inv}_0(C(X)\otimes M_n) : \det(a) = 1\}$.

(2) $N_C(n,d) < \infty$ *for* $n \geq 1$ *and* $0 \leq d < \infty$.

(3) *For fixed* $d < \infty$, *we have* $\lim_{n\to\infty} \sup N_C(n,d) \leq 6$.

(4) *If* X *is a compact manifold with boundary and* $n \geq 2$, *then*

$$\overline{\mathrm{rk}}_C(C(X) \otimes M_n) \geq \frac{\dim(X) - 2}{4(4n^2 - 1)} - 2.$$

Proof. Every multiplicative commutator of elements of $\mathrm{inv}_0(C(X) \otimes M_n)$ is clearly in $\mathrm{inv}_0(C(X) \otimes M_n)$ and has determinant 1. Thus

$$P_C(C(X) \otimes M_n) \subset \{a \in \mathrm{inv}_0(C(X) \otimes M_n) : \det(a) = 1\}.$$

The rest of part (1), and part (2), follows from [**28**] as in the proof of the corresponding parts of Theorem 2.3, because every elementary matrix is a commutator of elements of $\mathrm{inv}_0(C(X) \otimes M_n)$. (For $n \geq 3$, it is well known that every elementary matrix is actually a commutator of elementary matrices. For $n = 2$, see the proof of Theorem 5.4 of [**13**].)

Part (3) now follows from Theorem 2(d) of [**5**], using the fact that $\mathrm{sr}(C(X) \otimes M_n) < \infty$ ([**27**]).

Part (4) follows from Theorem 5.4 of [**13**], since the element u there is in $\mathrm{inv}_0(C(X) \otimes M_n)$. □

We note that part (1) of this result is in Proposition 2.4 of [**26**]. There is a similar result for products of commutators of arbitrary elements in

$\mathrm{inv}(C(X) \otimes M_n)$. Unfortunately, in (1) and (2) one must then apparently require

$$n \geq \mathrm{sr}(C(X)) + 1 = [\dim(X)/2] + 2,$$

is where [ ] is the greatest integer function.

3. Factorization of unitaries.

We start this section with the statement of the factorization theorem for exponentials of skewadjoint elements, essentially proved in [**13**]. We then prove factorization theorems for *-symmetries and for commutators of elements of $U_0(A)$. Unfortunately, triangular matrices are unitary only if they are diagonal, so the theorem of Dennis and Vaserstein used in the last section does not help here. We must therefore use more direct methods. (Presumably, these methods could also have been applied to some of the problems considered in the previous section.)

3.1 Theorem. *Let $\mathcal{C}$ be the class of exponentials of skewadjoint elements. Then:*

(1) $P_{\mathcal{C}}(C(X) \otimes M_n) = \overline{P}_{\mathcal{C}}(C(X) \otimes M_n) = U_0(C(X) \otimes M_n)$.

(2) *$N_{\mathcal{C}}(n,d) < \infty$ for $n \geq 1$ and $0 \leq d < \infty$.*

(3) *For fixed $d < \infty$, we have $\lim_{n\to\infty} \sup N_{\mathcal{C}}(n,d) \leq 4$.*

(4) *If X is a compact manifold with boundary and $n \geq 2$, then*

$$\overline{\mathrm{rk}}_{\mathcal{C}}(C(X) \otimes M_n) \geq \frac{\dim(X) - 2}{n^2 - 1} - 1.$$

Proof. (1) is well known (see [**2**, Proposition 3.4.5]), (2) is Corollary 3.2 of [**13**], (3) is Theorem 3.4 of [**13**], and (4) is Theorem 2.3 of [**13**]. □

As with exponentials of arbitrary elements, we will need a stronger statement than part (4) of this theorem. Its proof is the same as that of Proposition 2.2, and is omitted.

3.2 Proposition. *Let X be a compact manifold with boundary, and let $n \geq 2$. Let l be the least integer such that $l \geq [\dim(X) - 2]/[n^2 - 1] - 1$. Then there is a homotopically trivial $u \in C(X, SU_n)$ which is not a limit of products of fewer then l exponentials of skewadjoint elements of $C(X) \otimes M_n$.* □

3.3 Lemma. (Compare [**13**, Lemma 3.1].) *For integers $n \geq 2$ and $d \geq 0$, there is $M(n,d) < \infty$ such that for every compact metric space X of dimension at most d and every homotopically trivial $u \in C(X, SU_n)$,* *there exists a rectifiable path from u to 1 in $C(X, SU_n)$ with length at most $M(n, d)$.*

Proof. Let ρ be the metric induced on SU_n by some Riemannian metric (or by some realization of it as a finite simplicial complex). Apply Theorem 0.2 of [**3**] to obtain a number $b_d = b_d(SU_n) < \infty$. Choose $M(n, d)$ to be any real number greater than $b_d(SU_n)$. Then for every compact metric space X with $\dim(X) \leq d$, and every homotopically trivial $u \in C(X, SU_n)$, there is a homotopy $t \mapsto v_t$ from $v_0 = 1$ to $v_1 = u$ with width [**3**] less than $M(n, d)$. Corollary 3.6 of [**3**] gives a Lipschitz homotopy $t \mapsto u_t$ with Lipschitz constant less than $M(d, n)$; clearly the length of $t \mapsto u_t$ in $C(X) \otimes M_n$ is bounded by this Lipschitz constant. (Note that the existence of the geodesic equilocally convex structure on SU_n, needed implicitly in Corollary 3.6 of [**3**], is ensured by Lemma 1.2 of [**3**].) □

Part (2) of the following lemma is a quantitative improvement of Lemma 2.2 of [**26**]. We need only two commutators rather than n of them. (We also get a much larger value of ε_n in the proof.) It is interesting to note that at least part (2) can fail if $C(X) \otimes M_n$ is replaced by the section algebra of a locally trivial M_n-bundle. (We omit the example, since it would take us too far afield.)

3.4 Lemma. *For each $n \geq 1$ there exists $\varepsilon_n > 0$ such that whenever X is a compact space and $u : X \to SU_n$ is continuous and satisfies $\|u - 1\| < \varepsilon_n$, then:*

(1) *There exist 6 *-symmetries $s_1, \dots, s_6 \in C(X) \otimes M_n$ such that $u = s_1 s_2 s_3 s_4 s_5 s_6$.*

(2) *There exist 4 homotopically trivial elements $w_1, w_2, w_3, w_4 \in C(X, SU_n)$ such that $u = (w_1 w_2 w_1^* w_2^*)(w_3 w_4 w_3^* w_4^*)$.*

(3) *If $p \in C(X) \otimes M_n$ is a projection unitarily equivalent to a constant projection, then there exist 2 homotopically trivial elements $w_1, w_2 \in C(X, SU_n)$ such that $upu^* = (w_1 w_2 w_1^* w_2^*) p (w_1 w_2 w_1^* w_2^*)$.*

Proof. Conjugating everything by an appropriate unitary, we may assume in case (3) that p is a constant diagonal projection.

We now suppose n is even. Let $\xi = \exp(\pi/n)$, a primitive $2n$-th root of 1. Define $u_0 = \operatorname{diag}(\xi, \xi^3, \dots, \xi^{2n-1}) \in SU_n$, and let $u_0 \in C(X) \otimes M_n$ also denote the constant function with this value. Let $0 < \varepsilon_n < |1 - \xi| = 2\arcsin(1/(4n))$. Then the closed ε_n-balls about the eigenvalues of u_0 have disjoint intersections with S^1. By Theorem 13.6 of [**1**] (see Section 11 of [**1**] for the notation), if $v \in U(M_n)$ and $\|v - u_0\| \leq \varepsilon_n$, then the closed ε_n-ball about any eigenvalue of u_0 contains exactly one eigenvalue of v. In particular, if $\|u - 1\| \leq \varepsilon_n$, then uu_0 has distinct eigenvalues at each point of X. We claim that there exists $z \in U(C(X) \otimes M_n)$ such that zuu_0z^* is diagonal. To see this, apply the argument of the second paragraph of the proof of Lemma 2.4 of [**11**] to the unitary defined on $V = \{c \in U(M_n) : \|c - u_0\| \leq \varepsilon_n\}$ by $c \mapsto c$, to obtain $f : V \to U(M_n)$ such that $f(c)cf(c)^*$ is diagonal for all $c \in V$. Now define $z(x) = f(u(x)u_0(x))$ for $x \in X$.

Write $zuu_0z^* = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ with continuous functions $\lambda_1, \ldots, \lambda_n : X \to S^1$. Each of these functions has range contained in a ball of radius less than $2\arcsin(1/4)$ centered on the unit circle, and so is homotopically trivial. Furthermore, $\lambda_1(x) \cdots \lambda_n(x) = 1$ for all x.

Following the proof of Theorem 3 of [**21**], we factor zuu_0z^* continuously as

$$(*) \qquad \operatorname{diag}(\lambda_1, \overline{\lambda}_1, \lambda_1\lambda_2\lambda_3, \overline{\lambda}_1\overline{\lambda}_2\overline{\lambda}_3, \ldots) \operatorname{diag}(1, \lambda_1\lambda_2, \overline{\lambda}_1\overline{\lambda}_2, \lambda_1\lambda_2\lambda_3\lambda_4, \ldots).$$

(The last diagonal entry of the second factor is $\lambda_1(x) \cdots \lambda_n(x) = 1$.) The equation

$$(**) \qquad \begin{pmatrix} \alpha(x) & 0 \\ 0 & \overline{\alpha}(x) \end{pmatrix} = \begin{pmatrix} 0 & \alpha(x) \\ \overline{\alpha}(x) & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

applied to appropriate pairs of entries in (*), shows that each factor in (*) is a product of two *-symmetries. Therefore uu_0 is a product of four *-symmetries. We can further use (**) in the same way as above to show that u_0 is a product of two *-symmetries. Then $u = (uu_0)u_0^*$ is a product of six *-symmetries, proving (1).

To prove (2) and (3), we write

$$zuu_0z^* = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) = w_1w_2w_1^*w_2^*,$$

with

$$w_1 = \begin{pmatrix} 1 & & & & 0 \\ & \overline{\lambda}_1 & & & \\ & & \overline{\lambda}_1\overline{\lambda}_2 & & \\ & & & \ddots & \\ 0 & & & & \overline{\lambda}_1\overline{\lambda}_2 \cdots \overline{\lambda}_{n-1} \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

(This works since $\lambda_1(x) \cdots \lambda_n(x) = 1$. Compare with Lemma 2.1 of [**26**].) Therefore uu_0 is a commutator. The entries of w_1 are homotopically trivial. Therefore we may replace w_1 by $\det(w_1)^{-1/n}w_1$, and similarly for w_2, to express uu_0 as a commutator of homotopically trivial functions from X to SU_n. Our choices imply that u_0 commutes with p. Therefore $(uu_0)p(uu_0)^* =$

upu^*, and we have proved (3). To prove (2), we use the same trick as above to express u_0^* as a commutator.

If n is odd, we take $\xi = \exp(2\pi i/n)$, $u_0 = \operatorname{diag}(1, \xi, \dots, \xi^{n-1})$, and $0 < \varepsilon_n < |1 - \xi| = 2\arcsin(1/(2n))$ instead. Define λ_j as before. The proofs of (2) and (3) (involving commutators) are unchanged. For (1), we use the factorization (*) as before. This time, each of the two factors of (*), as well as u_0^*, has diagonal entries that match up in complex conjugate pairs, with one entry left over, which is equal to 1. Each is therefore again a product of two *-symmetries, proving (1). □

Combining Lemmas 3.3(1) and 3.4(1) and (2) yields the following result. The part of part (1) which applies to commutators has already been observed by Thomsen (in Proposition 2.4 of [**26**]).

3.5 Lemma. *If $\mathcal{C}$ is either the class of *-symmetries, or the class of commutators of homotopically trivial unitaries, then:*

(1) $P_{\mathcal{C}}(C(X) \otimes M_n) = \overline{P}_{\mathcal{C}}(C(X) \otimes M_n)$, *and is the set of homotopically trivial unitaries, with determinant ± 1 (depending on $x \in X$) for *-symmetries, determinant 1 for commutators.*

(2) $N_{\mathcal{C}}(n, d) < \infty$ *for* $n \geq 1$ *and* $0 \leq d < \infty$.

Proof. (1) We first observe that commutators of elements of $U_0(C(X) \otimes M_n)$, as well as *-symmetries, are homotopically trivial, that is, in $U_0(C(X) \otimes M_n)$. An argument similar to one in the proof of Theorem 3.4(1) shows that $\{u \in U_0(C(X) \otimes M_n) : \det(u) = 1\}$ is the set of homotopically trivial elements of $C(X, SU_n)$, and that

$$\{u \in U_0(C(X) \otimes M_n) : \det(u(x)) \in \{\pm 1\} \text{ for all } x\}$$

has exactly one path component corresponding to each continuous ± 1-valued function on X.

The previous lemma implies that the subgroups generated by the commutators and the *-symmetries both contain an ε-neighborhood of 1 in $C(X, SU_n)$. Furthermore, if $\lambda : X \to \{\pm 1\}$ is continuous, then $\operatorname{diag}(\lambda, 1, \dots, 1)$ is a *-symmetry contained in the corresponding path component of

$$\{u \in U_0(C(X) \otimes M_n) : \det(u(x)) \in \{\pm 1\} \text{ for all } x\}.$$

The claimed identifications of $P_{\mathcal{C}}(C(X) \otimes M_n)$ follow. Since these are closed subgroups of $\operatorname{inv}_0(C(X) \otimes M_n)$, we have $P_{\mathcal{C}}(C(X) \otimes M_n) = \overline{P}_{\mathcal{C}}(C(X) \otimes M_n)$ in both cases.

(2) With $M(n, d)$ as in Lemma 3.3(1) and ε_n as in Lemma 3.4, let $r(n, d)$ be the least integer exceeding $M(n, d)/\varepsilon_n$. Then $N_{\mathcal{C}}(n, d) \leq 2r(n, d)$ is clear

for commutators. For *-symmetries, one extra one suffices to get into any path component of $P_C(C(X) \otimes M_n)$, so $N_C(n,d) \leq 6r(n,d) + 1$. □

The previous lemma gives the first two parts of our standard factorization theorem for unitary commutators and for *-symmetries. Unfortunately what we have done so far does not give the third part. It turns out to be true that the numbers $M(n,d)$ of Lemma 3.3 can be chosen to satisfy $\lim_{n\to\infty} \sup M(n,d) \leq C$ for some fixed C independent of d. (See Remark 4.11.) This, however, does not help, since the numbers ε_n in the proof of Lemma 3.4 go to 0 as $n \to \infty$. Further progress requires another idea, which we take from **[15]**.

3.6 Lemma. *Let A be a unital* C**-algebra.*

(1) *Let $p, q, r \in A$ be unitarily equivalent projections with p orthogonal to q and q orthogonal to r. Then there exists a commutator $u = z_1 z_2 z_1^* z_2^*$, with $z_1, z_2 \in U_0(A)$, such that $upu^* = r$.*

(2) *Let $u \in U(A)$. Then $u \oplus u^* \in U_0(M_2(A))$ is a product of two *-symmetries in $M_2(A)$, and is also a commutator wvw^*v^* with $w, v \in U_0(M_2(A))$.*

Proof. (1) Let v be a unitary with $vpv^* = q$. Define

$$e = \frac{1}{2}(p + q + pv^*q + qvp) \quad \text{and} \quad w = 1 - p - q + \frac{1}{\sqrt{2}}(p + q - pv^*q + qvp).$$

Calculations show that e is a projection, w is a unitary, $wpw^* = e$, and

$$(*) \qquad (1 - 2e)p(1 - 2e)^* = q.$$

Furthermore, the unitary path

$$t \mapsto 1 + (\cos(t) - 1)(p + q) + \sin(t)(qvp - pv^*q),$$

for $t \in [0, \pi/4]$, shows that $w \in U_0(A)$.

In the same manner, we can construct a projection f and a unitary $x \in U_0(A)$ such that $xqx^* = f$ and

$$(1 - 2f)q(1 - 2f)^* = r.$$

Set $u = (1 - 2f)(1 - 2e)$; combining the previous equation with (*) yields $upu^* = r$.

Set $z_2 = w(1 - 2e)^* x^*$. Note that $z_2 f z_2^* = e$ by (*) and the choices of w and x. Furthermore, $z_2 \in U_0(A)$ because $w, x \in U_0(A)$ and $(1 - 2e)^*$ is a *-symmetry. Further set $z_1 = 1 - 2f$, which is in $U_0(A)$ because it is a *-symmetry. Then

$$z_1 z_2 z_1^* z_2^* = (1-2f)(1-2z_2 f z_2^*) = (1-2f)(1-2e) = u.$$

So u is a commutator of elements of $U_0(A)$.

(2) Define

$$v = \frac{1}{2}\begin{pmatrix} 1+u^* & 1-u^* \\ -1+u & 1+u \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & u^* \\ u & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Each factor on the right is easily checked to be a *-symmetry. Therefore $v \in U_0(M_2(A))$. Now one checks that

$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^* v^*.$$

This exhibits $u \oplus u^*$ as a commutator of elements of $U_0(M_2(A))$. Since the product of the last three factors is a conjugate of a *-symmetry, it also exhibits $u \oplus u^*$ as a product of two *-symmetries. □

3.7 Lemma. *Let X be compact metric, and let $n \geq 5\dim(X) + 3$. If $p, q \in C(X) \otimes M_n$ are homotopic projections which are unitarily equivalent to constant projections, then:*

(1) *There exists $v \in U_0(C(X) \otimes M_n)$, a product of at most 5 *-symmetries, such that $vpv^* = q$.*

(2) *There exists $v \in U_0(C(X) \otimes M_n)$, a product of at most 3 commutators of elements of $U_0(C(X) \otimes M_n)$, such that $vpv^* = q$.*

Proof. (1) This follows from Theorem 2.1 of [**15**] and the general relations between the C* projective length and rank, Theorem 2.4 of [**14**].

(2) Let ε_n be as in Lemma 3.4. Choose $\delta > 0$ such that whenever $e, f \in C(X_k) \otimes M_n$ are projections with $\|e - f\| < \delta$, then there exists $u \in U_0(C(X) \otimes M_n)$ such that $ueu^* = f$, $\|u - 1\| < \varepsilon_n/6$, and $\det(u(x)) = 1$ for all x. (A standard construction produces δ such that there is always a u satisfying all but the determinant condition. But if u is close enough to 1, with the required estimate depending only on n, one can construct $\det(u)^{-1/n}u$, and it will still be close to 1.)

Let $p, q \in C(X) \otimes M_n$ be homotopic projections. Write $C(X) \otimes M_n = \varinjlim_k C(X_k) \otimes M_n$, where the X_k are finite complexes of dimension at most $\dim(X)$, as in Lemma 1.8 of [**15**]. As the proof of Proposition 2.11 of [**14**], we can select a suitable term $C(Y) \otimes M_n$ from this direct system, and projections

$p_0, q_0 \in C(Y) \otimes M_n$, such that if $\varphi : C(Y) \otimes M_n \to C(X) \otimes M_n$ is the map to the direct limit, then $\|p - \varphi(p_0)\|, \|q - \varphi(q_0)\| < \delta$. Choosing $C(Y) \otimes M_n$ to be sufficiently far out in the direct system, we may further assume that p_0 and q_0 are homotopic.

We now work in $C(Y) \otimes M_n$. We consider one component of Y at a time; thus, we may assume Y is connected. Then p_0 and q_0 have constant rank. Replacing them by $1 - p_0$ and $1 - q_0$ if necessary, we assume they have rank at most $n/2$.

We follow the proof of Theorem 2.1 of [**15**], with $\Gamma(V) = C(Y) \otimes M_n$, and with p_0 and q_0 in place of p and q. Choose p_1, p_2, q_1, q_2 as in the second half of that proof, with ranks as chosen in the first half. Choose $\overline{p}_1$ and $\overline{q}_1$ as there, with $\|\overline{p}_1 - p_1\|, \|\overline{q}_1 - q_1\| < \delta$. Further choose f_1 as there. Then $\overline{p}_1, \overline{q}_1$, and f_1 are equivalent projections with $\overline{p}_1$ orthogonal to f_1 and f_1 orthogonal to $\overline{q}_1$. Lemma 3.6 (1) therefore provides a commutator z_1 such that $z_1 \overline{p}_1 z_1^* = \overline{q}_1$. The first paragraph of the proof provides unitaries c_1 and c_2 such that $\|c_i - 1\| < \varepsilon_n/6$ and $c_1 \overline{p}_1 c_1^* = p_1$, $c_2 \overline{q}_1 c_2^* = q_1$. Then $(c_2 z_1 c_1^*) p_1 (c_2 z_1 c_1^*)^* = q_1$.

Continuing to follow the proof of Theorem 2.1 of [**15**], let

$$A = (1 - q_1)[C(Y) \otimes M_n](1 - q_1).$$

Choose projections $\overline{p}_2, \overline{q}_2 \in A$ as there, with $\|\overline{p}_2 - (c_2 z_2 c_1^*) p_2 (c_2 z_1 c_1^*)^*\|$, $\|\overline{q}_2 - q_2\| < \delta$. Choosing f_2 as there, and applying Lemma 3.6 again, we obtain a commutator $w \in A$ such that $w \overline{p}_2 w^* = \overline{q}_2$. Let $z_2 = q_1 + w$, again a commutator of homotopically trivial unitaries. Further choose, again as in the first paragraph, unitaries c_3 and c_4 such that $\|c_i - 1\| < \varepsilon_n/6$ and

$$c_3(q_1 + \overline{p}_2)c_3^* = q_1 + (c_2 z_1 c_1^*) p_2 (c_2 z_1 c_1^*)^* \quad \text{and} \quad c_4(q_1 + \overline{q}_2)c_4^* = q_1 + q_2.$$

Then $u = c_4 z_2 c_3^* c_2 z_1 c_1^*$ satisfies

$$u p_0 u^* = u(p_1 + p_2)u^* = q_1 + q_2 = q_0.$$

Combining the choice made in the second paragraph of the proof with the result of the first paragraph, we can also find $c_5, c_6 \in U_0(C(X) \otimes M_n)$ such that

$$c_5 \varphi(p_0) c_5^* = p \qquad \text{and} \qquad c_6 \varphi(q_0) c_6^* = q.$$

Set $v_0 = c_6 \varphi(u) c_5^*$. Then $v_0 p v_0^* = q$. We can furthermore write

$$v_0 = [c_6 \varphi(c_4 c_3^* c_2 c_1^*) c_5^*] [(c_5 \varphi(c_1 c_2^* c_3)) \varphi(z_2) (c_5 \varphi(c_1 c_2^* c_3))^*] \cdot [(c_5 \varphi(c_1)) \varphi(z_1) (c_5 \varphi(c_1))^*].$$

Call the factors s, v_2, and v_3. Then v_2 and v_3 are commutators of homotopically trivial unitaries because z_2 and z_1 are. Furthermore, $\|s-1\| < \varepsilon_n$, since $\|c_i - 1\| < \varepsilon_n/6$ for $i = 1, \ldots, 6$. Lemma 3.4(3) therefore provides a commutator v_1 of homotopically trivial unitaries such that $v_1(v_2 v_3 p v_3^* v_2^*)v_1^* = s(v_2 v_3 p v_3^* v_2^*)s^* = q$. (Note that p is unitarily equivalent to a constant projection.) Therefore $v = v_1 v_2 v_3$ is the desired product of three commutators. □

Let $\langle \alpha \rangle$ denote the least integer l such that $l \geq \alpha$.

3.8 Lemma. *Let $n \geq 5\dim(X) + 3$. Let C_1 be the class of *-symmetries and let C_2 be the class of commutators of elements of $U_0(A)$. Let $b_i = \mathrm{rk}_{C_i}(C(X) \otimes M_n)$. Let $2n \leq k \leq 4n$. Then*

$$\mathrm{rk}_{C_1}(C(X) \otimes M_k) \leq 14 + \langle b_1/2 \rangle \quad \textit{and} \quad \mathrm{rk}_{C_2}(C(X) \otimes M_k) \leq 8 + \langle b_2/2 \rangle.$$

If $k = 2n$, then

$$\mathrm{rk}_{C_1}(C(X) \otimes M_k) \leq 7 + \langle b_1/2 \rangle \quad \textit{and} \quad \mathrm{rk}_{C_2}(C(X) \otimes M_k) \leq 4 + \langle b_2/2 \rangle.$$

Proof. We follow the ideas of Section 3 of [**15**]. Let $u \in U_0(C(X) \otimes M_k)$. Choose orthogonal projections p_1, q_1, p_2, and $q_2 \in M_k$ which sum to 1 and such that $\mathrm{rank}(p_i) = n$ and $0 \leq \mathrm{rank}(q_i) \leq n$. Let p_i, q_i also denote the corresponding constant functions in $C(X) \otimes M_k$. By Lemma 3.7, there exists $v_0 \in C(X) \otimes M_k$, either a product of 5 *-symmetries or of 3 commutators, such that $v_0^*(p_1 + q_1)v_0 = u^*(p_1 + q_1)u$, that is, uv_0^* commutes with $p_1 + q_1$. There further exists v_1, either a product of 5 *-symmetries or of 3 commutators, of the form $z_1 \oplus z_2$ with $z_i \in A_i = (p_i + q_i)(C(X) \otimes M_k)(p_i + q_i)$, such that $uv_0^* v_1^*$ comutes with each p_i and each q_i.

We now find v_2, a product of two *-symmetries and also a commutator, such that $uv_0^* v_1^* v_2^*$ still commutes with each p_i and each q_i, and such that $q_i(uv_0^* v_1^* v_2^*)q_i = q_i$. Again, $v_2 = z_1 \oplus z_2$ with $z_i \in A_i$. To construct z_i, let $r_i \leq p_i$ be a constant projection with the same rank as q_i. With respect to the decomposition of the identity $(1 - r_i) + r_i + q_i$ in A_i, we have the matrix representation

$$(p_i + q_i)(uv_0^* v_1^*)(p_i + q_i) = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & w \end{pmatrix},$$

and we take

$$z_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & w^* & 0 \\ 0 & 0 & w \end{pmatrix}.$$

By Lemma 3.6 (2), z_i is a commutator of homotopically trivial unitaries, and also a product of two *-symmetries.

The remainder of the proof takes place entirely within

$$(p_1 + p_2)(C(X) \otimes M_k)(p_1 + p_2) \cong M_2(C(X) \otimes M_n).$$

We write the elements as 2×2 matrices; elements of $C(X) \otimes M_k$ are gotten by adding $q_1 + q_2$ to everything. We know that deleting the summand $q_1 + q_2$ from $uv_0^* v_1^* v_2^*$ leaves a unitary u_0 of the form

$$u_0 = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}.$$

Note that $(s_1 s_2 \oplus 1) + q_1 + q_2 \in U_0(C(X) \otimes M_k)$ since it is homotopic to u. The topological stable rank $\operatorname{tsr}(C(X))$ is certainly less than $5\dim(X) + 3$, by Proposition 1.7 of [**22**], and so $s_1 s_2 \in U_0(C(X) \otimes M_n)$ by Theorem 2.10 of [**23**]. Furthermore,

$$\det(s_1 s_2) = \det \begin{pmatrix} s_1 s_2 & 0 \\ 0 & 1 \end{pmatrix} = \det(u_0) \det \begin{pmatrix} s_2 & 0 \\ 0 & s_2^* \end{pmatrix} = \det(u_0).$$

We now assume that $\det(u(x)) \in \{\pm 1\}$ for all x (in the *-symmetry case) or $\det(u) = 1$ (in the commutator case). Because the v_j are products of *-symmetries or commutators, it follows that $\det(uv_0^* v_1^* v_2^*)$, and so by the above $\det(u_0)$ and $\det(s_1 s_2)$, are ± 1 (in the *-symmetry case) or 1 (in the commutator case). Therefore $s_1 s_2$ is a product of at most b_1 *-symmetries or b_2 commutators, and we can write $s_1 s_2 = w_1 w_2$, where each w_i is a product of at most $\langle b_1/2 \rangle$ *-symmetries or $\langle b_2/2 \rangle$ commutators. One checks that $s_1 s_2 = w_1 w_2$ implies $u_0 = v_4 v_3$, where

$$v_4 = \begin{pmatrix} w_1^* s_1 & 0 \\ 0 & s_1^* w_1 \end{pmatrix} = \begin{pmatrix} w_1^* s_1 & 0 \\ 0 & (w_1^* s_1)^* \end{pmatrix}$$

is a product of two *-symmetries and a also commutator of homotopically trivial unitaries by Lemma 3.6 (2), and

$$v_3 = \begin{pmatrix} s_1^* w_1 s_1 & 0 \\ 0 & w_2 \end{pmatrix}$$

is a product of $\langle b_1/2 \rangle$ *-symmetries or $\langle b_2/2 \rangle$ commutators.

Replacing v_j by $v_j + q_1 + q_2$ for $j = 3, 4$, we get $u = v_4 v_3 v_2 v_1 v_0$, which is a product of $14 + \langle b_1/2 \rangle$ *-symmetries or of $8 + \langle b_2/2 \rangle$ commutators of elements of $U_0(C(X) \otimes M_k)$.

If $k = 2n$ then $q_1 = q_2 = 0$, and we can take $v_1 = v_2 = 1$. This leaves only $7 + \langle b_1/2 \rangle$ *-symmetries or $4 + \langle b_2/2 \rangle$ commutators. □

3.9 Theorem. *Let C be the class of *-symmetries. Then:*

(1) $P_C(C(X) \otimes M_n) = \overline{P}_C(C(X) \otimes M_n) = \{a \in U_0(C(X) \otimes M_n) : \det(u(x)) \in \{\pm 1\}$ *for all* $x\}$.

(2) $N_C(n, d) < \infty$ *for* $n \geq 1$ *and* $0 \leq d < \infty$.

(3) *For fixed* $d < \infty$, *we have* $\lim_{n \to \infty} \sup N_C(n, d) \leq 22$.

(4) *If X is a compact manifold with boundary, then*

$$\overline{\mathrm{rk}}_C(C(X) \otimes M_n) \geq 2\frac{\dim(X) - 2}{n^2 - 1} - 3.$$

Proof. Parts (1) and (2) are contained in Lemma 3.5. For part (3), fix d, and set $b = N_C(5d + 3, d)$, which is finite by (2). Define a sequence of integers b_r inductively by $b_0 = b$ and $b_{r+1} = 7 + \langle b/2 \rangle$. If $b_r > 15$ then $b_{r+1} < b$; therefore, there exists r such that $b_r \leq 15$. Furthermore, $b_s \leq 15$ for all $s \geq r$. Using induction and Lemma 3.8, we get $N_C(2^s(5d + 3), d) \leq 15$ for all $s \geq r$. Now let $k \geq 2^{r+1}(5d + 3)$. Then there exists $s \geq r$ such that $2 \cdot 2^s(5d + 3) \leq k \leq 4 \cdot 2^s(5d + 3)$, and Lemma 3.8 shows

$$N_C(k, d) \leq 14 + \langle N_C(2^s(5d + 3), d)/2 \rangle \leq 22.$$

For part (4), we claim that a product of two *-symmetries is a limit of exponentials of skewadjoint elements. The result will then follow from Proposition 3.2. To prove the claim, let s_1 and s_2 be *-symmetries. Then is_1 and $-is_2$ have spectrum contained in $\{\pm i\}$, and so can be written $is_1 = \exp(ih_1)$ and $-is_2 = \exp(ih_2)$ with h_1, h_2 selfadjoint and $\|h_1\| = \|h_2\| = \pi/2$. It follows from Corollary 2.2 of [**24**] that $\exp(ih_1)\exp((1 - \varepsilon)ih_2) = \exp(ih_\varepsilon)$ for some selfadjoint $h_\varepsilon \in A$, and clearly $\exp(ih_\varepsilon) \to (is_1)(-is_2) = s_1 s_2$ as $\varepsilon \to 0$. □

3.10 Theorem. *Let C be the class of commutators of homotopically trivial unitaries. Then:*

(1) $P_C(C(X) \otimes M_n) = \overline{P}_C(C(X) \otimes M_n) = \{a \in U_0(C(X) \otimes M_n) : \det(u) = 1\}$.

(2) $N_C(n, d) < \infty$ *for* $n \geq 1$ *and* $0 \leq d < \infty$.

(3) *For fixed* $d < \infty$, *we have* $\lim_{n \to \infty} \sup N_C(n, d) \leq 13$.

(4) *If X is a compact manifold with boundary, then*

$$\overline{\mathrm{rk}}_C(C(X) \otimes M_n) \geq \frac{\dim(X) - 2}{3(4n^2 - 1)} - 1.$$

Proof. Parts (1) and (2) are contained in Lemma 3.5. The argument for Theorem 3.9(3) applies here as well, with different numbers. Since $b > 9$ implies $4 + \langle b/2 \rangle < b$, we get

$$\lim_{n \to \infty} \sup N_{\mathcal{C}}(n, d) \leq 8 + \langle 9/2 \rangle = 13.$$

For (4), we use the unitary version of Theorem 5.4 of [**13**]. It asserts that if $n \geq 2$, and X is a compact manifold with boundary of dimension at least $m = 6(4n^2 - 1)l + 2$, then there is $u_0 \in U_0(C(X) \otimes M_n)$ which is a product of unitary commutators but not of $2l$ or fewer of them. To prove it, observe that, as in [**13**], we may take X to be the closed unit ball of $\mathbb{R}^m$. We use the same u_0 as in the proof of Theorem 5.4 of [**13**]. It is a product of unitary commutators by (1). Suppose it were a product of $2l$ of them. The formula

$$\begin{pmatrix} vwv^* & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} vw & 0 \\ 0 & (vw)^* \end{pmatrix} \begin{pmatrix} v^* & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} w^* & 0 \\ 0 & w \end{pmatrix}$$

would then show that $u = u_0 \oplus 1 \in C(X) \otimes M_{2n}$ is a product of $6l$ unitaries of the form $z \oplus z^*$. Each of these is a limit of exponentials of skewadjoint elements by Corollary 5 of [**12**]. So u would be a limit of products of $6l$ exponentials of skewadjoint elements, contradicting Theorem 1.10 of [**13**]. □

3.11 Remark. It seems very unlikely that $N_{\mathcal{C}}(n, d)$ is smaller when n is divisible by a large power of 2 than for other large n, when $\mathcal{C}$ is either the class of *-symmetries or of unitary commutators. Therefore we expect that one should be able to replace 22 by 15 in Theorem 3.9(3)and 13 by 9 in Theorem 3.10(3).

4. Factorization problems related to exponential length.

In this section we prove factorization theorems for the class $\mathcal{S}$ of positive-stable elements, the class $\mathcal{A}$ of accretive elements, and the class $\mathcal{U}$ of accretive unitaries. The most obvious formulations of these factorization problems yield behavior like that of the C* exponential length cel(A) introduced in [**24**]. (See Theorem 4.6.) However, restriction to elements of $C(X) \otimes M_n$ with determinant 1, a subset intrinsically characterized as the commutator subgroup, yields factorization theorems of the same sort as in the previous two sections. (See Theorems 4.12, 4.13, and 4.14.) Indeed, in Theorem 4.10 we see that, under this restriction, the C* exponential length itself behaves in the same way.

We point out that $\mathrm{rk}_{\mathcal{S}}(A)$ can be sensibly defined for Banach algebras A. Its behavior, as illustrated in Proposition 4.5 and Theorem 4.6, suggests that it is related to the Banach exponential rank in somewhat the same way that C* exponential length is related to C* exponential rank.

We begin this section with a few lemmas about the classes $\mathcal{S}, \mathcal{A}$, and $\mathcal{U}$. They are probably known, but we lack a reference. We next relate the associated ranks to $\mathrm{cel}(A)$, and prove the theorems for the unrestricted factorization problems. We then introduce the notation for the restricted problems, and prove the corresponding theorems.

Recall that in Definition 1.1 (5), we defined a to be accretive if $(a+a^*)/2$ is positive and invertible. This definition agrees with [**29**] but, as the following lemma shows, not with other commonly used definitions.

4.1 Lemma. *Let H be a Hilbert space, and let $a \in L(H)$. Then a is accretive if and only if there is $\varepsilon > 0$ such that the numerical range $W(a)$ is contained in $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq \varepsilon\}$.*

Proof. Recall ([**8**, Chapter 17]) that

$$W(a) = \{\langle a\xi, \xi\rangle : \xi \in H, \|\xi\| = 1\}.$$

Let $a = b + ic$ with b, c selfadjoint. Then clearly

$$\{\mathrm{Re}(\lambda) : \lambda \in W(a)\} = W(b),$$

and it is well known that a selfadjoint $b \in L(H)$ is positive and invertible if and only if there is $\varepsilon > 0$ such that $\langle b\xi, \xi\rangle \geq \varepsilon\|\xi\|^2$ for $\xi \in H$. ☐

At this point, we can contrast our definition with the more conventional definitions of accretive operators on Hilbert spaces which are used in the theory of semigroups. In Section V.3.10 of Kato's book [**9**], a not necessarily bounded operator a is called accretive essentially when

$$W(a) \subset \{\lambda \in \mathrm{C} : \mathrm{Re}(\lambda) \geq 0\}.$$

Fillmore ([**7**], page 87) uses a different sign convention in his semigroups, and therefore requires $W(a) \subset \{\lambda \in \mathrm{C} : \mathrm{Re}(\lambda) \leq 0\}$. Note that both definitions apply to unbounded operators, which we do not have in C*-algebras, and that even Kato's definition does not reduce to that of [**29**] for finite dimensional Hilbert spaces.

4.2 Corollary. *Let A be a unital C*-algebra. Then $a \in A$ is accretive if and only if* $\mathrm{Re}(\varphi(a)) > 0$ *for every state φ on A.*

Proof. Note that if a is accretive and π is a unital homomorphism, then $\pi(a)$ is accretive. So if $a \in A$ is accretive and φ is a state, we show $\mathrm{Re}(\varphi(a)) > 0$

by applying the Gelfand-Naimark-Segal construction to φ. Conversely, if a is not accretive, there is a state φ such that $\varphi((a + a^*)/2) \leq 0$. Then $\mathrm{Re}(\varphi(a)) \leq 0$. □

4.3 Corollary. *Let A be a unital* C*-*algebra. If $a \in A$ is accretive, then a is positive-stable, and, in particular, invertible.*

Proof. Represent A faithfully on a Hilbert space, and use the lemma together with the fact ([**8**, Problem 169]) that for $a \in L(H)$ we have $\mathrm{sp}(a) \subset \overline{W}(a)$. □

4.4 Lemma. *Let A be a unital* C*-*algebra, and let $u \in U(A)$. Then the following are equivalent:*

(1) *u is accretive.*

(2) *u is positive-stable.*

(3) $\|u - 1\| < \sqrt{2}$.

(4) $\mathrm{cel}(u) < \pi/2$.

Proof. (1) ⇔ (2): Represent A faithfully on a Hilbert space, and use the fact that for normal $a \in L(H)$, the convex hull of $\mathrm{sp}(a)$ is $\overline{W(a)}$ ([**8**, Problem 171]).

(2) ⇔ (3): Clearly u is positive-stable if and only if $\mathrm{sp}(u)$ is contained in $S^1 \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$. Now use the fact that the norm of $u - 1$ is equal to its spectral radius.

(3) ⇔ (4): See Proposition 2.4 of [**24**]. □

This lemma implies that $\mathrm{rk}_{\mathcal{U}}(A)$ is essentially a discrete version of the C* exponential length. The precise statement is:

$$\overline{\mathrm{rk}}_{\mathcal{U}}(A) = m \quad \text{if and only if} \quad \mathrm{cel}(A) \in (\pi(m-1)/2, \pi m/2].$$

(Compare with part (2) of the next proposition.)

Because of this relation, we would like to propose $\mathrm{rk}_{\mathcal{S}}(A)$ as the appropriate analog for Banach algebras of the C* exponential length. The direct analog is infinite even for M_2, as one sees by considering the matrix $\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)$ for $|\alpha|$ large. It doesn't help to restrict to elements of determinant 1. On the other hand, the definition of $\mathrm{rk}_{\mathcal{S}}(A)$ makes sense for unital Banach algebras A. We will see in the next proposition that $\mathrm{rk}_{\mathcal{S}}(A) \leq \mathrm{rk}_{\mathcal{U}}(A) + 1$ if A is a C*-algebra, just as $\mathrm{ber}(A) \leq \mathrm{cer}(A) + 1$. In the theorem following, we will also see that it behaves like cel on the algebras $C(X) \otimes M_n$, for which we already know that the behavior of ber is similar to that of cer.

4.5 Proposition. *Let A be any unital* C*-*algebra. Then:*

(1) $P_{\mathcal{A}}(A) = \overline{P}_{\mathcal{A}}(A) = P_{\mathcal{S}}(A) = \overline{P}_{\mathcal{S}}(A) = \mathrm{inv}_0(A)$ *and* $P_{\mathcal{U}}(A) = \overline{P}_{\mathcal{U}}(A) = U_0(A)$.

(2) $\mathrm{rk}_{\mathcal{S}}(A) \leq \overline{\mathrm{rk}}_{\mathcal{A}}(A) + 1, \mathrm{rk}_{\mathcal{A}}(A) \leq \overline{\mathrm{rk}}_{\mathcal{A}}(A) + 1,$ *and* $\mathrm{rk}_{\mathcal{U}}(A) \leq \overline{\mathrm{rk}}_{\mathcal{U}}(A) + 1.$

(3) $\mathrm{rk}_{\mathcal{S}}(A) \leq \overline{\mathrm{rk}}_{\mathcal{A}}(A) \leq \mathrm{rk}_{\mathcal{U}}(A)+1 \leq 2\,\mathrm{cel}(A)/\pi+2$ *and* $\mathrm{cel}(A) \leq \pi\,\mathrm{rk}_{\mathcal{U}}(A)/2.$

Proof. (1) The sets of accretive and positive-stable elements of A are both contained in $\mathrm{inv}_0(A)$ (in fact, in $\exp(A)$) and both contain neighborhoods of 1 in A. Similarly, the set of accretive unitaries is contained in $U_0(A)$ and contains a neighborhood of 1 in $U(A)$.

(2) This follows from the same openness properties as in the proof of (1).

(3) The first inequality is trivial. The second follows from polar decomposition. The third follows from the implication (4) $\Rightarrow$ (1) of the previous lemma, by choosing appropriate break points on a unitary path from 1 to $u \in U_0(A)$ of length at most $\mathrm{cel}(u) + \varepsilon$, and because the accretive unitaries contain a neighborhood of 1 in $U_0(A)$. The last inequality follows from the implication (1) $\Rightarrow$ (4) of the previous lemma. □

4.6 Theorem. *Let X be compact metric, let $n \geq 1$, and let $A = C(X)\otimes M_n$.*

(1) *If X is totally disconnected, then* $\mathrm{rk}_{\mathcal{S}}(A), \mathrm{rk}_{\mathcal{A}}(A) \leq 4, \mathrm{rk}_{\mathcal{U}}(A) \leq 3,$ *and* $\mathrm{cel}(A) \leq \pi$.

(2) *If X is not totally disconnected, then* $\overline{\mathrm{rk}}_{\mathcal{S}}(A) = \overline{\mathrm{rk}}_{\mathcal{A}}(A) = \overline{\mathrm{rk}}_{\mathcal{U}}(A) = \mathrm{cel}(A) = \infty$.

Proof. (1) If X is totally disconnected, then any unitary in A can be approximated arbitrarily closely by unitary functions having only finitely many values. So clearly $\mathrm{cel}(A) \leq \pi$. The rest now follows from the previous proposition.

(2) In view of the previous proposition, it suffices to show that $\overline{\mathrm{rk}}_{\mathcal{S}}(A) = \infty$. Since X is not totally disconnected, there exist distinct $x_1, x_2 \in X$ such that every closed and open subset of X which contains x_1 also contains x_2. Choose a continuous function $f : X \to \mathbb{R}$ such that $f(x_1) = 1$ and $f(x_2) = -1$. Define $u \in C(X) \otimes M_n$ by $u = \exp(iMf) \otimes 1$, where M is some positive real number. We will show that if $M > \pi l/2$, then u is not a limit of products of l positive-stable elements of A.

Suppose a is a product of l positive-stable elements of A, that is, $a = a_1 \cdots a_l$ with a_j positive-stable. Note that $1 - t + ta_j$ is also positive-stable for $t \in [0, 1]$. Define

$$b(t) = \prod_{j=1}^{l} (1 - t + ta_j).$$

We now calculate the continuous logarithm $\log(\det(b(t)))$ which is 0 when $t = 0$. Functional calculus with logarithms is taken with the continuous branch which is real on positive real numbers and undefined on negative real numbers. We have

$$\log(\det(b(t))) = \sum_{j=1}^{l} \log(\det(1 - t + ta_j)) = \sum_{j=1}^{l} \operatorname{tr}(\log(1 - t + ta_j)).$$

The second equality holds because both sides are continuous and agree when $t = 0$. Each term in the last sum has imaginary part in $[\pi n/2, \pi n/2]$. Therefore, putting $t = 1$, we get

$$(*) \qquad |\operatorname{Im}(\log(\det(a)))| \leq \pi n l/2.$$

One easily checks that even if a is only a limit of products of l positive-stable elements, its determinant must still have a logarithm satisfying (*). On the other hand, if f is a logarithm of $\det(u)$, then $f(x_1) - f(x_2) = 2Mni$. So u cannot be a limit of products of l positive-stable elements. □

The previous theorem shows that, at least on $C(X) \otimes M_n$, the quantities $\operatorname{rk}_{\mathcal{A}}$ and $\operatorname{rk}_{\mathcal{S}}$ also behave like C* exponential length. Nevertheless, behavior like that of exponential rank is hidden just beneath the surface. (This is even true for the C* exponential length itself.) To expose it, we restrict to factorizations of the elements of determinant 1, intrinsically characterized as the commutator subgroup. (See the characterizations of the commutator subgroups of $\operatorname{inv}_0(A)$ and $U_0(A)$ in Theorems 2.10(1) and 3.10(1).)

4.7 Definition. Let $\mathcal{C}$ be one of the clases $\mathcal{A}, \mathcal{S}$, or $\mathcal{U}$. For a topological group G we let G' denote the closed subgroup generated by the commutators. Let A be a unital C*-algebra, and define:

(1) $P'_{\mathcal{C}}(A) = P_{\mathcal{C}}(A) \cap \operatorname{inv}_0(A)'$ if $\mathcal{C} = \mathcal{A}$ or $\mathcal{S}$, and $P'_{\mathcal{C}}(A) = P_{\mathcal{C}}(A) \cap U_0(A)'$ if $\mathcal{C} = \mathcal{U}$.

(2) $\overline{P_{\mathcal{C}}}'(A)$ is the closure of $P'_{\mathcal{C}}(A)$ in $\operatorname{inv}_0(A)$.

(3) $\operatorname{rk}'_{\mathcal{C}}(A)$ is the smallest n (possibly ∞) such that every element of $P'_{\mathcal{C}}(A)$ is a product of at most n elements of the class $\mathcal{C}$.

(4) $\overline{\operatorname{rk}}'_{\mathcal{C}}(A)$ is the smallest n (possibly ∞) such that products of at most n elements of the class $\mathcal{C}$ are dense in $\overline{P_{\mathcal{C}}}'(A)$.

(5) $N'_{\mathcal{C}}(n, d) = \sup_{\dim(X) \leq d} \operatorname{rk}'_{\mathcal{C}}(C(X) \otimes M_n)$, where X runs through all compact metric spaces of dimension at most d.

We further set $\operatorname{cel}'(A) = \sup\{\operatorname{cel}(u) : u \in U_0(A)'\}$, and define

$$N'(n, d) = \sup_{\dim(X) \leq d} \operatorname{cel}'(C(X) \otimes M_n),$$

using the same spaces X as before.

Note that $\mathrm{rk}'_{\mathcal{C}}(C(X)) = \overline{\mathrm{rk}}'_{\mathcal{C}}(C(X)) = 1$ for $\mathcal{C}$ equal to any of $\mathcal{A}, \mathcal{S}$, and $\mathcal{U}$. We have thus recovered the behavior of Proposition 1.4(2).

4.8 Remark. For $A = C(X) \otimes M_n$, Theorems 2.10(1) and 3.10(1) imply that $\mathrm{inv}_0(A)' \cap U_0(A) = U_0(A)'$. So we could have used $\mathrm{inv}_0(A)'$ in the definition of $\mathrm{rk}'_{\mathcal{U}}(A)$ too.

We also have $\mathrm{inv}_0(A)' \cap U_0(A) = U_0(A)'$ for the algebras considered in [**4**] and [**26**]. In general, however, it seems to be unknown when this relation holds.

4.9 Remark. We can also analogously define $P'_{\mathcal{C}}(A), \mathrm{rk}'_{\mathcal{C}}(A)$, etc. for other classes mentioned in or before Definition 1.1. The analogs of the theorems in Section 2 and 3 still hold, with the trivial modification that one must impose the condition $\det(a) = 1$ in the characterizations of $P'_{\mathcal{C}}(A)$ and $\overline{P}_{\mathcal{C}}{}'(A)$. (Note that the elements used to prove the lower bounds in all of these theorems actually have determinant 1.)

We now present the main theorems, starting with the one for exponential length.

4.10 Theorem. (1) $\mathrm{cel}(u) < \infty$ *for all* $u \in U_0(C(X) \otimes M_n)'$.

(2) $N'(n, d) < \infty$ *for* $n \geq 1$ *and* $0 \leq d < \infty$.

(3) *For fixed* $d < \infty$, *we have* $\lim_{n\to\infty} \sup N'(n, d) \leq 4\pi$.

(4) *If* X *is a compact manifold with boundary and* $n \geq 2$, *then*

$$\mathrm{cel}'(C(X) \otimes M_n) > \pi\left(\frac{\dim(X) - 2}{n^2 - 1} - 2\right).$$

Proof. (1) Actually, $\mathrm{cel}(u) < \infty$ for any $u \in U_0(A)$ for any unital C*-algebra A. See [**24**].

(2) This is just Lemma 3.3.

(3) The proof of Theorem 3.3 of [**13**] gives, for each positive integer d and $\varepsilon > 0$, a number $M(d, \varepsilon)$ such that whenever $\dim(X) \leq d$ and u is a homotopically trivial element of $C(X, SU_n)$ with $n \geq M(d, \varepsilon)$, then there exist self-adjoint $h_1, h_2, h_3 \in C(X) \otimes M_n$ such that $\|u - \exp(ih_1)\exp(ih_2)\exp(ih_3)\| < \varepsilon$. (Note that the first step in this proof, which applies to an arbitrary $u \in U_0(C(X) \otimes M_n)$, is to reduce to the case $\det(u) = 1$.) An examination of the proof shows that the h_1 chosen in it satisfies $\|h_1\| \leq 2\pi$. Similarly, going back to [**12**] in Step 7 of this proof, we get $\|h_2\|, \|h_3\| \leq \pi$. Therefore $\mathrm{cel}(u) \leq 4\pi + 2\arcsin(\varepsilon/2)$, using Proposition 2.4 of [**24**]. So $\mathrm{cel}'(C(X) \otimes M_n) \leq 4\pi + 2\arcsin(\varepsilon/2)$ for all large enough n.

(4) Theorem 2.6 of [**24**] shows that if $\mathrm{cel}(u) \leq l\pi$, then u is a limit of products of l exponentials. Now use Proposition 3.2. □

4.11 Remark. The paths whose lengths are estimated in parts (2) and (3) of this theorem are not required to have determinant 1 except at the endpoints. It might seem more appropriate to require them to lie entirely in $U_0(A)'$. With this extra condition, (2) still holds, and still follows from Lemma 3.3. One can show that (3) holds with 4π replaced by 6π. We omit the details, but note that it is fairly easy to get the estimate 10π. One replaces each h_j by $h_j = -\frac{1}{n}\operatorname{tr}(h_j)\cdot 1$, which at most doubles the norm. One does the same thing with $h_4 = -i\log(\exp(-ih_3)\exp(-ih_2)\exp(-ih_1)u)$. From this, one gets a path of length not much more than 8π to a locally constant scalar element of $C(X, SU_n)$. Now observe that a scalar in SU_n can be connected to 1 by a path of length less than 2π.

4.12 Theorem. (1) $P'_{\mathcal{A}}(C(X)\otimes M_n) = \overline{P}'_{\mathcal{A}}(C(X)\otimes M_n) = \{a \in \operatorname{inv}_0(C(X)\otimes M_n) : \det(a) = 1\}$.

(2) *$N'_{\mathcal{A}}(n,d) < \infty$ for $n \geq 1$ and $0 \leq d < \infty$.*

(3) *For fixed $d < \infty$ we have $\lim_{n\to\infty}\sup N'_{\mathcal{A}}(n,d) \leq 10$.*

(4) *If X is a compact manifold with boundary and $n \geq 2$, then*

$$\overline{\operatorname{rk}}'_{\mathcal{A}}(C(X)\otimes M_n) \geq \frac{\dim(X) - 2}{n^2 - 1} - 1.$$

4.13 Theorem. (1) $P'_{\mathcal{S}}(C(X)\otimes M_n) = \overline{P}'_{\mathcal{S}}(C(X)\otimes M_n) = \{a \in \operatorname{inv}_0(C(X)\otimes M_n) : \det(a) = 1\}$.

(2) *$N'_{\mathcal{S}}(n,d) < \infty$ for $n \geq 1$ and $0 \leq d < \infty$.*

(3) *For fixed $d < \infty$ we have $\lim_{n\to\infty}\sup N'_{\mathcal{S}}(n,d) \leq 10$.*

(4) *If X is a compact manifold with boundary and $n \geq 2$, then*

$$\overline{\operatorname{rk}}'_{\mathcal{S}}(C(X)\otimes M_n) \geq \frac{\dim(X) - 2}{2(n^2 - 1)}.$$

4.14 Theorem. (1) $P'_{\mathcal{U}}(C(X)\otimes M_n) = \overline{P}'_{\mathcal{U}}(C(X)\otimes M_n) = \{a \in U_0(C(X)\otimes M_n) : \det(u) = 1\}$.

(2) *$N'_{\mathcal{U}}(n,d) < \infty$ for $n \geq 1$ and $0 \leq d < \infty$.*

(3) *For fixed $d < \infty$ we have $\lim_{n\to\infty}\sup N'_{\mathcal{U}}(n,d) \leq 9$.*

(4) *If X is a compact manifold with boundary and $n \geq 2$, then*

$$\overline{\operatorname{rk}}'_{\mathcal{U}}(C(X)\otimes M_n) \geq 2\frac{\dim(X) - 2}{n^2 - 1} - 3.$$

Proof of Theorems 4.12. 4.13. *and* 4.14. (1) This is immediate from Proposition 4.5 and the remark before Definition 4.7 in all three cases.

(2), (3) The proof of part (3) of Proposition 4.5 shows, in exactly the same way, that the same inequalities hold for rk'_C in place of rk_C and cel$'$ in place of cel. Part (2) now follows in all three cases from the corresponding part of Theorem 4.10. For (3), use Theorem 4.10(3) to choose N such that $n \geq N$ and $\dim(X) \leq d$ imply $\mathrm{cel}'(C(X) \otimes M_n) < 9\pi/2$. For such n and X,we have $\mathrm{rk}'_{\mathcal{A}}(C(X) \otimes M_n), \mathrm{rk}'_{\mathcal{S}}(C(X) \otimes M_n) < 11$ and $\mathrm{rk}'_{\mathcal{U}}(C(X) \otimes M_n) < 10$.

(4) In Theorem 4.12, we note that if $a, b \in C(X) \otimes M_n$ are accretive, then Theorem 2.22 of [**29**] implies that $\mathrm{sp}(a(x)b(x))$ contains no nonpositive real numbers. Therefore products of two accretive elements are exponentials. Now use Proposition 2.2. In Theorem 4.13, we simply observe that elements with spectrum in the right half plane have logarithms, and apply Proposition 2.2. In Theorem 4.14, we note that the product of two accretive unitaries has exponential length less than π by Lemma 4.4(4), and so is the exponential of a skewadjoint element. Use Proposition 3.2. □

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