COMMUTATORS AND INVARIANT DOMAINS FOR
SCHRÖDINGER PROPAGATORS

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We present an operator-theoretic approach to the problem of invariant domains for the Schrödinger evolution equation. The results are applied to the Hamiltonian operators with time-dependent potentials and electric fields.

1. Introduction.

This paper is concerned with the problem of invariant domains for the Schrödinger evolution equation

\[ i \frac{d}{dt} \varphi(t) = H(t) \varphi(t), \quad \varphi(s) = \varphi_s \]

where \( H(t), t \in \mathbb{R} \), is a family of self-adjoint operators acting on a Hilbert space \( \mathcal{H} \).

It is known that under suitable conditions on \( H(t) \) (see e.g. Kato [4], Reed-Simon [9] and Yajima [11]), there exists a unique unitary propagator \( U(t, s) \) on \( \mathcal{H} \), and a dense subspace \( \mathcal{D} \) of \( \mathcal{H} \) which is invariant under the propagator so that for each \( \varphi_s \in \mathcal{D} \), \( \varphi(t) = U(t, s)\varphi_s \) is strongly differentiable and satisfies (1).

The problem considered here has been studied by many authors; see Faris-Lavine [1], Fröhlich [2], Hunziker [3], Kuroda-Morita [5], Ozawa [6, 7], Radin-Simon [8] and Wilcox [10]. Most of them dealt with the time-independent case \( H(t) \equiv H \) in which the propagator \( U(t, s) = \exp[i(s - t)H] \) is given by the usual one-parameter unitary group. In a recent paper [7], Ozawa investigated the space-time behavior of \( U(t, s) \) for the Stark Hamiltonian \( H(t) = -\Delta + E \cdot x + V(x, t) \) on \( L^2(\mathbb{R}^n, dx) \). By using perturbation techniques and space-time estimates for the free propagator \( \exp[it(-\Delta + E \cdot x)] \), Ozawa established several results on the invariance property and smoothing effect for \( U(t, s) \) in certain weighted Sobolev spaces. For earlier related results in the case \( E = 0 \), see Kuroda-Morita [5].

We denote the domain of an operator \( A \) by \( \mathcal{D}(A) \), and if \( N \) is positive and self-adjoint, we denote its form domain by \( \mathcal{Q}(N) \). Given a positive self-adjoint operator \( N \), we are interested in conditions on \( H(t) \) for \( \mathcal{Q}(N) \) or
We study this problem in a general operator-theoretic setting in Section 2. Our approach is based on the commutator theorems of Faris and Lavine [1] and Fröhlich [2]. In Section 3, we apply the abstract theorems of Section 2 to Hamiltonians of the form

\[ H(t) = -\Delta + E(t) \cdot x + V(x, t) \]

with \( N = p^2 + x^2 \) or \( N = p^2 \), where \( p \) is the momentum operator \(-i \nabla\). Our results are related to some of those in [5, 7].

2. Abstract Theorems.

Let \( H(t), t \in \mathbb{R} \), be a family of self-adjoint operators acting on a Hilbert space \( \mathcal{H} \). Throughout this section, we will assume that \( \bigcap_t \mathcal{D}(H(t)) \supseteq \mathcal{D} \) for some dense subspace \( \mathcal{D} \) of \( \mathcal{H} \), and that \( H(t) \) generates a unitary propagator \( U(t, s) \) so that

\[
i \frac{d}{dt} U(t, s) \varphi = H(t) U(t, s) \varphi \quad \text{for all } \varphi \in \mathcal{D}.
\]

We denote by \( \mathcal{B}(\mathcal{H}) \) the space of all bounded linear operators on \( \mathcal{H} \) with the usual operator norm \( \| \cdot \| \). For a positive self-adjoint operator \( N \) on \( \mathcal{H} \) and \( \epsilon > 0 \), we define \( N_\epsilon = N(\epsilon N + 1)^{-1} \). Note that \( N_\epsilon \in \mathcal{B}(\mathcal{H}) \) is positive and self-adjoint. Concerning the invariance of the form domain \( \mathcal{Q}(N) = \mathcal{D}(N^{1/2}) \), we prove:

**Theorem 2.1.** Let \( N \) be a positive self-adjoint operator so that

(i) \( \mathcal{D}(N) \subseteq \bigcap_t \mathcal{D}(H(t)) \).

(ii) \( \pm i [H(t), N] \leq c(t) N \) for some \( c \in L_{\text{loc}}(\mathbb{R}) \); that is,

\[
\pm i \{ \langle H(t) \varphi, N \varphi \rangle - \langle N \varphi, H(t) \varphi \rangle \} \leq c(t) \langle \varphi, N \varphi \rangle \text{ for all } \varphi \in \mathcal{D}(N).
\]

Then \( U(t, s) [\mathcal{Q}(N)] = \mathcal{Q}(N) \) for all \( t, s \).

**Proof.** Fix \( s \) and set \( \varphi(t) = U(t, s) \varphi \) for \( \varphi \in \mathcal{H} \). Then we have for \( \varphi \in \mathcal{D} \)

\[
(d/dt) \langle \varphi(t), N_\epsilon \varphi(t) \rangle = \langle \varphi(t), i [H(t), N_\epsilon] \varphi(t) \rangle
 = \langle (\epsilon N + 1)^{-1} \varphi(t), i [H(t), N] (\epsilon N + 1)^{-1} \varphi(t) \rangle.
\]

The hypothesis (ii) now gives that

\[
|(d/dt) \langle \varphi(t), N_\epsilon \varphi(t) \rangle | \leq c(t) \langle (\epsilon N + 1)^{-1} \varphi(t), N(\epsilon N + 1)^{-1} \varphi(t) \rangle
 \leq c(t) \langle \varphi(t), N_\epsilon \varphi(t) \rangle.
\]

Integrating we obtain

\[
\langle \varphi(t), N_\epsilon \varphi(t) \rangle \leq \langle \varphi, N_\epsilon \varphi \rangle \exp \left| \int_s^t c(u) du \right|.
\]
Since $D$ is dense in $H$ and $N_\epsilon$ is bounded, this estimate holds for all $\varphi \in H$.
Now let $\varphi \in Q(N)$. Taking $\epsilon \to 0$, we find that $\varphi(t) \in Q(N)$ with

$$\|N^{1/2}\varphi(t)\|^2 \leq \|N^{1/2}\varphi\|^2 \exp \left| \int_s^t c(u)du \right|.$$  

This shows that $Q(N)$ is invariant under $U(t, s)$. Since $U(t, s)U(s, t) = I$, we conclude that $U(t, s)[Q(N)] = Q(N)$. □

Now for any positive integer $k$, we define (leaving aside the domain questions)

$$Z^k(t) = N^{k-1}[H(t), N]N^{-k} \quad \text{and} \quad Z^k_\epsilon(t) = N^{k-1}_\epsilon[H(t), N_\epsilon]N^{-k}_\epsilon.$$  

In our applications, these operators are defined on certain dense subspaces and extend to bounded operators on $H$. We also define

$$(\text{ad } N)H(t) = [N, H(t)] \quad \text{and} \quad (\text{ad } N)^kH(t) = \left[N, (\text{ad } N)^{k-1}H(t)\right].$$

As a preparation for our next theorem and further applications, we prove the following:

**Lemma 2.2.**

(a) $Z^k_\epsilon(t) = (\epsilon N + 1)^{-k} \sum_{j=0}^{k-1} \binom{k-1}{j} (\epsilon N)^j Z^{k-j}(t)$. In particular, if $Z^1(t), \ldots, Z^k(t) \in \mathcal{B}(H)$, then $Z^k_\epsilon(t) \in \mathcal{B}(H)$ and $\|Z^k_\epsilon(t)\| \leq \sum_{j=0}^{k-1} (k-1)\|Z^{k-j}(t)\|.$

(b) $\left\{ (\text{ad } N)^kH(t) \right\} N^{-k} = \sum_{j=0}^{k-1} (-1)^{j+1} \binom{k-1}{j} Z^{k-j}(t).$

**Proof.** Part (a) is obvious for $k = 1$. The general case follows by induction on $k$:

$Z^{k+1}_\epsilon(t) = N_\epsilon Z^k_\epsilon(t)N^{-1}_\epsilon$

$$= (\epsilon N + 1)^{-k} \sum_{j=0}^{k-1} \binom{k-1}{j} (\epsilon N)^j Z^{k-j}(t)N^{-1}_\epsilon$$

$$= (\epsilon N + 1)^{-k} \sum_{j=0}^{k-1} \binom{k-1}{j} (\epsilon N)^j N Z^{k-j}(t)N^{-1} (1 + \epsilon N)$$

$$= (\epsilon N + 1)^{-k} \sum_{j=0}^{k-1} \binom{k-1}{j} \left\{ (\epsilon N)^j Z^{k+1-j}(t) + (\epsilon N)^{j+1} Z^{k-j}(t) \right\}$$

$$= (\epsilon N + 1)^{-k} \sum_{j=0}^{k-1} \binom{k}{j} (\epsilon N)^j Z^{k+1-j}(t)$$
where we have used the identity \((\binom{k-1}{j} + \binom{k-1}{j+1}) = \binom{k}{j}\). The last statement of part (a) follows from the fact that \(\|(\epsilon N + 1)^{-k}(\epsilon N)\|^j \leq 1\) for \(0 \leq j \leq k - 1\).

Part (b) can also be proven by an induction argument. \(\Box\)

**Theorem 2.3.** Let \(N\) be a positive self-adjoint operator, and define \(Z^j(t)\) as in (2). Suppose that \(Z^j(t) \in \mathcal{B}(\mathcal{H})\) with \(\|Z^j(\cdot)\| \in L^1_{\text{loc}}(\mathbb{R})\) for each \(j = 1, 2, \ldots, k\). Then \(U(t, s) [D(N^k)] = D(N^k)\) for all \(t, s\).

**Proof.** As in the proof of Theorem 2.1, set \(\varphi(t) = U(t, s)\varphi\) for \(\varphi \in \mathcal{H}\). Then we have for \(\varphi \in D\)

\[
\frac{d}{dt} \langle N^k_\epsilon \varphi(t), N^k_\epsilon \varphi(t) \rangle = \langle \varphi(t), i [H(t), N^{2k}_\epsilon] \varphi(t) \rangle
\]

\[= i \sum_{j=0}^{2k-1} \langle \varphi(t), N^j_\epsilon [H(t), N^j_\epsilon] N^{2k-j-1}_\epsilon \varphi(t) \rangle
\]

\[= 2 \text{Im} \sum_{j=0}^{k-1} \langle N^{k-j-1}_\epsilon [H(t), N^j_\epsilon] N^j_\epsilon \varphi(t), N^k_\epsilon \varphi(t) \rangle
\]

where we have used

\[
[A, B^{2k}] = \sum_{j=0}^{2k-1} B^j [A, B] B^{2k-j-1}.
\]

Since \(Z^j(t)\) is bounded and \(\|Z^j(\cdot)\| \in L^1_{\text{loc}}(\mathbb{R})\) for \(1 \leq j \leq k\), Lemma 2.2 (a) implies that \(Z^j(t)\) is bounded for \(1 \leq j \leq k\) and that \(2 \sum_{j=1}^k \|Z^j(t)\| \leq \text{const.} \sum_{j=1}^k \|Z^j(t)\| = f_k(t)\), where \(f_k \in L^1_{\text{loc}}(\mathbb{R})\) and is independent of \(\epsilon\). It follows that

\[
\left| \frac{d}{dt} \| N^k_\epsilon \varphi(t) \|^2 \right| \leq 2 \sum_{j=0}^{k-1} \| N^{k-j-1}_\epsilon [H(t), N^j_\epsilon] N^j_\epsilon \varphi(t) \| \| N^k_\epsilon \varphi(t) \|
\]

\[\leq 2 \sum_{j=0}^{k-1} \|Z^{k-j}_\epsilon(t)\| \| N^k_\epsilon \varphi(t) \|^2
\]

\[\leq f_k(t) \| N^k_\epsilon \varphi(t) \|^2.
\]

Integrating we obtain

\[
\| N^k_\epsilon \varphi(t) \| \leq \| N^k_\epsilon \varphi \| \exp \left| \frac{1}{2} \int_s^t f_k(u) du \right|.
\]

We can now pass to the same argument as in the proof of Theorem 2.1 to conclude that \(U(t, s) [D(N^k)] = D(N^k)\). \(\Box\)
3. Applications.

In this section we want to give some applications of the results of Section 2 to the Schrödinger equation

\[ t \frac{d}{dt} \varphi(t) = H(t) \varphi(t), \quad \varphi(s) = \varphi_s \]

where \( H(t) \) is the time-dependent Hamiltonian acting on the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^n, dx) \).

We first consider Hamiltonians of the form

\[ H(t) = -\Delta + E(t) \cdot x + V(x, t). \]

We will restrict attention to electric fields \( E(t) : \mathbb{R} \rightarrow \mathbb{R}^n \) and potentials \( V(x, t) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R} \) obeying:

1. \( E(t) \) is differentiable.
2. \( |\nabla_x V(x, t)| \leq f(t) (|x| + 1) \) for some continuous function \( f \).
3. The mapping \( t \mapsto (x^2 + 1)^{-1} \frac{\partial V}{\partial t}(x, t) \in L^\infty(\mathbb{R}^n, dx) \) is continuous.

As for \( N \), we take \( N = p^2 + x^2 \), where \( p = -i \nabla \). Note that the operator \( N \geq 1 \) and is self-adjoint on \( \mathcal{D}(N) = \mathcal{D}(p^2) \cap \mathcal{D}(x^2) \). By Theorem 4 of Faris-Lavine [1], condition (ii) implies that \( H(t) \) is essentially self-adjoint on \( \mathcal{S}(\mathbb{R}^n) \), the space of \( C^\infty \)-functions on \( \mathbb{R}^n \) rapidly decreasing at infinity, with domain \( \mathcal{D}(H(t)) \supseteq \mathcal{D}(N) \). We remark that by the construction of the form domain, \( \mathcal{Q}(N) = \mathcal{D}(|p|) \cap \mathcal{D}(|x|) \). Also, one can prove that \( \mathcal{D}(N^k) = \mathcal{D}(p^{2k}) \cap \mathcal{D}(x^{2k}) \) by integration by parts.

Given two Banach spaces \( \mathcal{X} \) and \( \mathcal{Y} \), we denote by \( \mathcal{B}(\mathcal{X}, \mathcal{Y}) \) the space of all bounded linear operators with domain \( \mathcal{X} \) and range in \( \mathcal{Y} \). For a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \), where each \( \alpha_j \) is a nonnegative integer, and \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), we put \( |\alpha| = \alpha_1 + \cdots + \alpha_n \), \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) and \( \nabla^\alpha = (\frac{\partial}{\partial x})^{\alpha} = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n} \). Let \( B_m^\infty(\mathbb{R}^n) \) be the space of all \( m \)-times continuously differentiable functions \( \varphi \) on \( \mathbb{R}^n \) with bounded derivatives \( (\frac{\partial}{\partial x})^\alpha \varphi \) for \( 0 < |\alpha| \leq m \). Our result is:

**Theorem 3.1.** Let \( H(t) = -\Delta + E(t) \cdot x + V(x, t) \), where \( E(t) \) and \( V(x, t) \) obey conditions (i)-(iii) above, and let \( N = p^2 + x^2 \). Then there exists a unique unitary propagator \( U(t, s) \), \( t, s \in \mathbb{R} \), so that:

(a) for each \( \varphi_s \in \mathcal{D}(N) \), \( \varphi(t) = U(t, s) \varphi_s \) is strongly differentiable and satisfies (3).

(b) \( U(t, s) \) leaves \( \mathcal{Q}(N) \) and \( \mathcal{D}(N) \) invariant.
If, in addition, \( V(\cdot, t) \in B_{2k}^{2k}(\mathbb{R}^n) \) with \( \| (\frac{\partial}{\partial x})^\alpha V(x, \cdot) \|_\infty \in L^1_{loc}(\mathbb{R}) \) for \( 0 < |\alpha| \leq 2k \), then \( U(t, s) \) leaves \( \mathcal{D}(N^k) \) invariant.

**Proof.** To prove the existence of the propagator, we define for \( \varphi \in \mathcal{D} \equiv \mathcal{D}(N) \), \( \| \varphi \|_\mathcal{D} = \| \varphi \| + \| p^2 \varphi \| + \| x^2 \varphi \| \). Then \( (\mathcal{D}, \| \cdot \|_\mathcal{D}) \) forms a Banach space which is continuously and densely embedded in \( \mathcal{H} \). From (ii), we have \( |V(x, t)| \leq \frac{1}{5} f(t)x^2 + f(t)|x| + |V(0, t)| \). It follows by the continuity of \( E, V \) and \( f \) that on any compact interval \([ -T, T] \), there are constants \( a \) and \( b \) so that \( |E(t) \cdot x + V(x, t)| \leq ax^2 + b \) for all \( t \in [ -T, T] \). Since

\[
\| p^2 \varphi \|^2 + \| cx^2 \varphi \|^2 \leq \| (p^2 + cx^2)\varphi \|^2 + 2cn\| \varphi \|^2 \quad \text{for} \quad \varphi \in \mathcal{D},
\]

we see that if \( c > a \), then \( E(t) \cdot x + V(x, t) \) is \( (p^2 + cx^2) \)-bounded with relative bound less than one. Thus, by the Kato-Rellich theorem, \( H(t) + cx^2 \) is self-adjoint on \( \mathcal{D} \) for all \( t \in [ -T, T] \). Now, take \( S(t) = H(t) + cx^2 + i \).

Then \( S(t) \in B(\mathcal{D}, \mathcal{H}) \) is an isomorphism with \( S(t)H(t)S(t)^{-1} = H(t) + G(t) \), where \( G(t) = 2ci(p \cdot x + x \cdot p)S(t)^{-1} \in B(\mathcal{H}) \). By (i) and (iii), the mapping \( t \mapsto S(t) \in B(\mathcal{D}, \mathcal{H}) \) is strongly differentiable. Also, a simple computation gives that

\[
\| G(t) - G(u) \|_{B(\mathcal{H})} \leq \| G(t) \|_{B(\mathcal{H})} \| H(t) - H(u) \|_{B(\mathcal{D}, \mathcal{H})} \| S(u)^{-1} \|_{B(\mathcal{H}, \mathcal{D})}
\]

\[
\| H(t) - H(u) \|_{B(\mathcal{D}, \mathcal{H})} \leq |E(t) - E(u)| + \| (x^2 + 1)^{-1} [V(x, t) - V(x, u)] \|_{L^\infty(\mathbb{R}^n, dx)}.
\]

Thus, by (i) and (iii), the mapping \( t \mapsto H(t) \in B(\mathcal{D}, \mathcal{H}) \) and \( t \mapsto G(t) \in B(\mathcal{H}) \) are norm continuous. It follows from a classical result of Kato ([4], Theorem I) that there exists a unique unitary propagator \( U(t, s) \) leaving \( \mathcal{D} \) invariant so that (a) holds.

Next, we show that \( U(t, s) \) leaves \( \mathcal{Q}(N) \) invariant. We have seen that \( \mathcal{D}(H(t)) \supseteq \mathcal{D}(N) \) for all \( t \). So by Theorem 2.1, it suffices to show that \( \pm i [H(t), N] \leq c(t)N \) for some locally integrable function \( c(t) \). We compute

\[
\pm i [H(t), N]
\]

\[
= \pm i \{ [p^2, x^2] + [E(t) \cdot x, p^2] + [V(x, t), p^2] \}
\]

\[
= \pm \{ 2(p \cdot x + x \cdot p) - 2E(t) \cdot p - (p \cdot \nabla x V(x, t) + \nabla x V(x, t) \cdot p) \}
\]

\[
\leq 2(p^2 + x^2) + p^2 + |E(t)|^2 + p^2 + |\nabla x V(x, t)|^2
\]

\[
\leq \{ 4 + |E(t)|^2 + 4f(t)^2 \} N
\]

as required, where we have used condition (ii) and the fact that \( N \geq 1 \).

Finally, we prove the last statement of the theorem. Let

\[
\Gamma \equiv L^1_{loc}(\mathbb{R}, dt; B(\mathcal{H})).
\]
By Theorem 2.3, it suffices to show that if \( V(\cdot, t) \in B^{2k}(\mathbb{R}^n) \) with \( \| (\frac{\partial}{\partial x})^\alpha V(x, \cdot) \|_\infty \in L^1_{\text{loc}}(\mathbb{R}) \) for \( 0 < |\alpha| \leq 2k \), then
\[
Z^j = N^{j-1} [H(\cdot), N] N^{-j} \in \Gamma
\]
for \( 1 \leq j \leq k \). We prove this inductively. Let \( D = p \cdot x + x \cdot p \) be the dilation operator. Since
\[
Z^1(t) = [H(t), N] N^{-1}
\]
\[
= -2i \left\{ D - E(t) \cdot p - \nabla_x V(x, t) \cdot p + \frac{i}{2} \Delta_x V(x, t) \right\} N^{-1},
\]
the case \( k = 1 \) follows easily from the closed graph theorem and the hypotheses on \( E \) and \( V \). Now consider the case of general \( k \geq 2 \). By the induction hypothesis, we have \( Z^j \in \Gamma \) for \( 1 \leq j \leq k - 1 \). So, we need only prove that \( Z^k \in \Gamma \). By Lemma 2.2(b), it is sufficient to prove that \( \{(\operatorname{ad} N)^k H(\cdot)\} N^{-k} \in \Gamma \). We compute on \( S(\mathbb{R}^n) \):
\[
(\operatorname{ad} N)^2 H(t) = 4 \left\{ 2(p^2 - x^2) + E(t) \cdot x + \nabla_x V(x, t) \cdot x + \frac{i}{4} \Delta_x^2 V(x, t) \right\}
\]
\[
- \sum_{j=1}^n \left( \nabla_x \delta_{x_j} (x, t) \right) \cdot pp_j + i \nabla_x (\Delta_x V(x, t)) \cdot p
\]
where we have used the following basic identities:
\[
[N, D] = 4i(x^2 - p^2), \quad [N, E(t) \cdot p] = 2i E(t) \cdot x, \quad [N, E(t) \cdot x] = -2i E(t) \cdot p,
\]
\[
[p^2, W(x)] = -2i \nabla W \cdot p - \Delta W, \quad [x^2, \nabla W(x) \cdot p] = 2i \nabla W \cdot x,
\]
\[
[p^2, \nabla W(x) \cdot p] = -2i \sum_{j=1}^n \left( \nabla_x \delta_{x_j} \right) \cdot pp_j - \nabla (\Delta W) \cdot p.
\]
By repeated application of these formulas, we find that \((\operatorname{ad} N)^k H(t)\) is a linear combination of operators of the form:
\[
p^2 - x^2 \text{ (or } D), \quad E(t) \cdot x \text{ (or } E(t) \cdot p) \quad \text{ and } \quad \left( \frac{\partial}{\partial x} \right)^\alpha V(x, t) \cdot x^\beta p^\gamma
\]
where \( 0 < |\alpha| \leq 2k \), \( |\beta| \leq k/2 \) and \( |\gamma| \leq k \). Since \( x^\beta p^\gamma N^{-k} \) is bounded on \( \mathcal{H} \) so long as \( |\beta| \leq k \) and \( |\gamma| \leq k \), the hypotheses of \( E \) and \( V \) now imply that \( \{(\operatorname{ad} N)^k H(\cdot)\} N^{-k} \in \Gamma \). This completes the proof. \( \square \)

**Corollary 3.2.** In Theorem 3.1, if \( V(\cdot, t) \) is a \( C^\infty \)-function on \( \mathbb{R}^n \) with bounded derivatives and \( \| (\frac{\partial}{\partial x})^\alpha V(x, \cdot) \|_\infty \in L^1_{\text{loc}}(\mathbb{R}) \) for all \( \alpha \neq 0 \), then \( U(t, s) \) leaves \( S(\mathbb{R}^n) \) invariant.

**Proof.** The corollary follows immediately from the fact that
\[
S(\mathbb{R}^n) = \cap_{k=1}^\infty D(N^k).
\]
In the remainder of this section, we want to give an application to Hamiltonians of the form

\[ H(t) = -\Delta + V(x, t). \]

We will assume potentials \( V(x, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) obeying:

\[
\begin{align*}
(i) & \text{ for each } t, V(\cdot, t) \text{ is } \Delta\text{-bounded with relative bound less than one.} \\
(ii) & \text{ the mapping } t \mapsto \frac{\partial V}{\partial t}(x, t) \in L^\infty(\mathbb{R}^n, dx) \text{ is continuous.}
\end{align*}
\]

Notice that condition (i) and the Kato-Rellich theorem imply that \( H(t) \) is essentially self-adjoint on \( S(\mathbb{R}^n) \) with domain \( D(H(t)) = D(\Delta) \). Corresponding to Theorem 3.1, we have:

**Theorem 3.3.** Let \( H(t) = -\Delta + V(x, t) \), where \( V(x, t) \) obeys conditions (i) and (ii) above. Then there is a unique unitary propagator \( U(t, s), t, s \in \mathbb{R} \), leaving \( D(\Delta) \) invariant so that for each \( \varphi_s \in D(\Delta), \varphi(t) = U(t, s)\varphi_s \) is strongly differentiable and satisfies (3). Moreover,

(a) If \( |\nabla_x V(x, t)| \leq f(t) \) for some continuous \( f \), then \( U(t, s) \) leaves \( Q(-\Delta) \) invariant.

(b) If \( V(\cdot, t) \in B^2_k(\mathbb{R}^n) \) with \( \|(\frac{\partial}{\partial x})^\alpha V(x, \cdot)\|_\infty \in L^1_{\text{loc}}(\mathbb{R}) \) for \( 0 < |\alpha| \leq 2k \), then \( U(t, s) \) leaves \( D(\Delta^k) \) invariant.

**Proof.** The proof of the existence statement closely parallels the proof given in Theorem 3.1 except that we choose \( D = D(\Delta), S(t) = H(t) + i \) and define \( \|\varphi\|_D = \|\varphi\| + \|p^2\varphi\| \) so that \( S(t)H(t)S(t)^{-1} = H(t) \). Then one proves that the mapping \( t \mapsto S(t) \in B(D, \mathcal{H}) \) is strongly differentiable and that the mapping \( t \mapsto H(t) \in B(D, \mathcal{H}) \) is norm continuous as before. To prove (a) and (b), we take \( N = -\Delta + 1 \). In case (a), since

\[
\pm i [H(t), N] = \mp \{p \cdot \nabla_x V(x, t) + \nabla_x V(x, t) \cdot p\} \leq p^2 + |\nabla_x V(x, t)|^2 \leq \{1 + f(t)^2\} N,
\]

Theorem 2.1 implies that \( U(t, s) \) leaves \( Q(N) = Q(-\Delta) \) invariant. In case (b), the computations similar to those used in Theorem 3.1 show that \( \langle \text{ad } N \rangle^k H(t) \) is a linear combination of operators of the form:

\[
\left( \left( \frac{\partial}{\partial x} \right)^\alpha V(x, t) \right) p^\gamma,
\]

where \( 0 < |\alpha| \leq 2k \) and \( |\gamma| \leq k \). Thus by hypothesis, we have

\[
\left\{ \langle \text{ad } N \rangle^k H(\cdot) \right\} N^{-k} \in L^1_{\text{loc}}(\mathbb{R}, dt; B(\mathcal{H})).
\]

Again, following the proof of Theorem 3.1, we conclude that \( U(t, s) \) leaves \( D(N^k) = D(\Delta^k) \) invariant. \( \square \)
Acknowledgments. The author wishes to thank the referee for useful comments.

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Received December 20, 1993 and revised April 19, 1994.

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