Pacific Journal of Mathematics

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Volume 174 No. 2 June 1996

PACIFIC JOURNAL OF MATHEMATICS
Vol. 174, No. 2, 1996

DERIVATIONS OF C^*-ALGEBRAS AND ALMOST HERMITIAN REPRESENTATIONS ON Π_k-SPACES

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This paper studies almost Hermitian, J-symmetric representations of *-algebras on Π_k-spaces. It applies the results obtained to the theory of *-derivations δ of C^*-algebras implemented by symmetric operators S.

1. Introduction and preliminaries.

The work on representations of groups and algebras on spaces with invariant indefinite metric was strongly motivated by various applications to relativistic quantum mechanics and differential equations. Gelfand and Yaglom [**9**], Gelfand and Vilenkin [**8**], Naimark [**27**], Zhelobenko [**44**] and Ismagilov [**12**] investigated representations of the Lorentz group on Π_k-spaces. Representations of Lie groups were considered in a number of papers (see, for example, [**4**, **7**, **10**, **26**]) in relation to the study of massless particles. The Gupta-Bleuer triplets for indecomposable representations of groups were introduced and studied by Araki [**1**]. Rawnslew, Schmid and Wolf [**39**] investigated the indefinite harmonic theory of groups. Ismagilov [**13**] considered representation theory on Π_k-spaces for central extension groups.

Phillips [**34-36**] initiated the work on operator algebras on indefinite metric spaces. He applied the obtained results to the problem of extension of dissipative operators commuting with an operator algebra and to hyperbolic systems of differential equations.

Simultaneously with the growth of the area of applications of the theory, the process of its internal development has been taking place. Much work has been done on the investigation of the structure of operator algebras and of representations of Lie groups on Π_k-spaces (for extensive bibliography on this subject see Naimark and Ismagilov [**29**], Naimark, Loginov and Shulman [**31**] and Loginov and Shulman [**25**]).

The interrelation between representations on indefinite metric spaces and unbounded *-derivations δ of C^*-algebras $\mathfrak{A}$ was initially observed by Ota [**33**] and by Jorgensen and Muhly [**15**]. Using Phillips' results, Jorgensen

[‡]The authors are grateful to the University of North London for inviting them to Britain for a three months research visit.

and Muhly proved that if $\mathfrak{U}$ is commutative, δ is implemented by a symmetric operator S and at least one of the deficiency indices $n_-(S)$ or $n_+(S)$ of S is finite, then S extends to a maximal dissipative operator which implements δ. They applied this to study the Weyl canonical operator commutation relations.

In **[17]** it was shown that, for every pair (δ, S), the deficiency space $N(S)$ of the operator S is a *Krein space* and that there is a J-symmetric representation π_S^δ of the domain $D(\delta)$ of δ on $N(S)$. If $k = \min(n_\pm(S)) < \infty$, then $N(S)$ is a Π_k-space. Many questions about the pairs (δ, S), such as the values of $n_\pm(S)$ and the existence of maximal dissipative operators which implement δ, can be better understood and answered in terms of these representations.

Arveson **[2]**, Powers **[37]** and Powers and Robinson **[38]** studied the case when δ is the generator of a semigroup α_t of endomorphisms of $B(\mathfrak{H})$ and $-iS$ is the generator of a semigroup of isometries which intertwine α_t. In this case S is a maximal symmetric operator, $N(S)$ is a Hilbert space and π_S^δ is a *-representation. They introduced and investigated various notions of the index of α_t in terms of the representation π_S^δ.

The general case when $N(S)$ is not necessarily a Hilbert space was studied in **[15-21, 31]**. Jorgensen and Price **[16]** defined the V-index as the dimension of the Krein space of operators $V : \mathfrak{H} \to N(S)$, satisfying $VA = \pi_S^\delta(A)V, A \in D(\delta)$. In **[19]** a sextuple of integers $\operatorname{ind}(\delta, S)$ was associated with every pair (δ, S) and its stability under some perturbations of δ was shown. The representational indices of derivations were introduced in **[20]** and their uniqueness was studied in **[21]**.

This paper considers non-degenerate, almost Hermitian, J-symmetric representations of *-algebras on Π_k-spaces. Theorem 4 establishes the similarity of these representations to *-representations. (Almost Hermitian representations constitute probably the largest class of representations for which such similarity exists.) The most decisive step for proving Theorem 4 is the result obtained in Theorem 3: irreducible, uniformly closed J-symmetric operator algebras on Π_k-spaces contain the algebra of all compact operators.This last result was announced in **[24]** and its proof is based on Cuntz's theorem **[6]** and on some techniques developed in **[41]**.

It is well-known that not every bounded representation of a group on a Hilbert space is similar to a unitary one. It was established in **[40]** that the similarity problem for bounded representations of groups on Hilbert spaces is equivalent to the similarity problem for bounded J-unitary representations of groups on Krein spaces. From this it follows that there are bounded J-unitary representations of groups which are not similar to unitary representations. However, for Π_k-spaces the similarity problem is still open.In other words, it is unknown whether for every bounded representation of a group which preserves a quadratic form with a finite number k of negative squares, there always exists an invariant positive quadratic form. For $k = 1$ the positive answer was obtained in [**42**] with the use of methods of hyperbolic geometry. Making use of Theorem 4, we show in Theorem 7 that the problem has a positive solution for groups with almost Hermitian group algebras; the authors do not know examples of groups which do not possess this property.

Since all the domains $D(\delta)$ of derivations δ of $\mathfrak{U}$ are Hermitian algebras, it follows from Theorem 4 that their non-degenerate representations on Π_k-spaces are similar to *-representations and, therefore, extend to bounded representations of $\mathfrak{U}$. This allows us in section 5 to improve substantially on the results known previously (see [**16-19, 31**]) about symmetric implementations of δ. If, in particular, $\mathfrak{U}$ has no finite-dimensional representations, then, for every symmetric implementation S of δ, either $n_\pm(S) = \infty$ or $k = \min(n_\pm(S)) = 0$, so that S is maximal symmetric. If $\mathfrak{U}$ has finite-dimensional representations, then δ may have symmetric implementations S such that $0 < k < \infty$. In this case there are finite-dimensional representations $\{\pi_i\}_{i=1}^m$ of $\mathfrak{U}$ such that $k = \sum_{i=1}^m \dim \pi_i$ and all the results obtained in [**15, 18**] about the existence of maximal dissipative operators which extend S and implement δ, and about the Weyl commutation relations, are valid.

2. Irreducible uniformly closed J-symmetric algebras on Π_k-spaces.

We shall start this section by providing some information about J-symmetric representations on Π_k-spaces.

Let $H = H_- \oplus H_+$ be an orthogonal decomposition of a Hilbert space H with a scalar product (x, y). The involution $J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ defines an indefinite form $[x, y] = (Jx, y)$ on H and, with this form, H is called a *Krein space.* Let $k_\pm = \dim H_\pm$. If $k = \min(k_-, k_+) < \infty$, H is called a *Π_k-space.*

A subspace L in H is *non-negative* if $[x, x] \geq 0$; *positive* if $[x, x] > 0, x \neq 0$; *uniformly positive* if there is $r > 0$ such that $[x, x] \geq r(x, x)$ and *neutral* if $[x, x] = 0$, for all $x \in L$. *Non-positive, negative and uniformly negative* subspaces are introduced analogously. We shall call uniformly negative and uniformly positive subspaces *uniformly definite.*

If L is a subspace in H, the subspace

$$L^{[\perp]} = \{y \in H : [x, y] = 0 \quad \text{for all } x \in L\}$$

is the *J-orthogonal complement* of L. If H is a Π_k-space and $L \cap L^{[\perp]} = \{0\}$, then (see [**23**]) H can be decomposed in the direct and J-orthogonal sum

$$H = L\,[+]\,L^{[\perp]}. \tag{1}$$

For bounded operators B on H define an involution $\#$:

$$B^{\#} = JB^*J, \qquad \text{i.e., } [Bx, y] = \left[x, B^{\#}y\right], \qquad x, y \in H.$$

Then $\|B^{\#}\| = \|B\|$. A subalgebra $\mathcal{B}$ of $B(H)$ is *J-symmetric* if $B \in \mathcal{B}$ implies $B^{\#} \in \mathcal{B}$. Uniformly closed, J-symmetric subalgebras of $B(H)$ are Banach *-algebras.

A representation π of a *-algebra $\mathcal{A}$ on a Krein space H is called

$J-symmetric$ if $\pi(a^*) = \pi(a)^{\#}$, $a \in \mathcal{A}$, and
non-degenerate if π has no neutral invariant subspaces.

If a subspace L is invariant for π, $L^{[\perp]}$ is also invariant for π. By π_L we denote the restriction of π to L. If L is uniformly definite, it is a Hilbert space with respect to the scalar product

$$(x, y)_L = \begin{cases} -[x, y], & \text{if } L \text{ is uniformly negative,} \\ [x, y], & \text{if } L \text{ is uniformly positive,} \end{cases} \qquad x, y \in L.$$

The norm $\|x\|_L = (x, x)_L^{1/2}$ is equivalent to the original norm on L and π_L is a *-representation of $\mathcal{A}$ on L with respect to $(,)_L$. If $H = N[+]P$ where N and P are respectively uniformly negative and uniformly positive invariant subspaces, then H is a Hilbert space and π is a *-representation of $\mathcal{A}$ with respect to the scalar product

$$(2) \quad \langle x_1[+]y_1, x_2[+]y_2 \rangle = (x_1, x_2)_N + (y_1, y_2)_P, \quad x_i \in N,\ y_i \in P,\ i = 1, 2,$$

and the norm $\|\cdot\|_1 = \langle , \rangle^{1/2}$ is equivalent to the original norm on H. Let Q be the projection on N along P and $\mathfrak{J} = \mathbf{1}_H - 2Q$. Then $\mathfrak{J}$ is an involution on H with respect to $\langle , \rangle$, i.e., $\mathfrak{J}^* = \mathfrak{J}$ and $\mathfrak{J}^2 = \mathbf{1}_H$, $\mathfrak{J}|_N = -\mathbf{1}_N$, $\mathfrak{J}|_P = \mathbf{1}_P$ and

$$(3) \qquad [x, y] = \langle \mathfrak{J}x, y \rangle, \qquad x, y \in H.$$

Let L be a neutral subspace in a Π_k-space H, K be the orthogonal complement of $L^{[\perp]}$ in H and $M = L^{[\perp]} \ominus L$. Then $\dim L = \dim K \le k$, M is a Π_n-space, $n = k - \dim L$, and

$$(4) \qquad L^{[\perp]} = L \oplus M \quad \text{and} \quad H = L \oplus M \oplus K.$$

With respect to decomposition (4), J has the form $J = \begin{pmatrix} 0 & 0 & J_1 \\ 0 & J_2 & 0 \\ J_3 & 0 & 0 \end{pmatrix}$, where $J_1^* = J_3$, $J_1J_3 = \mathbf{1}_L$, $J_3J_1 = \mathbf{1}_K$ and J_2 is an involution on M. Let β be the orthoprojection on M. If L is a maximal neutral invariant subspace of a J-symmetric algebra of operators $\mathcal{B}$, then the algebra $\mathcal{B}_\beta = \{\beta\mathcal{B}\beta : B \in \mathcal{B}\}$ of operators on M is J_2-symmetric and has no neutral invariant subspaces.

The following lemma considers some simple conditions on normed spaces to be complete.

Lemma 1. *Let $(\mathfrak{X}, \|\cdot\|)$ be a normed space and L be a subspace of $\mathfrak{X}$ of finite codimension, i.e., there is a finite subspace K in $\mathfrak{X}$ such that $\mathfrak{X}$ is the direct sum of L and K.*

(i) *If $(L, \|\cdot\|)$ is complete, then $(\mathfrak{X}, \|\cdot\|)$ is complete.*

(ii) *Let $|\cdot|$ be another norm on $\mathfrak{X}$. If $(\mathfrak{X}, |\cdot|)$ and $(L, |\cdot|)$ are complete and the norms $\|\cdot\|$ and $|\cdot|$ are equivalent on L, then they are equivalent on $\mathfrak{X}$.*

(iii) *If T is a linear mapping from $\mathfrak{X}$ into a finite-dimensional Banach space $\mathcal{Y}$ and $\mathfrak{X}$ is complete with respect to the norm $|x| = \|x\| + \|Tx\|_{\mathcal{Y}}$, then the norms $\|\cdot\|$ and $|\cdot|$ are equivalent, $(\mathfrak{X}, \|\cdot\|)$ is complete and T is continuous on $(\mathfrak{X}, \|\cdot\|)$.*

Proof. Let $(L, \|\cdot\|)$ be complete. Since K is finite-dimensional, $(K, \|\cdot\|)$ is also complete. For $x = y + z$, $y \in L$ and $z \in K$, set

$$\|x\|_1 = \|y\| + \|z\|.$$

If there is a sequence $x_n = y_n + z_n$, $y_n \in L$, $z_n \in K$, such that $x_n \to 0$ and $\|z_n\| = 1$ as $n \to \infty$, then, since K is finite-dimensional, there is a subsequence $\{z_{n_k}\}$ which converges to $z \in K$ and $\|z\| = 1$. Hence $\|y_{n_k} + z\| \leq \|x_{n_k}\| + \|z - z_{n_k}\| \to 0$ so that $y_{n_k} \to -z \in K$. Since L is complete, we have a contradiction. From this it follows that there is $C > 0$ such that

$$\|z\| \leq C\|x\|, \quad \text{for } x = y + z,\ y \in L \text{ and } z \in K.$$

Therefore $\|y\| = \|x - z\| \leq (1 + C)\|x\|$ and

$$\|x\| = \|y + z\| \leq \|y\| + \|z\| = \|x\|_1 \quad \text{and} \quad \|x\|_1 = \|y\| + \|z\| \leq (1 + 2C)\|x\|.$$

Thus the norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent. Since $(\mathfrak{X}, \|\cdot\|_1)$ is complete, $(\mathfrak{X}, \|\cdot\|)$ is also complete. Part (i) is proved.

Since $(L, |\cdot|)$ is complete and the norms $\|\cdot\|$ and $|\cdot|$ are equivalent on L, $(L, \|\cdot\|)$ is complete and there is $C_1 > 0$ such that $|y| \leq C_1\|y\|$, $y \in L$. By (i), $(\mathfrak{X}, \|\cdot\|)$ and $(\mathfrak{X}, \|\cdot\|_1)$ are complete and the norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent. Since all norms on K are equivalent, there is $C_2 > 0$ such that $|z| \leq C_2\|z\|$, $z \in K$. Let $C = \max(C_1, C_2)$. Then, for $x = y + z$, $y \in L$, $z \in K$,

$$|x| \leq |y| + |z| \leq C(\|y\| + \|z\|) = C\|x\|_1.$$

Hence the identity operator from $(\mathfrak{X}, \|\cdot\|_1)$ onto $(\mathfrak{X}, |\cdot|)$ is bounded. Therefore the inverse operator is also bounded, i.e., there is $D > 0$ such that $\|x\|_1 \leq D|x|$, $x \in \mathfrak{X}$. Thus the norms $\|\cdot\|_1$ and $|\cdot|$ on $\mathfrak{X}$ are equivalent. Hence the norms $\|\cdot\|$ and $|\cdot|$ are also equivalent on $\mathfrak{X}$. Part (ii) is proved.

The mapping T is bounded with respect to $|\cdot|$. Hence Ker T is closed and, therefore, complete with respect to $|\cdot|$. Since $\|x\| = |x|$, for $x \in \operatorname{Ker} T$, and since Ker T has finite codimension in $\mathfrak{X}$, it follows from (ii) that the norms $\|\cdot\|$ and $|\cdot|$ are equivalent on $\mathfrak{X}$. Thus $(\mathfrak{X}, \|\cdot\|)$ is complete and T is continuous on $(\mathfrak{X}, \|\cdot\|)$. $\square$

A Banach *-algebra $(\mathcal{A}, \|\cdot\|, ^*)$ is called *C^*-equivalent* if there is another norm $|\cdot|$ on $\mathcal{A}$ equivalent to $\|\cdot\|$ such that $(\mathcal{A}, |\cdot|, ^*)$ is a C^*-algebra.

Theorem 2. *Let $(\mathcal{B}, \|\cdot\|, \#)$ be a uniformly closed J-symmetric commutative algebra of operators on a Π_k-space H.*

(i) *If $\mathcal{B}$ has no neutral invariant subspaces, then it is C^*-equivalent.*

(ii) *If $\mathcal{B}$ has no non-trivial finite rank operators, it is C^*-equivalent.*

Proof. Assume that $k = k_-$ and let $\mathcal{B}$ have no neutral invariant subspaces. By Naimark's theorem [**28**], $\mathcal{B}$ has a k -dimensional non-positive invariant subspace L. The subspace $L \cap L^{[\perp]}$ is neutral and invariant for $\mathcal{B}$. Since $\mathcal{B}$ has no such subspaces, $L \cap L^{[\perp]} = \{0\}$, so that, by (1), $H = L[+]L^{[\perp]}$. From the discussion before Lemma 1 it follows that $\mathcal{B}$ is a *-algebra with respect to the scalar product $\langle , \rangle$ defined in (2) as well as a J-symmetric algebra and $B^{\#} = B^*$. Since the norm $\|\cdot\|_1 = \langle , \rangle^{1/2}$ on H is equivalent to the original norm, the new norm $|B| = \sup_{x \in H} \dfrac{\|Bx\|_1}{\|x\|_1}$ on $\mathcal{B}$ is equivalent to the original norm on $\mathcal{B}$. Hence $(\mathcal{B}, |\cdot|, \#)$ is a C^*-algebra, so that the algebra $(\mathcal{B}, \|\cdot\|, \#)$ is C^*-equivalent. Part (i) is proved.

Let $\mathcal{B}$ have no non-trivial finite rank operators and let it have neutral invariant subspaces. Let L be a maximal such subspace. Then decomposition (4) holds. Since $\mathcal{B}_\beta = \beta\mathcal{B}\beta$ is a non-degenerate commutative J-symmetric operator algebra on M, we obtain as in (i) that $M = N[+]P$ where N and P are uniformly definite subspaces invariant for $\mathcal{B}_\beta$. Therefore (see (2) and (3)) there are a scalar product $\langle , \rangle$ and an involution $\mathfrak{J}_2$ on M such that N and M are orthogonal, $\mathfrak{J}_2|_N = -\mathbf{1}_N$, $\mathfrak{J}_2|_P = \mathbf{1}_P$ and $[x, y] = \langle \mathfrak{J}_2 x, y\rangle$, $x, y \in M$.

Define a new scalar product on H :

$$\{x_1 + y_1 + z_1,\, x_2 + y_2 + z_2\} = (x_1, x_2) + \langle y_1, y_2\rangle + (z_1, z_2), \quad x_i \in L,\, y_i \in M,\, z_i \in K.$$

Then

$$\begin{aligned}[x_1 + y_1 + z_1,\, x_2 + y_2 + z_2] &= (J(x_1 + y_1 + z_1),\, x_2 + y_2 + z_2) \\ &= (J_3 x_1, z_2) + (J_2 y_1, y_2) + (J_1 z_1, x_2) \\ &= \{J_3 x_1, z_2\} + [y_1, y_2] + \{J_1 z_1, x_2\} \\ &= \{J_3 x_1, z_2\} + \langle \mathfrak{J}_2 y_1, y_2 \rangle + \{J_1 z_1, x_2\} \\ &= \{\mathfrak{J}(x_1 + y_1 + z_1), x_2 + y_2 + z_2\}\end{aligned}$$

where $\mathfrak{J} = \begin{pmatrix} 0 & 0 & J_1 \\ 0 & \mathfrak{J}_2 & 0 \\ J_3 & 0 & 0 \end{pmatrix}$. Therefore, for $B \in B(H)$,

$$[Bx, y] = \{\mathfrak{J}Bx, y\} = \{\mathfrak{J}x, \mathfrak{J}B^*\mathfrak{J}y\} = [x, B^\# y]\,, \text{ so that } B^\# = \mathfrak{J}B^*\mathfrak{J}$$

where B^* is the adjoint of B with respect to $\{,\}$. Since the norm $\langle,\rangle^{1/2}$ on M is equivalent to the original norm, the new norm $\{,\}^{1/2}$ on H is equivalent to the original norm and

$$H = L\{+\}N\{+\}P\{+\}M.$$

Since L and $L^{[\perp]}$ are invariant for $\mathcal{B}$, since N and P are invariant for $\mathcal{B}_\beta$ and $\mathfrak{J}_2|_N = -\mathbf{1}_N$ and $\mathfrak{J}_2|_P = \mathbf{1}_P$, we have that with respect to the above decomposition of H

$$B = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ 0 & B_{22} & 0 & B_{24} \\ 0 & 0 & B_{33} & B_{34} \\ 0 & 0 & 0 & B_{44} \end{pmatrix}, \quad \text{for every } B \in \mathcal{B}, \qquad \text{and } \mathfrak{J} = \begin{pmatrix} 0 & 0 & 0 & J_1 \\ 0 & -\mathbf{1}_N & 0 & 0 \\ 0 & 0 & \mathbf{1}_P & 0 \\ J_3 & 0 & 0 & 0 \end{pmatrix}.$$

If $B_{33} = 0$, B is a finite rank operator. Since $\mathcal{B}$ has no such operators, the mapping $B \to B_{33}$ is a $\#$-isomorphism of $\mathcal{B}$ onto $\mathcal{B}_\gamma = \{\gamma B \gamma = B_{33} : B \in \mathcal{B}\}$ where γ is the orthogonal projection onto P. Set

$$T(B) = (\mathbf{1}_H \ominus \gamma)\, B\, (\mathbf{1}_H \ominus \gamma)\,.$$

Then T is a linear mapping from $\mathcal{B}$ into a subspace of operators on a finite-dimensional space $L\{+\}N\{+\}M$.

Together with the usual operator norm $\|\cdot\|$ on $\mathcal{B}$ we consider the following norms on $\mathcal{B}$:

$$\|B\|_1 = \|B_{33}\| + \|B_{13}\| + \|B_{34}\|, \qquad \|B\|_2 = \|B_{33}\|$$

and

$$|B| = \|B_{33}\| + \|B_{13}\| + \|B_{34}\| + \|T(B)\| = \|B\|_1 + \|T(B)\|.$$

Then $\mathcal{B}$ is a normed space with respect to all of them and a Banach space with respect to $\|\cdot\|$. Since $\|B\| \leq |B| \leq 4\|B\|$, $(\mathcal{B}, |\cdot|)$ is a Banach space. Hence, by Lemma 1(iii), the norms $|\cdot|$ and $\|\cdot\|_1$ are equivalent, $\mathcal{B}$ is complete with respect to the norm $\|\cdot\|_1$ and T is continuous on $(\mathcal{B}, \|\cdot\|_1)$. Therefore $(\operatorname{Ker} T, \|\cdot\|_1)$ is complete and there is $C > 0$ such that

$$\|B_{14}\| \leq \|T(B)\| \leq C\|B\|_1.$$

Let $B \in \operatorname{Ker} T$ and let $A = B^{\#}B = (\mathfrak{J}B^*\mathfrak{J})B$. Then

$$B_{i1} = B_{i2} = A_{i1} = A_{i2} = 0, \quad 1 \leq i \leq 4,$$
$$B_{14} = B_{23} = B_{24} = B_{43} = B_{44} = A_{23} = A_{24} = A_{43} = A_{44} = 0,$$

and

$$A_{13} = J_1 B_{34}^* B_{33}, \quad A_{14} = J_1 B_{34}^* B_{34}, \quad A_{33} = B_{33}^* B_{33}, \quad A_{34} = B_{33}^* B_{34}.$$

Therefore

$$\begin{aligned} \|A_{14}\| &= \|J_1 B_{34}^* B_{34}\| = \|B_{34}\|^2 \\ &\leq C\|A\|_1 = C(\|A_{33}\| + \|A_{13}\| + \|A_{34}\|) \\ &= C(\|B_{33}^* B_{33}\| + \|J_1 B_{34}^* B_{33}\| + \|B_{33}^* B_{34}\|) \\ &\leq C\left(\|B_{33}\|^2 + 2\|B_{33}\|\,\|B_{34}\|\right). \end{aligned}$$

For $B \neq 0$, $B_{33} \neq 0$. Hence $t = \|B_{34}\|/\|B_{33}\|$ satisfies the inequality

$$t^2 - 2Ct - C \leq 0.$$

Therefore $t \leq D$, where $D = C + (C^2 + C)^{1/2}$, so that $\|B_{34}\| \leq D\|B_{33}\|$. In a similar way, considering $BB^{\#}$, we obtain that $\|B_{13}\| \leq D\|B_{33}\|$. Thus, for $B \in \operatorname{Ker} T$,

$$\begin{aligned} \|B\|_2 = \|B_{33}\| \leq \|B\|_1 &= \|B_{33}\| + \|B_{13}\| + \|B_{34}\| \\ &\leq (1 + 2D)\|B_{33}\| = (1 + 2D)\|B\|_2, \end{aligned}$$

so that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent on $\operatorname{Ker} T$. Since $\operatorname{Ker} T$ has a finite codimension in $\mathcal{B}$ and since $(\mathcal{B}, \|\cdot\|_1)$ and $(\operatorname{Ker} T, \|\cdot\|_1)$ are complete, it follows from Lemma 1(ii) that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent on $\mathcal{B}$, so that the norms $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent on $\mathcal{B}$ and $(\mathcal{B}, \|\cdot\|_2)$ is complete. For $B \in \mathcal{B}$,

$$\|B^\#\|_2 = \|B_{33}^*\| = \|B_{33}\| = \|B\|_2$$

and

$$\|B^\# B\|_2 = \|B_{33}^* B_{33}\| = \|B_{33}\|^2 = \|B\|_2^2.$$

Therefore $(\mathcal{B}, \|\cdot\|_2, \#)$ is a C^*-algebra, so that $(\mathcal{B}, \|\cdot\|, \#)$ is C^*-equivalent. □

Making use of Theorem 2 we shall now prove the following main theorem of this section.

Theorem 3. *An irreducible, uniformly closed, J-symmetric operator algebra on a Π_k-space H contains the algebra $C(H)$ of all compact operators.*

Proof. Let $\mathcal{B}$ be a uniformly closed J-symmetric algebra on a Π_k-space $H, k \neq 0$, and let it be irreducible, i.e., it has no closed invariant subspaces. Suppose that $\mathcal{B}$ has no non-trivial finite rank operators. Then any uniformly closed J-symmetric commutative subalgebra of $\mathcal{B}$ also has no non-trivial finite rank operators. By Theorem 2, all these subalgebras are C^*-equivalent. Cuntz [**6**] proved that a Banach *-algebra is C^*-equivalent if the closed commutative *-subalgebra generated by any selfadjoint element of the algebra is C^*-equivalent. Applying this result to the algebra $\mathcal{B}$, we obtain that it is C^*-equivalent. Then it follows from [**40**] (cf. [**18**]) that $H = N[+]P$, where N and P are respectively uniformly negative and uniformly positive subspaces invariant for $\mathcal{B}$ and $\dim N = k$. This contradiction shows that $\mathcal{B}$ must have non-trivial finite rank operators. Therefore it follows from Barnes' theorem [**3**] that $\mathcal{B}$ contains all finite rank operators. Since $\mathcal{B}$ is closed, $C(H) \subseteq \mathcal{B}$ and Theorem 3 is proved. □

3. Almost Hermitian J-symmetric representations on Π_k-spaces.

A representation π of a *-algebra $\mathcal{A}$ on a Banach space is *Hermitian* if, for $a = a^* \in \mathcal{A}, \operatorname{Sp} \pi(a) \subset \mathbb{R}$. It is *almost Hermitian* if, for $a = a^* \in \mathcal{A}$ and $\varepsilon > 0$, there is $b = b^* \in \mathcal{A}$ such that

$$(5) \qquad \|\pi(a) - \pi(b)\| < \varepsilon \quad \text{and} \quad \operatorname{Sp} \pi(b) \subset \mathbb{R}.$$

A *-algebra is *Hermitian* if all selfadjoint elements have real spectrum. We say that a Banach *-algebra is *almost Hermitian* if selfadjoint elements with real spectrum are dense in the real part of the algebra. Clearly, all *-representations of Hermitian algebras are Hermitian and all continuous *-representations of almost Hermitian algebras are almost Hermitian.

A dense *-subalgebra $\mathcal{A}$ of a unitial Banach *-algebra $\mathfrak{U}$ is a *Q-subalgebra* if $\mathbf{1} \in \mathcal{A}$ and $\mathrm{Sp}_{\mathcal{A}}\, a = \mathrm{Sp}_{\mathfrak{U}}\, a$, for all $a \in \mathcal{A}$. The following theorem describes the structure of almost Hermitian non-degenerate representations on Π_k-spaces.

Theorem 4. *Let $\mathcal{A}$ be a *-algebra with identity.*

(i) *$\mathcal{A}$ has no almost Hermitian, J-symmetric irreducible representations on Π_k-spaces, $k \neq 0$.*

(ii) *Any almost Hermitian, J-symmetric and non-degenerate representation π of $\mathcal{A}$ on a Π_k-space H has decomposition $H = N[+]P$, where N and P are respectively uniformly negative and uniformly positive invariant subspaces, and, therefore, π is Hermitian and similar to a *-representation . If, in addition, $\mathcal{A}$ is Q-subalgebra of a Banach *-algebra, then π is automatically bounded.*

Proof. Let π be an almost Hermitian, J-symmetric and irreducible representation of a *-algebra $\mathcal{A}$ on a Π_k-space $H = H_- \oplus H_+$, $k \neq 0$. Choose $e \in H_-, f \in H_+$ and $\|e\| = \|f\| = 1$. Let $T \in B(H)$ be such that

$$Tz = (z,e)f - (z,f)e, \quad z \in H.$$

If L is the subspace of H generated by e and f, then $T|_L = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $T|_{L^\perp} = 0$, L is invariant for the involution J and $J|_L = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Therefore T is a finite rank operator, $T^\# = T$ and $\mathrm{Sp}\, T = \{0, \pm i\}$.

The uniform closure $\mathcal{B}$ of the algebra $\pi(\mathcal{A})$ satisfies the conditions of Theorem 3. Hence $T \in \mathcal{B}$ and there are $c_n \in \mathcal{A}$ such that $\pi(c_n) \to T$. Then $\pi(c_n^*) = \pi(c_n)^\# \to T^\# = T$. Set $a_n = (c_n + c_n^*)/2$. Then $a_n^* = a_n$ and $\pi(a_n) \to T$. Since π is almost Hermitian, it follows from (5) that there are $b_n = b_n^* \in \mathcal{A}$ such that $\pi(b_n) \to T$ and $\mathrm{Sp}\,\pi(b_n) \subset \mathbb{R}$.

Let $\mathcal{B}$ be a Banach algebra, $x \in \mathcal{B}$ and $x_n \to x$. Newburgh **[22]** showed that if W is a non-empty open and closed subset of $\mathrm{Sp}(x)$ and V is a neighbourhood of 0 in $\mathbb{C}$, then there is a positive N such that $\mathrm{Sp}(x_n) \cap (W + V) \neq \varnothing$ for all $n \geq N$, where $W + V = \{y + z : \ y \in W, z \in V\}$. From this result it follows that there is m such that, for $n > m, \mathrm{Sp}\,\pi(b_n)$ contain λ_n for which $|i - \lambda_n| < 1$. Hence $\lambda_n \notin \mathbb{R}$. This contradiction proves part (i).

Let now π be a J-symmetric, non-degenerate representation on H. Ismagilov **[11]** obtained the following decomposition of H :

$$H = N[+]H^1[+]\cdots[+]H^m[+]P,$$

where all the summands are invariant for π. The subspaces N and P are respectively uniformly negative and uniformly positive. All the subspaces $H^j, 1 \leq j \leq m$, are Π_{k_j}-spaces, $k_j \neq 0$, and the representations π_{H^j} are J-symmetric and irreducible. If π is almost Hermitian, all π_{H^j} are also almost Hermitian. From part (i) it follows that $\pi_{H^j} = 0$. Thus $H = N[+]P$.

Since π_N and π_P are *-representations of $\mathcal{A}$ with respect to $(,)_N$ and $(,)_P$, for any $a = a^* \in \mathcal{A}$, $\pi_N(a)$ and $\pi_P(a)$ are selfadjoint. Hence $\mathrm{Sp}\,\pi_N(a) \subset \mathbb{R}$ and $\mathrm{Sp}\,\pi_P(a) \subset \mathbb{R}$. Thus $\mathrm{Sp}\ \pi(a) = \mathrm{Sp}\,\pi_N(a) \cup \mathrm{Sp}\,\pi_P(a) \subset \mathbb{R}$ and π is Hermitian.

From the discussion at the beginning of Section 2 it follows that π is a *-representation of $\mathcal{A}$ with respect to the scalar product $\langle , \rangle$ on H defined in (2). Since the norm $\|\cdot\|_1 = \langle , \rangle^{1/2}$ is equivalent to the original norm $\|\cdot\|$ on H, the identity operator from $(H, \|\cdot\|)$ onto $(H, \|\cdot\|_1)$ is bounded and, therefore the representation π on $(H, \|\cdot\|)$ is similar to the *-representation π on $(H, \|\cdot\|_1)$. We also have that

$$\|\pi(a)\|_1^2 = \|\pi(a^*a)\|_1 = r(\pi(a^*a)) \leq r_{\mathcal{A}}(a^*a),$$

where r and $r_{\mathcal{A}}$ are the spectral radii on $B(H)$ and $\mathcal{A}$ respectively. Let $\mathcal{A}$ be a Q-subalgebra of a Banach *-algebra $\mathfrak{U}$. Then $r_{\mathcal{A}}(a^*a) = r_{\mathfrak{U}}(a^*a) \leq \|a\|^2$. Therefore π is bounded on $(H, \|\cdot\|_1)$, so that π is also bounded on $(H, \|\cdot\|)$. □

The following example shows that the condition in Theorem 4 that the representation π is almost Hermitian is absolutely essential.

Example. Let $\mathcal{A}$ be the algebra of all complex 2×2 matrices, H be the 2-dimensional Hilbert space and $J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ be an involution on H. Set

$$[x, y] = (Jx, y), \quad x, y \in H, \qquad \text{and} \qquad A^{\#} = JA^*J, \quad A \in \mathcal{A},$$

where A^* is the adjoint of A. Then H is a Π_1-space, $\#$ is an involution on $\mathcal{A}$ and the identity representation π of the *-algebra $(\mathcal{A}, \#)$ is J-symmetric and irreducible. For $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $A^{\#} = A$, so that A is selfadjoint with respect to $\#$ and $\mathrm{Sp}\ (A) = \{-i, i\}$. Thus π is not Hermitian and, since $\mathcal{A}$ is finite-dimensional, it is not almost Hermitian.

4. J-unitary representations of groups with almost Hermitian group algebras.

In this section we make use of Theorem 4 to show that all bounded continuous J-unitary representations on Π_k-spaces of locally compact groups with almost Hermitian group algebras are similar to unitary representations .

A representation $T : g \in G \to T(g) \in B(\mathfrak{X})$ of a group G on Banach space $\mathfrak{X}$ is bounded if there exists $C > 0$ such that $\|T(g)\| \leq C$ for $g \in C$.

The following lemma is a simple corollary of Johnson's theorem ([**14**], Theorem 3.4).

Lemma 5. *Let $T(g)$ be a bounded representation of a group G on a reflexive Banach space $\mathfrak{X}$ and let L be a closed subspace in $\mathfrak{X}$ invariant for T. If $\dim L < \infty$ or $\operatorname{codim} L < \infty$, then L has an invariant complement.*

Proof. Let M be any subspace of $\mathfrak{X}$ which complements L, so that $\mathfrak{X} = L \dotplus M$. Since $\mathfrak{X}$ is reflexive and $\dim L < \infty$ or $\operatorname{codim} L < \infty$, the Banach algebra $B(M, L)$ of all bounded operators from M to L is reflexive. Let Q be the projection onto M along L. With respect to the decomposition $\mathfrak{X} = L + M$ we write

$$T(g) = \begin{pmatrix} \lambda(g) & \xi(g) \\ 0 & \mu(g) \end{pmatrix}, \qquad g \in G,$$

where λ is the restriction of T to L, $\mu(g) = QT(g)Q$ is a representation of G on M and $\xi(g) \in B(M, L)$. Define a representation ρ of G on $B(M, L)$ by the formula:

$$\rho(g)Z = \lambda(g)Z\mu(g^{-1}), \qquad Z \in B(M, L).$$

Since T is a representation of G,

$$\xi(gh) = \lambda(g)\xi(h) + \xi(g)\mu(h), \qquad g, h \in G.$$

Set $\eta(g) = \xi(g)\mu(g^{-1})$. Then $\eta(g)$ is a bounded mapping from G into $B(M, L)$ and

$$\begin{aligned} \eta(gh) = \xi(gh)\mu\left((gh)^{-1}\right) &= [\lambda(g)\xi(h) + \xi(g)\mu(h)]\mu(h^{-1})\mu(g^{-1}) \\ &= \rho(g)\eta(h) + \eta(g). \end{aligned}$$

Thus $\eta(g)$ is a bounded ρ-cocycle. It follows from Johnson's theorem [**14**] that $\eta(g)$ is inner, i.e., there is $Z_0 \in B(M, L)$ such that $\eta(g) = \rho(g)Z_0 - Z_0$. Hence

$$\xi(g) = \lambda(g)Z_0 - Z_0\mu(g), \qquad g \in G,$$

so that the subspace $F = \{-Z_0x + x : x \in M\}$ is an invariant complement of L. □

A bounded operator U on a Krein space H is J-unitary if U has a bounded inverse and $[Ux, Uy] = [x, y]$, $x, y \in H$.

Lemma 6. *Let $T(g)$ be a bounded J-unitary representation of a group G on a Π_k-space H and let L be a maximal invariant neutral subspace in H. There are invariant subspaces R and $\mathcal{M}$ in H such that*

(i) $H = (L + R)[+]\mathcal{M}$ *is the direct sum of* L, R *and* $\mathcal{M}$, *and* $\dim L = \dim R \leq k$;

(ii) *the restriction of* $T(g)$ *to the subspace* $\mathcal{L} = L + R$ *is similar to a unitary representation of* G;

(iii) *the restriction of* $T(g)$ *to* $\mathcal{M}$ *is non-degenerate.*

Proof. As in (4), $H = L^{[\perp]} \oplus K$, where $\dim L = \dim K \leq k$. Since T is J-unitary, $L^{[\perp]}$ is invariant for T and codim $L^{[\perp]} = \dim K < \infty$. By Lemma 5, $L^{[\perp]}$ has an invariant complement R in H. Hence $\dim R = \dim K = \dim L$. The subspace $\mathcal{L} = L + R$ is invariant for T and $\mathcal{L}^{[\perp]} = L^{[\perp]} \cap R^{[\perp]}$.

Let $z \in \mathcal{L} \cap \mathcal{L}^{[\perp]}$. Then $z = x + y$ where $x \in L$ and $y \in R$. Since $z \in L^{[\perp]}$ and $L \subseteq L^{[\perp]}$, we have that $y = z - x \in L^{[\perp]} \cap R = \{0\}$. Thus $y = z - x = 0$, so that $z = x \in L \cap R^{[\perp]}$. Since R is a complement of $L^{[\perp]}$,

$$\{0\} = H^{[\perp]} = \left(L^{[\perp]} + R\right)^{[\perp]} = \left(L^{[\perp]}\right)^{[\perp]} \cap R^{[\perp]} = L \cap R^{[\perp]}.$$

Thus $z = 0$. Hence $\mathcal{L} \cap \mathcal{L}^{[\perp]} = \{0\}$ and it follows from (1) that $H = \mathcal{L}[+]\mathcal{L}^{[\perp]}$. Since T is J-unitary, $\mathcal{L}^{[\perp]}$ is invariant for T. Suppose that $\mathcal{L}^{[\perp]}$ has a neutral subspace N invariant for T. Then $L + N$ is a neutral invariant subspace larger than L. This contradiction shows that $\mathcal{M} = \mathcal{L}^{[\perp]}$ has no neutral invariant subspaces, so that the restriction of T to $\mathcal{M}$ is non-degenerate.

Let $V(g) = T(g)|_{\mathcal{L}}$ be the restriction of T to $\mathcal{L}$. Then V is a bounded finite-dimensional representation of G. It is well-known that V is similar to a unitary representation. □

Theorem 7. *Let* G *be a locally compact group with almost Hermitian group algebra* $L^1(G)$. *Then every bounded continuous J-unitary representation of* G *on a* Π_k*-space is similar to a unitary representation of* G.

Proof. By Lemma 6, $H = \mathcal{L}[+]\mathcal{M}$ where $\mathcal{L}$ and $\mathcal{M}$ are invariant for T, the representation $V(g) = T(g)|_{\mathcal{L}}$ is similar to a unitary representation and the representation $U(g) = T(g)|_{\mathcal{M}}$ is non-degenerate. Then (see **[23]**, **[34-36]**) $\mathcal{M}$ is a Π_n-space, $n \leq k$, and U is a J-unitary non-degenerate representation of G on $\mathcal{M}$. The representation U of G extends to a J-symmetric non-degenerate representation π of the group algebra $L^1(G)$ on $\mathcal{M}$. Since $L^1(G)$ is almost Hermitian, π is almost Hermitian and, by Theorem 4(ii), it is similar to a *-representation . Therefore U is similar to a unitary representation , so that T is also similar to a unitary representation. □

5. Closed derivations on C^*-algebras implemented by symmetric operators.

At the beginning of this section we consider briefly the link between *-derivations of C^*-algebras implemented by symmetric operators and J-symmetric representations of *-algebras on Krein spaces.

Let $\mathfrak{U}$ be a C^*-algebra of operators on a Hilbert space $\mathfrak{H}$ and let δ be a closed *-derivation from $\mathfrak{U}$ into $B(\mathfrak{H})$, i.e., δ is a closed mapping from a dense *-subalgebra $D(\delta)$ of $\mathfrak{U}$ into $B(\mathfrak{H})$ such that

$$\delta(AB) = \delta(A)B + A\delta(B) \quad \text{and} \quad \delta(A^*) = \delta(A)^*, \qquad A, B \in D(\delta).$$

Then $D(\delta)$ is a Banach *-algebra with respect to the norm $\|A\|_\delta = \|A\| + \|\delta(A)\|$.

An operator S on $\mathfrak{H}$ *implements* δ if $D(S)$ is dense in $\mathfrak{H}$ and

$$AD(S) \subseteq D(S) \quad \text{and} \quad \delta(A)|_{D(S)} = i(SA - AS)|_{D(S)}.$$

If T extends S and implements δ, then T is a *δ-extension of S*. If S is symmetric and has no symmetric δ-extensions, it is a *maximal symmetric implementation of δ.*

Let S be a symmetric operator, S^* be its adjoint, $N_\pm(S)$ be the deficiency spaces and $n_\pm(S) = \dim N_\pm(S)$ be the deficiency indices of S. Then $D(S^*)$ is a Hilbert space with respect to the scalar product

$$\langle x, y\rangle = (x, y) + (S^*x, S^*y), \qquad x, y \in D(S^*),$$

and it is the orthogonal sum of the subspaces $D(S), N_-(S)$ and $N_+(S)$:

$$D(S^*) = D(S) <+> N(S), \quad \text{where } N(S) = N_-(S) <+> N_+(S).$$

Let Q and Q_+ be respectively the projections on $N(S)$ and $N_+(S)$ in $D(S^*)$. Then $J = 2Q_+ - Q$ is an involution on $N(S)$ and $N(S)$ is a Krein space with respect to the indefinite form

$$[x, y] = \langle Jx, y\rangle, \qquad x, y \in N(S).$$

It decomposes into a J-orthogonal and orthogonal sum of uniformly positive and uniformly negative subspaces $N_+(S)$ and $N_-(S)$. If $k = \min(n_\pm(S)) < \infty$, then $N(S)$ is a Π_k-space.

If a symmetric operator S implements a *-derivation δ, then $D(S)$ is invariant for all operators $A \in D(\delta)$ and $D(S^*)$ is also invariant for all $A \in D(\delta)$. We define a representation π_S^δ of $D(\delta)$ on $N(S)$ by the formula:

$$\pi_S^\delta(A) = QAQ, \quad A \in D(\delta), \quad \text{i.e., } \pi_S^\delta(A)x = QAx, \quad x \in N(S).$$

Theorem 8 ([**17**]). *The representation π_S^δ of the Banach *-algebra $D(\delta)$ on $N(S)$ is J-symmetric and bounded:$\|\pi_S^\delta(A)\| \leq \|A\|_\delta$. There is a one-to-one correspondence between closed symmetric δ-extensions of S and neutral subspaces in $N(S)$ invariant for π_S^δ. There is a maximal symmetric δ-extension T of S and the representation π_T^δ is non-degenerate.*

An operator R is *dissipative* if $(Rx, x) + (x, Rx) \leq 0$, $x \in D(R)$, and *maximal dissipative* if, in addition, it is not a proper restriction of any other dissipative operator. A closed operator generates a strongly continuous semigroup of contractions if and only if it is a maximal dissipative [**34**].

Making use of Theorem 4, we shall now prove the following theorem.

Theorem 9. *Let $\mathfrak{U}$ be a unital C^*-algebra of operators on a Hilbert space $\mathfrak{H}$ and δ be a closed *-derivation from $\mathfrak{U}$ into $B(\mathfrak{H})$. Let S be a symmetric implementation of δ and $n = \min(n_\pm(S)) < \infty$.*

(i) *If $n \neq 0$, $\mathfrak{U}$ has irreducible *-representations $\{\pi_i\}_{i=1}^m$ such that $n = \sum_{i=1}^m \dim \pi_i$. If $\mathfrak{U}$ has no finite-dimensional representations, then $n = 0$, i.e., S is a maximal symmetric operator.*

(ii) *For all maximal symmetric δ-extensions T of S, the representations π_T^δ of $D(\delta)$ on $N(T)$ are J-equivalent, similar to *-representations of $D(\delta)$ and extend to bounded *-representations of $\mathfrak{U}$ on $N(T)$. For every T, there are maximal dissipative operators R and $W, R^* = W$, such that the operators iR and $-iW$ extend T and implement δ.*

Proof. By Theorem 8, π_S^δ is a J-symmetric representation of $D(\delta)$ on the Π_n-space $N(S)$. Let L be a maximal neutral subspace in $N(S)$ invariant for π_S^δ and $L^{[\perp]}$ be the J-orthogonal complement of L in $N(S)$. By Law of inertia [**23**], $\dim L \leq n$. By Lemma 2.5 [**21**], $\widehat{L} = L^{[\perp]}/L$ is a Π_k-space, $k = n - \dim L$, and the quotient representation $\widehat{\pi}_S^\delta$ of $D(\delta)$ on $\widehat{L}$ is J-symmetric and non-degenerate. It was proven in [**22**] that $D(\delta)$ is a Q-subalgebra of $\mathfrak{U}$. Therefore $D(\delta)$ is a Hermitian algebra and it follows from Theorem 4 that $\widehat{\pi}_S^\delta$ is bounded with respect to the norm on $\mathfrak{U}$ and that $\widehat{L} = N[+]P$, where N and P are respectively uniformly negative and uniformly positive subspaces invariant for $\widehat{\pi}_S^\delta$. The representations $(\widehat{\pi}_S^\delta)_N$ and $(\widehat{\pi}_S^\delta)_P$ of $D(\delta)$ on N and P are *representations and extend to *-representations of $\mathfrak{U}$. By Law of inertia, $k = \min(\dim N, \dim P)$. Hence there are irreducible *-representations $\{\pi_i\}_{i=1}^p$ of $\mathfrak{U}$ such that

$$k = \sum_{i=1}^p \dim \pi_i.$$

Let $(\pi_S^\delta)_L$ be the restriction of π_S^δ to L. In [**5**] (cf. [**22**]) it was shown that $\mathbf{1} \in D(\delta)$. Hence there is a nest $\{0\} = L_0 \subset L_1 \subset \cdots \subset L_q = L$

of subspaces invariant for $(\pi_S^\delta)_L$ such that the quotient representations of $D(\delta)$ on L_j/L_{j-1}, $1 \le j \le q$, are irreducible and non-trivial. It follows from Theorem 6 [**22**] that these representations extend to bounded representations of $\mathfrak{U}$ similar to *-representations τ_j of $\mathfrak{U}$. Hence

$$\dim L = \sum_{j=1}^{q} \dim(L_j/L_{j-1}) = \sum_{j=1}^{q} \dim \tau_j.$$

Setting $\tau_j = \pi_{p+j}$ and $m = p + q$, we obtain that

$$n = k + \dim L = \sum_{i=1}^{m} \pi_i.$$

Part (i) is proved.

Any derivation implemented by a symmetric operator S has infinitely many maximal symmetric implementations T which extend S. In general, the corresponding representations π_T^δ are not J-equivalent. However, it was shown in [**21**] that if $\min(n_\pm(S)) < \infty$, then all the representations π_T^δ are J-equivalent.

We have that $k = \min(n_\pm(T)) \le \min(n_\pm(S)) < \infty$. By Theorem 8, π_T^δ are J-symmetric, non-degenerate representations of $D(\delta)$ on the Π_k-spaces $N(T)$. Since $D(\delta)$ is a Q-subalgebra of $\mathfrak{U}$, it is a Hermitian algebra. From Theorem 4 it follows that π_T^δ are similar to *-representations and bounded with respect to the norm on $\mathfrak{U}$. Therefore they extend to bounded *-representations of $\mathfrak{U}$ on $N(T)$; this is exactly the sufficient condition of Theorem 3.2 [**18**] for the maximal dissipative operators R and W to exist. □

The Weyl canonical commutation relation [**43**] for unitary one-parameter groups $\{U(t) : t \in \mathbb{R}\}$ and $\{V(s) : s \in \mathbb{R}\}$ on $\mathfrak{H}$ is the operator identity:

$$U(t)V(s) = e^{ist}V(s)U(t), \qquad t, s \in \mathbb{R}.$$

Jorgensen and Muhly [**15**] considered the infinitesimal Weyl relation in the *strong sense* for $U(t)$ and for a densely defined symmetric operator S :

$$U(t)D(S) \subseteq D(S) \quad \text{and} \quad (SU(t) - U(t)S)|_{D(S)} = tU(t)|_{D(S)}, \quad t \in \mathbb{R}.$$

If S is selfadjoint, this is equivalent to the Weyl relation for $U(t)$ and for the unitary one-parameter group $V(s) = e^{-isS}$. However, if S is not selfjoint, it can easily fail to have any selfadjoint extensions satisfying the Weyl relation with respect to $U(t)$ even when S has equal deficiency indices ([**15**, Th. 2]). Using Phillips' rtesult [**36**], Jorgensen and Muhly showed that if $\min(n_\pm(S)) < \infty$, then S has a maximal dissipative extension R

which generates a strongly continuous semigroup of contractions $\{R(t)\}_{t\geq 0}$ on $\mathfrak{H}$ such that

$$U(t)R(s) = e^{ist}R(s)U(t), \quad \text{for all } -\infty < t < \infty \text{ and } 0 \leq s < \infty.$$

Let $\mathcal{A}_U$ be the commutative C^*-algebra generated by the group $U(t)$. The expression $\sigma(U(t)) = itU(t)$ defines an unbounded *-derivation σ on $\mathcal{A}_U$ and the operator S implements σ.

Suppose now that δ is a *-derivation of an arbitrary C^*-algebra of operators $\mathfrak{U}$ on $\mathfrak{H}$. Set

$$G = \{U \in D(\delta) : U \text{ is invertible in } \mathfrak{U} \text{ and } \delta(U) = \lambda(U)U, \ \lambda(U) \in \mathbb{C}\}.$$

If $U \in G$, it follows from Theorem 5 [**22**] that $U^{-1} \in D(\delta)$. Then

$$\delta(U^{-1}) = -U^{-1}\delta(U)U^{-1} = -\lambda(U)U^{-1},$$

so that $U^{-1} \in G$. For $U, V \in G$,

$$\delta(UV) = U\delta(V) + \delta(U)V = (\lambda(U) + \lambda(V))UV,$$

so that G is a group. One can easily show that G is a normed group with respect to $\|\cdot\|_\delta$ and that λ is a continuous character on G.

If a symmetric operator S implements δ and $\min(n_\pm(S)) < \infty$, it follows from Theorem 9(ii) that there is a maximal dissipative operator R such that iR extends S and implements δ. Let $\{R(t)\}_{t\geq 0}$ be the strongly continuous semigroup on $\mathfrak{H}$ generated by R. Then Theorem 4.2 [**18**] holds and we obtain the following generalization of the result of Jorgensen and Muhly.

Corollary 10. *Let an element $U \in D(\delta)$ be invertible in $\mathfrak{U}$. The operator $R + \delta(U)U^{-1}$ generates a one-parameter semigroup $T(t)$ of operators and $UR(t) = T(t)U$, $t \geq 0$. If $\delta(U)U^{-1}$ commutes with R, then*

$$UR(t) = R(t)e^{t\delta(U)U^{-1}}U, \qquad t \geq 0.$$

In particular, $UR(t) = e^{t\lambda(U)}R(t)U$, for $U \in G$ and $t \geq 0$.

References

[1] H. Araki, *Indecomposable Reprentations with Invariant Inner Product*, Commun. Math. Phys., **97** (1985), 149-159.

[2] W. Arveson, *Continuous analogues of Fock space*, Mem. of AMS, **80(409)** (1989).

[3] B.A. Barnes, *Density theorem for algebras of operators and annihilator Banach algebras*, Michigan Math. J., **19** (1972), 149-155.

[4] K. Bleuler, *Eine neue Methode zur Behandlung der longitudinalen und skalaren Photonen*, Helv. Phys. Acta, **23** (1950), 567-586.

[5] O. Bratteli and D.W. Robinson, *Unbounded derivations of C^*-algebras, I*, Comm. Math. Phys., **42** (1975), 253-268.

[6] J. Cuntz, *Locally C^*-equivalent algebras*, J. Funct. Anal., **23** (1976), 95-106.

[7] M Flato and C. Fronsdal, *Quantum field theory of singletons*, J. Math. Phys., **22** (1981), 1100-1105.

[8] I.M. Gelfand, M.I. Graev and N.Ya. Vilenkin, *Integral geometry and related problems of representation theory*, FizMatgiz, Moskow, 1962.

[9] I.M. Gelfand and A.M. Yaglom, *General relativistic invariant equations and infinite dimensional representations of the Lorentz group*, J. Exp. Th. Ph., **18** (1948), 703-733.

[10] S.N. Gupta, *Theory of longitudinal photons in quantum electrodynamics*, Proc. Phys. Soc., **63** (1950), 681-691.

[11] R.S. Ismagilov, *Rings of operators in a space with an indefinite metric*, Dokl. Acad. Nauk SSSR, **2** (1966)= Soviet Math. Dokl., **7(6)** (1966), 1460-1462.

[12] ———, *Unitary representations of the Lorentz group in spaces with indefinite metric*, Izv. Akad. Nauk SSSR, **3** (1966), 497-522.

[13] ———, *On the problem of extension of representations*, Matem. Zametki, **35(1)** (1984), 99-105.

[14] B.E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc., **127** (1972).

[15] P.E.T. Jorgensen and P.S. Muhly, *Self adjoint extensions satisfying the Weyl operator commutation relations*, J. Analyse Math., **37** (1980), 46-99.

[16] P.E.T. Jorgensen and G.L. Price, *Index theory and second quantization of boundary value problems*, J. Funct. Anal., **104** (1992), 243-290.

[17] E.V. Kissin, *Symmetric operator extensions of unbounded derivations of C^*-algebras*, J. Funct. Anal., **81** (1988), 38-53.

[18] ———, *Dissipative implementations of *-derivations of C^*-algebras and representations in indefinite metric spaces*, J. London Math. Soc., **43** (1991), 451-464.

[19] ———, *Indices of unbounded derivations of C^*-algebras*, Pac. J. Math., **152** (1992), 125-150.

[20] ———, *Representational indices of derivations of C^*-algebras and representations of *-algebras on Krein spaces*, J. reine angew. Math., **439** (1993), 71-92.

[21] ———, *On the uniqueness of representational indices of derivations of C^*-algebras*, Pac. J. Math., **162** (1994), 97-120.

[22] E.V. Kissin and V.S. Shulman, *Dense Q-subalgebras of Banach and C^*-algebras and unbounded derivations of Banach and C^*-algebras*, Proceedings of the Edinburgh Math. Soc., **36** (1993), 261-276.

[23] M.G. Krein, *Introduction to the geometry of indefinite J-spaces and to the theory of operators in those spaces*, Amer. Math. Soc. Transl., **93** (1970), 103-176.

[24] A.I. Loginov and V.S. Shulman, *Irreducible J-symmetric algebras of operators in spaces with an indefinite metric*, Dokl. Akad. Nauk SSSR, **240** **(1)** (1978)= Soviet Math. Dokl., **19(3)** (1978), 541-544.

[25] ———, *Invariant subspaces of operator algebras*, Mat. Anal. (Itogi Nauki. VINITI

Akad. Nauk SSSR), **26**, Moscow, 1988.

[26] G. Morchio, D. Pierotti and F. Strocchi, *Infrared and vacuum structure in two-dimensional quantum field theory models. The massless scalar field*, J. Math. Phys., **31** (1990), 1467-1477.

[27] M.A. Naimark, *Linear representations of the Lorentz group*, Fizmatgiz, 1958.

[28] ———, *On commuting unitary operators in spaces with indefinite metric*, Acta Sci. Math., **24** (1963), 177-189.

[29] M.A. Naimark and R.S. Ismagilov, *Representations of groups and algebras in spaces with indefinite metric*, Mat. Anal. 1968. (Itogi Nauki. VINITI Akad. Nauk SSSR), Moscow, (1969), 73-105.

[30] M.A. Naimark, A.I. Loginov and V.S. Shulman, *Non-selfadjoint operator algebras in Hilbert spaces*, Mat. Anal. (Itogi Nauki. VINITI Akad. Nauk SSSR), **12**, 413-465, Moscow, 1974 = J. Soviet Math., **5(2)** (1976), 250-278.

[31] H. Nakazato, *Indefinite inner product spaces and derivations*, Math. Japonica, **35** (1990), 1119-1124.

[32] J.D. Newburgh, *The variation of spectra*, Duke Math. J., **18** (1951), 165-176.

[33] S. Ota, *Certain operator algebras induced by *-derivations in C^*-algebras on an indefinite inner product space*, J. Funct. Anal., **30** (1978), 238-244.

[34] R.S. Phillips, *Dissipative operators and hyperbolic systems of partial differential equations*, Trans. Amer. Math. Soc., **90** (1959), 193-254.

[35] ———, *The extension of dual subspaces invariant under an algebra*, Proc. of the International symposium on Linear Spaces (Jerusalem 1960) (Academic Press, Jerusalem, 1961), 366-398.

[36] ———, *On dissipative operators*, Lecture series in differential equations, Vol II, Mathematics Studies 19 (ed. A.K. Aziz; von Nostrand, New York, 1969).

[37] R.T. Powers, *An index theory for semigroups of *-endomorphisms of $B(H)$ and type* II_1 *factors*, Canad. J. Math., **XL** (1988), 86-114.

[38] R.T. Powers and D.W. Robinson, *An index for continuous semigroups of $B(H)$*, J. Funct. Anal., **84** (1989), 85-96.

[39] J. Rawnsley, W. Schmid and J.A. Wolf, *Singular unitary representations and indefinite harmonic theory*, J. Funct. Anal., **51** (1983), 1-114.

[40] V.S. Shulman, *On representations of C^*-algebras on indefinite metric spaces*, Mat. Zametki, **22** (1977), 583-592 = Math. Notes, **22** (1977).

[41] ———, *Symmetric Banach algebras of operators in a space of type* Π_1, Mat. Sbornik, **89(131)** (1972), No 2 = Math. USSR Sbornik, **18(2)** (1972), 267-283.

[42] ———, *On fixed points of linear-fractional transformations*, Funct. Analis i ego prilogenia, **14** (1980), 93-94.

[43] H. Weyl, *The theory of groups and quantum mechanics*, Dover, New York, 1950.

[44] D.P. Zhelobenko, *Description of a certain class of representations of the Lorentz group*, Dokl. Akad. Nauk SSSR, **4** (1958), 586-590.

Received November 9, 1993 and revised March 22, 1994.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 174 No. 2 June 1996

0030-8730(1996)174:2;1-C