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**ENDPOINT INEQUALITIES FOR BOCHNER-RIESZ
MULTIPLIERS IN THE PLANE**

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ENDPOINT INEQUALITIES FOR BOCHNER-RIESZ MULTIPLIERS IN THE PLANE

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A weak-type inequality is proved for Bochner-Riesz means at the critical index, for functions in $L^p(\mathbb{R}^2)$, $1 \leq p < 4/3$.

1. Introduction.

For a Schwartz-function $f \in \mathcal{S}(\mathbb{R}^2)$ let $\hat{f}(\xi) = \int f(y)e^{-i\langle y, \xi \rangle} dy$ denote the Fourier transform and define the Bochner-Riesz means by

$$S_R^\lambda f(x) = \frac{1}{(2\pi)^2} \int_{|\xi| \leq R} \left(1 - \frac{|\xi|^2}{R^2}\right)^\lambda \hat{f}(\xi) e^{i\langle x, \xi \rangle} d\xi;$$

we set $S^\lambda = S_1^\lambda$. It is a classical theorem of Bochner that S^λ extends to a bounded operator on $L^p(\mathbb{R}^2)$, $1 \leq p \leq \infty$ if $\lambda > 1/2$. The theorem of Carleson and Sjölin [2] states that S^λ is bounded in $L^p(\mathbb{R}^2)$ if $0 < \lambda \leq \frac{1}{2}$ and $\frac{4}{3+2\lambda} < p < \frac{4}{1-2\lambda}$. It is well known that the L^p boundedness fails if $p \leq \frac{4}{3+2\lambda}$ and C. Fefferman [11] showed that S^0 is not bounded in $L^p(\mathbb{R}^2)$ if $p \neq 2$.

In this paper we are concerned with endpoint estimates for the critical exponent $p_0(\lambda) = \frac{4}{3+2\lambda}$. In [4, 5] M. Christ proved that S^λ is of weak type $(p_0(\lambda), p_0(\lambda))$ if $1/6 < \lambda \leq 1/2$ (for related results see also [6, 15]). A combination of L^2 -variants of Calderón-Zygmund theory (as used first by Fefferman [10]) and the $L^p \rightarrow L^2$ restriction theorem for the Fourier transform (valid for $p \leq 6/5 = p_0(1/6)$) is essential in Christ's analysis; this accounts for the restriction $\lambda > 1/6$. It had been an open problem whether the weak type inequality for the critical index $\lambda(p) = 2(1/p - 1/2) - 1/2$ is true for $6/5 \leq p < 4/3$ (although for radial functions this was proved by Chanillo and Muckenhoupt [3]).

Theorem 1.1. *Suppose that $0 < \lambda \leq 1/2$. Then for all $\alpha > 0$ there is the weak-type inequality*

$$|\{x \in \mathbb{R}^2 : |S^\lambda f(x)| > \alpha\}| \leq C \frac{\|f\|_{p_0}^{p_0}}{\alpha^{p_0}}, \quad p_0 = \frac{4}{3+2\lambda},$$

where C does not depend on f or α .

By scaling the same estimate holds for S_R^λ , uniformly in R , and a standard argument gives that $\lim_{R \rightarrow \infty} S_R^\lambda f = f$ in the topology of the weak type space $L^{p_0, \infty}$ provided that $f \in L^{p_0}(\mathbb{R}^2)$.

We shall also prove an L^p endpoint version of the Carleson-Sjölin theorem. Define

$$(1.1) \quad m_{\lambda, \gamma}(\xi) = \frac{(1 - |\xi|^2)_+^\lambda}{(1 - \log(1 - |\xi|^2))^\gamma}.$$

Theorem 1.2. *Suppose that $1 \leq p < 4/3$ and $\lambda(p) = 2\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2}$. Then $m_{\lambda(p), \gamma}$ is a Fourier multiplier of $L^p(\mathbb{R}^2)$ if and only if $\gamma > \frac{1}{p}$.*

The necessity of the condition $\gamma > 1/p$ was proved in [14], the sufficiency for $p \leq 6/5$ in [15].

In what follows c and C will always be positive numbers which may assume different values in different formulas.

2. Strong type estimates.

For an interval I on the real line denote by I^* the interval with same midpoint and double length. Suppose $\mathcal{I} = \{I_j\}_{j \geq 0}$ is a collection of intervals such that $I_j \subset (1/4, 4)$ and $2^{-j-3} \leq |I_j| \leq 2^{-j}$ and such that

$$I_j^* \cap I_{j'}^* = \emptyset \quad \text{if } j \neq j'.$$

For each $j \geq 0$ let ψ_j be a C^2 -function supported in I_j with bounds

$$\|\psi_j^{(\ell)}\|_\infty \leq 2^{j\ell}, \quad \ell = 0, 1, 2.$$

Let $\eta \in C_0^\infty(\mathbb{R}^2)$ such $\text{supp}(\eta) \subset \{\xi \in \mathbb{R}^2 : |\xi_1/\xi_2| \leq 10^{-1}, \xi_2 > 0\}$.

Define the operator T_j by

$$(2.1) \quad \widehat{T_j f}(\xi) = \eta(\xi)\psi_j(|\xi|)\widehat{f}(\xi).$$

T_j is a bounded operator on L^1 with operator norm $O(2^{j/2})$, and Córdoba [8] showed that the $L^{4/3}$ operator norm of T_j is $O(2^{j/4})$. We note that in order to prove results such as Theorem 1.2 for $p > 1$ it is not sufficient to derive sharp L^p bounds for the individual operators T_j . Our main result is

Theorem 2.1. *Suppose that $1 \leq p < 4/3$ and $\lambda(p) = 2\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2}$ and \mathcal{I}, T_j are as above. Then there is the inequality*

$$(2.2) \quad \left\| \sum_j T_j f_j \right\|_p \leq C \left(\sum_j [2^{j\lambda(p)} \|f_j\|_p]^p \right)^{\frac{1}{p}}.$$

In particular if

$$(2.3) \quad m = \sum_j 2^{-j\lambda(p)} a_j \eta(\xi) \psi_j(|\xi|)$$

then m is a Fourier multiplier of L^p if $\{a_j\} \in \ell^p$ (simply apply Theorem 2.1 with $f_j = a_j 2^{-j\lambda(p)} f$). It is easy to see that the multiplier $m_{\lambda,\gamma}$ in (1.1) is a finite sum of a smooth compactly supported function and rotates of multipliers of the form (2.3), with $a_j = cj^{-\gamma}$. Therefore Theorem 2.1 implies Theorem 1.2.

Proof of Theorem 2.1 By duality the inequality (2.2) is equivalent to

$$(2.4) \quad \left(\sum_j \left[2^{-j\lambda(q')} \|T_j f\|_q \right]^q \right)^{\frac{1}{q}} \leq C \|f\|_q, \quad q > 4.$$

As in [8] one decomposes each $\psi_j(|\cdot|)$ into pieces which are essentially supported in rectangles of dimensions $(c2^{-j/2}, c2^{-j})$. To this end let $\beta \in C_0^\infty(\mathbb{R})$ be supported in $(-1, 1)$ such that $\sum_{\nu=-\infty}^\infty \beta(s - \nu) = 1$ for all $s \in \mathbb{R}$. Then define T_j^ν by

$$\widehat{T_j^\nu f}(\xi) = \beta(2^{j/2}\xi_1 - \nu) \widehat{T_j f}(\xi).$$

For $n \leq j/2$ let

$$\mathfrak{J}_j^n = \{(\nu, \nu') \in \mathbb{Z}^2 : 2^{j/2-n-1} < |\nu - \nu'| \leq 2^{j/2-n}\}.$$

Notice that $T_j^\nu f T_j^{\nu'} f = 0$ if $(\nu, \nu') \in \mathfrak{J}_j^n$ and $n < 0$. Therefore

$$(2.5) \quad \begin{aligned} & \left(\sum_j \left[2^{-j\lambda(q')} \|T_j f\|_q \right]^q \right)^{\frac{1}{q}} \\ &= \left(\sum_j \left[2^{-2j\lambda(q')} \left\| \sum_{\nu} \sum_{\nu'} T_j^\nu f T_j^{\nu'} f \right\|_{\frac{q}{2}} \right]^{\frac{q}{2}} \right)^{\frac{1}{q}} \\ &\leq \sum_{n=0}^\infty \left(\sum_{j \geq 2n} \left[2^{-2j\lambda(q')} \left\| \sum_{(\nu, \nu') \in \mathfrak{J}_j^n} T_j^\nu f T_j^{\nu'} f \right\|_{\frac{q}{2}} \right]^{\frac{q}{2}} \right)^{\frac{1}{q}} \end{aligned}$$

We shall show that for $q \geq 4$ the n^{th} term in (2.5) is bounded by $C2^{-n(1/2-2/q)} \|f\|_q$ from which (2.4) immediately follows. This is contained in

Proposition 2.2. For $f, g \in \mathcal{S}(\mathbb{R}^2)$ let

$$\mathcal{B}_j^n(f, g) = \sum_{(\nu, \nu') \in \mathfrak{Z}_n^j} T_j^\nu f T_j^{\nu'} g.$$

Then for $q \geq 4$ there is the inequality

$$(2.6) \quad \left(\sum_{j \geq 2n} \left[2^{-2j\lambda(q')} \|\mathcal{B}_j^n(f, g)\|_{\frac{q}{2}} \right]^{\frac{q}{2}} \right)^{\frac{2}{q}} \leq C 2^{-n(1-\frac{4}{q})} \|f\|_q \|g\|_q.$$

Proof. The inequality follows by complex interpolation for bilinear mappings from the cases $q = 4$ and $q = \infty$. The correct interpretation of (2.6) for $q = \infty$ is of course

$$\sup_j 2^{-j} \left\| \sum_{(\nu, \nu') \in \mathfrak{Z}_n^j} T_j^\nu f T_j^{\nu'} g \right\|_\infty \leq C 2^{-n} \|f\|_\infty \|g\|_\infty.$$

But this is immediate since each operator T_j^ν is bounded on L^∞ with norm independent of j and ν and since the cardinality of \mathfrak{Z}_n^j is bounded by $C 2^{j/2} \times 2^{j/2-n} = C 2^{j-n}$.

We shall now prove the required estimate for $q = 4$ which is

$$(2.7) \quad \left(\sum_{j \geq 2n} \|\mathcal{B}_j^n(f, g)\|_2^2 \right)^{1/2} \leq C \|f\|_4 \|g\|_4$$

uniformly in n .

We first use Plancherel’s theorem and C. Fefferman’s basic observation ([12, 8]) that for fixed j the sets $\text{supp}(\widehat{T_j^\nu f}) + \text{supp}(\widehat{T_j^{\nu'} g})$ are essentially disjoint; that is each $\xi \in \mathbb{R}^2$ is contained in at most M of these sets where M is independent of j . This yields the inequality

$$(2.8) \quad \sum_{j \geq 2n} \|\mathcal{B}_j^n(f, g)\|_2^2 \leq C \sum_{j \geq 2n} \sum_{(\nu, \nu') \in \mathfrak{Z}_n^j} \|T_j^\nu f T_j^{\nu'} g\|_2^2$$

It is crucial for this proof that a finer decomposition can be made depending on how far apart the supports of $\widehat{T_j^\nu f}$ and $\widehat{T_j^{\nu'} g}$ are, that is, depending on n . We define operators $T_j^{\nu\mu}$ by

$$\widehat{T_j^{\nu\mu} f}(\xi) = \beta(2^{j-n}\xi_1 - \mu) \widehat{T_j^\nu f}(\xi)$$

so that $\widehat{T_j^{\nu\mu} f}$ is supported in a rectangle of dimensions $(C 2^{-j+n}, C 2^{-j})$. Again one can check that for fixed j and fixed $(\nu, \nu') \in \mathfrak{Z}_n^j$ each $\xi \in \mathbb{R}^2$ is contained in at most M of the sets $E_{j\nu\nu'}^{\mu\mu'} = \text{supp}(\widehat{T_j^{\nu\mu} f}) + \text{supp}(\widehat{T_j^{\nu'\mu'} g})$ where

M is independent of j, ν, ν' . Each $E_{j\nu\nu'}^{\mu\mu'}$ is contained in a rectangle of dimensions $(C'2^{-j+n}, C'2^{-j})$. For fixed j, ν, ν' there are no more than $C''2^{(j-2n)}$ of these rectangles and they form an essentially disjoint cover of $\text{supp}(\widehat{T_j^\nu f}) + \text{supp}(\widehat{T_j^{\nu'} g})$, the latter set being contained in a rectangle of dimensions $(C2^{-j/2}, C2^{-j/2-n})$. The disjointness property and Plancherel's theorem imply that

$$(2.9) \quad \sum_{j \geq 2n} \|\mathcal{B}_j^n(f, g)\|_2^2 \leq C \sum_{j \geq 2n} \sum_{\mu, \mu'} \sum_{(\nu, \nu') \in \mathfrak{Z}_j^n} \|T_j^{\nu\mu} f T_j^{\nu'\mu'} g\|_2^2.$$

For any integer κ with $|\kappa| \leq 2^n$ let

$$\mathfrak{W}_{j_n}^\kappa = \{\mu \in \mathbb{Z} : |2^{n-j}\mu - 2^{-n}\kappa| \leq 2^{-n}\}.$$

Then observe that

$$(2.10) \quad T_j^{\nu\mu} f T_j^{\nu'\mu'} g = 0 \quad \text{if } (\nu, \nu') \in \mathfrak{Z}_j^n, \mu \in \mathfrak{W}_{j_n}^\kappa, \mu' \in \mathfrak{W}_{j_n}^{\kappa'}, |\kappa - \kappa'| \geq 8.$$

Indeed, if $\mu \in \mathfrak{W}_{j_n}^\kappa, \mu' \in \mathfrak{W}_{j_n}^{\kappa'}, T_j^{\nu\mu} f T_j^{\nu'\mu'} g \neq 0$ then $|2^{n-j}\mu - 2^{-j/2}\nu| \leq 2^{-j/2+1}$ and $|2^{n-j}\mu' - 2^{-j/2}\nu'| \leq 2^{-j/2+1}$. If $(\nu, \nu') \in \mathfrak{Z}_j^n$ this implies that $|2^{n-j}(\mu - \mu')| \leq 2^{-j/2+2} + 2^{-n} \leq 5 \cdot 2^{-n}$ and therefore $|\kappa - \kappa'| \leq 7$, hence (2.10). Moreover we note that for $\mu \in \mathfrak{W}_{j_n}^\kappa$ the support of $\widehat{T_j^{\nu\mu} f}$ is essentially a rectangle with eccentricity 2^{-n} such that the directions of its sides depend on κ but not on μ .

By (2.9) and (2.10) we obtain that

$$\begin{aligned} & \sum_{j \geq 2n} \|\mathcal{B}_j^n(f, g)\|_2^2 \\ & \leq C \sum_{j \geq 2n} \sum_{\kappa} \sum_{\substack{\kappa' \\ |\kappa' - \kappa| < 8}} \left\| \left(\sum_{\mu \in \mathfrak{W}_{j_n}^\kappa} \sum_{\nu} |T_j^{\nu\mu} f|^2 \right)^{\frac{1}{2}} \left(\sum_{\mu' \in \mathfrak{W}_{j_n}^{\kappa'}} \sum_{\nu'} |T_j^{\nu'\mu'} g|^2 \right)^{\frac{1}{2}} \right\|_2^2 \\ & \leq C'' \sum_{j \geq 2n} \sum_{\kappa} \sum_{\substack{\kappa' \\ |\kappa' - \kappa| < 8}} \left\| \left(\sum_{\mu \in \mathfrak{W}_{j_n}^\kappa} \sum_{\nu} |T_j^{\nu\mu} f|^2 \right)^{\frac{1}{2}} \right\|_4^2 \left\| \left(\sum_{\mu' \in \mathfrak{W}_{j_n}^{\kappa'}} \sum_{\nu'} |T_j^{\nu'\mu'} g|^2 \right)^{\frac{1}{2}} \right\|_4^2 \\ & \leq C'' \left(\sum_{j \geq 2n} \sum_{\kappa} \left\| \left(\sum_{\mu \in \mathfrak{W}_{j_n}^\kappa} \sum_{\nu} |T_j^{\nu\mu} f|^2 \right)^{\frac{1}{2}} \right\|_4^4 \right)^{\frac{1}{2}} \\ & \quad \left(\sum_{j \geq 2n} \sum_{\kappa} \left\| \left(\sum_{\mu \in \mathfrak{W}_{j_n}^\kappa} \sum_{\nu} |T_j^{\nu\mu} g|^2 \right)^{\frac{1}{2}} \right\|_4^4 \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore the desired estimate (2.7) follows from the case $q = 4$ of the following lemma.

Lemma 2.3. *For $q \geq 2$ there is the inequality*

$$(2.11) \quad \left(\sum_{j \geq 2n} \sum_{\kappa} \left\| \left(\sum_{\mu \in \mathfrak{W}_{jn}^{\kappa}} \sum_{\nu} |T_j^{\nu\mu} f|^2 \right)^{\frac{1}{2}} \right\|_q^q \right)^{\frac{1}{q}} \leq C \|f\|_q$$

where C does not depend on n .

Proof. It suffices to prove (2.11) for $q = 2$ and $q = \infty$. Let $h_j^{\nu\mu}$ be the Fourier multiplier defining $T_j^{\nu\mu}$.

For fixed μ and j there are at most three ν such that $T_j^{\nu\mu} \neq 0$ and since the supports of the functions ψ_j are disjoint it follows that each $\xi \in \mathbb{R}^2$ is contained in at most 6 of the sets $\text{supp } h_j^{\mu\nu}$. Moreover for fixed μ and j there are at most two κ such that $\mu \in \mathfrak{W}_{jn}^{\kappa}$. Now (2.11) for $q = 2$ is an immediate consequence of Plancherel's theorem.

In order to check the required estimate for $q = \infty$ we consider for a fixed $\mathfrak{a} = \{a_{\nu\mu}\} \in \ell^2(\mathbb{Z}^2)$ the multiplier

$$m_{\mathfrak{a}}^{j\kappa}(\xi) = \sum_{\mu \in \mathfrak{W}_{jn}^{\kappa}} \sum_{\nu} a_{\nu\mu} h_j^{\nu\mu}(\xi)$$

and denote by $K_{\mathfrak{a}}^{j\kappa}$ its inverse Fourier transform.

Let $e_1^{\kappa} = (2^{-n}\kappa, \sqrt{1 - 2^{-2n}\kappa^2})$ and $e_2^{\kappa} = (-\sqrt{1 - 2^{-2n}\kappa^2}, 2^{-n}\kappa)$ and let L_{jn}^{κ} be the symmetric linear transformation in \mathbb{R}^2 with $L_{jn}^{\kappa} e_1^{\kappa} = 2^j e_1^{\kappa}$, $L_{jn}^{\kappa} e_2^{\kappa} = 2^{j-n} e_2^{\kappa}$. Then $h_j^{\nu\mu}(L_{jn}^{\kappa} \cdot)$ is supported in a cube $Q_j^{\nu\mu}$ of sidelength 10 and for fixed j the cubes $Q_j^{\nu\mu}$ have finite overlap, uniformly in j . Moreover it is easy to see that for $\mu \in \mathfrak{W}_{jn}^{\kappa}$

$$\left\| \frac{\partial^{\alpha}}{\partial \xi^{\alpha}} \left[h_j^{\nu\mu}(L_{jn}^{\kappa} \cdot) \right] \right\|_{\infty} \leq C, \quad |\alpha| \leq 2.$$

Since the Sobolev-space L_2^2 is a subspace of \widehat{L}^1 we obtain that

$$\begin{aligned} \|K_{\mathfrak{a}}^{j\kappa}\|_1 &= \|2^{-2j+n} K_{\mathfrak{a}}^{j\kappa}((L_{jn}^{\kappa})^{-1} \cdot)\|_1 \\ &\leq C \sum_{|\alpha| \leq 2} \left\| \sum_{\mu, \nu} a_{\nu\mu} \frac{\partial^{\alpha}}{\partial \xi^{\alpha}} \left[h_j^{\nu\mu}(L_{jn}^{\kappa} \cdot) \right] \right\|_2 \\ &\leq C' \left(\sum_{\mu, \nu} |a_{\nu\mu}|^2 \right)^{\frac{1}{2}} \end{aligned}$$

where C' does not depend on j , κ and \mathbf{a} . This implies

$$\begin{aligned} & \sup_{j \geq 2n} \sup_{\kappa} \left\| \left(\sum_{\mu \in \mathfrak{W}_{j_n}^{\kappa}} \sum_{\nu} |T_j^{\nu\mu} f|^2 \right)^{\frac{1}{2}} \right\|_{\infty} \\ &= \sup_{j \geq 2n} \sup_{\kappa} \sup_{x \in \mathbb{R}^2} \sup_{\|\mathbf{a}\|_{\ell^2(\mathbb{Z}^2)} \leq 1} |K_{\mathbf{a}}^{j\kappa} * f(x)| \\ &\leq \sup_{j \geq 2n} \sup_{\kappa} \sup_{\|\mathbf{a}\|_{\ell^2(\mathbb{Z}^2)} \leq 1} \|K_{\mathbf{a}}^{j\kappa}\|_1 \|f\|_{\infty} \leq C \|f\|_{\infty} \end{aligned}$$

which is the desired estimate for $q = \infty$. \square

Remarks.

(a) For $q = \infty$ the inequality (2.11) is closely related to an estimate on square-functions with respect to an equally spaced decomposition, see *e.g.* [9, 13]; in fact it can be obtained from these estimates.

(b) A variant of the above proof can be used to obtain the known sharp L^4 bound $\|T_j\|_{L^4 \rightarrow L^4} = O(j^{1/4})$ without making use of the sharp L^2 bounds for Keakeya-maximal functions.

(c) The observation concerning the overlapping properties of $\widehat{\text{supp}} T_j^{\nu\mu} f + \widehat{\text{supp}} T_j^{\nu'\mu'} g$ can be used to improve on some bounds for sectorial square-functions in Córdoba [9]. This has been observed by A. Carbery and the author.

(d) The decomposition in terms of the bilinear operators \mathcal{B}_j^n is related to a decomposition used by Carbery [1] in his work on weighted inequalities for the maximal Bochner-Riesz operator S_{*}^{λ} . The techniques above can be used to prove new weighted inequalities for S_{*}^{λ} .

3. Weak type estimates.

Let \mathcal{J} be a family of disjoint intervals as introduced in §2 and let T_j be as in (2.1). Define

$$T^{\lambda} f = \sum_{j \geq 0} 2^{-j\lambda} T_j f.$$

We shall prove the estimate

$$(3.1) \quad \left| \{x \in \mathbb{R}^2 : |T^{\lambda(p)} f(x)| > \alpha\} \right| \leq C \frac{\|f\|_p^p}{\alpha^p}, \quad p < \frac{4}{3}$$

where $\lambda(p) = 2(1/p - 1/2) - 1/2$ and C does not depend on f or α . Of course Theorem 1.1 is a consequence of (3.1).

As in [5] the proof is based on an interpolation. The argument uses Theorem 2.1 and known estimates previously obtained in the proof of weak-type (1,1) inequalities (see [4, 7, 15]).

Let $f \in L^p(\mathbb{R}^2)$ where $1 \leq p < \frac{4}{3}$ and let $\alpha > 0$. In order to estimate the quantity on the left hand side of (3.1) we apply the Calderón-Zygmund decomposition to $|f|^p$ at height α^p . We obtain a decomposition $f = g + b$ where $\|g\|_\infty \leq C\alpha$, $\|g\|_p \leq C\|f\|_p$, $b = \sum_Q b_Q$, $\text{supp } b_Q \subset Q$, the squares Q are pairwise disjoint, $\|b_Q\|_p^p \leq C\alpha^p|Q|$, $\sum_Q |Q| \leq C\alpha^{-p}\|f\|_p^p$; and as a consequence $\alpha^{p-2}\|g\|_2^2 + \|b\|_p^p \leq C\|f\|_p^p$.

Let $l(Q)$ be the sidelength of Q and $B_j = \sum_{Q:l(Q)=2^j} b_Q$ if $j > 0$ and $B_0 = \sum_{Q:l(Q) \leq 1} b_Q$. Then

$$\{x \in \mathbb{R}^2 : |T^{\lambda(p)}f(x)| > \alpha\} \subset \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5$$

where Ω_1 is the union of the double squares Q^* and

$$\begin{aligned} \Omega_2 &= \left\{ x \in \mathbb{R}^2 : |T^{\lambda(p)}g(x)| > \frac{\alpha}{5} \right\} \\ \Omega_3 &= \left\{ x \in \mathbb{R}^2 : \left| \sum_{s \geq 0} \sum_{j > s} 2^{-j\lambda(p)} T_j B_{j-s}(x) \right| > \frac{\alpha}{5} \right\} \\ \Omega_4 &= \left\{ x \in \mathbb{R}^2 : \left| \sum_{j \geq 0} 2^{-j\lambda(p)} T_j B_0(x) \right| > \frac{\alpha}{5} \right\} \\ \Omega_5 &= \left\{ x \in \mathbb{R}^2 \setminus \Omega_1 : \left| \sum_{\sigma > 0} \sum_{j \geq 0} 2^{-j\lambda(p)} T_j B_{j+\sigma}(x) \right| > \frac{\alpha}{5} \right\}. \end{aligned}$$

By the disjointness of the squares Q we have

$$|\Omega_1| \leq \sum_Q |Q^*| \leq C \frac{\|f\|_p^p}{\alpha^p}$$

and Chebyshev's inequality and the L^2 -boundedness of T^λ imply

$$|\Omega_2| \leq C \frac{\|T^\lambda g\|_2^2}{\alpha^2} \leq C' \frac{\|g\|_2^2}{\alpha^2} \leq C'' \frac{\|f\|_p^p}{\alpha^p}.$$

Next we choose r such that $p < r < 4/3$. We shall show that the following estimates hold with $\epsilon = \frac{1}{2}(\frac{r}{p} - 1)$.

$$(3.2) \quad \left\| \sum_{j > s} 2^{-j\lambda(p)} T_j B_{j-s} \right\|_r \leq C 2^{-\epsilon s} \alpha^{r-p} \|b\|_p^p, \quad s \geq 0,$$

$$(3.3) \quad \|2^{-j\lambda(p)}T_j B_0\|_r^r \leq C2^{-\epsilon j}\alpha^{r-p}\|b\|_p^p, \quad j \geq 0,$$

$$(3.4) \quad \left\| \sum_{j \geq 0} 2^{-j\lambda(p)}T_j B_{j+\sigma} \right\|_{L^p(\mathbb{R}^2 \setminus \Omega_1)}^p \leq C2^{-\epsilon\sigma}\|b\|_p^p, \quad \sigma \geq 0.$$

From (3.2-3.4) it follows by applications of Minkowski's and Chebyshev's inequalities that

$$|\Omega_3| + |\Omega_4| + |\Omega_5| \leq C \frac{\|b\|_p^p}{\alpha^p} \leq C' \frac{\|f\|_p^p}{\alpha^p}.$$

In order to prove (3.2-4) we use analytic interpolation (i.e. the Phragmen-Lindelöf principle) similarly as in [5]. For $\text{Re}(z) \in [0, 1]$ define

$$B_{j,z}(x) = |B_j(x)|^{p[(1-z)+z/r]} \text{sign}(B_j(x))$$

and

$$\gamma(z) = 2 \left(1 - z + \frac{z}{r} - \frac{1}{2} \right) - \frac{1}{2}.$$

Since $2^{-j\gamma(1+i\tau)}T_j$ is a bounded operator on L^1 with norm independent of j we obtain

$$(3.5) \quad \left\| \sum_{j>s} 2^{-j\gamma(1+i\tau)}T_j B_{j-s,1+i\tau} \right\|_1 \leq C \sum_{j>s} \|B_{j-s,1+i\tau}\|_1 \leq C'\|b\|_p^p$$

$$(3.6) \quad \|2^{-j\gamma(1+i\tau)}T_j B_{0,1+i\tau}\|_1 \leq C\|B_0\|_p^p \leq C'\|b\|_p^p.$$

From estimates in [7] (or [15]) it follows that

$$(3.7) \quad \left\| \sum_{j>s} 2^{-j\gamma(1+i\tau)}T_j B_{j-s,1+i\tau} \right\|_2^2 \leq C2^{-s/2}\alpha^p\|b\|_p^p$$

$$(3.8) \quad \|2^{-j\gamma(1+i\tau)}T_j B_{0,1+i\tau}\|_2^2 \leq C2^{-j/2}\|b\|_p^p$$

and also that

$$(3.9) \quad \left\| \sum_{j \geq 0} 2^{-j\gamma(1+i\tau)}T_j B_{j+\sigma,1+i\tau} \right\|_{L^1(\mathbb{R}^2 \setminus \Omega_1)} \leq C2^{-\sigma} \sum_{j \geq 0} \|B_{j+\sigma,1+i\tau}\|_1 \leq C'2^{-\sigma}\|b\|_p^p.$$

Using the inequality $\|F\|_r \leq C\|F\|_1^{\frac{2}{r}-1}\|F\|_2^{2-\frac{2}{r}}$ we get from (3.5), (3.7) and from (3.6), (3.8) that

$$(3.10) \quad \left\| \sum_{j>s} 2^{-j\gamma(1+i\tau)}T_j B_{j-s,1+i\tau} \right\|_r^r \leq C2^{-s\frac{r-1}{2}}\alpha^{p(r-1)}\|b\|_p^p$$

$$(3.11) \quad \|2^{-j\gamma(1+i\tau)}T_jB_{0,1+i\tau}\|_r^p \leq C2^{-j\frac{r-1}{2}}\alpha^{p(r-1)}\|b\|_p^p.$$

Now by Theorem 2.1 it follows that

$$(3.12) \quad \left\| \sum_{j>s} 2^{-j\gamma(i\tau)}T_jB_{j-s,i\tau} \right\|_r^r \leq C \sum_{j>s} \|B_{j-s,i\tau}\|_r^r \leq C'\|b\|_p^p$$

$$(3.13) \quad \|2^{-j\gamma(i\tau)}T_jB_{0,i\tau}\|_r^r \leq C\|B_{0,i\tau}\|_r^r \leq C'\|b\|_p^p$$

$$(3.14) \quad \left\| \sum_{j\geq 0} 2^{-j\gamma(i\tau)}T_jB_{j+\sigma,i\tau} \right\|_r^r \leq C \sum_{j\geq 0} \|B_{j+\sigma,i\tau}\|_r^r \leq C'\|b\|_p^p.$$

Now let h be arbitrary function in $L^{p'}$, $p' = p/(p-1)$, with $\|h\|_{p'} \leq 1$ and define

$$h_z(x) = |h(x)|^{z p'/r'} \operatorname{sign}(h(x)).$$

Moreover let g be an arbitrary function in $L^{r'}$ with $\|g\|_{r'} \leq 1$. We then apply the Phragmen-Lindelöf principle to the functions

$$\begin{aligned} z \mapsto W_{1,s}(z) &= \int \sum_{j>s} 2^{-j\gamma(z)}T_jB_{j-s,z}(x)g(x)dx \\ z \mapsto W_{2,j}(z) &= \int 2^{-j\gamma(z)}T_jB_{0,z}(x)g(x)dx \\ z \mapsto W_{3,\sigma}(z) &= \int \sum_{j\geq 0} 2^{-j\gamma(z)}T_jB_{j+\sigma,z}(x)h_z(x)dx \end{aligned}$$

and estimate these functions at $z = \theta$ chosen such that $1/p = (1-\theta) + \theta/r$. From (3.10), (3.12), from (3.11), (3.13) and from (3.9), (3.14) it follows that

$$\begin{aligned} |W_{1,s}(\theta)| &\leq C\alpha^{r-p}2^{-\frac{\sigma}{2}(\frac{r}{p}-1)}\|b\|_p^p \\ |W_{2,j}(\theta)| &\leq C\alpha^{r-p}2^{-\frac{j}{2}(\frac{r}{p}-1)}\|b\|_p^p \\ |W_{3,\sigma}(\theta)| &\leq C2^{-\sigma(\frac{r}{p}-1)}\|b\|_p^p \end{aligned}$$

and an application of the converse of Hölder's inequality yields (3.2), (3.3) and (3.4).

Remark. Endpoint versions for more general classes of multiplier transformations have been formulated in [15]. By combining arguments in this and the present paper one can prove similar results for radial Fourier multipliers of $L^p(\mathbb{R}^2)$, for the full range $1 \leq p < 4/3$.

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