Pacific Journal of Mathematics

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Volume 174 No. 2 June 1996

PACIFIC JOURNAL OF MATHEMATICS
Vol. 174, No. 2, 1996

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This paper will show that the twisted Alexander polynomial of a knot is the Reidemeister torsion of its knot exterior. As an application we obtain a proof that the twisted Alexander polynomial of a knot for an $SO(n)$-representation is symmetric.

Introduction.

In 1992, Wada [4] defined the twisted Alexander polynomial for finitely presentable groups. Let Γ be a finitely presentable group. We suppose that the abelianization $\Gamma/[\Gamma,\Gamma]$ is a free abelian group $T_r = \langle t_1,\ldots,t_r | t_i t_j = t_j t_i\rangle$ of rank r. Then we will assign a Laurent polynomial $\Delta_{\Gamma,\rho}(t_1,\ldots,t_r)$ with a unique factorization domain R-coefficients to each linear representation $\rho : \Gamma \to GL(n;R)$. We call it the twisted Alexander polynomial of Γ associated to ρ. For simplicity, we suppose that R is the real number field $\mathbf{R}$ and the image of ρ is included in $SL(n;\mathbf{R})$.

Because we are mainly interested in the case of the group of a knot, hereafter we suppose that Γ is a knot group. Let $K \subset S^3$ be a knot and E its exterior of K. We denote the canonical abelianization of Γ by

$$\alpha : \Gamma \to T = \langle t\rangle$$

and the twisted Alexander polynomial $\Delta_{\Gamma,\rho}(t)$ for $\Gamma = \pi_1 E$ by $\Delta_{K,\rho}(t)$. It is a generalization of the Alexander polynomial $\Delta_K(t)$ of K in the following sense. The Alexander polynomial $\Delta_K(t)$ of K is written as

$$\Delta_K(t) = (t-1)\Delta_{K,\mathbf{1}}(t)$$

where $\mathbf{1} : \Gamma \to \mathbf{R} - \{0\}$ is the 1-dimensional trivial representation of Γ.

On the other hand, Milnor [2] proved the following theorem about the connection between the Alexander polynomial and the Reidemeister torsion in 1962. We consider the abelianization

$$\alpha : \Gamma \to T$$

as a representation of Γ over $\mathbf{R}(t)$ where $\mathbf{R}(t)$ is the rational function field over $\mathbf{R}$. Then Milnor's theorem is the following.

Theorem (Milnor). *The Alexander polynomial $\Delta_K(t)$ of K is the Reidemeister torsion $\tau_\alpha(E)$ of E for α; that is,*

$$\Delta_K(t) = (t-1)\tau_\alpha E.$$

The Reidemeister torsion is a classical invariant for finite cell complexes using a representation of the fundamental group. In this paper we consider the following problem.

Problem. *Can we consider the twisted Alexander polynomial of K as a Reidemeister torsion of its exterior E of K.*

To state the main theorem, we define the tensor representation

$$\rho \otimes \alpha : \Gamma \to GL(n; \mathbf{R}(t))$$

by

$$(\rho \otimes \alpha)(x) = \rho(x)\alpha(x)$$

for $^\forall x \in \Gamma$. Then our main theorem is the following.

Theorem A. *The twisted Alexander polynomial $\Delta_{K,\rho}(t)$ associated to ρ is the Reidemeister torsion $\tau_{\rho\otimes\alpha}E$ for $\rho \otimes \alpha$; that is,*

$$\Delta_{K,\rho}(t) = \tau_{\rho\otimes\alpha}E.$$

As an application of this interpretation, we obtain the symmetry of the twisted Alexander polynomial in the following sense.

Theorem B. *If ρ is equivalent to an $SO(n)$-representation, then*

$$\Delta_{K,\rho}(t) = \Delta_{K,\rho}(t^{-1})$$

up to a factor ϵt^{mn} where $\epsilon \in \{\pm 1\}$ and $m \in \boldsymbol{Z}$.

Remark. If ρ is not equivalent to an $SO(n)$-representation, then it is an open problem to determine whether $\Delta_{K,\rho}(t)$ is always symmetric or not.

Now we describe the contents of this paper briefly. In Section 1 we review the theory of the twisted Alexander polynomial. We restrict the definition to the case of the group of a knot. In Section 2 we recall the necessary definition and results on the Reidemeister torsion for unimodular-representations. In

Section 3 we give a proof of Theorem A. In Section 4 as an application of Theorem A, we proof the symmetry of the twisted Alexander polynomial in our context.

The author would like to express his gratitude to Professor Shigeyuki Morita for his encouragement and much useful suggestions. He also would like to thank Professor Masaaki Wada for pointing out this topics and useful discussions.

1. Twisted Alexander polynomial.

Let us describe the definition of the twisted Alexander polynomial of a knot. See Wada [**4**] for details.

Let $K \subset S^3$ be a knot and Γ the knot group $\pi_1 E$. Let $F_k = \langle x_1, \ldots, x_k \rangle$ denote a free group of rank k and $T = \langle t \rangle$ an infinite cyclic group. The group ring of T over $\mathbf{Z}$ (resp. $\mathbf{R}$) is the Laurent polynomial ring $\mathbf{Z}[t^{\pm 1}]$ (resp. $\mathbf{R}[t^{\pm 1}]$). We choose and fix a Wirtinger presentation

$$P(\Gamma) = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_{k-1} \rangle$$

of Γ and

$$\phi : F_k \to \Gamma$$

the associated surjective homomorphism of the free group F_k to the knot group Γ. This ϕ induces a ring homomorphism

$$\tilde{\phi} : \mathbf{Z}[F_k] \to \mathbf{Z}[\Gamma].$$

The canonical abelianization

$$\alpha : \Gamma \to H_1(E; \mathbf{Z}) \cong T$$

is given by

$$\alpha(x_1) = \cdots = \alpha(x_k) = t.$$

Similarly α induces a ring homomorphism of the integral group ring

$$\tilde{\alpha} : \mathbf{Z}[\Gamma] \to \mathbf{Z}[t^{\pm 1}].$$

Let

$$\rho : \Gamma \to SL(n; \mathbf{R})$$

be a representation. The corresponding ring homomorphism of the integral ring $\mathbf{Z}[\Gamma]$ to the matrix algebra $M_n(\mathbf{R})$ is denoted by

$$\tilde{\rho} : \mathbf{Z}[\Gamma] \to M_n(\mathbf{R}).$$

The composition of the ring homomorphism $\tilde{\phi}$ and the tensor product homomorphism

$$\tilde{\rho} \otimes \tilde{\alpha} : \mathbf{Z}[\Gamma] \to M_n(\mathbf{R}[t^{\pm 1}])$$

will be used so often that we introduce a new symbol

$$\Phi = (\tilde{\rho} \otimes \tilde{\alpha}) \circ \tilde{\phi} : \mathbf{Z}[F_k] \to M_n(\mathbf{R}[t^{\pm 1}]).$$

Let us consider the $(k-1) \times k$ matrix $A_{\rho\otimes\alpha}$ whose (i,j)-component is the $n \times n$ matrix $\Phi(\frac{\partial r_i}{\partial x_j}) \in M_n(\mathbf{R}[t^{\pm 1}])$. This matrix $A_{\rho\otimes\alpha}$ is called the generalized Alexander matrix of the presentation $P(\Gamma)$ associated to the representation ρ. By the definition, the classical Alexander matrix A is $A_{\mathbf{1}\otimes\alpha}$ where $\mathbf{1}$ is a 1-dimensional trivial representation of Γ. For $1 \leq {}^\forall j \leq k$, let us denote by $A^j_{\rho\otimes\alpha}$ the $(k-1) \times (k-1)$ matrix obtained from $A_{\rho\otimes\alpha}$ by removing the j-th column. Now regard $A^j_{\rho\otimes\alpha}$ as a $(k-1)n \times (k-1)n$ matrix with coefficients in $\mathbf{R}[t^{\pm 1}]$. The following two lemmas are the foundation of our definition of the twisted Alexander polynomial.

Lemma 1.1. $\det \Phi(x_j - 1) \neq 0$ *for* $1 \leq {}^\forall j \leq k$.

Proof. Since we fix a Wirtinger presentation $P(\Gamma)$ as a presentation of Γ, we have

$$\alpha(x_j) = t \neq 1$$

for $1 \leq {}^\forall j \leq k$. Then $\det \Phi(x_j - 1) = \det(t\rho(x_j) - I)$ is the characteristic polynomial of $\rho(x_j)$ where I is the unit matrix. This completes the proof of Lemma 1.1.

Lemma 1.2. $\det A^j_{\rho\otimes\alpha} \det \Phi(x_{j'} - 1) = \pm \det A^{j'}_{\rho\otimes\alpha} \det \Phi(x_j - 1)$ *for* $1 \leq {}^\forall j < {}^\forall j' \leq k$.

Proof. We may assume that $j = 1$ and $j' = 2$ without the loss of generality. Since any relator $r_i = 1$ in $\mathbf{Z}[\Gamma]$, it is easy to see that

$$\sum_{l=1}^{k} \frac{\partial r_i}{\partial x_l}(1 - x_l) = 0$$

in $\mathbf{Z}[\Gamma]$. Then apply the homomorphism Φ to this, we have

$$\sum_{l=1}^{k} \Phi\left(\frac{\partial r_i}{\partial x_l}\right) \Phi(x_l - 1) = 0.$$

Let $\tilde{A}^2_{\rho\otimes\alpha}$ be the matrix obtained from $A^2_{\rho\otimes\alpha}$ by replacing the first column

$$^t\left(\Phi\left(\frac{\partial r_1}{\partial x_1}\right), \Phi\left(\frac{\partial r_2}{\partial x_1}\right), \ldots, \Phi\left(\frac{\partial r_{k-1}}{\partial x_1}\right)\right)$$

with

$$^t\left(\Phi\left(\frac{\partial r_1}{\partial x_1}\right)\Phi\left(x_1-1\right),\Phi\left(\frac{\partial r_2}{\partial x_1}\right)\Phi\left(x_1-1\right),\ldots,\Phi\left(\frac{\partial r_{k-1}}{\partial x_1}\right)\Phi\left(x_1-1\right)\right).$$

Then we have

$$\det \tilde{A}^2_{\rho\otimes\alpha} = \pm\det A^2_{\rho\otimes\alpha}\det\Phi(x_1-1).$$

Since

$$\begin{aligned}\Phi\left(\frac{\partial r_i}{\partial x_1}\right)\Phi\left(x_1-1\right) &= -\sum_{l=2}^{k}\Phi\left(\frac{\partial r_i}{\partial x_l}\right)\Phi\left(x_l-1\right)\\ &= -\Phi\left(\frac{\partial r_i}{\partial x_2}\right)\Phi\left(x_2-1\right)-\sum_{l=3}^{k}\Phi\left(\frac{\partial r_i}{\partial x_l}\right)\Phi\left(x_l-1\right),\end{aligned}$$

we can reduce the matrix $\tilde{A}^2_{\rho\otimes\alpha}$ to $\tilde{A}^1_{\rho\otimes\alpha}$ where the matrix $\tilde{A}^1_{\rho\otimes\alpha}$ can be obtained by multiplying the first column of the matrix $A^1_{\rho\otimes\alpha}$ by $\Phi(x_2-1)$. Therefore we have

$$\begin{aligned}\det \tilde{A}^2_{\rho\otimes\alpha} &= \pm\det \tilde{A}^1_{\rho\otimes\alpha}\\ &= \pm\det A^1_{\rho\otimes\alpha}\det\Phi(x_2-1).\end{aligned}$$

This completes the proof of this lemma.

By Lemma 1.1 and Lemma 1.2, we can define the twisted Alexander polynomial of K associated to the representation ρ to be the rational expression

$$\Delta_{K,\rho}(t) = \frac{\det A^1_{\rho\otimes\alpha}}{\det\Phi(x_1-1)}.$$

Theorem 1.3 (Wada). *The twisted Alexander polynomial $\Delta_{K,\rho}(t)$ is well-defined up to a factor ϵt^{mn} as an invariant of the oriented knot type of K where $\epsilon\in\{\pm1\}$, $m\in\boldsymbol{Z}$ and n is a degree of ρ.*

Remark. Two representations ρ and ρ' are said to be equivalent if there is an element $g\in GL(n;\mathbf{R})$ such that $\rho'(x) = g\cdot\rho(x)\cdot g^{-1}$ in $SL(n;\mathbf{R})$ for $^\forall x\in\Gamma$. Then the twisted Alexander polynomials for ρ and ρ' are the same ;

$$\Delta_{K,\rho}(t) = \Delta_{K,\rho'}(t)$$

up to a factor ϵt^{mn} where $\epsilon\in\{\pm1\}$ and $m\in\mathbf{Z}$.

2. Reidemeister torsion.

Let us describe the definition of the Reidemeister torsion over a field $\mathbf{F}$. See Johnson [**1**] and Milnor [**2**], [**3**], for details.

Let V denote an n-dimensional vector space over $\mathbf{F}$. Let $\mathbf{b}=(b_1,\ldots,b_n)$ and $\mathbf{c}=(c_1,\ldots,c_n)$ be two bases for V. Setting $c_i = \sum_{j=1}^{n} a_{ij}b_j$, we obtain a nonsingular matrix $A = (a_{ij})$ with entries in $\mathbf{F}$. Let $[\mathbf{b}/\mathbf{c}]$ denote the determinant of A.

Suppose

$$C_* : 0 \longrightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

is an acyclic chain complex of finite dimensional vector spaces over $\mathbf{F}$.

We assume that a preferred basis $\mathbf{c}_q$ for $C_q(C_*)$ is given for $\forall q$. Choose any basis $\mathbf{b}_q$ for $B_q(C_*)$ and take a lift of it in $C_{q+1}(C_*)$, which we denote by $\tilde{\mathbf{b}}_q$.

Since

$$B_q(C_*) \to Z_q(C_*)$$

is an isomorphism, the basis $\mathbf{b}_q$ can serve as a basis for $Z_q(C_*)$. Similarly the sequence

$$0 \to Z_q(C_*) \to C_q(C_*) \to B_{q-1}(C_*) \to 0$$

is exact and the vectors $(\mathbf{b}_q, \tilde{\mathbf{b}}_{q-1})$ is a basis for $C_q(C_*)$. It is easily shown that $[\mathbf{b}_q, \tilde{\mathbf{b}}_{q-1}/\mathbf{c}_q]$ is independent of the choices of $\tilde{\mathbf{b}}_{q-1}$. Hence we simply denote it by $[\mathbf{b}_q, \mathbf{b}_{q-1}/\mathbf{c}_q]$.

Definition 2.1. The torsion of the chain complex C_* is given by the alternating product

$$\prod_{q=0}^{m}[\mathbf{b}_q, \mathbf{b}_{q-1}/\mathbf{c}_q]^{(-1)^{q+1}}$$

and we denote it by $\tau(C_*)$.

Remark. The torsion $\tau(C_*)$ depends only on the bases $\mathbf{c}_0,\ldots,\mathbf{c}_m$.

Now we apply this torsion invariant of chain complexes to the following geometric situations. Let X be a finite cell complex and $\tilde{X}$ a universal covering of X with the fundamental group $\pi_1 X$ acting on it from the right-side as deck transformations. Then the chain complex $C_*(\tilde{X};\mathbf{Z})$ has a structure of a chain complex of right free $\mathbf{Z}[\pi_1 X]$-modules. Let

$$\rho : \pi_1 X \to SL(n;\mathbf{F})$$

be a representation. We may consider V as a $\pi_1 X$-module by using this representation ρ and denote it by V_ρ. Define the chain complex $C_*(X;V_\rho)$ by $C_*(\tilde{X};\mathbf{Z}) \otimes_{\mathbf{Z}[\pi_1 X]} V_\rho$ and choose a preferred basis

$$\{\sigma_1 \otimes e_1, \sigma_1 \otimes e_2, \ldots, \sigma_1 \otimes e_n, \ldots, \sigma_{k_q} \otimes e_1, \ldots, \sigma_{k_q} \otimes e_n\}$$

of $C_q(X; V_\rho)$ where $\{e_1, e_2, \ldots, e_n\}$ is a basis of V and $\sigma_1, \ldots, \sigma_{k_q}$ are q-cells giving the preferred basis of $C_q(\tilde{X}; \mathbf{Z})$.

Now we consider the following situation. That is $C_*(X; V_\rho)$ is acyclic, namely all homology groups vanish : $H_*(X; V_\rho) = 0$. In this case we call ρ an acyclic representation.

Definition 2.2. Let $\rho : \pi_1 X \to SL(n; \mathbf{F})$ be an acyclic representation. Then the Reidemeister torsion of X with V_ρ-coefficients is defined by the torsion of the chain complex $C_*(X; V_\rho)$. We denote it by $\tau(X; V_\rho)$ or simply $\tau_\rho(X)$.

Remark.

1. It is well known that the Reidemeister torsion is invariant under subdivision of the cell decomposition up to a factor $\epsilon \in \{\pm 1\}$. Hence the Reidemeister torsion is a piecewise linear invariant. See Milnor [**2**], [**3**].
2. In general let $\rho : \Gamma \to GL(n; \mathbf{F})$ be an acyclic representation. Then the Reidemeister torsion is well-defined up to a factor $d \in Im(\det \circ \rho) \subset \mathbf{F} - 0$.

3. Proof of Theorem A.

In this section, let $\mathbf{F}$ be the rational function field $\mathbf{R}(t)$ and V the n-dimensional vector space over $\mathbf{R}(t)$. We recall a Wirtinger presentation $P(\Gamma)$ of the knot group Γ of K is given by as follows ;

$$P(\Gamma) = \langle x_1, x_2, \ldots, x_k \mid r_1, r_2, \ldots, r_{k-1} \rangle$$

where r_i is the crossing relation for each i.

Let W be a 2-dimensional complex constructed from one 0-cell p, k 1-cells $x_1, \ldots, x_k$ and $(k-1)$ 2-cells $D_1, \ldots, D_{k-1}$ with attaching maps given by $r_1, \ldots, r_{k-1}$. It is well-known that the exterior E of K collapses to the 2-dimensional complex W. If an acyclic representation

$$\rho : \Gamma \to SL(n; \mathbf{R})$$

is fixed, we have the following by the simple homotopy invariance of the Reidemeister torsion ;

$$\tau(E; V_{\rho\otimes\alpha}) = \tau(W; V_{\rho\otimes\alpha})$$

up to a factor ϵt^{mn} where $\epsilon \in \{\pm 1\}$ and $m \in \mathbf{Z}$. In this case, we show that

$$\tau(W; V_{\rho\otimes\alpha}) = \frac{\det A^1_{\rho\otimes\alpha}}{\det \Phi(x_1 - 1)}.$$

By easy computation, this chain complex $C_*(W;V_{\rho\otimes\alpha})$ is as follows;

$$0 \longrightarrow V_{\rho\otimes\alpha}^{k-1} \xrightarrow{\partial_2} V_{\rho\otimes\alpha}^{k} \xrightarrow{\partial_1} V_{\rho\otimes\alpha} \longrightarrow 0$$

where

$$\begin{aligned}\partial_2 &= {}^tA_{\rho\otimes\alpha}\\ &= \begin{pmatrix} \Phi(\frac{\partial r_1}{\partial x_1}) & \dots & \Phi(\frac{\partial r_{k-1}}{\partial x_1}) \\ \vdots & \ddots & \vdots \\ \Phi(\frac{\partial r_1}{\partial x_k}) & \dots & \Phi(\frac{\partial r_{k-1}}{\partial x_k}) \end{pmatrix},\end{aligned}$$

$$\partial_1 = \Big(\Phi(x_1-1)\ \Phi(x_2-1)\ \dots\ \Phi(x_k-1)\Big).$$

Here we briefly denote by $V_{\rho\otimes\alpha}^l$ the l-times direct sum of $V_{\rho\otimes\alpha}$.

Proposition 3.1. *All homology groups vanish :* $H_*(W;V_{\rho\otimes\alpha})=0$ *if and only if* $\det A_{\rho\otimes\alpha}^1 \neq 0$. *In this case, we have*

$$\tau(W;V_{\rho\otimes\alpha}) = \frac{\det A_{\rho\otimes\alpha}^1}{\det \Phi(x_1-1)}.$$

Proof. It is obvious that $H_0(W;V_{\rho\otimes\alpha})$ is trivial because $\det\Phi(x_1-1)\neq 0$ and hence the boundary map ∂_1 is surjective. For a canonical basis $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ of V, we choose lifts

$$\begin{aligned}\tilde{\mathbf{e}}_1 &= {}^t(\Phi(x_1-1)^{-1}\mathbf{e}_1,\mathbf{0},\dots,\mathbf{0}),\\ &\vdots\\ \tilde{\mathbf{e}}_n &= {}^t(\Phi(x_1-1)^{-1}\mathbf{e}_n,\mathbf{0},\dots,\mathbf{0})\end{aligned}$$

in V^n. Define the $kn\times kn$ matrix M whose first $(kn-n)$ columns are ${}^tA_{\rho\otimes\alpha}$ and last n columns are $\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n$. The matrix M takes the form

$$M = \begin{pmatrix} * & \\ & \tilde{\mathbf{e}}_1 \dots \tilde{\mathbf{e}}_n \\ {}^tA_{\rho\otimes\alpha}^1 & \end{pmatrix}.$$

It is obvious that $\det M \neq 0$ if and only if $\det A_{\rho\otimes\alpha}^1 \neq 0$. If all homology groups vanish : $H_*(W;V_{\rho\otimes\alpha})=0$, then

$$\begin{aligned}\operatorname{rank} A_{\rho\otimes\alpha} &= \operatorname{rank} A_{\rho\otimes\alpha}^1\\ &= kn-n.\end{aligned}$$

Hence we have

$$\det A^1_{\rho\otimes\alpha} \neq 0.$$

In this case the Reidemeister torsion is given by

$$\begin{aligned}\tau(W; V_{\rho\otimes\alpha}) =& \det M \\ =& \frac{\det A^1_{\rho\otimes\alpha}}{\det \Phi(x_1 - 1)}.\end{aligned}$$

It is clear that the contrary is also true. Namely if $\det A^1_{\rho\otimes\alpha} \neq 0$, then $H_*(W; V_{\rho\otimes\alpha})$ is trivial. This completes the proof.

By the above propositions, we have the proof of Theorem A.

4. Symmetry of the twisted Alexander polynomial.

Hereafter we suppose that ρ is conjugate to an $SO(n)$-representation of Γ. For simplicity, we may suppose that ρ is an $SO(n)$-representation. We fix a structure of the simplicial complex in the exterior E of K and assume that each simplex of E has a dual cell. For a q-simplex of E we can define not only the dual $(3-q)$-cell in E, but also the dual $(2-q)$-cell in the boundary ∂E. Taking the cells of both types, we obtain a dual complex E' with subcomplex $\partial E'$. We denote the universal covering complex of E by $\tilde{E}$ and the one of E' by $\tilde{E}'$. Let $\langle c', c\rangle$ denote the algebraic intersection number of $c' \in C_{3-q}(\tilde{E}', \partial\tilde{E}'; \mathbf{Z})$ and $c \in C_q(\tilde{E}; \mathbf{Z})$. Next lemma is well-known fact (see Milnor [**2**]).

Lemma 4.1. *The left $\mathbf{Z}[\Gamma]$-module $C_{3-q}(\tilde{E}', \partial\tilde{E}'; \mathbf{Z})$ is canonically isomorphic to the dual of $C_q(\tilde{E}, \mathbf{Z})$ and the dual pairing*

$$[\quad,\quad] : C_{3-q}(\tilde{E}', \partial\tilde{E}'; \mathbf{Z}) \times C_q(\tilde{E}; \mathbf{Z}) \to \mathbf{Z}[\Gamma]$$

is given by

$$[c', c] = \sum_{x\in\Gamma} \langle c', cx^{-1}\rangle x$$

for $\forall c' \in C_{n-q}(\tilde{E}', \partial\tilde{E}'; \mathbf{Z})$ and $\forall c \in C_q(\tilde{E}; \mathbf{Z})$.

Now let us apply this duality to the torsion invariant. Let $V^*_{\rho\otimes\alpha}$ denote the dual vector space of $V_{\rho\otimes\alpha}$. A structure of left $\mathbf{Z}[\Gamma]$-module in $V^*_{\rho\otimes\alpha}$ is given by

$$(x \cdot u^*)(v) = u^*({}^t(\rho\otimes\alpha)(x)^{-1} \cdot v)$$

for $\forall x \in \Gamma, \forall u^* \in V^*_{\rho\otimes\alpha}$, and $\forall v \in V_{\rho\otimes\alpha}$. Then we denote this dual representation space by $V^*_{\rho\otimes\alpha}$ and define the dual pairing

$$C_{3-q}(E', \partial E'; V^*_{\rho\otimes\alpha}) \times C_q(E; V_{\rho\otimes\alpha}) \to \mathbf{R}$$

by

$$(c' \otimes u^*, c \otimes v) = u^*([c', c]v)$$

for ${}^\forall c' \otimes u^* \in C_{3-q}(E', \partial E'; V^*_{\rho\otimes\alpha})$ and ${}^\forall c \otimes v \in C_q(E; V_{\rho\otimes\alpha})$. Hence it is straightforward that $C_{3-q}(E', \partial E'; V^*_{\rho\otimes\alpha})$ is isomorphic to the dual of $C_q(E; V_{\rho\otimes\alpha})$.

Lemma 4.2. *Let C_* be an acyclic chain complex with preferred basis $\{c_i\}$ and C^* the dual complex with preferred basis $\{c_i^*\}$. Then we have*

$$\tau(C_*) = \tau(C^*)$$

up to a factor $\epsilon \in \{\pm 1\}$.

This lemma is also well-known. By this lemma and the invariance of the Reidemeister torsion for the subdivision of the cell complex, we have

$$\tau(E; V_{\rho\otimes\alpha}) = \tau(E, \partial E; V^*_{\rho\otimes\alpha}).$$

We define a representation

$$\bar\alpha : \Gamma \to T$$

by

$$\bar\alpha(x) = \alpha(x)^{-1}.$$

For the tensor representation $\rho \otimes \alpha$, because ρ is an $SO(n)$-representation, the dual representation

$$(\rho \otimes \alpha)^* : \Gamma \to GL(n; \mathbf{R}(t))$$

is given by

$$\begin{aligned}(\rho \otimes \alpha)^*(x) &= {}^t\rho(x)^{-1}\alpha(x)^{-1}\\ &= \rho(x)\bar\alpha(x)\\ &= (\rho \otimes \bar\alpha)(x)\end{aligned}$$

for ${}^\forall x \in \Gamma$. Therefore the representation space $V^*_{\rho\otimes\alpha}$ is equivalent to $V_{\rho\otimes\bar\alpha}$. Hence from the above observation, we have

$$\tau(E; V_{\rho\otimes\alpha}) = \tau(E, \partial E; V_{\rho\otimes\bar\alpha}).$$

Similarly it is easy to show that

$$\tau(E; V_{\rho\otimes\bar\alpha}) = \tau(E, \partial E; V_{\rho\otimes\alpha}).$$

The following lemma is also well-known to the experts. See Milnor [**6**].

Lemma 4.3. *Let $0 \to C'_* \to C_* \to C''_* \to 0$ be an exact sequence of n-dimensional chain complexes with preferred bases $\{\mathbf{c}'_i\}, \{\mathbf{c}_i\}$, and $\{\mathbf{c}''_i\}$ such that $[\mathbf{c}'_i, \mathbf{c}''_i/\mathbf{c}_i] = 1$ for any i. Suppose any two of the complexes are acyclic. Then the third one is also acyclic and the Reidemeister torsion of the three complexes are all well-defined. Moreover the next formula holds.*

$$\tau(C_*) = \tau(C'_*)\tau(C''_*).$$

Apply the above lemma to the short exact sequence :

$$0 \to C_*(\partial E; V_{\rho\otimes\alpha}) \to C_*(E; V_{\rho\otimes\alpha}) \to C_*(E, \partial E; V_{\rho\otimes\alpha}) \to 0,$$

we have

$$\tau(E; V_{\rho\otimes\alpha}) = \tau(\partial E; V_{\rho\otimes\alpha})\tau(E, \partial E; V_{\rho\otimes\alpha}).$$

Then we compute the Reidemeister torsion of ∂E first.

Proposition 4.4. *Let $\rho : \pi_1(\partial E) \to SL(n; \boldsymbol{R})$ be a representation. Then the Reidemeister torsion is given by*

$$\tau(\partial E; V_{\rho\otimes\alpha}) = 1$$

up to a factor ϵt^{mn} where $\epsilon \in \{\pm 1\}$ and $m \in \boldsymbol{Z}$.

Proof. Let x, y be generators of $\pi_1(\partial E)$ such that $x = x_1$ in $\pi_1 E$ and y is the canonical longitude. We assume that a cell structure of ∂E are given by :

(0) one 0-cell b,
(1) two 1-cells x and y,
(2) one 2-cell w,

with the attaching map given by $\partial w = xyx^{-1}y^{-1}$. To compute the local homology of ∂E, we compute boundary operators of this chain complex.

$$0 \longrightarrow w \otimes V \xrightarrow{\partial_2} x \otimes V \oplus y \otimes V \xrightarrow{\partial_1} p \otimes V \longrightarrow 0$$

where

$$\partial_2 = \begin{pmatrix} -\Phi(y-1) & \Phi(x-1) \end{pmatrix},$$

$$\partial_1 = \begin{pmatrix} \Phi(x-1) \\ \Phi(y-1) \end{pmatrix}.$$

It is obvious that this chain complex is acyclic because $\det \Phi(x-1) \neq 0$. Then the Reidemeister torsion $\tau(\partial E; V_{\rho\otimes\alpha})$ is defined as a rational function over $\mathbf{R}$. By the definition of the Reidemeister torsion,

$$\tau(\partial E; V_{\rho\otimes\alpha}) = [\mathbf{b}_1/\mathbf{c}_2]^{-1}[\mathbf{b}_1, \mathbf{b}_0/\mathbf{c}_1][\mathbf{b}_0/\mathbf{c}_0]^{-1}.$$

By straightforward computation, we have

$$\tau(\partial E; V_{\rho\otimes\alpha}) = 1.$$

This completes the proof.

Hence combine the above lemmas,

$$\begin{aligned}\tau(E; V_{\rho\otimes\alpha}) &= \tau(E, \partial E; V_{\rho\otimes\alpha}) \\ &= \tau(E; V_{\rho\otimes\bar{\alpha}}).\end{aligned}$$

By the definition of the twisted Alexander polynomial and Theorem A, it is obvious that

$$\tau(E; V_{\rho\otimes\bar{\alpha}}) = \Delta_{K,\rho}(t^{-1}).$$

Therefore we have

$$\Delta_{K,\rho}(t) = \Delta_{K,\rho}(t^{-1}).$$

This completes the proof of Theorem B.

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Received January 10, 1994 and revised March 23, 1994.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 174 No. 2 June 1996
