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This paper will show that the twisted Alexander polynomial of a knot is the Reidemeister torsion of its knot exterior. As an application we obtain a proof that the twisted Alexander polynomial of a knot for an $SO(n)$-representation is symmetric.

Introduction.

In 1992, Wada [**4**] defined the twisted Alexander polynomial for finitely presentable groups. Let Γ be a finitely presentable group. We suppose that the abelianization $\Gamma/[\Gamma,\Gamma]$ is a free abelian group $T_r = \langle t_1,\ldots,t_r | t_i t_j = t_j t_i\rangle$ of rank r. Then we will assign a Laurent polynomial $\Delta_{\Gamma,\rho}(t_1,\ldots,t_r)$ with a unique factorization domain R-coefficients to each linear representation $\rho : \Gamma \to GL(n;R)$. We call it the twisted Alexander polynomial of Γ associated to ρ. For simplicity, we suppose that R is the real number field $\mathbf{R}$ and the image of ρ is included in $SL(n;\mathbf{R})$.

Because we are mainly interested in the case of the group of a knot, hereafter we suppose that Γ is a knot group. Let $K \subset S^3$ be a knot and E its exterior of K. We denote the canonical abelianization of Γ by

$$\alpha : \Gamma \to T = \langle t\rangle$$

and the twisted Alexander polynomial $\Delta_{\Gamma,\rho}(t)$ for $\Gamma = \pi_1 E$ by $\Delta_{K,\rho}(t)$. It is a generalization of the Alexander polynomial $\Delta_K(t)$ of K in the following sense. The Alexander polynomial $\Delta_K(t)$ of K is written as

$$\Delta_K(t) = (t-1)\Delta_{K,\mathbf{1}}(t)$$

where $\mathbf{1} : \Gamma \to \mathbf{R} - \{0\}$ is the 1-dimensional trivial representation of Γ.

On the other hand, Milnor [**2**] proved the following theorem about the connection between the Alexander polynomial and the Reidemeister torsion in 1962. We consider the abelianization

$$\alpha : \Gamma \to T$$

as a representation of Γ over $\mathbf{R}(t)$ where $\mathbf{R}(t)$ is the rational function field over $\mathbf{R}$. Then Milnor's theorem is the following.

Theorem (Milnor). *The Alexander polynomial $\Delta_K(t)$ of K is the Reidemeister torsion $\tau_\alpha(E)$ of E for α; that is,*

$$\Delta_K(t) = (t-1)\tau_\alpha E.$$

The Reidemeister torsion is a classical invariant for finite cell complexes using a representation of the fundamental group. In this paper we consider the following problem.

Problem. *Can we consider the twisted Alexander polynomial of K as a Reidemeister torsion of its exterior E of K.*

To state the main theorem, we define the tensor representation

$$\rho \otimes \alpha : \Gamma \to GL(n; \mathbf{R}(t))$$

by

$$(\rho \otimes \alpha)(x) = \rho(x)\alpha(x)$$

for $^\forall x \in \Gamma$. Then our main theorem is the following.

Theorem A. *The twisted Alexander polynomial $\Delta_{K,\rho}(t)$ associated to ρ is the Reidemeister torsion $\tau_{\rho\otimes\alpha}E$ for $\rho \otimes \alpha$; that is,*

$$\Delta_{K,\rho}(t) = \tau_{\rho\otimes\alpha}E.$$

As an application of this interpretation, we obtain the symmetry of the twisted Alexander polynomial in the following sense.

Theorem B. *If ρ is equivalent to an $SO(n)$-representation, then*

$$\Delta_{K,\rho}(t) = \Delta_{K,\rho}(t^{-1})$$

up to a factor ϵt^{mn} where $\epsilon \in \{\pm 1\}$ and $m \in \mathbf{Z}$.

Remark. If ρ is not equivalent to an $SO(n)$-representation, then it is an open problem to determine whether $\Delta_{K,\rho}(t)$ is always symmetric or not.

Now we describe the contents of this paper briefly. In Section 1 we review the theory of the twisted Alexander polynomial. We restrict the definition to the case of the group of a knot. In Section 2 we recall the necessary definition and results on the Reidemeister torsion for unimodular-representations. In

Section 3 we give a proof of Theorem A. In Section 4 as an application of Theorem A, we proof the symmetry of the twisted Alexander polynomial in our context.

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1. Twisted Alexander polynomial.

Let us describe the definition of the twisted Alexander polynomial of a knot. See Wada [**4**] for details.

Let $K \subset S^3$ be a knot and Γ the knot group $\pi_1 E$. Let $F_k = \langle x_1, \ldots, x_k \rangle$ denote a free group of rank k and $T = \langle t \rangle$ an infinite cyclic group. The group ring of T over $\mathbf{Z}$ (resp. $\mathbf{R}$) is the Laurent polynomial ring $\mathbf{Z}[t^{\pm 1}]$ (resp. $\mathbf{R}[t^{\pm 1}]$). We choose and fix a Wirtinger presentation

$$P(\Gamma) = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_{k-1} \rangle$$

of Γ and

$$\phi : F_k \to \Gamma$$

the associated surjective homomorphism of the free group F_k to the knot group Γ. This ϕ induces a ring homomorphism

$$\tilde{\phi} : \mathbf{Z}[F_k] \to \mathbf{Z}[\Gamma].$$

The canonical abelianization

$$\alpha : \Gamma \to H_1(E; \mathbf{Z}) \cong T$$

is given by

$$\alpha(x_1) = \cdots = \alpha(x_k) = t.$$

Similarly α induces a ring homomorphism of the integral group ring

$$\tilde{\alpha} : \mathbf{Z}[\Gamma] \to \mathbf{Z}[t^{\pm 1}].$$

Let

$$\rho : \Gamma \to SL(n; \mathbf{R})$$

be a representation. The corresponding ring homomorphism of the integral ring $\mathbf{Z}[\Gamma]$ to the matrix algebra $M_n(\mathbf{R})$ is denoted by

$$\tilde{\rho} : \mathbf{Z}[\Gamma] \to M_n(\mathbf{R}).$$

The composition of the ring homomorphism $\tilde{\phi}$ and the tensor product homomorphism

$$\tilde{\rho} \otimes \tilde{\alpha} : \mathbf{Z}[\Gamma] \to M_n(\mathbf{R}[t^{\pm 1}])$$

will be used so often that we introduce a new symbol

$$\Phi = (\tilde{\rho} \otimes \tilde{\alpha}) \circ \tilde{\phi} : \mathbf{Z}[F_k] \to M_n(\mathbf{R}[t^{\pm 1}]).$$

Let us consider the $(k-1) \times k$ matrix $A_{\rho\otimes\alpha}$ whose (i,j)-component is the $n \times n$ matrix $\Phi(\frac{\partial r_i}{\partial x_j}) \in M_n(\mathbf{R}[t^{\pm 1}])$. This matrix $A_{\rho\otimes\alpha}$ is called the generalized Alexander matrix of the presentation $P(\Gamma)$ associated to the representation ρ. By the definition, the classical Alexander matrix A is $A_{\mathbf{1}\otimes\alpha}$ where $\mathbf{1}$ is a 1-dimensional trivial representation of Γ. For $1 \leq {}^\forall j \leq k$, let us denote by $A^j_{\rho\otimes\alpha}$ the $(k-1) \times (k-1)$ matrix obtained from $A_{\rho\otimes\alpha}$ by removing the j-th column. Now regard $A^j_{\rho\otimes\alpha}$ as a $(k-1)n \times (k-1)n$ matrix with coefficients in $\mathbf{R}[t^{\pm 1}]$. The following two lemmas are the foundation of our definition of the twisted Alexander polynomial.

Lemma 1.1. $\det \Phi(x_j - 1) \neq 0$ *for* $1 \leq {}^\forall j \leq k$.

Proof. Since we fix a Wirtinger presentation $P(\Gamma)$ as a presentation of Γ, we have

$$\alpha(x_j) = t \neq 1$$

for $1 \leq {}^\forall j \leq k$. Then $\det \Phi(x_j - 1) = \det(t\rho(x_j) - I)$ is the characteristic polynomial of $\rho(x_j)$ where I is the unit matrix. This completes the proof of Lemma 1.1.

Lemma 1.2. $\det A^j_{\rho\otimes\alpha} \det \Phi(x_{j'} - 1) = \pm \det A^{j'}_{\rho\otimes\alpha} \det \Phi(x_j - 1)$ *for* $1 \leq {}^\forall j < {}^\forall j' \leq k$.

Proof. We may assume that $j = 1$ and $j' = 2$ without the loss of generality. Since any relator $r_i = 1$ in $\mathbf{Z}[\Gamma]$, it is easy to see that

$$\sum_{l=1}^{k} \frac{\partial r_i}{\partial x_l}(1 - x_l) = 0$$

in $\mathbf{Z}[\Gamma]$. Then apply the homomorphism Φ to this, we have

$$\sum_{l=1}^{k} \Phi\left(\frac{\partial r_i}{\partial x_l}\right) \Phi(x_l - 1) = 0.$$

Let $\tilde{A}^2_{\rho\otimes\alpha}$ be the matrix obtained from $A^2_{\rho\otimes\alpha}$ by replacing the first column

$${}^t\left(\Phi\left(\frac{\partial r_1}{\partial x_1}\right), \Phi\left(\frac{\partial r_2}{\partial x_1}\right), \ldots, \Phi\left(\frac{\partial r_{k-1}}{\partial x_1}\right)\right)$$

with

$$^t\left(\Phi\left(\frac{\partial r_1}{\partial x_1}\right)\Phi\left(x_1-1\right),\Phi\left(\frac{\partial r_2}{\partial x_1}\right)\Phi\left(x_1-1\right),\ldots,\Phi\left(\frac{\partial r_{k-1}}{\partial x_1}\right)\Phi\left(x_1-1\right)\right).$$

Then we have

$$\det \tilde{A}^2_{\rho\otimes\alpha}=\pm\det A^2_{\rho\otimes\alpha}\det\Phi(x_1-1).$$

Since

$$\begin{aligned}\Phi\left(\frac{\partial r_i}{\partial x_1}\right)\Phi\left(x_1-1\right)&=-\sum_{l=2}^{k}\Phi\left(\frac{\partial r_i}{\partial x_l}\right)\Phi\left(x_l-1\right)\\&=-\Phi\left(\frac{\partial r_i}{\partial x_2}\right)\Phi\left(x_2-1\right)-\sum_{l=3}^{k}\Phi\left(\frac{\partial r_i}{\partial x_l}\right)\Phi\left(x_l-1\right),\end{aligned}$$

we can reduce the matrix $\tilde{A}^2_{\rho\otimes\alpha}$ to $\tilde{A}^1_{\rho\otimes\alpha}$ where the matrix $\tilde{A}^1_{\rho\otimes\alpha}$ can be obtained by multiplying the first column of the matrix $A^1_{\rho\otimes\alpha}$ by $\Phi(x_2-1)$. Therefore we have

$$\begin{aligned}\det\tilde{A}^2_{\rho\otimes\alpha}&=\pm\det\tilde{A}^1_{\rho\otimes\alpha}\\&=\pm\det A^1_{\rho\otimes\alpha}\det\Phi(x_2-1).\end{aligned}$$

This completes the proof of this lemma.

By Lemma 1.1 and Lemma 1.2, we can define the twisted Alexander polynomial of K associated to the representation ρ to be the rational expression

$$\Delta_{K,\rho}(t)=\frac{\det A^1_{\rho\otimes\alpha}}{\det\Phi(x_1-1)}.$$

Theorem 1.3 (Wada). *The twisted Alexander polynomial $\Delta_{K,\rho}(t)$ is well-defined up to a factor ϵt^{mn} as an invariant of the oriented knot type of K where $\epsilon\in\{\pm1\}$, $m\in\mathbf{Z}$ and n is a degree of ρ.*

Remark. Two representations ρ and ρ' are said to be equivalent if there is an element $g\in GL(n;\mathbf{R})$ such that $\rho'(x)=g\cdot\rho(x)\cdot g^{-1}$ in $SL(n;\mathbf{R})$ for $^\forall x\in\Gamma$. Then the twisted Alexander polynomials for ρ and ρ' are the same ;

$$\Delta_{K,\rho}(t)=\Delta_{K,\rho'}(t)$$

up to a factor ϵt^{mn} where $\epsilon\in\{\pm1\}$ and $m\in\mathbf{Z}$.

2. Reidemeister torsion.

Let us describe the definition of the Reidemeister torsion over a field $\mathbf{F}$. See Johnson [**1**] and Milnor [**2**], [**3**], for details.

Let V denote an n-dimensional vector space over $\mathbf{F}$. Let $\mathbf{b}=(b_1,\ldots,b_n)$ and $\mathbf{c}=(c_1,\ldots,c_n)$ be two bases for V. Setting $c_i = \sum_{j=1}^{n} a_{ij}b_j$, we obtain a nonsingular matrix $A = (a_{ij})$ with entries in $\mathbf{F}$. Let $[\mathbf{b}/\mathbf{c}]$ denote the determinant of A.

Suppose

$$C_* : 0 \longrightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

is an acyclic chain complex of finite dimensional vector spaces over $\mathbf{F}$.

We assume that a preferred basis $\mathbf{c}_q$ for $C_q(C_*)$ is given for $^\forall q$. Choose any basis $\mathbf{b}_q$ for $B_q(C_*)$ and take a lift of it in $C_{q+1}(C_*)$, which we denote by $\tilde{\mathbf{b}}_q$.

Since

$$B_q(C_*) \to Z_q(C_*)$$

is an isomorphism, the basis $\mathbf{b}_q$ can serve as a basis for $Z_q(C_*)$. Similarly the sequence

$$0 \to Z_q(C_*) \to C_q(C_*) \to B_{q-1}(C_*) \to 0$$

is exact and the vectors $(\mathbf{b}_q, \tilde{\mathbf{b}}_{q-1})$ is a basis for $C_q(C_*)$. It is easily shown that $[\mathbf{b}_q, \tilde{\mathbf{b}}_{q-1}/\mathbf{c}_q]$ is independent of the choices of $\tilde{\mathbf{b}}_{q-1}$. Hence we simply denote it by $[\mathbf{b}_q, \mathbf{b}_{q-1}/\mathbf{c}_q]$.

Definition 2.1. The torsion of the chain complex C_* is given by the alternating product

$$\prod_{q=0}^{m} [\mathbf{b}_q, \mathbf{b}_{q-1}/\mathbf{c}_q]^{(-1)^{q+1}}$$

and we denote it by $\tau(C_*)$.

Remark. The torsion $\tau(C_*)$ depends only on the bases $\mathbf{c}_0,\ldots,\mathbf{c}_m$.

Now we apply this torsion invariant of chain complexes to the following geometric situations. Let X be a finite cell complex and $\tilde{X}$ a universal covering of X with the fundamental group $\pi_1 X$ acting on it from the right-side as deck transformations. Then the chain complex $C_*(\tilde{X};\mathbf{Z})$ has a structure of a chain complex of right free $\mathbf{Z}[\pi_1 X]$-modules. Let

$$\rho : \pi_1 X \to SL(n;\mathbf{F})$$

be a representation. We may consider V as a $\pi_1 X$-module by using this representation ρ and denote it by V_ρ. Define the chain complex $C_*(X;V_\rho)$ by $C_*(\tilde{X};\mathbf{Z}) \otimes_{\mathbf{Z}[\pi_1 X]} V_\rho$ and choose a preferred basis

$$\{\sigma_1 \otimes e_1, \sigma_1 \otimes e_2, \ldots, \sigma_1 \otimes e_n, \ldots, \sigma_{k_q} \otimes e_1, \ldots, \sigma_{k_q} \otimes e_n\}$$

of $C_q(X;V_\rho)$ where $\{e_1,e_2,\dots,e_n\}$ is a basis of V and $\sigma_1,\dots,\sigma_{k_q}$ are q-cells giving the preferred basis of $C_q(\tilde{X};\mathbf{Z})$.

Now we consider the following situation. That is $C_*(X;V_\rho)$ is acyclic, namely all homology groups vanish : $H_*(X;V_\rho)=0$. In this case we call ρ an acyclic representation.

Definition 2.2. Let $\rho:\pi_1 X\to SL(n;\mathbf{F})$ be an acyclic representation. Then the Reidemeister torsion of X with V_ρ-coefficients is defined by the torsion of the chain complex $C_*(X;V_\rho)$. We denote it by $\tau(X;V_\rho)$ or simply $\tau_\rho(X)$.

Remark.

1. It is well known that the Reidemeister torsion is invariant under subdivision of the cell decomposition up to a factor $\epsilon\in\{\pm1\}$. Hence the Reidemeister torsion is a piecewise linear invariant. See Milnor [**2**], [**3**].
2. In general let $\rho:\Gamma\to GL(n;\mathbf{F})$ be an acyclic representation. Then the Reidemeister torsion is well-defined up to a factor $d\in Im(\det\circ\rho)\subset\mathbf{F}-0$.

3. Proof of Theorem A.

In this section, let $\mathbf{F}$ be the rational function field $\mathbf{R}(t)$ and V the n-dimensional vector space over $\mathbf{R}(t)$. We recall a Wirtinger presentation $P(\Gamma)$ of the knot group Γ of K is given by as follows ;

$$P(\Gamma)=\langle x_1,x_2,\dots,x_k \mid r_1,r_2,\dots,r_{k-1}\rangle$$

where r_i is the crossing relation for each i.

Let W be a 2-dimensional complex constructed from one 0-cell p, k 1-cells $x_1,\dots,x_k$ and $(k-1)$ 2-cells $D_1,\dots,D_{k-1}$ with attaching maps given by $r_1,\dots,r_{k-1}$. It is well-known that the exterior E of K collapses to the 2-dimensional complex W. If an acyclic representation

$$\rho:\Gamma\to SL(n;\mathbf{R})$$

is fixed, we have the following by the simple homotopy invariance of the Reidemeister torsion ;

$$\tau(E;V_{\rho\otimes\alpha})=\tau(W;V_{\rho\otimes\alpha})$$

up to a factor ϵt^{mn} where $\epsilon\in\{\pm1\}$ and $m\in\mathbf{Z}$. In this case, we show that

$$\tau(W;V_{\rho\otimes\alpha})=\frac{\det A^1_{\rho\otimes\alpha}}{\det\Phi(x_1-1)}.$$

By easy computation, this chain complex $C_*(W; V_{\rho\otimes\alpha})$ is as follows;

$$0 \longrightarrow V_{\rho\otimes\alpha}^{k-1} \xrightarrow{\partial_2} V_{\rho\otimes\alpha}^{k} \xrightarrow{\partial_1} V_{\rho\otimes\alpha} \longrightarrow 0$$

where

$$\begin{aligned}\partial_2 &= {}^tA_{\rho\otimes\alpha} \\ &= \begin{pmatrix} \Phi(\frac{\partial r_1}{\partial x_1}) & \dots & \Phi(\frac{\partial r_{k-1}}{\partial x_1}) \\ \vdots & \ddots & \vdots \\ \Phi(\frac{\partial r_1}{\partial x_k}) & \dots & \Phi(\frac{\partial r_{k-1}}{\partial x_k}) \end{pmatrix},\end{aligned}$$

$$\partial_1 = \begin{pmatrix} \Phi(x_1 - 1) \ \Phi(x_2 - 1) \dots \ \Phi(x_k - 1) \end{pmatrix}.$$

Here we briefly denote by $V_{\rho\otimes\alpha}^l$ the l-times direct sum of $V_{\rho\otimes\alpha}$.

Proposition 3.1. *All homology groups vanish : $H_*(W; V_{\rho\otimes\alpha}) = 0$ if and only if* $\det A_{\rho\otimes\alpha}^1 \neq 0$. *In this case, we have*

$$\tau(W; V_{\rho\otimes\alpha}) = \frac{\det A_{\rho\otimes\alpha}^1}{\det \Phi(x_1 - 1)}.$$

Proof. It is obvious that $H_0(W; V_{\rho\otimes\alpha})$ is trivial because $\det\Phi(x_1 - 1) \neq 0$ and hence the boundary map ∂_1 is surjective. For a canonical basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of V, we choose lifts

$$\begin{aligned}\tilde{\mathbf{e}}_1 &= {}^t(\Phi(x_1 - 1)^{-1}\mathbf{e}_1, \mathbf{0}, \dots, \mathbf{0}), \\ &\vdots \\ \tilde{\mathbf{e}}_n &= {}^t(\Phi(x_1 - 1)^{-1}\mathbf{e}_n, \mathbf{0}, \dots, \mathbf{0})\end{aligned}$$

in V^n. Define the $kn \times kn$ matrix M whose first $(kn - n)$ columns are ${}^tA_{\rho\otimes\alpha}$ and last n columns are $\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_n$. The matrix M takes the form

$$M = \begin{pmatrix} * & \\ & \tilde{\mathbf{e}}_1 \dots \tilde{\mathbf{e}}_n \\ {}^tA_{\rho\otimes\alpha}^1 & \end{pmatrix}.$$

It is obvious that $\det M \neq 0$ if and only if $\det A_{\rho\otimes\alpha}^1 \neq 0$. If all homology groups vanish : $H_*(W; V_{\rho\otimes\alpha}) = 0$, then

$$\begin{aligned}\operatorname{rank} A_{\rho\otimes\alpha} &= \operatorname{rank} A_{\rho\otimes\alpha}^1 \\ &= kn - n.\end{aligned}$$

Hence we have

$$\det A^1_{\rho\otimes\alpha} \neq 0.$$

In this case the Reidemeister torsion is given by

$$\begin{aligned}\tau(W; V_{\rho\otimes\alpha}) =& \det M \\ =& \frac{\det A^1_{\rho\otimes\alpha}}{\det \Phi(x_1 - 1)}.\end{aligned}$$

It is clear that the contrary is also true. Namely if $\det A^1_{\rho\otimes\alpha} \neq 0$, then $H_*(W; V_{\rho\otimes\alpha})$ is trivial. This completes the proof.

By the above propositions, we have the proof of Theorem A.

4. Symmetry of the twisted Alexander polynomial.

Hereafter we suppose that ρ is conjugate to an $SO(n)$-representation of Γ. For simplicity, we may suppose that ρ is an $SO(n)$-representation. We fix a structure of the simplicial complex in the exterior E of K and assume that each simplex of E has a dual cell. For a q-simplex of E we can define not only the dual $(3 - q)$-cell in E, but also the dual $(2 - q)$-cell in the boundary ∂E. Taking the cells of both types, we obtain a dual complex E' with subcomplex $\partial E'$. We denote the universal covering complex of E by $\tilde{E}$ and the one of E' by $\tilde{E}'$. Let $\langle c', c\rangle$ denote the algebraic intersection number of $c' \in C_{3-q}(\tilde{E}', \partial\tilde{E}'; \mathbf{Z})$ and $c \in C_q(\tilde{E}; \mathbf{Z})$. Next lemma is well-known fact (see Milnor [**2**]).

Lemma 4.1. *The left* $\boldsymbol{Z}[\Gamma]$*-module* $C_{3-q}(\tilde{E}', \partial\tilde{E}'; \boldsymbol{Z})$ *is canonically isomorphic to the dual of* $C_q(\tilde{E}, \boldsymbol{Z})$ *and the dual pairing*

$$[\quad,\quad] : C_{3-q}(\tilde{E}', \partial\tilde{E}'; \boldsymbol{Z}) \times C_q(\tilde{E}; \boldsymbol{Z}) \to \boldsymbol{Z}[\Gamma]$$

is given by

$$[c', c] = \sum_{x\in\Gamma} \langle c', cx^{-1}\rangle x$$

for $^\forall c' \in C_{n-q}(\tilde{E}', \partial\tilde{E}'; \boldsymbol{Z})$ *and* $^\forall c \in C_q(\tilde{E}; \boldsymbol{Z})$.

Now let us apply this duality to the torsion invariant. Let $V^*_{\rho\otimes\alpha}$ denote the dual vector space of $V_{\rho\otimes\alpha}$. A structure of left $\mathbf{Z}[\Gamma]$-module in $V^*_{\rho\otimes\alpha}$ is given by

$$(x \cdot u^*)(v) = u^*({}^t(\rho\otimes\alpha)(x)^{-1} \cdot v)$$

for $^\forall x \in \Gamma, {}^\forall u^* \in V^*_{\rho\otimes\alpha}$, and $^\forall v \in V_{\rho\otimes\alpha}$. Then we denote this dual representation space by $V^*_{\rho\otimes\alpha}$ and define the dual pairing

$$C_{3-q}(E', \partial E'; V^*_{\rho\otimes\alpha}) \times C_q(E; V_{\rho\otimes\alpha}) \to \mathbf{R}$$

by

$$(c' \otimes u^*, c \otimes v) = u^*([c', c]v)$$

for $^\forall c' \otimes u^* \in C_{3-q}(E', \partial E'; V^*_{\rho\otimes\alpha})$ and $^\forall c \otimes v \in C_q(E; V_{\rho\otimes\alpha})$. Hence it is straightforward that $C_{3-q}(E', \partial E'; V^*_{\rho\otimes\alpha})$ is isomorphic to the dual of $C_q(E; V_{\rho\otimes\alpha})$.

Lemma 4.2. *Let C_* be an acyclic chain complex with preferred basis $\{c_i\}$ and C^* the dual complex with preferred basis $\{c_i^*\}$. Then we have*

$$\tau(C_*) = \tau(C^*)$$

up to a factor $\epsilon \in \{\pm 1\}$.

This lemma is also well-known. By this lemma and the invariance of the Reidemeister torsion for the subdivision of the cell complex, we have

$$\tau(E; V_{\rho\otimes\alpha}) = \tau(E, \partial E; V^*_{\rho\otimes\alpha}).$$

We define a representation

$$\bar{\alpha} : \Gamma \to T$$

by

$$\bar{\alpha}(x) = \alpha(x)^{-1}.$$

For the tensor representation $\rho \otimes \alpha$, because ρ is an $SO(n)$-representation, the dual representation

$$(\rho \otimes \alpha)^* : \Gamma \to GL(n; \mathbf{R}(t))$$

is given by

$$\begin{aligned}(\rho \otimes \alpha)^*(x) &= {}^t\rho(x)^{-1}\alpha(x)^{-1} \\ &= \rho(x)\bar{\alpha}(x) \\ &= (\rho \otimes \bar{\alpha})(x)\end{aligned}$$

for $^\forall x \in \Gamma$. Therefore the representation space $V^*_{\rho\otimes\alpha}$ is equivalent to $V_{\rho\otimes\bar{\alpha}}$. Hence from the above observation, we have

$$\tau(E; V_{\rho\otimes\alpha}) = \tau(E, \partial E; V_{\rho\otimes\bar{\alpha}}).$$

Similarly it is easy to show that

$$\tau(E; V_{\rho\otimes\bar{\alpha}}) = \tau(E, \partial E; V_{\rho\otimes\alpha}).$$

The following lemma is also well-known to the experts. See Milnor [**6**].

Lemma 4.3. *Let $0 \to C'_* \to C_* \to C''_* \to 0$ be an exact sequence of n-dimensional chain complexes with preferred bases $\{\boldsymbol{c}'_i\}, \{\boldsymbol{c}_i\}$, and $\{\boldsymbol{c}''_i\}$ such that $[\boldsymbol{c}'_i, \boldsymbol{c}''_i/\boldsymbol{c}_i] = 1$ for any i. Suppose any two of the complexes are acyclic. Then the third one is also acyclic and the Reidemeister torsion of the three complexes are all well-defined. Moreover the next formula holds.*

$$\tau(C_*) = \tau(C'_*)\tau(C''_*).$$

Apply the above lemma to the short exact sequence :

$$0 \to C_*(\partial E; V_{\rho\otimes\alpha}) \to C_*(E; V_{\rho\otimes\alpha}) \to C_*(E, \partial E; V_{\rho\otimes\alpha}) \to 0,$$

we have

$$\tau(E; V_{\rho\otimes\alpha}) = \tau(\partial E; V_{\rho\otimes\alpha})\tau(E, \partial E; V_{\rho\otimes\alpha}).$$

Then we compute the Reidemeister torsion of ∂E first.

Proposition 4.4. *Let $\rho : \pi_1(\partial E) \to SL(n; \boldsymbol{R})$ be a representation. Then the Reidemeister torsion is given by*

$$\tau(\partial E; V_{\rho\otimes\alpha}) = 1$$

up to a factor ϵt^{mn} where $\epsilon \in \{\pm 1\}$ and $m \in \boldsymbol{Z}$.

Proof. Let x, y be generators of $\pi_1(\partial E)$ such that $x = x_1$ in $\pi_1 E$ and y is the canonical longitude. We assume that a cell structure of ∂E are given by :

(0) one 0-cell b,
(1) two 1-cells x and y,
(2) one 2-cell w,

with the attaching map given by $\partial w = xyx^{-1}y^{-1}$. To compute the local homology of ∂E, we compute boundary operators of this chain complex.

$$0 \longrightarrow w \otimes V \xrightarrow{\partial_2} x \otimes V \oplus y \otimes V \xrightarrow{\partial_1} p \otimes V \longrightarrow 0$$

where

$$\partial_2 = \begin{pmatrix} -\Phi(y-1) & \Phi(x-1) \end{pmatrix},$$

$$\partial_1 = \begin{pmatrix} \Phi(x-1) \\ \Phi(y-1) \end{pmatrix}.$$

It is obvious that this chain complex is acyclic because $\det \Phi(x-1) \neq 0$. Then the Reidemeister torsion $\tau(\partial E; V_{\rho\otimes\alpha})$ is defined as a rational function over $\mathbf{R}$. By the definition of the Reidemeister torsion,

$$\tau(\partial E; V_{\rho\otimes\alpha}) = [\mathbf{b}_1/\mathbf{c}_2]^{-1}[\mathbf{b}_1, \mathbf{b}_0/\mathbf{c}_1][\mathbf{b}_0/\mathbf{c}_0]^{-1}.$$

By straightforward computation, we have

$$\tau(\partial E; V_{\rho\otimes\alpha}) = 1.$$

This completes the proof.

Hence combine the above lemmas,

$$\begin{aligned}\tau(E; V_{\rho\otimes\alpha}) &= \tau(E, \partial E; V_{\rho\otimes\alpha}) \\ &= \tau(E; V_{\rho\otimes\bar{\alpha}}).\end{aligned}$$

By the definition of the twisted Alexander polynomial and Theorem A, it is obvious that

$$\tau(E; V_{\rho\otimes\bar{\alpha}}) = \Delta_{K,\rho}(t^{-1}).$$

Therefore we have

$$\Delta_{K,\rho}(t) = \Delta_{K,\rho}(t^{-1}).$$

This completes the proof of Theorem B.

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