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A necessary and sufficient condition, in terms of asymptotic properties of the sequence, is given for the inductive limit of a sequence of finite direct sums of matrix algebras over commutative C*-algebras to have real rank zero (i.e., for each self-adjoint element to be approximible by one with finite spectrum).

1. Introduction.

In this paper we will consider C*-algebra inductive limits $A = \varinjlim A_n$ where the C*-algebras A_n have the form

$$A_n = \bigoplus_{j=1}^{r_n} \mathrm{C}(\Omega_{n,j}\,,\, \mathrm{M}_j)\ , \tag{1.1}$$

with each $\Omega_{n,j}$ a compact metrizable space (possibly empty, but let us assume that at least Ω_{n,r_n} is non-empty), r_n finite and M_j the C*-algebra of $j \times j$ complex matrices.

(We could just as well consider non-compact spaces, but the resulting increased generality in Theorem 1.1, below, would be illusory — one could reduce easily to the compact case.)

The spectrum Ω_n of A_n is the disjoint union of the spaces $\Omega_{n,j}$, $j = 1, \cdots, r_n$. If Λ is a clopen (closed and open) subset of Ω_n, let $A_n(\Lambda)$ denote the corresponding sub-C*-algebra of A_n, i.e.,

$$A_n(\Lambda) = \bigoplus_{j=1}^{r_n} \mathrm{C}(\Omega_{n,j} \cap \Lambda\,,\, \mathrm{M}_j)\ .$$

If Λ_1 is a clopen subset of Ω_m, let $\Phi_{(m,\Lambda_1)(n,\Lambda)} = \Phi_{\Lambda_1\Lambda}$ denote the homomorphism $A_n(\Lambda) \to A_m(\Lambda_1)$ obtained from $\Phi_{m,n}$ by cutting down by the unit of $A_m(\Lambda_1)$ in A_m and restricting to $A_n(\Lambda)$. Let $\mathcal{P}(\Omega_n)$ denote the set of partitions of Ω_n into clopen sets which are refinements of the partition

$$\{\Omega_{n,1}, \Omega_{n,2}, \cdots, \Omega_{n,r_n}\}\ .$$

For any compact metrizable space Ω, let $\dim \Omega$ denote the covering dimension of Ω; see [**Eng**]. We will assume throughout that $\dim \Omega_n < +\infty$ for each n. (Our results also hold without this restriction, but in a vacuous way.)

In [**BBEK**] and [**BDR**] it was proved that A has real rank zero, i.e., that any self-adjoint element in A can be approximated by self-adjoint elements with finite spectrum, in several situations. For instance, this was proved in [**BBEK**] under the conditions that A be simple and unital, each $\Omega_{n,j}$ be the union of a finite number of connected clopen subsets, $\dim \Omega_n \leq 2$ for all n, and the projections of A separate the tracial states of A. By using techniques from [**DNNP**], the condition that $\dim \Omega_n \leq 2$ in this result was replaced in [**BDR**] by the much weaker condition that A have slow dimension growth, i.e., that

$$\lim_{n\to\infty} \max_j \left\{ \frac{\dim \Omega_{n,j}}{j} \right\} = 0 .$$

Since simplicity together with the presence of a unit implies that

$$\lim_{n\to\infty} \min \{ j \mid \Omega_{n,j} \neq \emptyset \} = +\infty ,$$

this condition is automatically fulfilled if the dimensions of the Ω_n are uniformly bounded.

Another condition on the sequence $A_1 \to A_2 \to \cdots$ considered in [**BBEK**] was small eigenvalue variation. Although this is a condition which is somewhat complicated to state compared to the conditions in the preceding paragraph, it is much easier to check in concrete examples of inductive limits, unless one knows for some extraneous reason that the inductive limit has, for example, a unique trace. It is also a condition which is convenient to work with in connection with the classification of inductive limits. (See [**Ell0**], [**Su**], [**EG**], and [**BEEK**].)

In [**BBEK**] it was assumed that the spaces Ω_n were finite unions of connected clopen sets, and hence we have to cast the definition of small eigenvalue variation in a slightly different form from that of [**BBEK**]. In the case that the Ω_n are finite unions of connected clopen sets our present definition reduces to the previous one.

If $x = x^* \in A_n$ and $P \in \mathcal{P}(\Omega_n)$ then by the eigenvalue list of x (relative to P) we will mean the collection of functions λ_m on each of the spaces $\Lambda \in P$, with $\lambda_m(\omega)$ the mth lowest eigenvalue of $x(\omega)$, counted with multiplicity. Here $m = 1, 2, \cdots, j$ if $\Lambda \subseteq \Omega_{n,j}$.

By the eigenvalue variation of x, denoted by $\mathrm{EV}(x)$, we will mean the quantity

$$\inf_{P\in\mathcal{P}(\Omega_n)} \max_{\Lambda\in P} \max_m \sup_{\omega,\omega'\in\Lambda} |\lambda_m(\omega) - \lambda_m(\omega')| ,$$

where all allowed values of m are considered, namely, $m = 1, 2, \cdots, j$ if $\Lambda \subseteq \Omega_{n,j}$. In the case that Ω_n is a finite union of connected clopen sets, this infimum is attained at the corresponding partition of Ω_n (i.e., the finest partition), and we recover the definition of [**BBEK**].

Let us say that A has small eigenvalue variation if

$$\lim_{m \to \infty} \mathrm{EV}(\Phi_{m,n}(x)) = 0 \text{ for all } x = x^* \in A_n \text{ and all } n .$$

It is clear that if A has real rank zero, then A has small eigenvalue variation. Conversely, it was proved in [**BBEK**] that if $\dim \Omega_{n,j} \leq 2$ for all n and j, and A has small eigenvalue variation, then A has real rank zero. The proof is a simple consequence of the fact (see [**CE**]) that if Ω is a compact metrizable space of dimension at most two, then any function from Ω into the self-adjoint matrices in M_n can be approximated by a function into the self-adjoint matrices with nondegenerate spectrum at each point. This fails if $\dim \Omega \geq 3$, but, as was first pointed out in [**DNNP**], the difficulty presented by higher dimensionality can be overcome by use of the Michael selection theorem. We shall use this theorem in a way which is slightly different from the way it was used in [**DNNP**] (and in [**BDR**]), namely, to construct approximate eigenprojections in $\mathrm{C}(\Omega, \mathrm{M}_n)$.

Using this technique, we have obtained a necessary and sufficient condition for real rank zero.

We shall say that $A = \varinjlim A_n$ has very small eigenvalue variation if each algebra A_n contains a set D_n of self-adjoint elements such that the union of the images of the sets D_n in A is dense in the set of self-adjoint elements of A (although D_n is not assumed to be dense in the self-adjoint elements of A_n), and such that, for any self-adjoint element x of D_n and any $\varepsilon > 0$, there exist $m > n$ and $P \in \mathcal{P}(\Omega_m)$ such that the image h of x in any of the direct summands $\mathrm{C}(\Lambda, \mathrm{M}_k)$ of A_m with $\Lambda \in P$ has eigenvalue variation at most ε and, in addition, if $[a, b]$ is any interval of length at least ε inside the spectrum, $\mathrm{Sp}(h)$, of h then $\mathrm{Sp}(h(\omega)) \cap [a, b]$ contains at least $\dim \Lambda$ points (counted with multiplicity) for each $\omega \in \Lambda$.

A simple argument shows that, in this definition, we may equivalently replace $\dim \Lambda$ by $\delta \dim \Lambda$ where δ is any fixed positive number. (Break up the interval of length ε into smaller intervals.)

The reason for introducing this apparently somewhat unwieldy condition is that not only does it readily follow from the known sufficient conditions for real rank zero, but in fact it is also necessary. It also turns out to be sufficient:

Theorem 1.1. *Adopt the general assumptions above on* $A = \varinjlim A_n$.

The following properties are equivalent.

1. *A has very small eigenvalue variation.*
2. *A has real rank zero.*

Before stating a number of corollaries of this result, let us remark that the proof of $2 \Rightarrow 1$ is surprisingly simple — in fact, trivial: If A has real rank zero, then D_n may just be taken to be the set of self-adjoint elements in A_n with finite spectrum.

We will defer the proof of $1 \Rightarrow 2$ to Section 4.

Let us consider various corollaries of Theorem 1.1.

Corollary 1.2. *Adopt the general assumptions above on* $A = \varinjlim A_n$.

The following properties are equivalent.

1. *Each* A_n *contains a set* D_n *of self-adjoint elements such that the union of the images of the sets* D_n *in* A *is dense in the set of self-adjoint elements of* A, *and such that for any* n *and any self-adjoint element* $x \in D_n$,

$$\lim_{m\to\infty} \inf_{P\in\mathcal{P}(\Omega_m)} \max_{\Lambda\in P} \{\dim(\Lambda)\,\mathrm{EV}(\Phi_{\Lambda,\Omega_n}(x))\} = 0\ .$$

2. *A has real rank zero.*

Proof. The implication $2 \Rightarrow 1$ is trivial just as in Theorem 1.1. To prove $1 \Rightarrow 2$, we have only to prove that the present condition 1 implies the condition 1 of Theorem 1.1, with the D_n's in the definition of very small eigenvalue variation being the present D_n's. So, let $x = x^* \in D_n$ and $\varepsilon > 0$ be given, and use the present condition 1 to choose m such that

$$\dim(\Omega_{m,j})\,\mathrm{EV}\,(\Phi_{m,j,n}(x)) < \frac{\varepsilon}{3}$$

for $j = 1, \cdots, r_m$. Fix j, and put $\Omega = \Omega_{m,j}$, $h = \Phi_{m,j,n}(x)$. If $\dim\Omega = 0$, then h can be approximated arbitrarily closely by a self-adjoint element with finite spectrum, and there is nothing to prove. If $\dim(\Omega) \geq 1$, then

$$\mathrm{EV}(h) < \frac{\varepsilon}{3\dim\Omega}\ ,$$

i.e., after passing to a possibly finer partition, each eigenvalue in the eigenvalue list of $h(\omega)$ varies by at most $\varepsilon/3\dim\Omega$ as ω varies over Ω. As $\mathrm{Sp}(h) = \bigcup_{\omega\in\Omega} \mathrm{Sp}(h(\omega))$ (without closure since Ω is compact), it follows that, in order that an interval $[a, b]$ of length at least ε be contained in the spectrum of h (i.e., contain no spectral gap), $h(\omega)$ must contain at least

$$\frac{\varepsilon}{\mathrm{EV}(h)} - 2 > 3\dim\Omega\ -\ 2 \geq \dim\Omega$$

eigenvalues inside $[a,b]$ for each ω. This shows that A has very small eigenvalue variation.

Corollary 1.3. *Adopt the general assumptions above on $A = \varinjlim A_n$, but assume also that $\dim \Omega_{m,j}$ is bounded uniformly in m and j.*

The following properties are equivalent.

1. *A has small eigenvalue variation.*
2. *A has real rank zero.*

Proof. The implication $1 \Rightarrow 2$ follows immediately from Corollary 1.2 (with $D_n = A_n^{\mathrm{s.a.}}$), and the implication $2 \Rightarrow 1$ is trivial (cf. [**BBEK**]).

In the case that A is simple and unital, a stronger form of this result was proved in [**BDR**] — with bounded dimension replaced by slow dimension growth (as defined above). (Actually, in place of the condition 1 the result of [**BDR**] had the condition that traces should be separated by projections, but in [**BBEK**] this was shown to be equivalent to the present condition 1 — assuming that A is simple and unital, but with no restriction on the dimensions.)

Using Theorem 1.1, we can establish a similar strengthening of Corollary 1.3 in the general case (i.e., A not necessarily either simple or unital).

This requires an extension of the definition of slow dimension growth to the general case. Let us say that A has slow dimension growth (with respect to the sequence $A_1 \to A_2 \to \cdots$) if, whenever A is replaced by eAe where e is a projection in A_k for some k, the condition

$$\lim_{n\to\infty} \max_j \left\{ \frac{\dim \Omega_{n,j}}{j} \right\} = 0$$

still holds.

Corollary 1.4. *Adopt the general assumptions above on $A = \varinjlim A_n$. Assume also that A has slow dimension growth (see above).*

The following properties are equivalent.

1. *A has small eigenvalue variation.*
2. *A has real rank zero.*

Proof. $2 \Rightarrow 1$ is trivial (cf. [**BBEK**]).

In order to prove $1 \Rightarrow 2$, we state a lemma which is proved in [**Stev**]. (In the case that A is simple the lemma was proved in [**Ell1**].)

Lemma 1.5. *Adopt the general assumptions above on $A = \varinjlim A_n$. Suppose that every closed two-sided ideal of A is generated (as an ideal) by its projections. If $n \in \mathbb{N}$, $\varepsilon > 0$ and $x = x^* \in A_n$ are given, then there exist $\Lambda > 0$*

and $m_0 \in \mathbb{N}$, and $P \in \mathcal{P}(\Omega_{m_0})$, such that for each $k \in \mathbb{Z}$ and each $\Lambda \in P$, if h denotes the image of x in the direct summand of A_{m_0} with spectrum Λ, then either the spectrum of h is disjoint from the interval $[k\varepsilon, (k+1)\varepsilon]$, or at least the fraction δ of the non-zero eigenvalues of $h(\omega)$ are contained in $[(k-1)\varepsilon, (k+2)\varepsilon]$ for each $\omega \in \Lambda$.

Let us use this lemma to show that if A has small eigenvalue variation then A in fact has very small eigenvalue variation, with respect to the sequence of subsets $D_n = A_n^{\text{s.a.}}$.

First, we must show that if A has small eigenvalue variation, then the hypothesis of Lemma 1.5 holds. (This is vacuous in the simple case.) Let I be a closed two-sided ideal of A, and denote by I_n the pre-image of I in A_n. Let x be a positive element of I_n, and let us show that for every $\varepsilon > 0$ there exists $m > n$ such that the image of x in A_m is within ε of a direct summand of A_m contained in I_m.

Choose $m > n$ such that the eigenvalue variation of the image of x in A_m is strictly less than ε. Then there exists a partition $P \in \mathcal{P}(\Omega_m)$ such that, for each $\Lambda \in P$, each eigenvalue of x in the direct summand of A_m with spectrum Λ has variation (as a function) at most ε. Hence, either the image of x in this direct summand is invertible — so that the direct summand belongs to I_m — or it has norm at most ε, as desired.

Let x be a self-adjoint element of A_n, and let $\varepsilon > 0$ be given. Let $\delta > 0$ and $m_0 \in \mathbb{N}$ be as given by Lemma 1.5. Using small eigenvalue variation, replacing m_0 by a larger number we may suppose that for any $m \geq m_0$ the image of x in A_m has eigenvalue variation at most ε.

Let $P_0 \in \mathcal{P}(\Omega_{m_0})$ be as given by Lemma 1.5. Using slow dimension growth, choose $m \geq m_0$ such that, for every $\Lambda \in P_0$, after replacing A by $e(\Lambda)Ae(\Lambda)$ where $e(\Lambda) \in A_{m_0}$ is the unit of the direct summand of A_{m_0} with spectrum Λ, we have

$$\max_j \frac{\dim \Omega_{m,j}}{j} \leq \delta \, .$$

(Of course, we still keep the original A; we use this form of expression only for convenience.)

Now choose $P \in \mathcal{P}(\Omega_m)$ such that for every $\Lambda \in P_0$, the image of the projection $e(\Lambda) \in A_{m_0}$ in A_m has central support a union of central projections corresponding to clopen sets in the partition P.

By the choice of m_0 and P_0, for each $k \in \mathbb{Z}$ and each $\Lambda \in P_0$, with h_0 the image of x in $e(\Lambda)A_{m_0}$, either the spectrum of h_0 is disjoint from the interval $[k\varepsilon, (k+1)\varepsilon]$ or at least the fraction δ of the non-zero eigenvalues of $h_0(\omega)$ are contained in $[(k-1)\varepsilon, (k+2)\varepsilon]$ for each $\omega \in \Lambda$.

Fix $k \in \mathbb{Z}$. Denote by $e \in A_{m_0}$ the sum of the projections $e(\Lambda)$ with $\Lambda \in P_0$ such that the interval $[k\varepsilon, (k+1)\varepsilon]$ is not disjoint from the spectrum of the image of x in $e(\Lambda)A_{m_0}$. Then the interval $[k\varepsilon, (k+1)\varepsilon]$ is disjoint from the spectrum of the image of x in $(1-e)A_{m_0}(1-e)$ and at least the fraction δ of the non-zero eigenvalues of $h_0(\omega)$ are contained in $[(k-1)\varepsilon, (k+2)\varepsilon]$ for each ω in the spectrum of $eA_{m_0}e$, where h_0 denotes the image of x in $eA_{m_0}e$. It follows that this is also true with m in place of m_0. (The condition that the fraction of non-zero eigenvalues of $h_0(\omega)$ be at least δ for all ω in the spectrum of $eA_{m_0}e$ (or eA_me) is just the condition that all normalized traces evaluated on certain continuous functions of h_0 be at least δ, and every normalized trace of eA_me restrict to a normalized trace of $eA_{m_0}e$; cf. **[Ell1]**, **[Stev]**.)

By the choice of m, the actual number of these non-zero eigenvalues for ω in the spectrum of eA_me and also in $\Omega_{m,j}$ is at least

$$\delta j \geq \dim \Omega_{m,j} .$$

This shows that, for every $\Lambda_1 \in P$, for every $k \in \mathbb{Z}$, either the interval $[k\varepsilon, (k+1)\varepsilon]$ is disjoint from the spectrum of the image h of x in the direct summand of A_m with spectrum Λ_1, or else at least $\dim \Lambda_1$ eigenvalues of $h(\omega)$ are contained in $[(k-1)\varepsilon, (k+2)\varepsilon]$ for every $\omega \in \Lambda_1$. (With j such that $\Omega_{m,j} \supseteq \Lambda_1$, one has $\dim \Omega_{m,j} \geq \dim \Lambda_1$.)

Hence, for any $\Lambda_1 \in P$, if $[a, b]$ is any interval of length 4ε inside the spectrum of h (necessarily containing some interval $[(k-1)\varepsilon, (k+2)\varepsilon]$), $\mathrm{Sp}(h(\omega)) \cap [a, b]$ contains at least $\dim \Lambda_1$ points (counting multiplicity).

This shows that A has very small eigenvalue variation. Hence by Theorem 1.1, A has real rank zero.

2. Connectedness of frame spaces.

Let $\mathrm{V}_k(\mathbb{C}^n)$ denote the Stiefel manifold of k-orthogonal frames in $\mathbb{C}^n$ and $\mathrm{G}_k(\mathbb{C}^n)$ the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$. Consider the canonical map

$$\pi : \mathrm{V}_k(\mathbb{C}^n) \to \mathrm{G}_k(\mathbb{C}^n)$$

which to each frame associates the linear span of the vectors in it.

Let $\mathbb{C}^p \subseteq \mathbb{C}^n$ denote the p-dimensional supspace of $\mathbb{C}^n$ spanned by the p first coordinate vectors, and let $\mathbb{C}^{n-p} \subseteq \mathbb{C}^n$ denote the subspace spanned by the $n-p$ last coordinate vectors. Assume that $p \leq k \leq n$. There is an inclusion

$$i_p : \mathrm{G}_{k-p}(\mathbb{C}^{n-p}) \to \mathrm{G}_k(\mathbb{C}^n)$$

given by $i_p(V) = \mathbb{C}^p \oplus V$. Set

$$B = \{F \in \mathrm{V}_k(\mathbb{C}^n) \mid \pi(F) \in \ \mathrm{Im}\ i_p\} .$$

Thus, B is the space of k-frames whose associated subspace contains $\mathbb{C}^p$.

Let π_i denote the ith homotopy group.

Lemma 2.1. $\pi_i(B) = 1$ *for*

$$i < 2\min(k-p, n-k) .$$

Proof. The standard frame in $\mathbb{C}^p$, combined with a $(k-p)$-frame in $\mathbb{C}^{n-p}$, gives a k-frame in B. Thus, the natural fibration

$$\mathrm{U}(k-p) \to \mathrm{V}_{k-p}(\mathbb{C}^{n-p}) \to \mathrm{G}_{k-p}(\mathbb{C}^{n-p})$$

maps in a natural way into the fibration

$$\mathrm{U}(k) \to B \to \mathrm{G}_{k-p}(\mathbb{C}^{n-p}) ,$$

where the map $B \to \mathrm{G}_{k-p}(\mathbb{C}^{n-p})$ consists in taking the orthogonal complement of $\mathbb{C}^p$ in $\pi(F)$ for $F \in B$.

The map $\mathrm{G}_{k-p}(\mathbb{C}^{n-p}) \to \mathrm{G}_{k-p}(\mathbb{C}^{n-p})$ is the identity.

In the associated map of long exact sequences

$$\begin{tikzcd}
\pi_{i+1}(\mathrm{G}_{k-p}(\mathbb{C}^{n-p})) \arrow[r] \arrow[d] & \pi_i(\mathrm{U}(k-p)) \arrow[r] \arrow[d] & \pi_i(\mathrm{V}_{k-p}(\mathbb{C}^{n-p})) \arrow[r] \arrow[d] & \pi_i(\mathrm{G}_{k-p}(\mathbb{C}^{n-p})) \cdots \arrow[d] \\
\pi_{i+1}(\mathrm{G}_{k-p}(\mathbb{C}^{n-p})) \arrow[r] & \pi_i(\mathrm{U}(k)) \arrow[r] & \pi_i(B) \arrow[r] & \pi_i(\mathrm{G}_{k-p}(\mathbb{C}^{n-p})) \cdots ,
\end{tikzcd}$$

every third vertical map is therefore an isomorphism.

If $i < 2(k-p)$, by Theorem I.7.4.1 of [**Hus**] the map $\pi_i(\mathrm{U}(k-p)) \to \pi_i(\mathrm{U}(k))$ is also an isomorphism, and hence by the five lemma the map $\pi_i(\mathrm{V}_{k-p}(\mathbb{C}^{n-p})) \to \pi_i(B)$ is an isomorphism. By Theorem I.7.5.1 of [**Hus**], the group $\pi_i(\mathrm{V}_{k-p}(\mathbb{C}^{n-p}))$ is trivial if $i < 2((n-p)-(k-p)) = 2(n-k)$. It follows that $\pi_i(B)$ is trivial for $i < 2\min(k-p, n-k)$.

3. The selection theorem.

If Ω is a compact Hausdorff space, and F is a map from Ω into the space of self-adjoint elements of M_n, we shall say that F is upper semicontinuous if for every vector $\xi \in \mathbb{C}^n$ the real-valued function

$$\omega \mapsto (F(\omega)\xi \mid \xi)$$

is upper semicontinuous. We shall say that F is lower semicontinuous if these functions are lower semicontinuous.

Clearly, F is continuous if and only if F is both upper and lower semicontinuous.

If $x = x^* \in \mathrm{C}(\Omega, \mathrm{M}_n)$, i.e., x is a continuous function from Ω into the self-adjoint elements of M_n, and $P(]-\infty, \lambda[)$ and $P(]-\infty, \lambda])$ denote the spectral projections of x corresponding to $\lambda \in \mathbb{R}$, then $\omega \mapsto P(]-\infty, \lambda[)(\omega)$ is lower semicontinuous, as the corresponding characteristic function is lower semicontinuous, and, similarly, $\omega \mapsto P(]-\infty, \lambda])(\omega)$ is upper semicontinuous.

Our selection theorem is as follows.

Theorem 3.1. *Let Ω be a compact metrizable Hausdorff space of dimension d, and let P and Q be maps from Ω into the projections of M_n such that P is lower semicontinuous and Q is upper semicontinuous. Suppose that*

$$P(\omega) \geq Q(\omega)$$

for each $\omega \in \Omega$, and that, furthermore, there exists a natural number k such that

$$\dim P(\omega) > k + \frac{1}{2}(d+1)$$

for all $\omega \in \Omega$, and

$$\dim Q(\omega) < k - \frac{1}{2}(d+1)$$

for all $\omega \in \Omega$.

It follows that there exists a continuous map $\omega \in \Omega \mapsto R(\omega)$, from Ω into the k-dimensional projections of M_n, such that

$$P(\omega) \geq R(\omega) \geq Q(\omega)$$

for all $\omega \in \Omega$.

Proof. We will partly follow [**DNNP**] and use Michael's selection theorem, Theorem 1.2 of [**Mi**], which states: If Ω is a compact metrizable space of dimension d, T is a complete metric space, and Y is a map from Ω to the nonempty closed subsets of T such that

(a) Y is lower semicontinuous, i.e., for each open subset U of T the set $\{\omega \in \Omega \mid Y(\omega) \cap U \neq \emptyset\}$ is open,

(b) each $Y(\omega)$ is $(d+1)$-connected, i.e., $\pi_i(Y(\omega)) = 0$ for $i = 0, 1, \cdots, d+1$, and

(c) there is an $\varepsilon > 0$ such that for any $0 < r < \varepsilon$ and $\omega \in \Omega$, the intersection of $Y(\omega)$ with any closed ball of radius r in T is a contractible space,

then there is a continuous map $p : \Omega \to T$ such that $p(\omega) \in Y(\omega)$ for all $\omega \in \Omega$.

We will apply this selection theorem in the case $T = \mathrm{V}_k(\mathbb{C}^n)$ = the Stiefel manifold of k-orthogonal frames in $\mathbb{C}^n$, where k, n are as in the statement of the theorem, and

$$Y(\omega) = \{F \in T \mid Q(\omega) \le \pi(F) \le P(\omega)\} \ .$$

Let us verify the hypotheses of Michael's selection theorem.

Clearly, each $Y(\omega)$ is a nonempty closed subset of T.

Ad (a). If U is an open subset of $\mathrm{V}_k(\mathbb{C}^n)$, then $\pi(U)$ is an open subset of the set of k-dimensional projections on $\mathbb{C}^n$. Since $\Omega \mapsto Q(\omega)$ is upper semicontinuous and $\omega \mapsto P(\omega)$ is lower semicontinuous, it follows that the set

$$\{\omega \in \Omega \mid \ \exists\, q \in \pi(U)\,,\ Q(\omega) \le q \le P(\omega)\}$$

is open, and is equal to the set

$$\{\omega \in \Omega \mid Y(\omega) \cap U \neq \emptyset\} \ .$$

Ad (b). Lemma 2.1 implies that

$$\pi_i\left(Y(\omega)\right) = 0$$

for $i < 2\min(k - \dim Q(\omega), \dim P(\omega) - k)$. Since $\min(k - \dim Q(\omega), \dim P(\omega) - k) > \frac{1}{2}(d+1)$, it follows in particular that $\pi_i(Y(\omega)) = 0$ for $0 \le i \le d+1$, and so (b) holds.

Ad (c). The local connectivity of $Y(\omega)$ is trivial in our case.

Applying Michael's selection theorem, we obtain a continuous map $F : \Omega \to \mathrm{V}_k(\mathbb{C}^n)$. Set

$$R(\omega) = \pi\left(F(\omega)\right) \ .$$

Then $\omega \mapsto R(\omega)$ is a continuous map from Ω into the self-adjoint k-dimensional projections in M_n such that

$$P(\omega) \ge R(\omega) \ge Q(\omega)$$

for all $\omega \in \Omega$, as desired.

4. Sufficiency of very small eigenvalue variation.

In this section we shall prove the remaining implication $1 \Rightarrow 2$ in Theorem 1.1. Suppose that A has very small eigenvalue variation, and let $x = x^* \in A$. We wish to approximate x by a self-adjoint element with finite spectrum, and to begin with we may approximate x by an element of some D_n (where

D_n is as specified just before the statement of Theorem 1.1). Thus, we may assume that $x \in D_n$. By the definition of very small eigenvalue variation, for any $\varepsilon > 0$ there is an $m > n$ such that the image h of x in any of the direct summands $\mathrm{C}(\Omega, \mathrm{M}_k)$ of A_m has eigenvalue variation strictly less than $\varepsilon/3$, and, in addition, if $[a, b]$ is any interval of length at least $\varepsilon/3$ inside the spectrum of h, then $\mathrm{Sp}(h(\omega)) \cap [a, b]$ contains at least $\dim(\Omega) + 2$ points. (This is provided $\dim \Omega \neq 0$; in the trivial case $\dim \Omega = 0$, h may easily be approximated by elements of finite spectrum.) We need to approximate h by an element y with finite spectrum. Choose $a \in \mathbb{R}$ and $j \in \mathbb{N}$ such that $\mathrm{Sp}(h) \subseteq [a, a + j\varepsilon]$, and for each $i = 1, 2, \cdots, j$ set

$$\begin{aligned} P_i^- = {} & \text{spectral projection of } h \\ & \text{corresponding to the interval} \\ &]-\infty, a + (i-1)\varepsilon]\ , \\ P_i^+ = {} & \text{spectral projection of } h \\ & \text{corresponding to the interval} \\ &]-\infty, a + i\varepsilon[\ . \end{aligned}$$

Thus,

$$P_1^- \leq P_1^+ \leq P_2^- \leq \cdots \leq P_j^+ ,$$

the map $\omega \mapsto P_i^-(\omega)$ is upper semicontinuous from Ω into the space of projections in M_k, and the map $\omega \mapsto P_i^+(\omega)$ is lower semicontinuous.

Given $i \in \{1, 2, \cdots, j\}$, there are two possibilities.

Case 1. $[a + (i-1)\varepsilon + \varepsilon/3, a + i\varepsilon - \varepsilon/3] \not\subseteq \mathrm{Sp}(h)$.

In this case, choose λ in this interval which is not in $\mathrm{Sp}(h)$, and consider the spectral projection P_i of h corresponding to the interval $]-\infty, \lambda]$. Then $\omega \mapsto P_i(\omega)$ is continuous, i.e., $P_i \in \mathrm{C}(\Omega, \mathrm{M}_k)$, and

$$P_i^- \leq P_i \leq P_i^+ .$$

Case 2. $I := [a + (i-1)\varepsilon + \varepsilon/3\,,\ a + i\varepsilon - \varepsilon/3] \subseteq \mathrm{Sp}(h)$.

In this case, $\mathrm{Sp}(h(\omega)) \cap I$ contains at least $\dim(\Omega) + 2$ points for each $\omega \in \Omega$. But since each eigenvalue of $h(\omega)$ varies by strictly less than $\varepsilon/3$ as ω varies over Ω, it follows that there exists an integer l such that

$$\dim P_i^-(\omega) < l - \frac{1}{2}(\dim(\Omega) + 1)$$

and

$$\dim P_i^+(\omega) > l + \frac{1}{2}(\dim(\Omega) + 1)$$

for each $\omega \in \Omega$. From Theorem 3.1 (the selection theorem), it follows that there exists a continuous map $\omega \in \Omega \mapsto P_i(\omega)$ from Ω into the l-dimensional projections in M_k such that

$$P_i^-(\omega) \leq P_i(\omega) \leq P_i^+(\omega)$$

for all $\omega \in \Omega$. Then $P_i \in C(\Omega, \mathrm{M}_k)$ and

$$P_i^- \leq P_i \leq P_i^+ .$$

Now, set

$$y = \sum_{i=1}^{j+1} (a + (i-1)\varepsilon)\,(P_i - P_{i-1}) ,$$

with the conventions $P_0 = 0$, $P_{j+1} = 1$. Then $y = y^* \in \mathrm{C}(\Omega, \mathrm{M}_k)$ and y has finite spectrum. Set

$$z = \sum_{i=1}^{j+1} \left(a + \left(i - \frac{1}{2}\right)\varepsilon\right)(P_i^+ - P_{i-1}^+) ,$$

again with the conventions $P_0^+ = 0$, $P_{j+1}^+ = 1$. By spectral theory,

$$\|z - h\| \leq \frac{\varepsilon}{2} .$$

But since

$$P_0^+ \leq P_1 \leq P_1^+ \leq P_2 \leq P_2^+ \leq \cdots \leq P_j^+ \leq P_{j+1} ,$$

it follows by using spectral theory in the finite dimensional abelian C^*-algebra generated by $P_i, P_i^+, i = 0, \cdots, j+1$, that

$$\|y - z\| \leq \frac{\varepsilon}{2} .$$

We conclude that

$$\|y - h\| \leq \varepsilon .$$

This ends the proof of Theorem 1.1.

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References

[BBEK] B. Blackadar, O. Bratteli, G. A. Elliott, and A. Kumjian, *Reduction of real rank in inductive limits of C^*-algebras*, Math. Ann., **292** (1992), 111–126.

[BDR] B. Blackadar, M. Dădărlat, and M. Rørdam, *The real rank of inductive limit C^*-algebras*, Math. Scand., **69** (1991), 211–216.

[BEEK] O. Bratteli, G.A. Elliott, D.E. Evans, and A. Kishimoto, *On the classification of C^*-algebras of real rank zero,* III: *The infinite case, Operator Algebras and their Applications* (editors, P.A. Fillmore and J.A. Mingo), Fields Institute Communications, American Mathematical Society, to appear.

[CE] M.-D. Choi and G.A. Elliott, *Density of the self-adjoint elements with finite spectrum in an irrational rotation C^*-algebra*, Math. Scand., **167** (1990), 73–86.

[DNNP] M. Dădărlat, G. Nagy, A. Némethi, and C. Pasnicu, *Reduction of topological stable rank in inductive limits of C^*-algebras*, Pacific J. Math., **153** (1992), 267–276.

[Ell0] G.A. Elliott, *On the classification of C^*-algebras of real rank zero*, J. Reine Angew. Math., **443** (1993), 179–219.

[Ell1] ———, *A classification of certain simple C^*-algebras, Quantum and Non-Commutative Analysis* (editors, H. Araki et al.), Kluwer, Dordrecht, 1993, 373–385.

[EG] G.A. Elliott and G. Gong, *On the classification of C^*-algebras of real rank zero*, II, Ann. of Math., to appear.

[Eng] R. Engelking, *Dimension Theory*, North Holland, Amsterdam, 1978.

[Hus] D. Husemoller, *Fibre Bundles*, McGraw-Hill, New York, 1966.

[Mi] E. Michael, *Continuous selections* II, Ann. of Math., **164** (1956), 562–580.

[Stev] K.H. Stevens, *The classification of certain non-simple approximate interval algebras*, Ph.D. thesis, University of Toronto, 1994.

[Su] H. Su, *On the classification of C^*-algebras of real rank zero: inductive limits of matrix algebras over non-Hausdorff graphs*, Ph.D. thesis, University of Toronto, 1992, and Memoirs Amer. Math. Soc., **547** (1995).

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