ON THE EXISTENCE OF EXTREMAL METRICS

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We study the well known variational problem proposed by Calabi: Minimize the functional \( \int_M s^2_g dv_g \) among all metrics in a given Kahler class. We are able to establish the existence of the extremal when the closed Riemann surface has genus different from zero. We have also given a different proof of the result originally proved by Calabi that: On a closed Riemann surface, the extremal metric has constant scalar curvature on a closed Riemann surface, the extremal metric has constant scalar curvature, which originally is proved by Calabi.

1. Introduction.

In the early 80's, E. Calabi [C1, C2] proposed the following variational problem. Let \( M \) be a compact, connected, complex \( n \)-dimensional manifold without boundary and assume that \( M \) admits a Kahler metric \( g \) locally expressible in the form \( ds^2 = 2g_{\alpha\beta}dz^\alpha dz^\beta \). Let us fix the deRham cohomology class \( \Omega \) of the real valued, closed exterior \((1,1)\) form \( \omega = \sqrt{-1} g_{\alpha\beta}dz^\alpha \Lambda dz^\beta \) associated to the metric \( g \), and denote by \( C_\Omega \) the function space of all differentiable Kahler metrics \( g \) with the Kahler form \( \omega \in \Omega \). On this function space, Calabi introduces the (non negative) real valued functional \( \Phi \) which assigns to each \( g \) the integral

\[
\Phi(g) = \int_M s^2_g dv_g
\]

where \( dv_g = (\sqrt{-1})^n \det(g_{\alpha\beta}) \Lambda_{\alpha=1}^n (dz^\alpha \Lambda dz^\alpha) \) denotes the volume element in \( M \) associated with the e Kahler metric \( g \), and

\[
s_g = -g^{\alpha\beta} \frac{\partial^2}{\partial z^\alpha \partial z^\beta} \log \det(g_{\lambda\mu})
\]

the scalar curvature.

The variational problem proposed by Calabi is that of minimizing the functional \( \Phi(g) \) over all \( g \in C_\Omega \). The motivation for considering this is the fact that, as \( g \) varies in \( C_\Omega \), both the volume

\[
V = V_g = \int_M dv_g
\]
and the total scalar curvature

\[ S_g = \int_M s_g dv_g \]

remain constants. Thus by the virtue of the Schwartz inequality, the functional \( \Phi(g) \) has a nonnegative lower bound \( S_g^2 V \), we wish that the latter can be achieved if and only if there exists a \( g \in C_\Omega \) with constant scalar curvature.

As M. Levine \([L]\) points out, if we call the critical metric of \( \Phi \) the extremal metric, the extremal metric does not necessarily have constant scalar curvature if the dimension \( n > 1 \). For \( n = 1 \), E. Calabi is able to show that the extremal metric always has constant scalar curvature if the extremal metric exists (see \([C1]\) and also §5 of this paper).

The problem of finding extremal metrics is quite nature but quite difficult. There are several results about the non-existence ((\([C1],[C2],[L],[BB]\)). However, in the past decade, there has almost been no progress on the existence of extremal metrics.

The propose of this paper is to show:

**Main Theorem.** If \( n = 1 \) with \( \chi(M) \leq 0 \), then the extremal metric exists.

Remember that this is not so surprising at all since there are several methods to reach this conclusion: Poincaré’s classical uniformization theorem \([P]\); M. Berger’s minimization method \([A],[B]\); R. Hamilton’s Ricci flow \([CH],[H1],[H2]\); B. Osgood, R. Phillips and P. Sarnack’s minimizing the log determinant of the Laplace operator \([OPS]\).

What it is new in this paper is that, we exactly follow the Calabi’s original idea, by using the direct method, to show that the minimizer of the Calabi functional can be achieved. Since the Kähler class and the conformal class for \( n = 1 \) are equivalent, our setting is in the conformal class. The main difficulty is to get \( H^2 \) norm bound of the conformal factors in terms of the volume bound and the bound on the Calabi functional.

The organization of this paper is as follows: after some preliminaries(§2), we will give the proof of our main theorem for the case \( \chi(M) < 0(\S 3) \). §4 will simply indicate the case \( \chi(M) = 0 \). Since our setting is in the conformal class, we will show, in this setting, that the extremal metrics have constant scalar curvatures (§5). Clearly this is an alternative proof of one of Calabi’s theorems \([C1]\).

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2. Preliminaries.

Consider a compact connected complex $n$—dimensional manifold $M$ without boundary. Assume that $M$ admits a Kähler metric $g$ locally expressible in the form

$$ds^2 = 2g_{\alpha\bar{\beta}}dz^{\alpha}d\bar{z}^{\beta}$$

where and thereafter the Einstein convention is used. It is well known that there is a real valued, closed exterior (1, 1) form $\omega = \sqrt{-1}g_{\alpha\bar{\beta}}dz^{\alpha}d\bar{z}^{\beta}$ associated to the metric $g$. This (1, 1) form usually is called Kähler form. By the deRham theory, $\omega$ determines a deRham cohomology class $\Omega$. Now let us consider the change of the Kähler metric,

$$g'_{\lambda\bar{\mu}} = g_{\lambda\bar{\mu}} + \partial_{\lambda\bar{\mu}}\varphi$$

where $\varphi \in C^\infty$ is such that $g'$ is positive definite. Obviously, $g'$ is a Kähler metric. Also it is clear that the Kähler form $\omega'$ associated to the metric $g'$ determines the same deRham cohomology class as $\omega$ does. Note that under this change of the metric, new metric $g'$ is also a Riemannian metric since it is Kähler. We will denote all such functions $\varphi$ by $C_\Omega$.

For a given Kähler metric $g$, we have the volume element defined in the local coordinate by $dv_g = (\sqrt{-1})^n \det(g_{\alpha\bar{\beta}})\Lambda_{\alpha=1}^{n}(dz^{\alpha}d\bar{z}^{\bar{\alpha}})$. And the scalar curvature associated to the metric $g$ is defined by

$$s_g = -g^{\alpha\bar{\beta}}\frac{\partial^2}{\partial z^{\alpha}\partial \bar{z}^{\beta}}(\log \det(g_{\lambda\bar{\mu}})).$$

For a $\varphi \in C_\Omega$, the Calabi functional can be written as

$$\Phi(\varphi) = \int_{M} s_{g'}^{2}dv_{g'}$$

where $g'$ and $\varphi$ are related by (2.1).

From now on, we only consider $M$ a Riemann surface without boundary, that is, a compact complex 1—dimensional manifold without boundary. It is known that on a complex 1—dimensional manifold, any Riemannian metric is conformally flat [Be, Theorem 1.169]. Thus any two Riemannian metrics on $M$ are conformally equivalent. If metrics $g$ and $g'$ are related by (2.1), there exists a $C^\infty$ function $u$ such that $g' = e^{2u}g$. In this form, we know that $dv_{g'} = e^{2u}dv_{g}$ and the scalar curvatures satisfy the relation

$$\Delta u + k_{g'}e^{2u} = k_{g}$$

where $k_{g'}$ and $k_{g}$ are the Gaussian curvatures of the metrics $g'$ and $g$ respectively, and $\Delta$ is the Laplace operator associated to the metric $g$. The relation
between the Gaussian curvature and the scalar curvature is that \( s_g = 2k_g \). Thus, up to a constant multiple, the Calabi functional \( \Phi(\varphi) \) is equivalent to

\[
J(u) = \int_M (k_g - \Delta u)^2 e^{-2u} dv_g
\]

with the constrain

\[
\int_M e^{2u} dv_g = \int_M dv_g.
\]

Our main propose of this section is going to show that the functional \( J \) has the following properties.

**Proposition 2.1.**

(a) \( J \) is continuously differentiate on \( H^2_2 \) where \( H^2_2 \) denotes the Hilbert space as usual.

(b) The first variation of \( J \) at a point \( u \) in a direction \( \varphi \) is given by

\[
J'(u)(\varphi) = -2 \int_M [e^{2u}(k_g - \Delta u)^2 + \Delta(e^{2u}(k_g - \Delta u))] \varphi dv_g.
\]

(c) The Euler equation associated to \( J \) under the constrain (1.6) is given by

\[
\Delta[e^{-2u}(k_g - \Delta u)] + e^{-2u}(k_g - \Delta u)^2 = \lambda^2 e^{2u}
\]

for some constant \( \lambda \geq 0 \).

*Proof.* (c) can easily follow from (b), Lagrange multiplier and the fact that if we set \( G(u) = \int_M e^{2u} dv_g \), \( G'(u)(\varphi) = \int_M e^{2u} \varphi dv_g \). (b) will follow from the proof of (a). Before we are going to prove the part (a), let us recall two simple facts.

**Fact 1.** If the real dimension of a manifold is two, there exists a constant \( C > 0 \) such that

\[
\max |u| \leq C\|u\|_{H^2_2}
\]

for all \( u \in H^2_2 \).

It can be proved as follows. Suppose \( G \) is a Green function for \( \Delta \). Then it is known that \( \int_M G^2 dv_g \) is finite and for any \( u \in H^2_2 \), \( u(p) = \int_M u dv_g - \int_M G \Delta u dv_g \). Thus the Hölder inequality implies that (2.9) holds with \( C = [(\int_M dv_g)^{1/2} + (\int_M G^2 dv_g)^{1/2}] \).

**Fact 2.** We can choose \( \varepsilon_0 > 0 \) such that if \( 0 < t < \varepsilon_0 \), then \( |e^{-2t} - 1 + 2t| \leq 4t^2 \) and \( |e^{-2t} - 1| \leq 100t \) where numbers 4 and 100 are not so important.
In order to simplify the notation, from now on, we will denote the norm $\|u\|_{H^2}^2$ by $\|u\|$.

Now we are in position to give the proof of the part (a).

Let us now assume that $u \in H^2_2$ such that $J(u) < \infty$. Let $\varphi$ be any function in $H^2_2$ such that $\max|\varphi| < \epsilon_0$. Then we have

$$\|\varphi\|^{-1} \left| J(u + \varphi) - J(u) + 2 \int_M [e^{-2u(k_g - \Delta u)^2 + \Delta(e^{-2u(k_g - \Delta u)})}] \varphi dv_g \right|$$

$$= \|\varphi\|^{-1} \left| \int_M (k_g - \Delta u)^2 e^{-2u} [e^{-2u} + 2\varphi - 1] dv_g \right.$$ 

$$+ \int_M (\Delta \varphi)^2 e^{-2(u+\varphi)} dv_g - 2 \int_M \Delta \varphi [e^{-2u(k_g - \Delta u)}(e^{-2\varphi} - 1)]dv_g \right|$$

$$\leq \|\varphi\|^{-1} [4C^2 \|\varphi\|^2 J(u) + C_1 \exp[2C(\|u\| + \|\varphi\|)] \cdot \|\varphi\|^2$$

$$+ 200C_2 \exp[C \|u\|]J(u) \frac{1}{2} \cdot \|\varphi\|^2$$

$$= [4C^2 J(u) + C_1 \exp[2C(\|u\| + \|\varphi\|)] + 200C_2 \exp[C \|u\|] J(u) \frac{1}{2}\|\varphi\|,$$

where $C$ is given in (2.9) and $C_1$ and $C_2$ are constants. This proves (b) and the half of (a). For the rest of the part (a), we can argue as follows: for a fixed function $u \in H^2_2$, if $v \in H^2_2$ is such that $\max|u - v| < \epsilon_0$, then we have

$$|J'(u)(w) - J'(v)(w)|$$

$$= 2 \left| \int_M [e^{-2u}(k_g - \Delta u)^2 + \Delta(e^{-2u}(k_g - \Delta u))] w dv_g \right.$$ 

$$- \int_M [e^{-2v}(k_g - \Delta v)^2 + \Delta(e^{-2v}(k_g - \Delta v))] w dv_g \right|$$

$$= 2 \left| \int_M (e^{-2u} - e^{-2v})(k_g - \Delta u)^2 w dv_g \right.$$ 

$$+ \int_M e^{-2v} [(k_g - \Delta u)^2 - (k_g - \Delta v)^2] w dv_g$$

$$+ \int_M (e^{-2u} - e^{-2v}) k_g \Delta w dv_g - \int_M (e^{-2u} \Delta u - e^{-2v} \Delta v) \Delta w dv_g \right|$$

$$\leq 200 \max|u - v| J(u) \max|w| + 2e^{2\max|u|} \int_M |\Delta(u - v)||\Delta w| dv_g$$

$$+ 2e^{2\max|v|} \max|w| \int_M |\Delta(u - v)||2k_g - \Delta(u + v)| dv_g$$

$$+ 400e^{\max|v|} \int_M |u - v||\Delta w| dv_g \max |k_g|$$

$$+ 2e^{2\max|u|} \max|u - v| \cdot |v| \cdot |w|$$

$$\leq C \|u - v\| \cdot |w|,$$
where $C > 0$ depends on $||u||, ||v||, J(u)$ and $\max |k_g|$. This proves the result. \hfill \Box

Also we note that if we define $H = \{u \in H^2_2 | \int_M e^{2u}dv_g = \int_M dv_g \}$, we have

**Proposition 2.2.** $H$ is a weakly closed subset in $H^2_2$.

**Proof.** Weakly convergent subsequences in $H^2_2$ is strongly convergent sequences in $H^2_1$. $\int_M e^{2u}dv_g$ is a weakly continuous functional on $H^2_1$ by Moser’s inequality. Thus it is a weakly continuous functional on $H^2_2$. We are done. \hfill \Box

3. Proof of Main Theorem for $\chi(M) < 0$.

The proof of the main theorem in this case will consist of several lemmas.

**Lemma 3.1.** Let $(M, g)$ be a closed Riemann surface with $\chi(M) < 0$. If the Gaussian curvature $k_g$ is positive somewhere, there exists a function $u \in H^2_2$ so that the metric $\tilde{g} = e^{2u}g$ has nonpositive Gaussian curvature $k_{\tilde{g}}$ and $\int_M k_{\tilde{g}}^2dv_{\tilde{g}} \leq \int_M k^2dv_g$.

**Proof.** Choose

$$s = \begin{cases} -k_g & k_g \geq 0 \\ k_g & k_g \leq 0. \end{cases}$$

Then it is clear that $s \leq 0$. Let us consider

$$\Delta u + se^{2u} = k_g. \tag{3.1}$$

Clearly, if $\varphi = 0$, then

$$\Delta \varphi + se^{2\varphi} - k_g = \begin{cases} -2k_g & k_g \geq 0 \\ 0 & k_g \leq 0 \end{cases}$$

which is always nonpositive. Let $\phi$ be a solution of $\Delta \phi = k_g - \int_M k_g dv_g$. By standard elliptic theory, up to add a constant to $\phi$, $\phi$ is unique. We fix a solution by requiring $\int_M \phi dv_g = 0$. Choose $N > 0$ large enough so that

$$se^{2\phi - N} - \int_M k_g dv_g > 0 \text{ and } \phi - N/2 < 0.$$ 

It is clear that such an $N$ exists. Then set $v = \phi - N/2$, we can get

$$\Delta v + se^{2v} - k_g = se^{2\phi - N} - \frac{\int_M k_g dv_g}{\int_M dv_g} > 0$$
and \( v < 0 \). Therefore \( \phi = 0 \) is a supper solution and \( v \) is a sub-solution of the equation (3.1). By standard elliptic theory, equation (3.1) has a \( H^2 \) solution \( w \) since \( s \) and \( k_g \) are in \( L^2(M) \), [S, Theorem 2.4; Chapter 1].

Now choose a constant \( \alpha \) such that \( \int_M e^{2(w+\alpha)} \, dv_g = \int_M dv_g \), and set \( k = se^{-2\alpha} \leq 0 \). Since \( w \leq 0 \) and \( -2\alpha = \log \frac{\int_M e^{2w} \, dv_g}{\int_M dv_g} \), by the convexity of the exponential function, \( \alpha > 0 \). Also \( \int_M (k)^2 e^{2(w+\alpha)} \, dv_g = \int_M s^2 e^{-4\alpha} e^{2(w+\alpha)} \, dv_g = \int_M s^2 e^{2(w-\alpha)} \, dv_g \leq \int_M s^2 dv_g = \int_M k_g^2 \, dv_g \). Thus if we set \( u = w + \alpha \), then \( \tilde{g} = e^{2u}g \) will satisfy the requirement. This completes the proof of Lemma 3.1.

**Lemma 3.2.** If \( u \in H^2, J(u) \leq C_0, k_g - \Delta u \leq 0 \) and \( \int_M e^{2u} \, dv_g = \int_M dv_g \), there exist constants \( C_1, C_2 \) depending only \( C_0 \) and \( k_g \) and \( \int_M dv_g \) such that the following inequality holds: \( C_1 < u < C_2 \).

**Proof.** Let \( \varphi \) be a solution of the equation \( \Delta \varphi = k_g - \int_M k_g \, dv_g \) with \( \int_M \varphi \, dv_g = 0 \). Since \( k_g \) is continuous and \( \varphi \) is in \( C^2 \), \( \varphi \) is in \( H^2 \). There exist constants \( \alpha \) and \( \beta \) such that \( \alpha < \varphi < \beta \) where \( \alpha \) and \( \beta \) only depend on \( k_g \).

Now from \( J(u) \leq C_0 \), we have

\[
C_0 \geq J(u) = \int_M e^{2u} (k_g - \Delta u)^2 \, dv_g \\
\geq \int_M e^{2(\varphi-\beta)} (k_g - \Delta u)^2 e^{-2u} \, dv_g \\
= e^{-2\beta} \int_M e^{-2(u-\varphi)} (k_g - \Delta u)^2 \, dv_g \\
= e^{-2\beta} \int_M e^{-2(u-\varphi)} \left( k_g - \frac{\int_M k_g \, dv_g}{\int_M dv_g} + \frac{\int_M k_g \, dv_g}{\int_M dv_g} - \Delta u \right)^2 \, dv_g \\
= e^{-2\beta} \int_M e^{-2(u-\varphi)} \left( \frac{\int_M k_g \, dv_g}{\int_M dv_g} - \Delta (u - \varphi) \right)^2 \, dv_g \\
= e^{-2\beta} \left[ \left( \frac{\int_M k_g \, dv_g}{\int_M dv_g} \right)^2 \int_M e^{-2(u-\varphi)} \, dv_g \\
- 2 \frac{\int_M k_g \, dv_g}{\int_M dv_g} \int_M e^{-2(u-\varphi)} \Delta (u - \varphi) \, dv_g \\
+ \int_M e^{-2(u-\varphi)} \Delta (u - \varphi)^2 \, dv_g \right] \\
= e^{-2\beta} \left[ \left( \frac{\int_M k_g \, dv_g}{\int_M dv_g} \right)^2 \int_M e^{-2(u-\varphi)} \, dv_g \right].
\]
\[-4 \int_M k_g dv_g \int_M e^{-2(u-\varphi)}|\nabla (u-\varphi)|^2 dv_g \\
+ \int_M e^{-2(u-\varphi)}(\Delta (u-\varphi))^2 dv_g \].

The above inequality and the fact that \( \int_M k_g dv_g = \chi(M) \int_M dv_g < 0 \) imply that

\[
\int_M e^{-2(u-\varphi)} dv_g \leq \frac{[C_0 e^{2\beta} (\int_M dv_g)^2]}{(-4 \int_M k_g dv_g)^2}, \tag{3.2}
\]

\[
\int_M e^{-2(u-\varphi)}|\nabla (u-\varphi)|^2 dv_g \leq \frac{[C_0 e^{2\beta} \int_M dv_g]}{(-4 \int_M k_g dv_g)}, \tag{3.3}
\]

\[
\int_M e^{-2(u-\varphi)}|\Delta (u-\varphi)|^2 \leq C_0 e^{2\beta}. \tag{3.4}
\]

Simply (3.2) gives us

\[
e^{2\alpha} \int_M e^{-2u} dv_g \leq \int_M e^{-2u+2\varphi} dv_g \leq \frac{[C_0 e^{2\beta} (\int_M dv_g)^2]}{(\int_M k_g dv_g)^2},
\]

or equivalently,

\[
\int_M e^{-2u} dv_g \leq \frac{C_0 e^{2(\beta-\alpha)} (\int_M dv_g)^2}{(\int_M k_g dv_g)^2} := C_3. \tag{3.5}
\]

By using the convexity of the exponential function, we can get that

\[-\int_M 2udv_g \leq \log C_3 \text{ from (3.5).} \]

Since \( \int_M e^{2u} dv_g = \int_M dv_g \), \( 2 \int_M udv_g \leq \log \int_M dv_g := C_4 \).

From now on we will assume that the volume \( \int_M dv_g = 1 \).

Thus \( C_4 = 0 \). Anyway, we get \( |\int_M udv_g| \leq \frac{\log C_3}{2} \). Now

\[
u(p) - \int_M udv_g = -\int_M G\Delta udv_g \tag{3.6}
\]

where \( G \) is Green’s function associated to Laplace operator \( \Delta \). We can choose \( G \) such that it is positive everywhere [A, Theorem 4.14]. These give us the estimate on \( u \) as follows

\[
u(p) - \int_M udv_g = \int_M G(k_g - \Delta u) dv_g - \int_M k_g G dv_g \\
\leq \left( \int_M (k_g)^2 dv_g \right)^{1/2} \left( \int_M G^2 dv_g \right)^{1/2} \\
:= C_5.
\]
This implies that
\[ u(p) \leq \left( \int_M u \, dv_g \right) + \left( \int_M (k_g)^2 \, dv_g \right)^{1/2} \left( \int_M G^2 \, dv_g \right)^{1/2} \leq \frac{\log C_3}{2} + C_5 := C_2. \]

As \( p \) is arbitrary on \( M \), \( u \leq C_2 \).

Apply (3.6) with \( u \) replaced by \( e^{-u} \) to get
\[
e^{-u} - \int_M e^{-u} \, dv_g = - \int_M G \Delta e^{-u} \, dv_g
\]
\[ (3.7) \]
\[ = \int_M e^{-u} G (\Delta u - |\nabla u|^2) \, dv_g \]
\[ \leq \int_M e^{-u} G \Delta u \, dv_g \]
\[ \leq \left( \int_M e^{-2u}(\Delta u)^2 \, dv_g \right)^{1/2} \left( \int_M G^2 \, dv_g \right)^{1/2}, \]

where we have used the fact that \( \Delta (e^{-u}) = e^{-u}(|\nabla u|^2 - \Delta u) \). But from (3.3), we obtain
\[
C_0 e^{2\beta} \geq \int_M e^{-2(u-\phi)}(\Delta(u - \phi))^2 \, dv_g
\]
\[ = \int_M e^{-2(u-\phi)}[(\Delta u)^2 - 2\Delta u \Delta \phi + (\Delta \phi)^2] \, dv_g \]
\[ \geq \int_M e^{-(u-\phi)}(\Delta u)^2 \, dv_g - 1/2 \int_M (\Delta u)^2 e^{-2(\omega \phi)} \, dv_g
\]
\[ - 2 \int_M (\Delta \phi)^2 e^{-2(u-\phi)} \, dv_g + \int_M (\Delta \phi)^2 e^{-2(u-\phi)} \, dv_g
\]
\[ = 1/2 \int_M e^{-2(u-\phi)}(\Delta u)^2 \, dv_g - \int_M (\Delta \phi)^2 e^{-2(u-\phi)} \, dv_g. \]

This simply implies that
\[ (3.8) \]
\[
\frac{1}{2} \int_M e^{-2u}(\Delta u)^2 \, dv_g e^{2\alpha}
\]
\[ \leq \frac{1}{2} \int_M e^{-2(u-\phi)}(\Delta u)^2 \, dv_g
\]
\[ \leq \int_M e^{-2(u-\phi)}(\Delta(u - \phi))^2 \, dv_g + \int_M e^{-2(u-\phi)}(\Delta \phi)^2 \, dv_g
\]
\[ \leq C_0 e^{2\beta} + \int_M \left( \int_M k_g \, dv_g - k_g \right)^2 \cdot e^{-2u} \, dv_g \cdot e^{2\beta}
\]
\[ \leq \left[ C_0 + 2 \left( \int_M k_g \, dv_g \right)^2 \int_M e^{-2u} \, dv_g + 2 \int_M k_g^2 e^{-2u} \, dv_g \right] e^{2\beta} \]
by using (3.5) and (3.7). Thus this implies that

\[ e^{-u(p)} \leq C_3^{1/2} + \left( \int_M G^2 \right)^{1/2} (2C_6)^{1/2} e^{-\alpha} \]

\[ := e^{-C_1}. \]

Hence, by taking log on both sides of (3.9), we have \( u(p) \geq C_1 \). As before, \( p \) is a general point on \( M \), we have what we need to show. \( \square \)

**Lemma 3.3.** If \( u \in H^2_2, J(u) \leq C_0, k_g - \Delta u \leq 0 \) and \( \int_M e^{2u} dv_g = \int_M dv_g \), there exists a constant \( C_7 \) depending on \( C_0, k_g \) and \( \int_M dv_g \) only such that \( ||u|| \leq C_7 \).

**Proof.** It is clear from (3.8) and Lemma 3.2 that

\[ \frac{1}{2} e^{-2C_2} \cdot e^{2\alpha} \int_M (\Delta u)^2 dv_g \leq \frac{1}{2} e^{2\alpha} \int_M e^{-2u} (\Delta u)^2 dv_g \leq C_6. \]

If we define \( C_7 \) to be \( 2\{2C_6 e^{2(C_2-\alpha)} + (C_1^2 + C_2^2)\} \), combining with Lemma 3.2, the result follows. \( \square \)

**Lemma 3.4.** The weak solution of the equation (2.8) exists.

**Proof.** Let \( \alpha_0 = \inf_{u \in H} J(u) \). Suppose that \( \{u_j\} \) is a minimizing sequence in \( H^2_2 \) for the functional \( J \) with \( \int_M e^{2u} dv_g = \int_M dv_g \). Without loss of generality by applying Lemma 3.1, we can choose a minimizing sequence \( \{u_j\} \) such that \( k_g - \Delta u_j \leq 0 \). Then Lemma 3.3 implies that \( \{u_j\} \) is bounded in \( H^2_2 \). Thus there exists a subsequence of \( \{u_j\} \) still denoted by \( \{u_j\} \) and a function \( u_0 \in H^2_2 \) such that \( u_j \) is weakly convergent to \( u_0 \) in \( H^2_2 \) and \( u_j \) is pointwise almost everywhere convergent to \( u_0 \) because \( H^2_2 \) is reflexive and the Proposition 3.43 of [A]. Now we have to show that \( J(u_0) = \alpha_0 \) in order to show that \( u_0 \) weakly satisfies the equation (2.8). On the one hand, since the subset \( H \) of \( H^2_2 \) is weakly closed by Proposition 2.2, \( u_0 \) is in the subset \( H \). By definition of \( \alpha_0 \), we can easily see that \( \alpha_0 \leq J(u_0) \). On the other hand, we have

\[ J(u_n) - J(u_0) = \int_M e^{-2u_n} (k_g - \Delta u_n)^2 dv_g - \int_M e^{-2u_0} (k_g - \Delta u_0)^2 dv_g \]
As \( n \) goes to \( \infty \), I and VI go to zero because \( e^{-2u_n} \) goes to \( e^{-2u_0} \) pointwise almost everywhere and the Dominated convergence theorem (Theorem 3.32 of [A]) can be applied; III and V go to zero by the definition of the weakly convergence in \( H_2^2 \). For II,

\[
\left| \int_M k_g (e^{-2u_n} - e^{-2u_0}) \Delta u_n d v_g \right| \\
\leq \max |k_g| \left[ \int_M (\Delta u_n)^2 d v_g \right]^{1/2} \left[ \int_M (e^{-2u_n} - e^{-2u_0})^2 d v_g \right]^{1/2} \\
\rightarrow 0
\]

by the Hölder inequality and the Dominated convergence theorem again since \( \|u_n\| \) is uniformly bounded. Similarly we can estimate the term IV, we will leave it to reader.

By letting \( n \rightarrow \infty \) in (3.10), we have that \( \alpha_0 - J(u_0) \geq 0 \), i.e., \( J(u_0) \leq \alpha_0 \). The Lemma is proved.

**Remark.** In fact, we can show that \( J(u) \) is weakly lower semicontinuous on \( H_2^2 \). Since we do not need this, we will not give a proof here.

**Lemma 3.5.** The weak solution of the equation (2.8) is smooth if \( k_g \) is smooth.

**Proof.** Suppose \( u_0 \) satisfies (2.8). Set \( s = e^{-2u_0}(k_g - \Delta u_0) \) and \( \lambda^2 - s^2 = f \). Then it is clear that equation (2.8) can be written as

\[
\Delta s = e^{2u_0}f.
\]
Since \( u_0 \in H^2_2 \), \( s \) must be square integrable. If we set \( v = s - \int_M s dv_g \), above equation looks like:

\[
\Delta v = e^{2u_0} f.
\]

By applying Theorem 3.67 of [A] on p. 91, we can get that \( \int_M |\nabla v|^2 dv_g \) is finite. Then \( v \) is in \( \mathbb{L}^p \) for all \( p \) by applying same theorem. Now the above equation tells us that \( \Delta v \) is square integrable. Thus \( v \), so \( s \), is continuous. By using the definition of \( s \), we can see that \( u_0 \) is in \( C^2 \). Then \( s \) is in \( C^2 \) which will imply that \( u_0 \) is in \( C^4 \). Repeating this argument, we can easily see that \( u_0 \) is smooth. This finishes the proof. In fact, we will see later that \( s \) is a constant.

\[\Box\]

4. \( \chi(M) = 0 \) case.

In this case, for a given metric \( g \) on a Riemann surface with \( \int_M k_g dv_g = 0 \), we can choose a function \( \phi \) such that \( \Delta \phi = k \) and \( \int_M e^{2\phi} dv_g = \int_M dv_g \) by standard elliptic theory. Now \( J(\phi) = \int_M e^{-2\phi} (k_g - \Delta \phi)^2 dv_g = \int_M e^{-2\phi} (k_g - k_g)^2 dv_g = 0 \). Thus \( \phi \) is a minimizer for \( J \). Of course, \( \phi \) is smooth.

5. The scalar curvatures of the extremal metrics.

In this section, we will prove the following theorem due to Calabi:

**Theorem.** For a Riemann surface, the extremal metrics have constant scalar curvatures.

The proof will follow easily from the following

**Lemma 5.1.** Let \((M, g)\) be a compact Riemann surface. If the Gaussian curvature \( k \) of the metric \( g \) satisfies

\[
\Delta k + k^2 = \lambda^2
\]

for some constant \( \lambda \), then \( \nabla k \) is a conformal vector field.

**Proof.** The \( \nabla k \) is a conformal vector field if and only if its components satisfy

\[
k_{i,j} + k_{j,i} = 2f g_{ij}
\]

in local coordinates for some function \( f \). That is to say that we only need to show that the hessian of the Gaussian curvature is proportional to the metric \( g \). The standard Ricci identity shows that

\[
\frac{1}{2} \Delta |\nabla k|^2 = \sum k_{ij}^2 + \langle \nabla \Delta k, \nabla k \rangle + R_{ij} k_i k_j.
\]
But for a Riemann surface, $R_{ij} = kg_{ij}$. Thus
\[
\frac{1}{2} \Delta |\nabla k|^2 = |\text{Hess}(k)|^2 + (\nabla \Delta, \nabla k) + k|\nabla k|^2.
\]

By using (5.1), we get
\[
\frac{1}{2} \Delta |\nabla k|^2 = |\text{Hess}(k)|^2 - k|\nabla k|^2
\]
\[
= |\text{Hess}(k)|^2 - \frac{(\Delta k)^2}{2} + \frac{(\Delta k)^2}{2} - k|\nabla k|^2.
\]

(5.2)

Also by using integration by parts and (5.1) we obtain
\[
\int_M k|\nabla k|^2 d\nu_g = \int_M k \cdot \nabla k \cdot \nabla k d\nu_g
\]
\[
= -\frac{1}{2} \int_M k^2 \Delta k d\nu_g
\]
\[
= \frac{1}{2} \int_M (\lambda^2 - k^2) \Delta k d\nu_g
\]
\[
= \frac{1}{2} \int_M (\Delta k)^2 d\nu_g.
\]

Now, we integrate both sides of (5.2) to get
\[
0 = \int_M |\text{Hess} k|^2 d\nu_g - \int_M \frac{(\Delta k)^2}{2} d\nu_g
\]
\[
= \int_M \left| \text{Hess}(k) - \frac{\Delta k}{2} g \right|^2 d\nu_g.
\]

The last equality holds because of a well known identity. The Lemma can be seen easily from this.

\[\square\]

**Proof of Theorem.** Set $g' = e^{2u_0}g$. Then we have $\Delta g' = e^{-2u_0}\Delta g$ and $k_{g'} = e^{-2u_0}(k_g - \Delta u_0)$. Thus the equation (2.8) can be written as
\[
(5.3) \quad \Delta g'k_{g'} + k_{g'}^2 = \lambda^2.
\]

Thus Lemma 5.1 can be applied to show that $\nabla k_{g'}$ is a conformal vector field with respect to metric $g'$. But a conformal vector field $X$ on a manifold $M$ satisfies the identity
\[
\int_M X \cdot s_g d\nu_g = 0
\]
where $s_g$ is the scalar curvature of the metric $g ([\text{BE}],[X])$. In our case, $s_g = 2k_{g'}$. Thus we have
\[
2 \int_M |\nabla k_{g'}|^2 d\nu_{g'} = 0
\]
Therefore, $4|\nabla k'|^2 = 0$. That is, $k'$ is a constant. The theorem is proved.

References


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