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## SHARING VALUES AND A PROBLEM DUE TO C.C. YANG

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In this paper, we proved a unicity theorem for meromorphic functions with one sharing pair and a condition on deficiency. An example shows that the condition on deficiency is best possible. This result gives a general answer to the problem due to C.C. Yang (1977).

### 1. Introduction.

In this paper, by meromorphic function we always mean a function which is meromorphic in the plane. Let  $f(z)$  be meromorphic. We shall use the following standard notations in Nevanlinna theory:

$$T(r, f), \quad m(r, f), \quad N(r, f), \dots$$

(see Gross [5]). We denote by  $S(r, f)$  any function satisfying

$$S(r, f) = o\{T(r, f)\}$$

as  $r \rightarrow +\infty$ , possibly outside a set of finite Lebesgue measure. A meromorphic function  $a(z)$  is said to be a small function of  $f$  if

$$T(r, a) = S(r, f).$$

In this case, we define

$$\delta(a, f) = 1 - \lim_{r \rightarrow \infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)},$$

and  $a(z)$  is said to be a deficient function of  $f$  if  $\delta(a, f) > 0$ .

Let  $g(z)$ ,  $a_1(z)$  and  $a_2(z)$  be meromorphic functions. If the two functions  $f(z) - a_1(z)$  and  $g(z) - a_2(z)$  assume the same zeros with the same multiplicities, then we call that  $f$  and  $g$  share the pair  $(a_1, a_2)$  CM. In particular, if  $a_1 = a_2 = a$ , then the word "the pair" is replaced by "the value" or "the function" provided that  $a$  is a constant or  $a$  is a function respectively (cf. Frank-Ohlenroth [4], Gundersen [6], etc.). In addition, if

$$N(r, (f = a_1)\Delta(g = a_2)) = \min\{S(r, f), S(r, g)\},$$

then we say that  $f$  and  $g$  almost share the pair  $(a_1, a_2)$  CM. Here,  $N(r, (f = a_1)\Delta(g = a_2))$  is the counting function of those points which satisfy one of the following three cases: (i)  $f = a_1$  but  $g \neq a_2$ ; (ii)  $f \neq a_1$  but  $g = a_2$ ; (iii)  $f = a_1$  and  $g = a_2$  but the multiplicities are not the same.

In 1977, Yang [9] proved the following result.

**Theorem A.** *Suppose that  $F$  is a family of the functions which are of the form  $\alpha_1(z)e^{\mu(z)} + \alpha_2(z)$ , where  $\mu(z)$  is an entire functions with finite order,  $\alpha_j(z)$  ( $j = 1, 2$ ) are meromorphic functions of finite order,  $\alpha_1 \not\equiv 0$ ,  $\alpha_2 \not\equiv \text{const.}$ , the order of  $\alpha_j$  ( $j = 1, 2$ ) is less than the order of  $\mu$ . Let  $c_1$  and  $c_2$  be two distinct constants, and let  $f \in F$ ,  $g \in F$ . If  $f$  and  $g$  share the two values  $c_1$  and  $c_2$  CM, then  $f \equiv g$  or*

$$\left(f - \frac{c_2 - c_1\lambda(z)}{1 - \lambda(z)}\right) \left(g + \frac{c_2 - c_1\lambda(z)}{1 - \lambda(z)}\right) = -\frac{(c_2 - c_1)^2\lambda(z)}{(1 - \lambda(z))^2},$$

where  $\lambda(z)$  is a nonconstant meromorphic function.

Based on this result, Yang [9] proposed the following problem.

**Yang's problem.** *Whether can we omit the restrictions on the order in the family  $F$ ?*

It is easy to see from the hypotheses of Theorem A that, if  $f = \alpha_1(z)e^{\mu(z)} + \alpha_2(z) \in F$  and  $g = \alpha_3(z)e^{\nu(z)} + \alpha_4(z) \in F$ , then  $N(r, \frac{1}{f-\alpha_2}) = o\{T(r, f)\}$  and  $N(r, \frac{1}{g-\alpha_4}) = o\{T(r, g)\}$ . Thus

$$(1) \quad \delta(\alpha_2, f) = \delta(\alpha_4, g) = 1,$$

$$(2) \quad \delta(\infty, f) = \delta(\infty, g) = 1.$$

These observations lead to our main result.

**Theorem 1.** *Let  $f(z)$ ,  $g(z)$ ,  $a(z)$ ,  $b(z)$ ,  $\alpha(z)$  and  $\beta(z)$  be meromorphic functions in the plane, where  $a(z)$  and  $\alpha(z)$  are small functions of  $f$ ,  $b(z)$  and  $\beta(z)$  are small functions of  $g(z)$ ,  $a(z) \not\equiv \alpha(z)$ ,  $b(z) \not\equiv \beta(z)$ . Suppose that  $f$  and  $g$  share the pair  $(a, b)$  CM and*

$$(3) \quad \delta = \delta(\alpha, f) + \delta(\beta, g) + \delta(\infty, f) + \delta(\infty, g) > 3.$$

Then either

$$(4) \quad \frac{f - \alpha}{a - \alpha} = \frac{g - \beta}{b - \beta}$$

or

$$(5) \quad \frac{f - \alpha g - \beta}{a - \alpha b - \beta} = 1.$$

**Remark 1.** The number 3 in the inequality (3) is sharp.

For example, let  $P$  and  $Q$  be two nonzero polynomials,  $f = e^{2z} - Qe^z$ ,  $g = \frac{e^{2z}P}{e^z + Q}$ . Then one can check that  $f$  and  $g$  share the pair  $(P, Q)$  CM and

$$\delta(0, f) + \delta(0, g) + \delta(\infty, f) + \delta(\infty, g) = \frac{1}{2} + 1 + 1 + \frac{1}{2} = 3.$$

However,  $\frac{f}{P} \not\equiv \frac{g}{Q}$  and  $\frac{f}{P} \frac{g}{Q} \not\equiv 1$ .

**Remark 2.** Note that Theorem A needs two shared values. However, in our theorem 1, we only need one shared pair.

**Remark 3.** The topic on unicity theorem concerning deficiency were studied by Ozawa [7], Ueda [8] etc. The case that  $f$  and  $g$  are entire functions and  $a(z) = b(z) = 1$  was considered by Yi [10].

**Remark 4.** From the proof of Theorem 1 we see that the word “share” can be replaced by “almost share”.

As an application, we obtain the following

**Corollary.** *The answer to Yang’s problem is affirmative.*

## 2. Some Symbols.

For the sake of convenience, we shall use some symbols introduced by Chuang [1] and Chuang-Hua [2].

For meromorphic function  $f(z)$  and a point  $z$ , according as  $z$  is a pole of  $f$  or not, we denote by  $\omega(f, z)$  the multiplicity of  $z$  or 0 and by  $\bar{\omega}(f, z)$  the value 1 or 0. For three meromorphic functions  $f, g$  and  $h$ , we divide the set of the poles of  $f$  and  $g$  on  $\{|z| \leq r\}$  into five pairwise disjoint subsets as follows:

$$\begin{aligned} V_1 &=: \{z : f(z) \neq \infty, g(z) = \infty\}, \\ V_2 &=: \{z : f(z) = \infty, g(z) \neq \infty\}, \\ V_3 &=: \{z : f(z) = \infty, g(z) = \infty, h(z) = \infty\}, \\ V_4 &=: \{z : f(z) = \infty, g(z) = \infty, h(z) \neq 0, \infty\}, \\ V_5 &=: \{z : f(z) = \infty, g(z) = \infty, h(z) = 0\}. \end{aligned}$$

Furthermore, for each  $j \in \{1, \dots, 5\}$ , we denote by  $n_j(f)$  and  $n_j(g)$  the number of the poles of  $f$  and  $g$  in the set  $V_j$  respectively, with due count of multiplicity. The corresponding counting functions are denoted by  $N_j(f)$  and  $N_j(g)$  respectively. Obviously,

$$(6) \quad N(r, f) = N_2(f) + N_3(f) + N_4(f) + N_5(f),$$

$$(7) \quad N(r, g) = N_1(g) + N_3(g) + N_4(g) + N_5(g).$$

### 3. One Basic Lemma.

For the proof of our results, we need the following lemma which can be found in Chuang-Yang [3, p. 39] or Gross [5, pp. 70-73].

**Lemma 1.** *Let  $f_j$  ( $j = 1, \dots, n \geq 2$ ) be  $n$  linearly independent meromorphic functions. If  $f_1 + \dots + f_n \equiv 1$ , then we have*

$$\begin{aligned} T(r, f_1) \leq & \sum_{j=1}^n N\left(r, \frac{1}{f_j}\right) + N(r, W) - N\left(r, \frac{1}{W}\right) \\ & - \sum_{j=2}^n N(r, f_j) + S(r, f_1) + \dots + S(r, f_n), \end{aligned}$$

where  $W = W(z)$  is the Wronskian of  $f_1, \dots, f_n$ .

### 4. Proof of Theorem 1.

Let

$$\begin{aligned} F_1 & =: \{z : a(z) = \infty\} \cup \{z : b(z) = \infty\} \cup \{z : \alpha(z) = \infty\} \cup \{z : \beta(z) = \infty\}, \\ F_2 & =: \{z : a(z) = \alpha(z)\} \cup \{z : b(z) = \beta(z)\}. \end{aligned}$$

Set

$$F =: F_1 \cup F_2,$$

the corresponding counting function is denoted by  $N_F(r)$ . Put

$$(8) \quad h(z) =: \frac{f(z) - a(z)}{g(z) - b(z)}.$$

Since  $a(z)$  and  $b(z)$  are small functions of  $f$  and  $g$  respectively, we know that  $h(z) \not\equiv 0, \infty$ . Let  $z_o$  be a pole of  $h$  with  $z_o \notin F$ . Since  $f$  and  $g$  share the

pair  $(a, b)$  CM, we have  $z_o \in V_2 \cup V_3$ . If  $z_o \in V_2$ , then  $\omega(h, z_o) = \omega(f, z_o)$ ; If  $z_o \in V_3$ , then  $\omega(h, z_o) = \omega(f, z_o) - \omega(g, z_o)$ . Thus

$$(9) \quad N_2(f) + N_3(f) - N_3(g) - N_F(r) \leq N(r, h) \leq N_2(f) + N_3(f) - N_3(g) + N_F(r).$$

Similarly we have

$$(10) \quad N\left(r, \frac{1}{h}\right) \leq N_1(g) + N_5(g) - N_5(f) + N_F(r).$$

Let

$$f_1 =: \frac{f - \alpha}{a - \alpha}, \quad f_2 =: -\frac{g - \beta}{a - \alpha}h, \quad f_3 =: \frac{b - \beta}{a - \alpha}h.$$

Then

$$(11) \quad N\left(r, \frac{1}{f_1}\right) \leq N\left(r, \frac{1}{f - \alpha}\right) + N_F(r),$$

$$(12) \quad N\left(r, \frac{1}{f_3}\right) \leq N\left(r, \frac{1}{h}\right) + N_F(r),$$

$$(13) \quad N(r, h) - N_F(r) \leq N(r, f_3) \leq N(r, h) + N_F(r).$$

From (8) it is easy to see that any zero of  $h$  which is not in the set  $F$  is not a zero of  $f_2$ . Thus

$$(14) \quad N\left(r, \frac{1}{f_2}\right) \leq N\left(r, \frac{1}{g - \beta}\right) + N_F(r).$$

Now for any pole  $z_o$  of  $f_2$  with  $z_o \notin F$ , we know that  $z_o$  is a pole of  $g$  or  $h$ . If  $z_o \in V_1$ , then  $\omega(g, z_o) = \omega(\frac{1}{h}, z_o)$ , and so,  $\omega(f_2, z_o) = 0$ ; If  $z_o \in V_2$ , then  $\omega(f_2, z_o) = \omega(h, z_o) = \omega(f, z_o)$ ; If  $z_o \in V_3$ , then  $\omega(h, z_o) = \omega(f, z_o) - \omega(g, z_o)$ , and so,  $\omega(f_2, z_o) = \omega(g, z_o) + \omega(h, z_o) = \omega(f, z_o)$ ; If  $z_o \in V_4$ , then  $h(z_o) \neq 0, \infty$  and  $\omega(f_2, z_o) = \omega(g, z_o)$ ; If  $z_o \in V_5$ , then  $\omega(\frac{1}{h}, z_o) = \omega(g, z_o) - \omega(f, z_o)$  and  $\omega(f_2, z_o) = \omega(g, z_o) - \omega(\frac{1}{h}, z_o) = \omega(f, z_o)$ . Combining all these facts we get

$$(15) \quad N_2(f) + N_3(f) + N_4(g) + N_5(f) - N_F(r) \leq N(r, f_2) \\ \leq N_2(f) + N_3(f) + N_4(g) + N_5(f) + N_F(r).$$

Next we rewrite (8) in the form

$$(16) \quad f_1 + f_2 + f_3 = 1.$$

Without loss of generality, we suppose that there exists a set  $I$  with infinite measure such that

$$(17) \quad T(r, g) \leq T(r, f), \quad r \in I.$$

(Otherwise, we only need to consider  $T(r, g)$  instead of  $T(r, f)$  in the following discussions.) Thus  $N_F(r) = S(r, f)$ ,  $r \in I$ .

In the sequel, we always let  $r \in I$ . Now we prove the following lemma.

**Lemma 2.**  $f_1, f_2$  and  $f_3$  are linearly dependent.

*Proof.* Suppose on the contrary that the  $f$ 's are linearly independent. By lemma 1,

$$\begin{aligned} T(r, f_1) &\leq \sum_{j=1}^3 N\left(r, \frac{1}{f_j}\right) + N(r, W) - N\left(r, \frac{1}{W}\right) - N(r, f_2) - N(r, f_3) \\ &\quad + S(r, f_1) + S(r, f_2) + S(r, f_3) \end{aligned}$$

where  $W$  is the Wronskian of  $f_1, f_2, f_3$ , i.e.,

$$W = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = - \begin{vmatrix} f_1' & f_3' \\ f_1'' & f_3'' \end{vmatrix}$$

by (16). Now by (10), (11), (12) and (14),

$$\begin{aligned} \sum_{j=1}^3 N\left(r, \frac{1}{f_j}\right) &\leq N\left(r, \frac{1}{f - \alpha}\right) + N\left(r, \frac{1}{g - \beta}\right) \\ &\quad + N_1(g) + N_5(g) - N_5(f) + 4N_F(r). \end{aligned}$$

In addition, by the inequalities on the left hand sides of (9), (13) and (15),

$$N(r, f_2) + N(r, f_3) \geq 2N_2(f) + 2N_3(f) + N_4(f) + N_5(f) - N_3(g) - 3N_F(r).$$

Combining the three inequalities above we get

$$\begin{aligned} T(r, f_1) &\leq N(r, W) - N\left(r, \frac{1}{W}\right) + N\left(r, \frac{1}{f - \alpha}\right) + N\left(r, \frac{1}{g - \beta}\right) \\ &\quad + N_1(g) + N_3(g) + N_5(g) \\ &\quad - 2N_2(f) - 2N_3(f) - N_4(f) - 2N_5(f) \\ &\quad + 7N_F(r) + S(r, f_1) + S(r, f_2) + S(r, f_3) \end{aligned}$$

$$\begin{aligned}
&= N(r, W) - N\left(r, \frac{1}{W}\right) + N\left(r, \frac{1}{f - \alpha}\right) + N\left(r, \frac{1}{g - \beta}\right) \\
&\quad + N_1(g) + N_3(g) + N_4(g) + N_5(g) \\
&\quad - 2N_2(f) - 2N_3(f) - 2N_4(f) - 2N_5(f) \\
&\quad - N_4(g) + N_4(f) \\
&\quad + 7N_F(r) + S(r, f_1) + S(r, f_2) + S(r, f_3).
\end{aligned}$$

Substituting (6) and (7) into the above inequality and using the facts that

$$N_4(f) = N_4(g), \quad T(r, f) = T(r, f_1) + S(r, f),$$

$$T(r, a), \quad T(r, b), \quad T(r, \alpha), \quad T(r, \beta) = S(r, f),$$

$$N_F(r), \quad S(r, f_j) = S(r, f), \quad (j = 1, \dots, 3)$$

we obtain

$$T(r, f) \leq N\left(r, \frac{1}{f - \alpha}\right) + N\left(r, \frac{1}{g - \beta}\right) + N(r, W) - N\left(r, \frac{1}{W}\right)$$

$$(18) \quad + N(r, g) - 2N(r, f) + S(r, f).$$

Next we estimate the term  $N(r, W) - N\left(r, \frac{1}{W}\right)$ . Since

$$(19) \quad W = -(f_1' f_3'' - f_1'' f_3'),$$

from the expressions of  $f_1$  and  $f_3$  we see that the poles of  $W$  only occur at the poles of  $f$  and the points in  $F$ . Let  $z_0$  be a pole of  $f$  with  $z_0 \notin F$ .

If  $z_0 \in V_2$ , then near  $z = z_0$ ,

$$f_1 = \frac{1}{(z - z_0)^{\omega(f, z_0)}} \{x + \mathcal{O}(z - z_0)\}, \quad f_3 = \frac{1}{(z - z_0)^{\omega(f, z_0)}} \{y + \mathcal{O}(z - z_0)\},$$

where  $x$  and  $y$  are nonzero constants. If  $\omega(f, z_0) \geq 2$ , then

$$f_1' f_3'' = \frac{1}{(z - z_0)^{2\omega(f, z_0) + 3}} \{-\omega(f, z_0)^2 (\omega(f, z_0) + 1) xy + \mathcal{O}(z - z_0)\},$$

$$f_1'' f_3' = \frac{1}{(z - z_0)^{2\omega(f, z_0) + 3}} \{-\omega(f, z_0)^2 (\omega(f, z_0) + 1) xy + \mathcal{O}(z - z_0)\},$$

and so,

$$f_1' f_3'' - f_1'' f_3' = \mathcal{O}\left\{\frac{1}{(z - z_0)^{2\omega(f, z_0) + 2}}\right\}.$$

If  $\omega(f, z_0) = 1$ , then

$$f_1' f_3'' = \frac{-2xy}{(z - z_0)^5} + \frac{\mathcal{O}(1)}{(z - z_0)^3} + \dots,$$

$$f_1'' f_3' = \frac{-2xy}{(z - z_0)^5} + \frac{\mathcal{O}(1)}{(z - z_0)^3} + \dots,$$

and so,

$$f_1' f_3'' - f_1'' f_3' = \mathcal{O}\left\{\frac{1}{(z - z_0)^3}\right\}.$$

Thus

$$\begin{aligned} \omega(W, z_0) &\leq \begin{cases} 2\omega(f, z_0) + 2, & \text{if } \omega(f, z_0) \geq 2 \\ 3, & \text{if } \omega(f, z_0) = 1 \end{cases} \\ &\leq 3\omega(f, z_0). \end{aligned}$$

If  $z_0 \in V_3$ , then  $\omega(g, z_0) \geq 1$  and  $\omega(f, z_0) \geq 2$ . Thus, by (19),

$$\begin{aligned} \omega(W, z_0) &\leq 2\omega(f, z_0) + 3 - \omega(g, z_0) \\ &\leq 2\omega(f, z_0) + 2 \leq 3\omega(f, z_0). \end{aligned}$$

If  $z_0 \in V_4$ , then  $\omega(f, z_0) = \omega(g, z_0)$ , and so,  $\omega(f_3, z_0) = 0$ . By (19), we get

$$\omega(W, z_0) \leq \omega(f, z_0) + 2 \leq 3\omega(f, z_0).$$

If  $z_0 \in V_5$ , and if  $z_0$  is a pole of  $W$ , then by (19),

$$\begin{aligned} \omega(W, z_0) &\leq \omega(f, z_0) + 2 \\ &\leq 3\omega(f, z_0). \end{aligned}$$

Combining all the cases above and noting (6), we deduce that

$$N(r, W) \leq 3N(r, f) + N_F(r).$$

This and (18) give

$$(20) \quad T(r, f) \leq N\left(r, \frac{1}{f - \alpha}\right) + N\left(r, \frac{1}{g - \beta}\right) + N(r, f) + N(r, g) + S(r, f).$$

Now by the definition of deficiency, for  $\epsilon = \frac{\delta - 3}{8} > 0$ , where  $\delta$  is the sum in (3), there exists  $r_0 > 0$  such that

$$N\left(r, \frac{1}{f - \alpha}\right) \leq (1 - \delta(\alpha, f) + \epsilon)T(r, f),$$

$$\begin{aligned} N\left(r, \frac{1}{g-\beta}\right) &\leq (1 - \delta(\beta, g) + \epsilon)T(r, g), \\ N(r, f) &\leq (1 - \delta(\infty, f) + \epsilon)T(r, f) \end{aligned}$$

and

$$N(r, g) \leq (1 - \delta(\infty, g) + \epsilon)T(r, g)$$

hold for  $r \in I$  and  $r > r_0$ . Substituting all these inequality into (20) and noting (17), we get  $\delta \leq 3$ , which contradicts our hypothesis. This completes the proof of the lemma.

Now by Lemma 2, there exist three constants  $c_1, c_2$  and  $c_3$  with

$$(21) \quad |c_1| + |c_2| + |c_3| \neq 0$$

and

$$(22) \quad c_1 f_1 + c_2 f_2 + c_3 f_3 = 0.$$

If  $c_1 = 0$ , then  $c_2 c_3 \neq 0$  and  $f_2 = -\frac{c_3}{c_2} f_3$ . This leads to  $g = \frac{c_3}{c_2} b(z) + \left(1 - \frac{c_3}{c_2}\right) \beta(z)$ , which contradicts the assumptions that  $b(z)$  and  $\beta(z)$  are small functions of  $g$ . Thus,  $c_1 \neq 0$ . We may suppose  $c_1 = -1$ , and (22) reads  $f_1 = c_2 f_2 + c_3 f_3$ . Combining this and (16) we obtain

$$(23) \quad (1 + c_2) f_2 + (1 + c_3) f_3 = 1.$$

Next we consider two cases.

(i)  $1 + c_2 = 0$ . Then  $1 + c_3 \neq 0$  and  $(1 + c_3) f_3 = 1$ . It follows from (8) and the definition of  $f_3$  between (10) and (11) that

$$\begin{aligned} (24) \quad f - \frac{c_3 a + \alpha}{1 + c_3} &= f - a + a - \frac{c_3 a + \alpha}{1 + c_3} \\ &= \left(\frac{1}{1 + c_3}\right) \frac{a - \alpha}{b - \beta} (g - b) + a - \frac{c_3 a + \alpha}{1 + c_3} \\ &= \left(\frac{1}{1 + c_3}\right) \frac{a - \alpha}{b - \beta} (g - \beta). \end{aligned}$$

If  $c_3 \neq 0$ , then  $\frac{c_3 a + \alpha}{1 + c_3} \neq \alpha$ . By the Nevanlinna “three-functions theorem” we deduce that

$$\begin{aligned} T(r, f) &\leq N(r, f) + N\left(r, \frac{1}{f - \alpha}\right) + N\left(r, \frac{1}{f - \frac{c_3 a + \alpha}{1 + c_3}}\right) + S(r, f) \\ &= N(r, f) + N\left(r, \frac{1}{f - \alpha}\right) + N\left(r, \frac{1}{g - \beta}\right) + S(r, f). \end{aligned}$$

This is impossible by the same reasoning as in the proof of Lemma 2. Therefore  $c_3 = 0$  and (24) reads

$$\frac{f - \alpha}{g - \beta} = \frac{a - \alpha}{b - \beta}.$$

This is what we need.

(ii)  $1 + c_2 \neq 0$ . It follows from (8), (23) and the definitions of  $f_2$  and  $f_3$  between (10) and (11) that  $-(1 + c_2)\frac{g-\beta}{a-\alpha} + (1 + c_3)\frac{b-\beta}{a-\alpha} = \frac{g-b}{f-a}$ , which can be written as

$$(25) \quad f - \frac{c_2 a + \alpha}{1 + c_2} = \left( \frac{c_2 - c_3}{(1 + c_2)^2} \right) \frac{(a - \alpha)(b - \beta)}{g - \frac{1+c_3}{1+c_2}b - \frac{c_2-c_3}{1+c_2}\beta}.$$

If  $c_2 \neq 0$ , then  $\frac{c_2 a + \alpha}{1 + c_2} \neq \alpha$ . By the “three-functions theorem”, we have

$$\begin{aligned} T(r, f) &\leq N(r, f) + N\left(r, \frac{1}{f - \alpha}\right) + N\left(r, \frac{1}{f - \frac{c_2 a + \alpha}{1 + c_2}}\right) + S(r, f) \\ &\leq N(r, f) + N\left(r, \frac{1}{f - \alpha}\right) + N(r, g) + S(r, f). \end{aligned}$$

By the same reasoning as in the proof of Lemma 2, we can get a contradiction. Thus  $c_2 = 0$ , and (25) reads

$$(26) \quad f - \alpha = -c_3 \frac{(a - \alpha)(b - \beta)}{g - (1 + c_3)b + c_3\beta}.$$

If  $c_3 = -1$ , then

$$(f - \alpha)(g - \beta) = (a - \alpha)(b - \beta),$$

as asserted. If  $c_3 \neq -1$ , then  $\frac{\alpha + c_3 a}{1 + c_3} \neq \alpha$  and (26) can be written as

$$f - \frac{\alpha + c_3 a}{1 + c_3} = -\left( \frac{c_3}{1 + c_3} \right) \frac{(a - \alpha)(g - \beta)}{g - (1 + c_3)b + c_3\beta}.$$

Thus, the “three-functions theorem” gives

$$\begin{aligned} T(r, f) &\leq N(r, f) + N\left(r, \frac{1}{f - \alpha}\right) + N\left(r, \frac{1}{f - \frac{\alpha + c_3 a}{1 + c_3}}\right) + S(r, f) \\ &\leq N(r, f) + N\left(r, \frac{1}{f - \alpha}\right) + N\left(r, \frac{1}{g - \beta}\right) + S(r, f). \end{aligned}$$

By the same reasoning as in the proof of Lemma 2 we obtain a contradiction.

This completes the proof of the theorem.  $\square$

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### References

- [1] C.T. Chuang, *Une généralisation d'une négativité de Nevanlinna*, Sci. Sinica, **13** (1964), 887-895.
- [2] C.T. Chuang and X.H. Hua, *On the growth of meromorphic functions*, Sci. Sinica (Sci. in China), **33** (1990), 1025-1033.
- [3] C.T. Chuang and C.C. Yang, *Fix-points and factorization of meromorphic functions*, World Sci. Publishing, Singapore 1990.
- [4] G. Frank and W. Ohlenroth, *Meromorphe Funktionen, die mit einer ihrer Ableitungen Werte teilen*, Complex Variables, **6** (1986), 23-27.
- [5] F. Gross, *Factorization of meromorphic functions*, U. S. Government Printing Office 1972.
- [6] G.G. Gundersen, *Meromorphic functions that share three or four values*, J. London Math. Soc., **20** (1979), 457-466.
- [7] M. Ozawa, *Unicity theorems for entire functions*, J. Analyse Math., **30** (1976), 411-420.
- [8] H. Ueda, *Unicity theorems for entire functions*, Kodai Math. J., **3** (1980), 212-223.
- [9] C.C. Yang, *On meromorphic functions taking the same values at the same points*, Kodai Math Sem. Rep., **28** (1977), 300-309.
- [10] H.X. Yi, *Meromorphic functions with two deficient values*, Acta Math. Sinica, **30** (1987), 588-597.

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# PACIFIC JOURNAL OF MATHEMATICS

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Homogeneous Ricci positive 5-manifolds	1
DIMITRI ALEKSEEVSKY, ISABEL DOTTI DE MIATELLO and CARLOS J. FERRARIS	
On the structure of tensor products of $\ell_p$ -spaces	13
ALVARO ARIAS and JEFFREY D. FARMER	
The closed geodesic problem for compact Riemannian 2-orbifolds	39
JOSEPH E. BORZELLINO and BENJAMIN G. LORICA	
Small eigenvalue variation and real rank zero	47
OLA BRATTELI and GEORGE A. ELLIOTT	
Global analytic hypoellipticity of $\square_b$ on circular domains	61
SO-CHIN CHEN	
Sharing values and a problem due to C. C. Yang	71
XIN-HOU HUA	
Commutators and invariant domains for Schrödinger propagators	83
MIN-JEI HUANG	
Chaos of continuum-wise expansive homeomorphisms and dynamical properties of sensitive maps of graphs	93
HISAO KATO	
Some properties of Fano manifolds that are zeros of sections in homogeneous vector bundles over Grassmannians	117
OLIVER KÜCHLE	
On polynomials orthogonal with respect to Sobolev inner product on the unit circle	127
XIN LI and FRANCISCO MARCELLAN	
Maximal subfields of $\mathbf{Q}(i)$ -division rings	147
STEVEN LIEDAHL	
Virtual diagonals and $n$ -amenability for Banach algebras	161
ALAN L. T. PATERSON	
Rational Pontryagin classes, local representations, and $K^G$ -theory	187
CLAUDE SCHOCHET	
An equivalence relation for codimension one foliations of 3-manifolds	235
SANDRA SHIELDS	
A construction of Lomonosov functions and applications to the invariant subspace problem	257
ALEKSANDER SIMONIČ	
Complete intersection subvarieties of general hypersurfaces	271
ENDRE SZABÓ	



0030-8730(1996)175:1;1-C