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Let $\mathcal{Y}_1(n, \mu_0)$ be the class of compact connected smooth n -manifolds M ($n \geq 3$) with Yamabe metrics g of unit volume which satisfy

$$\mu(M, [g]) \geq \mu_0 > 0,$$

where $[g]$ and $\mu(M, [g])$ denote the conformal class of g and the Yamabe invariant of $(M, [g])$, respectively. The purpose of this paper is to prove several convergence theorems for compact Riemannian manifolds in $\mathcal{Y}_1(n, \mu_0)$ with integral bounds on curvature. In particular, we present a pinching theorem for flat conformal structures of positive Yamabe invariant on compact 3-manifolds.

1. Introduction.

Let M be a compact connected smooth manifold of dimension $n \geq 3$. The Yamabe functional I on a conformal class C of M is defined by

$$I(g) = \frac{\int_M S_g dv_g}{V_g^{(n-2)/n}} \quad \text{for } g \in C,$$

where S_g , dv_g and V_g denote the scalar curvature, the volume element and the volume $\text{vol}(M, g)$ of (M, g) , respectively. The infimum of this functional is denoted by $\mu(M, C)$, i.e.,

$$\mu(M, C) = \inf_{g \in C} I(g)$$

and called the Yamabe invariant of (M, C) . The following so-called Yamabe problem was solved affirmatively by the work of Yamabe [Ym], Trudinger [T], Aubin [Au1] and Schoen [S1, SY2]:

Given a conformal class C on a compact manifold M of dimension $n \geq 3$, find a metric g which minimizes the Yamabe functional I on the conformal class C .

We call a metric, which is a solution of the Yamabe problem, simply a *Yamabe metric*. It is well-known that the scalar curvature of every Yamabe metric is constant. Moreover, a Yamabe metric of positive scalar curvature satisfies a volume estimate of geodesic balls from below (see §2). In [K1, K2] and [S3], a differential-topological invariant $\mu(M)$ of M was independently introduced by Kobayashi and Schoen, which is defined as the supremum of $\mu(M, C)$ of all conformal structures C on M , i.e.,

$$\mu(M) = \sup\{\mu(M, C); C \text{ is a conformal structure on } M\}.$$

They also studied properties of $\mu(M)$ and proposed some problems (for recent remarkable developments see [L1, L2]). In particular, Schoen conjectured affirmatively the following problem. That is *whether the positive constant curvature metrics h on non-simply connected quotient spaces S^n/Γ of the standard n -sphere S^n achieve the supremum of $\mu(S^n/\Gamma, C)$, i.e., $\mu(S^n/\Gamma, [h]) = \mu(S^n/\Gamma)$, where $[h]$ denotes the conformal class of h .*

On the other hand, Gromov and Lawson [GL] (cf. [SY1]) proved that a irreducible oriented compact 3-manifold M^3 admitting a metric of positive scalar curvature is diffeomorphic to a quotient space of a homotopy 3-sphere. Moreover, if M^3 admits a conformally flat metric, then M^3 is diffeomorphic to a quotient space of S^3 (cf. [Ku]). Under these situation, the following naive problem arises naturally; *for a quotient manifold X^3 of a homotopy 3-sphere which admits a metric of positive scalar curvature, is there a positive constant curvature metric h such that $\mu(X^3, [h]) = \mu(X^3)$?* It also should be pointed out that, which was proved by Izezi [I], if a compact 3-manifold N^3 admits a conformally flat metric of positive scalar curvature, then a finite cover of N^3 is diffeomorphic to S^3 or a connected sum $k(S^1 \times S^2)$ of k -copies of $S^1 \times S^2$.

Let A_g denote the tensor field $(A_{ijk}) = (\nabla_k R_{ij} - \nabla_j R_{ik})$ of type (0,3), where (R_{ij}) stands for the Ricci curvature Ric_g of a metric g and ∇ the Levi-Civita connection of g . We note that if (M, g) is a conformally flat manifold of constant scalar curvature, then A_g vanishes identically (i.e., h is of *harmonic curvature*). Conversely, under the condition $\dim M = 3$, $A_g \equiv 0$ implies that (M, g) is a conformally flat manifold of constant scalar curvature. Throughout this paper, we always assume that p, q and μ_0 are positive constants satisfying $p > \frac{n}{2}$, $\max\{1, \frac{n}{4}\} < q < \frac{n}{2}$ and $\mu_0 < n(n-1) \text{vol}(S^n(1))^{2/n}$, where $S^n(1)$ denotes the Euclidean n -sphere of radius 1. We also always denote by $\alpha = 2 - \frac{n}{p} (> 0)$ and $\beta = 4 - \frac{n}{q} (> 0)$.

Let $\mathcal{Y}_1(n, \mu_0)$ be the class of compact connected smooth n -manifolds M ($n \geq 3$) with Yamabe metrics g of unit volume which satisfy

$$\mu(M, [g]) \geq \mu_0 > 0.$$

Inspired by the above problems, we obtain the following convergence theorem.

Theorem 1.1. *Let $\{(M_i, g_i)\}_{i \in \mathbb{N}}$ be a sequence in $\mathcal{Y}_1(n, \mu_0)$ which satisfy*

$$\int_{M_i} |R_{g_i}|^{n/2} dv_{g_i} \leq \Lambda_1, \quad \int_{M_i} |\nabla A_{g_i}|^q dv_{g_i} \leq \Lambda_2$$

with some positive constants Λ_1, Λ_2 , where $|R_{g_i}|$ denotes the norm of Riemann curvature tensor R_{g_i} of g_i . Then either of the following two cases must be hold.

(1°) (M_i, g_i) converges to a point in the Hausdorff distance.

(2°) There exist a subsequence $\{j\} \subset \{i\}$, a compact connected metric space (M_∞, d_∞) with positive diameter and a finite subset $\mathcal{S} = \{x_1, \dots, x_k\} \subset M_\infty$ (possibly empty) such that:

(2°.1) (M_j, g_j) converges to (M_∞, d_∞) in the Hausdorff distance.

(2°.2) $M_\infty \setminus \mathcal{S}$ has a structure of smooth n -manifold and a $C^\beta \cap L^{2, nq/(n-2q)}_{loc}$ metric g_∞ of positive constant scalar curvature S_{g_∞} such that g_∞ is compatible with the distance d_∞ on $M_\infty \setminus \mathcal{S}$ and that

$$0 < \mu_0 \leq S_{g_\infty} \leq n(n-1) \text{vol}(S^n(1))^{2/n}.$$

(2°.3) For each compact subset $K \subset M_\infty \setminus \mathcal{S}$, there exists an into diffeomorphism $\Phi_j : K \rightarrow M_j$ for j sufficiently large such that the pull-back metrics $(\Phi_j)^* g_j$ converges to g_∞ in the $C^{\beta'}$ topology for $\beta' < \beta$ and weakly in the $L^{2, nq/(n-2q)}$ topology on K .

(2°.4) $\lim_{j \rightarrow \infty} S_{g_j} = S_{g_\infty}$.

(2°.5) For every $x_a \in \mathcal{S} (a = 1, \dots, k)$ and j , there exist $x_{a,j} \in M_j$ and positive number r_j such that:

(2°.5 a) $B_\varepsilon(x_{a,j})$ converges to $B_\varepsilon(x_a)$ in the Hausdorff distance for all $\varepsilon > 0$.

(2°.5 b) $\lim_{j \rightarrow \infty} r_j = \infty$.

(2°.5 c) $((M_j, r_j g_j), x_{a,j})$ converges to $((N_a, h_a), x_{a,\infty})$ in the pointed Hausdorff distance, where (N_a, h_a) is a complete noncompact, scalar-flat, non-flat C^∞ Riemannian n -manifold which satisfies

$$\sup_{N_a} |R_{h_a}| < \infty, \quad 0 < \int_{N_a} |R_{h_a}|^{n/2} dv_{h_a} < \infty,$$

and

$$\text{vol}(B_r(x); h_a) \geq (5 \cdot 2^n (n-1))^{-n/2} (n-2)^{n/2} \mu_0^{n/2} r^n$$

for $x \in N_a$ and $r > 0$. In particular, when $n = 3$ (resp. $n = 4, 5$) each (N_a, h_a) is conformally flat (resp. of harmonic curvature).

(2°.5 d) For every $r > 0$, there exists an into diffeomorphism $\Psi_j : B_r(x_{a,\infty}) \rightarrow M_j$ for j sufficiently large such that $(\Psi_j)^*(r_j g_j)$ converges to h_a in the $C^{1,\sigma}$ topology for $\sigma < 1$ and weakly in the $L^{2,s}$ topology for $s > n$ on $B_r(x_{a,\infty})$.

(2°.6) It holds

$$\lim_{j \rightarrow \infty} \int_{M_j} |R_{g_j}|^{n/2} dv_{g_j} \geq \int_{M_\infty} |R_{g_\infty}|^{n/2} dv_{g_\infty} + \sum_{a=1}^k \int_{N_a} |R_{h_a}|^{n/2} dv_{h_a}.$$

Remark 1.2.

- (1) Theorem 1.1 is a generalization of Theorem 1.2 in [Ak].
- (2) Since the metric g_∞ is of class $C^\beta \cap L^{2,nq/(n-2q)}_{loc}$, then its curvature tensors R_{g_∞} , Ric_{g_∞} and S_{g_∞} make sense in $L^{nq/(n-2q)}_{loc}$.
- (3) It would be conjectured that only the second case (2°) holds, and when $n = 3$ each conformally flat, scalar-flat 3-manifold (N_a, h_a) in (2°.5 c) is asymptotically locally Euclidean (cf. [BKN]). However, when solving them, a technical difficulty arises in obtaining volume estimates of geodesic balls from above. Moreover, when $n = 3$ and M_i is a quotient 3-manifold M^3 of a homotopy 3-sphere for all i , then it would be also conjectured that $\mathcal{S} = \phi$ (see [B, AC1] for reconstruction of manifolds).

The following result includes a pinching theorem for flat conformal structures of positive Yamabe invariant on compact 3-manifolds (see Remark 2.2).

Theorem 1.3. For given positive constants $s(> \frac{n}{3})$ and Λ , there exists a positive constant $\varepsilon_0 = \varepsilon_0(\mu_0, n, p, s, \Lambda)$ such that if a compact Riemannian n -manifold $(M, g) \in \mathcal{Y}_1(n, \mu_0)$ satisfies

$$\int_M |R_g|^p dv_g \leq \Lambda, \quad \int_M |\nabla A_g|^s dv_g \leq \varepsilon_0,$$

then M admits a Yamabe metric h of harmonic curvature with

$$V_h = 1, \quad \mu(M, [h]) \geq \mu_0 > 0.$$

In particular, when $n = 3$ (M^3, h) is conformally flat, and a finite cover of M^3 is diffeomorphic to S^3 or a connected sum $k(S^1 \times S^2)$ for some $k \in \mathbb{N}_+$,

In §2 we give basic known facts on Yamabe metrics and the Hausdorff distance. These contain a geometric inequality for Yamabe metrics of positive scalar curvature, which plays a key role in our proofs. In §3 we give the notion of $L^{2,p}$ harmonic radius. We then summarize convergence results

for manifolds and a priori estimates for $L^{2,p}$ harmonic radius, which were developed mainly by Anderson [An2, 3, AC2]. The proofs of our theorems are essentially based on these results. For the proofs of Theorem 1.1 and Theorem 1.3, we also give another a priori estimate for $L^{2,p}$ harmonic radius. In §4, using these results, we prove compactness and pinching theorems for Yamabe metrics of positive scalar curvature, and we also prove Theorem 1.3. Finally, in §5 we give the proof of Theorem 1.1.

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2. Preliminaries.

In this section, first we shall give several known properties for Yamabe metrics. Let M be a compact n -manifold. Since a Yamabe metric g on M is a minimizer of the Yamabe functional $I : [g] \rightarrow \mathbf{R}$, then the first variational formula shows the following equation (cf. [Au2], [LP]):

$$(2.1) \quad S_g = \mu(M, [g])V_g^{-2/n} \equiv \text{const..}$$

Moreover, the following inequality is due to Aubin [Au1].

$$(2.2) \quad \mu(M, [g]) \leq n(n - 1)\text{vol}(S^n(1))^{2/n}.$$

Now let g be a Yamabe metric of positive scalar curvature on M . The Yamabe invariant $\mu(M, [g])$ is rewritten as

$$(2.3) \quad \mu(M, [g]) = \inf_{\substack{u \in L^{1,2}(M) \\ u \neq 0}} \frac{4 \frac{n-1}{n-2} \int_M |\nabla u|^2 dv_g + \int_M S_g u^2 dv_g}{\left(\int_M |u|^{2n/(n-2)} dv_g \right)^{(n-2)/n}},$$

where $L^{1,2}(M)$ denotes the Sobolev space of functions on M with L^2 first derivatives (cf. [Au2, GT]). It then follows from (2.1), (2.3) and the positivity of $\mu(M, [g])$ that

$$(2.4) \quad \left(\int_M |u|^{2n/(n-2)} dv_g \right)^{(n-2)/n} \leq \frac{1}{c_g} \int_M |\nabla u|^2 dv_g + \frac{1}{V_g^{2/n}} \int_M u^2 dv_g$$

for $u \in L^{1,2}(M)$, where $c_g = \frac{n-2}{4(n-1)}\mu(M, [g]) > 0$. By the Sobolev inequality (2.4), we can prove the following geometric inequalities for Yamabe metrics of positive scalar curvature (cf. [Ak]). In particular, the first inequality (2.5) is of essential use in the proofs of our theorems.

Proposition 2.1. *Let g be a Yamabe metric of positive scalar curvature on M . Then*

$$(2.5) \quad \text{vol}(B_r(x)) \geq (5 \cdot 2^{n-2})^{-n/2} c_g^{n/2} r^n \quad \text{for } x \in M \text{ and } r \leq \sqrt{\frac{V_g^{2/n}}{c_g}},$$

$$(2.6) \quad \text{diam}(M, g) \leq 2(5 \cdot 2^{n-2})^{n/2} \sqrt{\frac{V_g^{2/n}}{c_g}},$$

where $B_r(x) = B_r(x; g)$ denotes the geodesic ball of radius r centered at x and $\text{vol}(B_r(x)) = \text{vol}(B_r(x); g)$ the volume of $B_r(x)$ with respect to g .

Remark 2.2. Let $\mathcal{M}_1(n, \mu_0, p, S_0)$ (resp. $\mathcal{M}_1(n, \mu_0)$) denote the class of compact Riemannian n -manifolds (M, g) of unit volume which satisfy

$$\begin{aligned} \mu(M, [g]) \geq \mu_0 > 0, \quad \int_M (S_g^+)^p dv_g \leq S_0, \\ (\text{resp. } \mu(M, [g]) \geq \mu_0 > 0), \end{aligned}$$

where $S_g^+ = \max\{S_g, 0\}$ and $p > \frac{n}{2}$. For an element $(M, g) \in \mathcal{M}_1(n, \mu_0, p, S_0)$, Kasue and Kumura [KK] proved geometric inequalities similar to (2.5) and (2.6). In Theorem 2.3 below, if we replace $\mathcal{Y}_1(n, \mu_0)$ by $\mathcal{M}_1(n, \mu_0, p, S_0)$, then the same conclusion holds. We can also prove convergence theorems for Riemannian manifolds in $\mathcal{M}_1(n, \mu_0, p, S_0)$ with integral bounds on curvature, similar to those in this paper. In particular, if we only replace $\mathcal{Y}_1(n, \mu_0)$ by $\mathcal{M}_1(n, \mu_0)$ in Theorems 1.3 and 4.1, then similar conclusions hold.

Next, we recall the definition of the Hausdorff distance on the set \mathcal{MET} of all isometry classes of compact metric spaces introduced by Gromov [Gr] (cf. [F]). Let X and Y be compact metric spaces. A map $f : X \rightarrow Y$ (not necessarily continuous) is said to be an ε -Hausdorff approximation if the following two conditions are satisfied.

The ε -neighborhood of $f(X)$ in Y is equal to Y .

$$|d_X(x, y) - d_Y(f(x), f(y))| < \varepsilon \quad \text{for } x, y \in X.$$

The Hausdorff distance $d_H(X, Y)$ between X and Y is defined to be the infimum of all positive numbers ε such that there exist ε -Hausdorff approximations from X to Y and from Y to X . Unfortunately $d_H(\cdot, \cdot)$ does not satisfy the triangle inequality. However the inequality (2.7) below holds and then shows that it gives a metrizable complete uniform structure on the set \mathcal{MET} . Thus we treat $d_H(\cdot, \cdot)$ as if it is a distance function.

$$(2.7) \quad d_H(X, Z) \leq 2\{d_H(X, Y) + d_H(Y, Z)\},$$

for $X, Y, Z \in \mathcal{MET}$.

For noncompact metric spaces, we also recall the definition of the pointed Hausdorff distance. Let (X, x) and (Y, y) be pointed metric spaces (possibly compact). A map $f : (X, x) \rightarrow (Y, y)$ is said to be an ε -pointed Hausdorff approximation if

$$\begin{aligned} f(x) &= y, \\ f(B_{1/\varepsilon}(x)) &\subset B_{1/\varepsilon}(y), \\ f|_{B_{1/\varepsilon}(x)} : B_{1/\varepsilon}(x) &\rightarrow B_{1/\varepsilon}(y) \text{ is an } \varepsilon\text{-Hausdorff approximation.} \end{aligned}$$

The pointed Hausdorff distance $d_{p,H}((X, x), (Y, y))$ between pointed metric spaces (X, x) and (Y, y) is the infimum of all positive numbers ε such that there exist ε -pointed Hausdorff approximations from (X, x) to (Y, y) and from (Y, y) to (X, x) . $d_{p,H}(\cdot, \cdot)$ also defines a distance on the set \mathcal{MET}_0 of all isometry classes of pointed metric spaces whose metric balls are all precompact.

By (2.2), (2.5) and (2.6), we can prove the following precompactness theorem for Yamabe metric of positive scalar curvature (cf. [Ak]), which is also of use in the proof of Theorem 1.1.

Theorem 2.3. *The set $\mathcal{Y}_1(n, \mu_0)$ is precompact in \mathcal{MET} with respect to the Hausdorff distance.*

3. $L^{2,p}$ harmonic radius.

In this section, we first give the notion of $L^{2,p}$ harmonic radius (cf. [An2, 3], [AC2]). Let (M, g) be a complete Riemannian n -manifold (without boundary). For given $p(> \frac{n}{2})$ and $L(> 0)$, the $L^{2,p}$ harmonic radius at $x \in M$ is the radius $r_H(x) = r_H(x; g, p, L)$ of the largest geodesic ball $B_{r_H(x)}(x) \subset M$ centered at x , on which there exist harmonic coordinates $U = \{u^i\}_{i=1}^n : B_{r_H(x)}(x) \rightarrow \mathbf{R}^n$ (cf. [J, DK]) such that the metric components $g_{ij} = g(\partial/\partial u^i, \partial/\partial u^j)$ are bounded in the $L^{2,p}$ norm on $\tilde{B} = U(B_{r_H(x)}(x))$, i.e.,

$$(3.1) \quad e^{-L} \cdot \delta_{ij} \leq g_{ij} \leq e^L \cdot \delta_{ij} \quad (\text{as bilinear forms}),$$

$$(3.2) \quad r_H(x)^{\alpha-1} \|\partial g_{ij}\|_{L^p(\tilde{B})} + r_H(x)^\alpha \|\partial\partial g_{ij}\|_{L^p(\tilde{B})} \leq L$$

for $i, j = 1, \dots, n$, where $\alpha = 2 - \frac{n}{p} (> 0)$.

Remark 3.1.

- (1) By the Sobolev embedding theorem $L^{2,p}(\tilde{B}) \subset C^\alpha(\tilde{B})$, the $L^{2,p}$ harmonic radius controls the C^α norm of g_{ij} on \tilde{B} .

- (2) For the rescaled metric $h = \lambda^2 \cdot g$ ($\lambda \equiv \text{const.} > 0$), the $L^{2,p}$ harmonic radius at x changes as follows;

$$r_H(x; h) = \lambda \cdot r_H(x; g).$$

Next we shall summarize part of convergence results for manifolds and a priori estimates for $L^{2,p}$ harmonic radius, which were developed by Anderson [An1–3, AC2] and also many mathematicians (cf. [BKN, Ga1–2, GW, Gr, Ka, N, P, Yn1, 3]). These are also of essential use in our proofs of §4 and §5.

Theorem 3.2. *Let $\{(M_i, g_i)\}_{i \in \mathbf{N}}$ be a sequence of compact C^∞ Riemannian n -manifolds which satisfy, for each $i \in \mathbf{N}$*

$$(3.3) \quad r_H(x) \geq r_0 \quad \text{for } x \in M_i, \quad V_{g_i} \leq V_0$$

*with some positive constants r_0, V_0 . Then there exist a subsequence $\{j\} \subset \{i\}$, a compact C^∞ n -manifold M_∞ with $C^\alpha \cap L^{2,p}$ metric g_∞ and a diffeomorphism $\Phi_j : M_\infty \rightarrow M_j$ for each j such that $(\Phi_j)^*g_j$ converges to g_∞ in the $C^{\alpha'}$ topology for $\alpha' < \alpha$ and weakly in the $L^{2,p}$ topology on M_∞ .*

Remark 3.3. In Theorem 3.2, we should remark the following fact. For a point $x \in M_\infty$ and fix it. From (3.3), for each $h(j) = (\Phi_j)^*g_j$, there exist harmonic coordinates

$$U_j = \{u^a\}_{a=1}^n (= \{u(j)^a\}_{a=1}^n) : B_{r_0}(\subset M_\infty) \rightarrow \mathbf{R}^n \quad \text{with } U_j(x) = \mathbf{0} \in \mathbf{R}^n$$

such that the metric components $h(j)_{ab} = h(j)(\partial/\partial u^a, \partial/\partial u^b)$ satisfy (3.1) and (3.2). Then, by (3.1), (3.2), the L^p estimates for elliptic differential equations (cf. [GT]) and the construction of Φ_j and M_∞ , there exists a coordinate system $\{V_\lambda\}_{\lambda=1}^n$ of class C^∞ on M_∞ such that

$$\|U_j \circ V_\lambda^{-1}\|_{L^{4,p}} \leq C, \quad \|U_j \circ V_\lambda^{-1}\|_{C^{2,\alpha}} \leq C$$

where C is independent of j and $x \in M_\infty$ (cf. [AC2, Ka]). Similar results hold in the following Theorems 3.4 and 3.5.

Theorem 3.4. *Let $\{(M_i, g_i)\}_{i \in \mathbf{N}}$ be a sequence of compact C^∞ Riemannian n -manifolds and $\Omega_i \subset M_i$ open subsets (possibly disconnected) which satisfy, for each $i \in \mathbf{N}$*

$$r_H(x) \geq r_0 \quad \text{for } x \in \Omega_i \subset M_i, \quad 0 < c_1 \leq \text{vol}(\Omega_i) \leq c_2$$

and that each $\Omega_i(2\varepsilon) = \{x \in \Omega_i; \text{dist}_{g_i}(x, \partial\Omega_i) > 2\varepsilon\}$ is nonempty, where r_0, c_1, c_2 and ε denote positive constants. Then there exist a subsequence

$\{j\} \subset \{i\}$, a compact C^∞ n -manifold $\Omega_\infty(\varepsilon)$ (possibly disconnected but only finitely many components) with $C^\alpha \cap L^{2,p}$ metric g_∞ and an into diffeomorphism $\Phi_j : \Omega_\infty(\varepsilon) \rightarrow \Omega_j(\frac{\varepsilon}{2})$ with $\Phi_j(\Omega_\infty(\varepsilon)) \supset \Omega_j(\frac{3}{2}\varepsilon)$ for each j such that $(\Phi_j)^*g_j$ converges to g_∞ in the $C^{\alpha'}$ topology for $\alpha' < \alpha$ and weakly in the $L^{2,p}$ topology on $\Omega_\infty(\varepsilon)$.

Theorem 3.5. *Let $\{(M_i, g_i, b_i)\}_{i \in \mathbb{N}}$ be a sequence of pointed compact C^∞ Riemannian n -manifolds which satisfy:*

$$\sup_{M_j} |R_{g_j}| \leq \Lambda \quad \text{for } i \in \mathbb{N}$$

with some constant Λ ,

$$\text{diam}(M_i, g_i) \rightarrow \infty \quad (i \rightarrow \infty),$$

and for any $R > 0$ there exist $i_R \in \mathbb{N}$ and $r_0 (> 0)$ such that

$$\text{inj}_{(M_i, g_i)}(x) \geq r_0 > 0 \quad \text{for } x \in B_R(b_i; g_i), \quad i \geq i_R,$$

where $\text{inj}_{(M_i, g_i)}(x)$ denotes the injectivity radius of (M_i, g_i) at x . Then there exist a subsequence $\{j\} \subset \{i\}$ and a noncompact complete C^∞ pointed n -manifold (N, b_∞) with $C^{1,\sigma}$ ($0 < \sigma < 1$) metric h such that (M_j, g_j, b_j) converges to (N, h, b_∞) in the pointed Hausdorff distance. Moreover, for each $r > 0$ there exists an into diffeomorphism $\Phi_j : B_r(b_\infty; h) \subset N \rightarrow M_j$ with $b_j \in \Phi_j(B_r(b_\infty; h))$ for j sufficiently large such that $(\Phi_j)^*g_j$ converges to h in the $C^{1,\sigma}$ topology on $B_r(b_\infty; h)$ and that $\lim_{j \rightarrow \infty} \Phi_j^{-1}(b_j) = b_\infty$.

In order to state a priori estimates for $L^{2,p}$ harmonic radius, we set, for a given $\delta > 0$

$$v^\delta(x) = \sup\{r > 0; \text{vol}(B_s(y)) \geq \delta \cdot s^n \quad \text{for all } B_s(y) \subset B_r(x)\},$$

which was introduced by Anderson [An3]. We note that, for each $\delta < \omega_n$, $v^\delta(x)$ is positive for any $x \in M$, where ω_n denotes the volume of the unit ball in \mathbb{R}^n . Let $\mathcal{M}_1(n)$ denote the space of all compact C^∞ Riemannian n -manifolds (M, g) with unit volume $V_g = 1$.

Theorem 3.6. *For $(M, g) \in \mathcal{M}_1(n)$ which satisfies*

$$\int_M |R_g|^p dv_g \leq \Lambda$$

with some positive constant Λ . Then there exists a positive constant $c_0 = c_0(\Lambda, n, p, \delta, L)$ such that

$$r_H(x) \geq c_0 \cdot v^\delta(x) \quad \text{for } x \in M.$$

Theorem 3.7. *Let (M, g) be an element in $\mathcal{M}_1(n)$ which satisfies*

$$\int_{B_r} |R_g|^p dv_g \leq \Lambda$$

with some positive constant Λ , where B_r denotes a geodesic ball of radius $r > 0$ in M . Then there exists a positive constant $c_0 = c_0(\Lambda, n, p, \delta, L)$ such that

$$\gamma_H(x) \geq c_0 \cdot \nu^\delta(x) \quad \text{for } x \in B_r,$$

where $\gamma_H(x) = \min\{r_H(x), \text{dist}_g(x, \partial B_r)\}$ and $\nu^\delta(x) = \min\{v^\delta(x), \text{dist}_g(x, \partial B_r)\}$.

Finally, for the proofs of Theorem 1.1 and Theorem 1.3, we shall prove another a priori estimate for $L^{2,p}$ harmonic radius. To start with, we prove the following lemma (cf. [An1, SU, S2]).

Lemma 3.8. *Let (M, g) be a compact C^∞ Riemannian n -manifold which satisfies the following Sobolev inequality*

$$(3.4) \quad \left(\int_M |u|^{2n/(n-2)} dv_g \right)^{(n-2)/n} \leq \frac{1}{c_s} \int_M |\nabla u|^2 dv_g + \frac{1}{V_g^{2/n}} \int_M u^2 dv_g$$

for $u \in L^{1,2}(M)$ with some positive constant c_s , and

$$(3.5) \quad \int_M |\nabla A_g|^q \leq \Lambda$$

with some constant Λ . Then there exist positive constants $\varepsilon_0 = \varepsilon_0(n, q, c_s)$ and $c_1 = c_1(n, q, c_s, V_g, \Lambda, r)$ such that, if

$$(3.6) \quad \int_{B_r} |R_g|^{n/2} dv_g \leq \varepsilon_0,$$

then

$$(3.7) \quad \int_{B_{\frac{r}{2}}} |R_g|^{nq/(n-2q)} dv_g \leq c_1,$$

where $B_r = B_r(x)$ is a geodesic ball of radius $r > 0$ in M .

Proof. We first note that the Riemann curvature tensor $R_g = (R_{ijkl})$ satisfies the following equation

$$(3.8) \quad \Delta_g R_{ijkl} = -\nabla_i A_{jkl} + \nabla_j A_{ikl} + (R_g * R_g)_{ijkl},$$

where $R_g * R_g$ denotes a linear combination of contractions of the tensor $R_g \otimes R_g$ by the metric g and $\Delta_g = g^{ij} \nabla_i \nabla_j$ the (nonpositive) Laplacian of g , respectively. From (3.8), we then obtain the following differential inequality

$$(3.9) \quad \Delta_g |R_g| \geq -c_2 |\nabla A_g| - c_3 |R_g|^2,$$

where c_2 and c_3 are positive constants depending only on n .

Let $u = |R_g|$ and ξ a cut-off function satisfying $\xi = 1$ on $B_{\frac{r}{2}}$ and $\xi = 0$ on $M \setminus B_r$ with $|\nabla \xi| \leq \frac{4}{r}$. Using Hölder's inequality in (3.5), we may particularly assume $q \leq \frac{n}{4(n-1)}$. Multiply both sides of (3.9) by $\xi^2 u^\tau$, where $\tau = \frac{n(q-1)}{n-2q} (\max\{1, \frac{2n-4}{n}\} < \tau + 1 \leq \frac{n}{2})$. Integrating by part, we obtain

$$(3.10) \quad \begin{aligned} & \frac{4\tau}{(\tau+1)^2} \int \xi^2 \left| \nabla \left(u^{(\tau+1)/2} \right) \right|^2 dv_g - 2 \int \xi u^\tau |\nabla \xi| |\nabla u| dv_g \\ & \leq c_4 \int \xi^2 u^\tau (|\nabla A_g| + u^2) dv_g, \end{aligned}$$

where $c_4 = \max\{c_2, c_3\}$. The Young inequality implies

$$(3.11) \quad \xi u^\tau |\nabla \xi| |\nabla u| \leq u^{\tau+1} |\nabla \xi|^2 + \frac{\xi^2}{(\tau+1)^2} \left| \nabla \left(u^{(\tau+1)/2} \right) \right|^2.$$

Using (3.10) in (3.11) then gives

$$(3.12) \quad \begin{aligned} & \int \xi^2 \left| \nabla \left(u^{(\tau+1)/2} \right) \right|^2 dv_g \\ & \leq \frac{(\tau+1)^2}{(4\tau-2)} \left[c_4 \int \xi^2 u^\tau (|\nabla A_g| + u^2) dv_g + 2 \int u^{\tau+1} |\nabla \xi|^2 dv_g \right]. \end{aligned}$$

From (3.4) and (3.12), we obtain

$$(3.13) \quad \begin{aligned} & \left(\int \left(\xi u^{(\tau+1)/2} \right)^{2n/(n-2)} dv_g \right)^{(n-2)/n} \\ & \leq \frac{c_5 \tau}{c_s} \int (\xi^2 u^{\tau+2} + \xi^2 u^\tau |\nabla A_g| + |\nabla \xi|^2 u^{\tau+1}) dv_g + \frac{1}{V_g^{2/n}} \int \xi^2 u^{\tau+1} dv_g, \end{aligned}$$

where c_5 is a positive constant depending only on n .

By Hölder's inequality, we note

$$(3.14) \quad \int \xi^2 u^{\tau+2} dv_g \leq \left(\int_{B_r} u^{n/2} dv_g \right)^{2/n} \left(\int \left(\xi u^{(\tau+1)/2} \right)^{2n/(n-2)} dv_g \right)^{(n-2)/n}.$$

Taking ε_0 in (3.6) satisfying $\varepsilon_0^{2/n} \leq \frac{c_s}{2c_5\tau}$, it then follows from (3.5), (3.13), (3.14) and Hölder's inequality again that

$$\begin{aligned}
 (3.15) \quad & \left(\int \xi^{2n/(n-2)} |R_g|^{nq/(n-2q)} dv_g \right)^{(n-2)/n} \\
 & \leq \left(\int \left(\xi u^{(\tau+1)/2} \right)^{2n/(n-2)} dv_g \right)^{(n-2)/n} \\
 & \leq \frac{2c_5\tau}{c_s} \int (|\nabla \xi|^2 u^{\tau+1} + \xi^2 u^\tau |\nabla A_g|) dv_g + \frac{2}{V_g^{2/n}} \int \xi^2 u^{\tau+1} dv_g \\
 & \leq c_6 \left[\int (|\xi|^2 + |\nabla \xi|^2) u^{\tau+1} dv_g \right. \\
 & \quad \left. + \left(\int_{B_r} |\nabla A_g|^q dv_g \right)^{1/q} \cdot \left(\int (\xi^2 u^\tau)^{q/(q-1)} dv_g \right)^{(q-1)/q} \right] \\
 & \leq c_7 \left(1 + \frac{1}{r^2} \right) \left(\int_{B_r} |R_g|^{n/2} dv_g \right)^{2(\tau+1)/n} \\
 & \quad + c_6 \left(\int_M |\nabla A_g|^q dv_g \right)^{1/q} \cdot \left(\int \xi^{2n/(n-2)} u^{nq/(n-2q)} dv_g \right)^{(q-1)/q} \\
 & \leq c_7 \left(1 + \frac{1}{r^2} \right) \varepsilon_0^{2(\tau+1)/n} + c_6 \Lambda^{1/q} \cdot \left(\int \xi^{2n/(n-2)} |R_g|^{nq/(n-2q)} dv_g \right)^{(q-1)/q},
 \end{aligned}$$

where $c_6 = c_6(n, c_s, V_g)$ and $c_7 = c_7(n, q, c_s, V_g)$.

Set $X = \int \xi^{2n/(n-2)} |R_g|^{nq/(n-2q)} dv_g$. The inequality (3.15) implies

$$(3.16) \quad X^{(n-2)/n} - c_6 \Lambda^{1/q} \cdot X^{(q-1)/q} - c_7 \left(1 + \frac{1}{r^2} \right) \varepsilon_0^{2q(n-2)/n(n-2q)} \leq 0.$$

From $q < \frac{n}{2}$ we note

$$(3.17) \quad 0 < \frac{q-1}{q} < \frac{n-2}{n}.$$

It then follows from (3.16) and (3.17) that there exists a positive constant $c_1 = c_1(n, q, c_s, V_g, \Lambda, r)$ such that

$$\int_{B_{\frac{r}{2}}} |R_g|^{nq/(n-2q)} dv_g \leq X \leq c_1.$$

This completes the proof of Lemma 3.8. \square

Proposition 3.9. *Let (M, g) be an element in $\mathcal{M}_1(n)$ which satisfies the Sobolev inequality (3.4) and (3.5) in Lemma 3.8. Let B_r denote a geodesic*

ball of radius $r > 0$ in M . Then there exist positive constants $\varepsilon_0 = \varepsilon_0(n, q, c_s)$ and $c_0 = c_0(n, q, c_s, \Lambda, r, \delta, L)$ such that if

$$(3.18) \quad \int_{B_{2r}} |R_g|^{n/2} dv_g \leq \varepsilon_0,$$

then

$$\gamma_H(x) \geq c_0 \cdot \nu^\delta(x) \quad \text{for } x \in B_r,$$

where $\gamma_H(x) = \min\{r_H(x), \text{dist}_g(x, \partial B_r)\}$, $\nu^\delta(x) = \min\{v^\delta(x), \text{dist}_g(x, \partial B_r)\}$ and $r_H(x)$ denotes the $L^{2, nq/(n-2q)}$ harmonic radius at x .

Proof. We take the same ε_0 in (3.18) as in (3.6). By Lemma 3.8 we obtain the following estimate

$$(3.19) \quad \int_{B_r} |R_g|^{nq/(n-2q)} dv_g \leq \tilde{c}_1,$$

where $\tilde{c}_1 = \tilde{c}_1(n, q, c_s, \Lambda, r)$. It then follows from (3.19) and Theorem 3.7 that there exists a positive constant $c_0 = c_0(n, q, c_s, \Lambda, r, \delta, L)$ such that

$$\gamma_H(x) \geq c_0 \cdot \nu^\delta(x) \quad \text{for } x \in B_r.$$

This completes the proof of Proposition 3.9. □

4. Compactness and pinching theorems for Yamabe metrics.

In this section, we shall prove compactness and pinching theorems for Yamabe metrics of positive scalar curvature with integral bounds on curvature, and we also prove Theorem 1.3.

Theorem 4.1. *Let $\{(M_i, g_i)\}_{i \in \mathbb{N}}$ be a sequence in $\mathcal{Y}_1(n, \mu_0)$ such that each (M_i, g_i) satisfies*

$$(4.1) \quad \int_{M_i} |R_{g_i}|^p dv_{g_i} \leq \Lambda$$

with some positive constant Λ . Then there exist a subsequence $\{j\} \subset \{i\}$, a compact C^∞ n -manifold M_∞ with $C^\alpha \cap L^{2,p}$ metric g_∞ of positive constant scalar curvature S_{g_∞} and a diffeomorphism $\Phi_j : M_\infty \rightarrow M_j$ for each j such that the following hold.

- (1) $V_{g_\infty} = 1, \quad 0 < \mu_0 \leq S_{g_\infty} \leq n(n-1) \text{vol}(S^n(1))^{2/n}.$
- (2) $(\Phi_j)^* g_j$ converges to g_∞ in the $C^{\alpha'}$ topology for $\alpha' < \alpha$ and weakly in the $L^{2,p}$ topology on M_∞ .

Moreover, there are only finitely many diffeomorphism types of compact n -manifolds M , which satisfy that there exists a Yamabe metric g on each M such that $(M, g) \in \mathcal{Y}_1(n, \mu_0)$ with (4.1).

Corollary 4.2. *There exists a positive constant $\varepsilon_0 = \varepsilon_0(n, p, \mu_0)$ such that if a compact Riemannian manifold $(M, g) \in \mathcal{Y}_1(n, \mu_0)$ satisfies*

$$\int_M |Z_g|^p dv_g \leq \varepsilon_0,$$

then M admits a C^∞ metric h of positive constant curvature with

$$(4.2) \quad V_h = 1, \quad \mu(M, [h]) \geq \mu_0 > 0,$$

where $Z_g = (Z_{ijkl})$ denotes the concircular curvature tensor of g , i.e.,

$$Z_{ijkl} = R_{ijkl} - \frac{S_g}{n(n-1)}(g_{ik}g_{jl} - g_{il}g_{jk}).$$

Moreover,

$$\#(\pi_1(M)) \leq \left(\frac{n(n-1) \text{vol}(S^n(1))^{2/n}}{\mu_0} \right)^{n/2}.$$

Remark 4.3. Under more general setting, compactness and pinching results similar to Theorem 4.1 and Corollary 4.2 have been already proved in [An3] and [Yn2]. However, we make only minimal assumptions for Yamabe metrics of positive scalar curvature and conclude some additional results.

Proof of Theorem 4.1. Set $\delta = (5 \cdot 2^{n-2})^{-n/2} \left(\frac{n-2}{4(n-1)}\right)^{n/2} \mu_0^{n/2} > 0$. From (2.2), (2.5) and $V_g = 1$, we obtain

$$(4.3) \quad v^\delta(x) \geq \sqrt{\frac{4}{n(n-2) \text{vol}(S^n(1))^{2/n}}} > 0 \quad \text{for } x \in M.$$

Using (4.1) and (4.3) in Theorem 3.6, then there exists a positive constant $c_0 = c_0(\Lambda, n, p)$ such that the following estimate for $L^{2,p}$ harmonic radius holds

$$(4.4) \quad r_H(x) \geq c_0 \sqrt{\frac{4}{n(n-2) \text{vol}(S^n(1))^{2/n}}} > 0 \quad \text{for } x \in M.$$

It then follows from (4.4) and Theorem 3.2 that there exist a subsequence $\{j\} \subset \{i\}$, a compact C^∞ n -manifold M_∞ with $C^\alpha \cap L^{2,p}$ metric g_∞ and a

diffeomorphism $\Phi_j : M_\infty \rightarrow M_j$ for each j such that $(\Phi_j)^*g_j$ converges to g_∞ in the $C^{\alpha'}$ topology for $\alpha' < \alpha$ and weakly in the $L^{2,p}$ topology on M_∞ , and then $V_{g_\infty} = 1$.

Moreover, taking a subsequence if necessary, we may assume that

$$\lim_{j \rightarrow \infty} S_{g_j} = S_{g_\infty} \equiv \text{const.},$$

then we also obtain

$$0 < \mu_0 \leq S_{g_\infty} \leq n(n-1) \text{vol}(S^n(1))^{2/n}.$$

This completes the proof of Theorem 4.1. □

Proof of Corollary 4.2. Our assertion will be done by contradiction. If the assertion does not hold, then there exist sequences $\{\varepsilon_i\}_{i \in \mathbb{N}}$ of positive constants and $\{(M_i, g_i)\}_{i \in \mathbb{N}} \subset \mathcal{Y}_1(n, \mu_0)$ satisfying $\varepsilon_1 > \varepsilon_2 > \dots \rightarrow 0$ and

$$(4.5) \quad \int_{M_i} |Z_{g_i}|^p dv_{g_i} \leq \varepsilon_i$$

for all $i \in \mathbb{N}$ such that each M_i never admits a metric of positive constant curvature.

From (2.2) and $|R_{g_i}|^2 = |Z_{g_i}|^2 + \frac{2}{n(n-1)}S_{g_i}^2$, we first note that

$$(4.6) \quad |R_{g_i}| \leq |Z_{g_i}| + \sqrt{2n(n-1) \text{vol}(S^n(1))^{2/n}}$$

for $i \in \mathbb{N}$. Combining (4.5) and (4.6) with Minkowski's inequality, we have

$$(4.7) \quad \begin{aligned} \int_{M_i} |R_{g_i}|^p dv_{g_i} &\leq \left[\left(\int_{M_i} |Z_{g_i}|^p dv_{g_i} \right)^{1/p} + \sqrt{2n(n-1) \text{vol}(S^n(1))^{2/n}} \right]^p \\ &\leq \left[\varepsilon_i^{1/p} + \sqrt{2n(n-1) \text{vol}(S^n(1))^{2/n}} \right]^p \end{aligned}$$

for $i \in \mathbb{N}$. It then follows from (4.7) and Theorem 4.1 that there exist a subsequence $\{j\} \subset \{i\}$, a compact C^∞ n -manifold M_∞ with $C^\alpha \cap L^{2,p}$ metric h of positive constant scalar curvature S_h and a diffeomorphism $\Phi_j : M_\infty \rightarrow M_j$ for each j such that the following hold.

$$(4.8) \quad V_h = 1, \quad 0 < \mu_0 \leq S_h \leq n(n-1) \text{vol}(S^n(1))^{2/n}.$$

$$(4.9) \quad h(j) = (\Phi_j)^*g_j \rightarrow h \quad (j \rightarrow \infty)$$

in the $C^{\alpha'}$ topology for $\alpha' < \alpha$ and weakly in the $L^{2,p}$ topology on M_∞ .

$$(4.10) \quad \lim_{j \rightarrow \infty} S_{g_j} = S_h \equiv \text{const.}$$

Using (4.7) in (4.1) of Theorem 4.1 then gives the following estimate for $L^{2,p}$ harmonic radius of each $h(j)$

$$(4.11) \quad r_H(x; h(j)) \geq r_0 > 0 \quad \text{for } x \in M_\infty,$$

where $r_0 = r_0(n, p, \mu_0)$. From (4.9) and (4.11), we can cover M_∞ by a finite collection of geodesic balls $\{B_{\frac{r_0}{2}}(y_k; h); y_k \in M_\infty\}_{k=1}^l$ with respect to h . It then follows from (4.11) and Remark 3.3 that there exist harmonic coordinates

$$U_{j,k} = \{u^a\}_{a=1}^n (= \{u(j)^a\}_{a=1}^n) : B_{r_0}(y_k; h(j)) (\subset M_\infty) \longrightarrow \mathbf{R}^n$$

with $U_{j,k}(y_k) = \mathbf{0} \in \mathbf{R}^n$

and a coordinate system $\{V_\lambda\}_{\lambda=1}^m$ of class C^∞ on M_∞ such that the metric components $h(j)_{ab} = h(j)(\partial/\partial u^a, \partial/\partial u^b)$ satisfy (3.1) and (3.2) on $\tilde{B}_{j,k} = U_{j,k}(B_{r_0}(y_k; h(j)))$, and that

$$(4.12) \quad \|U_{j,k} \circ V_\lambda^{-1}\|_{L^{4,p}} \leq C, \quad \|U_{j,k} \circ V_\lambda^{-1}\|_{C^{2,\alpha}} \leq C$$

where C is independent of j and k . From (4.12), taking a subsequence if necessary, we may assume that there exists a coordinate system $\{U_{\infty,k}\}_{k=1}^l$ of class $C^{2,\alpha} \cap L^{4,p}$ such that, for each k , $U_{j,k}$ converges to $U_{\infty,k}$ in the $C^{2,\alpha'}$ topology for $\alpha' < \alpha$ and weakly in the $L^{4,p}$ topology. We also assume that a collection of geodesic balls $\{B_{r_0}(y_k; h(j))\}_{k=1}^l$ covers M_∞ and that $U_{j,k}(B_{r_0}(y_k; h(j))) \supset \tilde{B}_{\infty,k}(1/2)$ for all j and k , where $\tilde{B}_{\infty,k}(1/2) = U_{\infty,k}(B_{\frac{r_0}{2}}(y_k; h))$.

On the other hand, from (4.5) we obtain

$$(4.13) \quad \int_{M_\infty} |\widehat{\text{Ric}}_{h(j)}|^p dv_{h(j)} = \int_{M_j} |\widehat{\text{Ric}}_{g_j}|^p dv_{g_j}$$

$$\leq \left(\frac{n-2}{4}\right)^{p/2} \int_{M_j} |Z_{g_j}|^p dv_{g_j} \longrightarrow 0 \quad (j \longrightarrow \infty),$$

where $\widehat{\text{Ric}}_g$ denotes the traceless part of Ric_g , i.e., $\widehat{\text{Ric}}_g = \text{Ric}_g - \frac{S_g}{n} \cdot g$. In each harmonic coordinates $U_{j,k}$, by using (3.1) in (4.13), then the components $\widehat{R}(j)_{ab}$ of $\widehat{\text{Ric}}_{h(j)}$ converge to 0 strongly in the L^p topology on $\tilde{B}_{\infty,k}(1/2)$. Combining (4.9) with this fact, we obtain

$$(4.14) \quad R(j)_{ab} \longrightarrow \frac{S_h}{n} \cdot h_{ab} \quad \text{strongly in } L^p(\tilde{B}_{\infty,k}(1/2)),$$

where $R(j)_{ab}$ denote the components of $\text{Ric}_{h(j)}$. In terms of this harmonic coordinates, $R(j)_{ab}$ are expressed as follows (cf. [J, DK]).

$$(4.15) \quad h(j)^{cd} \frac{\partial^2 h(j)_{ab}}{\partial u^c \partial u^d} + Q_{ab}(\partial h(j)) = -2R(j)_{ab} \quad \text{on } \tilde{B}_{j,k},$$

where Q is a quadratic term in the first derivatives $\partial h(j)$ of $h(j)$. It then follows from (4.9), (4.14) and (4.15) that the metric h in terms of the coordinates $\{U_{\infty,k}\}_{k=1}^l$ is a weak $C^\alpha \cap L^{2,p}$ solution to the following equation

$$(4.16) \quad h^{cd} \frac{\partial^2 h_{ab}}{\partial u^c \partial u^d} + Q_{ab}(\partial h) = -\frac{2}{n} S_h \cdot h_{ab} \quad \text{on } \tilde{B}_{\infty,k}(1/2).$$

Applying the elliptic regularity theory (cf. [Gi, GT]) to the equation (4.16), we obtain that h is an Einstein metric of class C^∞ on M_∞ , and then $\{U_{\infty,k}\}_{k=1}^l$ is a harmonic coordinate system of class C^∞ compatible with $\{V_\lambda\}_{\lambda=1}^m$. From (4.5), we also obtain that h is a positive constant curvature metric on M_∞ . Here we remark that an Einstein metric is a Yamabe metric (cf. [O, S3]). Combining (4.8) and (4.10) with these facts then contradict our assumption.

Since there exists a C^∞ metric h of positive constant curvature on M satisfying (4.2), then from (2.1) we have

$$\mu_0 \leq \mu(M, [h]) = n(n-1) \left(\frac{\text{vol}(S^n(1))}{\#(\pi_1(M))} \right)^{2/n}.$$

This completes the proof of Corollary 4.2. □

Proof of Theorem 1.3. Our assertion will be also done by contradiction. If the assertion does not hold, then there exist sequences $\{\varepsilon_i\}_{i \in \mathbb{N}}$ of positive constants and $\{(M_i, g_i)\}_{i \in \mathbb{N}} \subset \mathcal{Y}_1(n, \mu_0)$ of compact Riemannian n -manifolds satisfying $\varepsilon_1 > \varepsilon_2 > \dots \rightarrow 0$ and

$$(4.17) \quad \int_{M_i} |R_{g_i}|^p dv_{g_i} \leq \Lambda, \quad \int_{M_i} |\nabla A_{g_i}|^s dv_{g_i} \leq \varepsilon_i$$

for all $i \in \mathbb{N}$ such that each M_i never admits a C^∞ metric of harmonic curvature.

Since each M_i satisfies the estimate (4.17), we can apply volume estimates of geodesic balls from above due to Yang [Yn3]. Then, there exist positive constants $c_0 = c_0(n)$ and $\rho_0 = \rho_0(n, p, \Lambda)$ such that

$$(4.18) \quad \text{vol}(B_r(x; g_i)) \leq c_0 r^n \quad \text{for } r \leq \rho_0 \quad \text{and } x \in M_i.$$

From (4.17), (4.18) and Hölder’s inequality, we have

$$\begin{aligned}
 (4.19) \quad & \int_{B_r(x)} |R_{g_i}|^{n/2} dv_{g_i} \\
 & \leq \left(\int_{B_r(x)} |R_{g_i}|^p dv_{g_i} \right)^{n/2p} \cdot \text{vol}(B_r(x; g_i))^{(2p-n)/2p} \\
 & \leq \Lambda^{n/2p} (c_0 r^n)^{(2p-n)/2p} \quad \text{for } r \leq \rho_0 \quad \text{and } x \in M_i.
 \end{aligned}$$

Taking $\rho_1 = \rho_1(n, p, \Lambda)$ satisfying $\Lambda^{n/2p} (c_0 \rho_1^n)^{(2p-n)/2p} \leq \varepsilon_0$ and $\rho_1 \leq \rho_0$, where ε_0 is the same constant as in (3.6) with $c_s = \frac{n-2}{4(n-1)} \mu_0 > 0$. By Hölder’s inequality and $V_{g_i} = 1$, we may particularly assume that $s < \frac{n}{2}$. It then follows from (2.4), (4.17), (4.19) and Lemma 3.8 that

$$(4.20) \quad \int_{B_{\frac{\rho_1}{2}}(x)} |R_{g_i}|^{ns/(n-2s)} dv_{g_i} \leq c_1 \quad \text{for } x \in M_i,$$

where $c_1 = c_1(n, p, s, \mu_0, \Lambda)$. Here, by (2.5) and $V_{g_i} = 1$, there exists a finite subset $\{x_a(i)\}_{a=1}^{m_0} \subset M_i$ for each $i \in \mathbb{N}$ such that $\{B_{\frac{\rho_1}{2}}(x_a(i))\}_{i=1}^{m_0}$ is a covering of M_i , where $m_0 = m_0(n, p, \mu_0, \Lambda)$. Combining (4.20) with this fact then gives

$$(4.21) \quad \int_{M_i} |R_{g_i}|^{ns/(n-2s)} dv_{g_i} \leq \sum_{a=1}^{m_0} \int_{B_{\frac{\rho_1}{2}}(x_a(i))} |R_{g_i}|^{ns/(n-2s)} dv_{g_i} \leq m_0 c_1 \quad \text{for } i \in \mathbb{N}.$$

Set $p = \frac{ns}{n-2s} (> 2s)$ and $\alpha = 4 - \frac{n}{s} (> 1)$ in the proof of Corollary 4.2, respectively. From (4.21), Theorem 4.1 and Remark 3.3, the same results as (4.8)–(4.12) and (4.15) hold, and then we will use the same notation as in the proof of Corollary 4.2 except for p and α . In terms of each harmonic coordinates $U_{j,k}$, the Ricci curvature $\text{Ric}_{h(j)}$ satisfies the following equation beside (4.15)

$$\begin{aligned}
 (4.22) \quad & h(j)^{cd} \frac{\partial^2 R(j)_{ab}}{\partial u^c \partial u^d} + (\partial h(j) * \partial \text{Ric}_{h(j)})_{ab} \\
 & = [(\partial \partial h(j) + \partial h(j) * \partial h(j) + \text{Ric}_{h(j)}) * \text{Ric}_{h(j)}]_{ab} \\
 & \quad + h(j)^{cd} [\nabla_c A(j)_{adb} - \nabla_a A(j)_{cbd}] \\
 & (\equiv T(j)_{ab}) \quad \text{on } \tilde{B}_{j,k},
 \end{aligned}$$

where $A(j) = A_{h(j)}$. Here we note

$$(4.23) \quad \|h(j)_{ab}\|_{L^2, \frac{ns}{n-2s}}(\tilde{B}_{j,k}) \leq C_1, \quad \|h(j)_{ab}\|_{C^{1,3-\frac{n}{s}}}(\tilde{B}_{j,k}) \leq C_1$$

for all j, k and $a, b = 1, \dots, n$, where C_1 and C_2, C_3 below are independent of j . Combining (4.17), (4.21) and (4.23) with (4.22) then gives

$$(4.24) \quad \|T(j)_{ab}\|_{L^s(\tilde{B}_{j,k})} \leq C_2$$

for all j, k and $a, b = 1, \dots, n$. From (4.23), (4.24) and the L^p estimate (cf. [GT]) for (4.22), we have for $B \in \tilde{B}_{j,k}$

$$(4.25) \quad \|R(j)_{ab}\|_{L^{2,s}(B)} \leq C_3$$

for all j, k and $a, b = 1, \dots, n$.

From (4.17), (4.23) and (4.25), there exist a $C^{1,3-n/s} \cap L^{2,ns/(n-2s)}$ metric h and a $L^{2,s}$ symmetric tensor P of type $(0,2)$ on M_∞ such that we obtain the following.

$$(4.26) \quad h(j)_{ab} \longrightarrow h_{ab} \quad (j \longrightarrow \infty)$$

in the $C^{1,\alpha'}$ topology for $\alpha' < 3 - \frac{n}{s}$ and weakly in the $L^{2,ns/(n-2s)}$ topology on $\tilde{B}_{\infty,k}(1/2)$.

$$(4.27) \quad R(j)_{ab} \longrightarrow P_{ab} \quad (j \longrightarrow \infty)$$

weakly in the $L^{2,s}$ topology on $\tilde{B}_{\infty,k}(1/2)$, where P_{ab} denote the components of P .

$$(4.28) \quad \nabla_a A(j)_{bcd} \longrightarrow 0 \quad \text{strongly in } L^s(\tilde{B}_{\infty,k}(1/2)).$$

It then follows from (4.12), (4.15), (4.22) and (4.26)-(4.28) that, in terms of the $C^{3,3-n/s} \cap L^{4,ns/(n-2s)}$ coordinates $\{U_{\infty,k}\}_{k=1}^l$, h is a weak $C^{1,3-n/s} \cap L^{2,ns/(n-2s)}$ solution and P a weak $L^{2,s}$ solution to the following equations, on each $\tilde{B}_{\infty,k}(1/2)$

$$(4.29) \quad h^{cd} \frac{\partial^2 h_{ab}}{\partial u^c \partial u^d} + Q_{ab}(\partial h) = -2P_{ab},$$

$$(4.30) \quad h^{cd} \frac{\partial^2 P_{ab}}{\partial u^c \partial u^d} + (\partial h * \partial P)_{ab} = [(\partial \partial h + \partial h * \partial h + P) * P]_{ab}.$$

Applying the elliptic regularity theory (cf. [Gi, GT]) to the equations (4.29) and (4.30), we obtain that h is a C^∞ metric of $S_h \equiv \text{const.}$ on M_∞ , and then $P = \text{Ric}_h, \{U_{\infty,k}\}_{k=1}^l$ is a harmonic coordinate system of class C^∞ compatible with $\{V_\lambda\}_{\lambda=1}^m$. From (4.28) we also note that $\nabla A_h \equiv 0$. By the definition of A_h , we then obtain

$$(4.31) \quad \int_{M_\infty} |A_h|^2 dv_h \leq 2 \int_{M_\infty} |\text{Ric}_h| \cdot |\nabla A_h| dv_h = 0.$$

From (4.31) we have $A_h \equiv 0$, i.e., h is of harmonic curvature. Since a sequence of Yamabe metrics $h(j) = (\Phi_j)^*g_j$ converges to the C^∞ metric h in the $C^{1,\alpha'}$ topology for $\alpha' < 3 - \frac{\varepsilon}{n}$ and weakly in the $L^{2,ns/(n-2s)}$ topology on M_∞ , then h is also a Yamabe metric on M_∞ . In fact, unless h is a Yamabe metric, one can show that $h(j)$ is not a Yamabe metric for j sufficiently large. Combining this fact with the following

$$V_h = 1, \quad \mu(M_\infty, [h]) \geq \mu_0 > 0,$$

it then contradicts our assumption.

When $n = 3$, $A_h \equiv 0$ implies that (M^3, h) is conformally flat. Now the compact 3-manifold M^3 admits a conformally flat metric h of positive scalar curvature. Then, by Izeki's theorem [I], the rest assertion is immediate. This completes the proof of Theorem 1.3. □

5. Proof of Theorem 1.1.

In this section we shall prove Theorem 1.1.

By Theorem 2.3, if the first case (1°) in Theorem 1.1 does not hold, then there exist subsequence $\{j\} \subset \{i\}$ and a connected compact metric space (M_∞, d_∞) with $\text{diam}(M_\infty, d_\infty) = D_0 > 0$ such that

$$\lim_{j \rightarrow \infty} d_H((M_j, g_j), (M_\infty, d_\infty)) = 0.$$

Taking a subsequence if necessary, we may assume that

$$(5.1) \quad \text{diam}(M_j, g_j) \geq \frac{1}{2}D_0 > 0$$

for all j and that there exists a $(1/j)$ -Hausdorff approximation

$$\varphi_j : (M_j, g_j) \longrightarrow (M_\infty, d_\infty)$$

for each j . For each $y \in M_\infty$, we can find $y_j \in M_j$ such that

$$d_\infty(y, \varphi_j(y_j)) < \frac{1}{j}.$$

We define the singular set \mathcal{S} by

$$\mathcal{S} = \bigcap_{0 < r < D_0} \left\{ y \in M_\infty; \liminf_{j \rightarrow \infty} \int_{B_{2r}(y_j)} |R_{g_j}|^{n/2} dv_{g_j} \geq \varepsilon_0 \right. \\ \left. \text{for arbitrary } \{y_j\}_{j \in \mathbb{N}} \text{ as above} \right\},$$

where $\varepsilon_0 = \varepsilon_0(n, q, \mu_0)$ denotes a constant similar to ε_0 in Proposition 3.9.

To prove Theorem 1.1, we first note the following.

Lemma 5.1. *\mathcal{S} is a finite set.*

Proof. Take a small constant $r > 0$ and fix it. Then we can cover \mathcal{S} by a finite collection of metric balls $\{B_{2r}(x_a); x_a \in \mathcal{S}\}_{a \in \Gamma}$ with respect to d_∞ such that the collection $\{B_r(x_a)\}_{a \in \Gamma}$ is disjoint. Since $x_a \in \mathcal{S}$, for j sufficiently large there exists a point $x_{a,j} \in M_j$ such that

$$(5.2) \quad \int_{B_{\frac{r}{2}}(x_{a,j})} |R_{g_j}|^{n/2} dv_{g_j} \geq \frac{\varepsilon_0}{2},$$

$$(5.3) \quad d_\infty(x_a, \varphi_j(x_{a,j})) < \frac{1}{j} \quad \text{for } a \in \Gamma,$$

$$(5.4) \quad B_{\frac{r}{2}}(x_{a,j}) \cap B_{\frac{r}{2}}(x_{b,j}) = \emptyset \quad \text{for } a \neq b.$$

It follows from (5.2)–(5.4) that

$$(5.5) \quad \begin{aligned} \#(\Gamma) &\leq 2\varepsilon_0^{-1} \sum_{a \in \Gamma} \int_{B_{\frac{r}{2}}(x_{a,j})} |R_{g_j}|^{n/2} dv_{g_j} \\ &\leq 2\varepsilon_0^{-1} \int_{M_j} |R_{g_j}|^{n/2} dv_{g_j} \leq 2\Lambda_1 \varepsilon_0^{-1}. \end{aligned}$$

Since $2\Lambda_1 \varepsilon_0^{-1}$ is independent of r , letting $r \rightarrow 0$ in (5.5), we then obtain

$$\#(\mathcal{S}) \leq 2\Lambda_1 \varepsilon_0^{-1}.$$

This completes the proof of Lemma 5.1. □

Next we give a proof of (2° .2) – (2° .4). Fix a point $y \in M_\infty \setminus \mathcal{S}$.

Lemma 5.2. *There exist r satisfying $0 < r \leq \frac{1}{4}d_\infty(y, \mathcal{S})$, a point $y_j \in M_j$ for each j and a positive constant $r_0 = r_0(n, q, \mu_0, \Lambda_2, r)$ such that*

$$(5.6) \quad \gamma_H(x) \geq r_0 > 0 \quad \text{for } x \in B_r(y_j) \subset M_j,$$

where $\gamma_H(x) = \min \{r_H(x), \text{dist}_{g_j}(x, \partial B_r(y_j))\}$ and $r_H(x)$ denotes the $L^{2, nq/(n-2q)}$ harmonic radius at x .

Proof. Taking a subsequence if necessary, we can find r satisfying $0 < r \leq \frac{1}{4}d_\infty(y, \mathcal{S})$ and $y_j \in M_j$ for each j such that $d_\infty(y, \varphi_j(y_j)) < \frac{1}{j}$ and

$$(5.7) \quad \int_{B_{2r}(y_j)} |R_{g_j}|^{n/2} dv_{g_j} \leq \varepsilon_0 \quad \text{for all } j.$$

Using (2.4), (5.7) and $\int_{M_j} |\nabla A_{g_j}|^q dv_{g_j} \leq \Lambda_2$ in Proposition 3.9, then gives

$$(5.8) \quad \gamma_H(x) \geq c_0 \cdot \nu^\delta(x) \quad \text{for } x \in B_r(y_j) \quad \text{and all } j,$$

where $c_0 = c_0(n, q, \mu_0, \delta, \Lambda_2, C)$ and $\nu^\delta(x) = \min \{v^\delta(x), \text{dist}_{g_j}(x, \partial B_r(y_j))\}$. Now set $\delta = (5 \cdot 2^{n-2})^{-n/2} (\frac{n-2}{4(n-1)})^{n/2} \mu_0^{n/2} > 0$. It then follows from (2.5) and (5.8) that

$$\gamma_H(x) \geq r_0 > 0 \quad \text{for } x \in B_r(y_j) \quad \text{and all } j.$$

This completes the proof of Lemma 5.2. □

By (5.6), for each j there exist harmonic coordinates $U_j : B_{r_0}(y_j) \rightarrow \mathbf{R}^n$ with $U_j(B_{r_0}(y_j)) \supset B_{2\rho} = \{x \in \mathbf{R}^n; |x| < 2\rho\}$ for some $\rho > 0$ independent of j , such that the metric components of g_j satisfy (3.1) and (3.2) for $p = \frac{nq}{n-2q}$ and $\alpha = 4 - \frac{n}{q} (= \beta > 0)$. Now $\varphi_j \circ U_j^{-1}|_{B_\rho}$ is a $(\frac{1}{j})$ -Hausdorff approximation from (B_ρ, γ_j) to a neighborhood of y in M_∞ equipped with d_∞ , where $\gamma_j = (U_j^{-1})^* g_j$. From (3.1) and (3.2), taking a subsequence if necessary, there exists a $C^\beta \cap L^{2,nq/(n-2q)}$ metric $\gamma_{y,\infty}$ on B_ρ such that $\varphi_j \circ U_j^{-1}|_{B_\rho}$ converges to an isometry $H_y : (B_\rho, \gamma_{y,\infty}) \rightarrow (O_y, d_\infty)$, where O_y is also a neighborhood of y . Moreover, for any $y, z \in M_\infty \setminus \mathcal{S}$, $H_z^{-1} \circ H_y : (H_y^{-1}(O_y \cap O_z), \gamma_{y,\infty}) \rightarrow (H_z^{-1}(O_y \cap O_z), \gamma_{z,\infty})$ is also an isometry unless $O_y \cap O_z = \emptyset$. Since each metric $\gamma_{y,\infty}$ is of class C^β , then $H_z^{-1} \circ H_y$ is of class $C^{1,\beta}$ unless $O_y \cap O_z = \emptyset$ (cf. [CH]). By Whitney's theorem, there exists a unique C^∞ structure on $M_\infty \setminus \mathcal{S}$ compatible with the $C^{1,\beta}$ structure $\{(H_y^{-1}, O_y)\}_{y \in M_\infty \setminus \mathcal{S}}$. Thus $\{\gamma_{y,\infty}\}_{y \in M_\infty \setminus \mathcal{S}}$ also gives a C^β metric g_∞ on $M_\infty \setminus \mathcal{S}$ compatible with d_∞ .

For each $x_a \in \mathcal{S} = \{x_1, \dots, x_k\}$, let $\{x_{a,j}\}_{j \in \mathbf{N}}$ be points same as in the proof of Lemma 5.1. For each $m \in \mathbf{N}$, define the open subsets $D_j(2^{-m})$ in M_j and $D_\infty(2^{-m})$ in M_∞ by

$$D_j(2^{-m}) = \{x \in M_j; \text{dist}_{g_j}(x, x_{a,j}) > 2^{-m} \quad \text{for } a = 1, \dots, k\}$$

and

$$D_\infty(2^{-m}) = \{x \in M_\infty; d_\infty(x, \mathcal{S}) > 2^{-m}\},$$

respectively. From (5.1) and (5.6), each $D_j(2^{-m})$ is nonempty for m sufficiently large. Also from (3.1), (5.6) and $V_{g_j} = 1$, we have for all j

$$(5.9) \quad 0 < c_1 \leq \text{vol}(D_j(2^{-m}), g_j) \leq 1$$

for some constant $c_1 = c_1(m)$ independent of j . Replace Ω_i and ε in Theorem 3.4 by $D_j(2^{-m-1})$ and 2^{-m-1} respectively. It then follows from (5.6), (5.9) and Theorem 3.4 that, for each m , there exist a subsequence $\{j_m\} \subset \{j\}$, a

C^∞ n -manifold \mathcal{D}_∞^m with $C^\beta \cap L^{nq/(n-2q)}$ metric g_∞^m and an into diffeomorphism $F_{j_m} : \mathcal{D}_\infty^m \rightarrow D_{j_m}(3 \cdot 2^{-m-2})$ with $F_{j_m}(\mathcal{D}_\infty^m) \supset D_{j_m}(5 \cdot 2^{-m-2})$ for each j_m such that $(F_{j_m})^*g_{j_m}$ converges to g_∞^m in the $C^{\beta'}$ topology for $\beta' < \beta$ and weakly in the $L^{2,nq/(n-2q)}$ topology on \mathcal{D}_∞^m . Moreover we assume that $\{j_{m+1}\} \subset \{j_m\}$ for all m .

Now we remark that $\varphi_{j_m} \circ F_{j_m} : \mathcal{D}_\infty^m \rightarrow M_\infty$ converges to an into isometry $G^m : (\mathcal{D}_\infty^m, g_\infty^m) \rightarrow (D_\infty(3 \cdot 2^{-m-2}), g_\infty)$ with $G^m(\mathcal{D}_\infty^m) \supset D_\infty(5 \cdot 2^{-m-2})$ for each m . Take the diagonal sequence $\{j_j\}$ of $\{j_m\}_{j,m \in \mathbb{N}}$. We shall rewrite the index “ j_m ” by “ j ” again. Then we obtain that, for each m , there exists an into diffeomorphism $\Phi_j^m = F_j \circ (G^m)^{-1}|_{D_\infty(5 \cdot 2^{-m-2})} : D_\infty(5 \cdot 2^{-m-2}) \rightarrow M_j$ for j sufficiently large such that $(\Phi_j^m)^*g_j$ converges to g_∞ in the $C^{\beta'}$ topology for $\beta' < \beta$ and weakly in the $L^{2,nq/(n-2q)}$ topology on $D_\infty(5 \cdot 2^{-m-2})$. We also note that g_∞ is a $L_{loc}^{2,nq/(n-2q)}$ metric on $M_\infty \setminus \mathcal{S}$. Moreover, taking a subsequence if necessary, we may assume that

$$\lim_{j \rightarrow \infty} S_{g_j} = S_{g_\infty} \equiv \text{const.},$$

then we obtain

$$0 < \mu_0 \leq S_{g_\infty} \leq n(n-1) \text{vol}(S^n(1))^{2/n}.$$

For a compact subset $K \subset M_\infty \setminus \mathcal{S}$, there exists $m \in \mathbb{N}$ such that $K \subset D_\infty(5 \cdot 2^{-m-2})$. Thus we can take $\Phi_j = \Phi_j^m|_K$ as in (2° .3).

Finally we give a proof of (2° .5) and (2° .6). Fix a point $x_a \in \mathcal{S}$. There exists $x_{a,j} \in M_j$ such that $d_\infty(\varphi_j(x_{a,j}), x_a) < \frac{1}{j}$. Since \mathcal{S} is a finite set, we can take $\rho > 0$ so that $(B_{2\rho}(x_a) \setminus \{x_a\}) \cap \mathcal{S} = \emptyset$. For each j , we define the positive number r_j in (2° .5) by

$$r_j = \sup_{B_\rho(x_{a,j})} |R_{g_j}|.$$

By the definition of \mathcal{S} ,

$$(5.10) \quad r_j \rightarrow \infty \quad (j \rightarrow \infty).$$

Moreover we may assume that $|R_{g_j}|$ takes a local maximum value r_j at $x_{a,j}$. We consider the new sequence of pointed Riemannian manifolds $((M_j, \tilde{g}_j), x_{a,j})$, where $\tilde{g}_j = r_j g_j$. From (2.1), (2.2), (2.5), (5.1), (5.10) and the condi-

tions in Theorem 1.1, this sequence satisfies:

$$(5.11) \quad \sup_{B_{\sqrt{r_j \cdot \rho}}(x_{a,j}; \tilde{g}_j)} |R_{\tilde{g}_j}| = 1, \quad |R_{\tilde{g}_j}|(x_{a,j}) = 1,$$

$$(5.12) \quad \text{diam}(M_j, \tilde{g}_j) = \sqrt{r_j} \text{diam}(M_j, g_j) \longrightarrow \infty \quad (j \longrightarrow \infty),$$

$$(5.13) \quad V_{\tilde{g}_j} = r_j^{n/2} V_{g_j} = r_j^{n/2} \longrightarrow \infty \quad (j \longrightarrow \infty),$$

$$(5.14) \quad S_{\tilde{g}_j} = r_j^{-1} S_{g_j} = r_j^{-1} \mu(M_j, [g_j]) \longrightarrow 0 \quad (j \longrightarrow \infty),$$

$$(5.15) \quad \int_{B_{\sqrt{r_j \cdot \rho}}(x_{a,j}; \tilde{g}_j)} |R_{\tilde{g}_j}|^{n/2} dv_{\tilde{g}_j} = \int_{B_\rho(x_{a,j}; g_j)} |R_{g_j}|^{n/2} dv_{g_j} \leq \Lambda_1,$$

$$(5.16) \quad \int_{B_{\sqrt{r_j \cdot \rho}}(x_{a,j}; \tilde{g}_j)} |\nabla A_{\tilde{g}_j}|^q dv_{\tilde{g}_j} \\ = r_j^{-2q+n/2} \int_{B_\rho(x_{a,j}; g_j)} |\nabla A_{g_j}|^q dv_{g_j} \longrightarrow 0 \quad (j \longrightarrow \infty),$$

$$(5.17) \quad \text{vol}(B_r(x); \tilde{g}_j) \geq (5 \cdot 2^n (n-1))^{-n/2} (n-2)^{n/2} \mu_0^{n/2} r^n$$

for $x \in B_{c_2 \sqrt{r_j}}(x_{a,j}; \tilde{g}_j)$, where $c_2 = c_2(n) > 0$. By using (5.11) and (5.17) in Theorem 5.3 below due to Cheeger-Gromov-Taylor [CGT], then for any $R > 0$, there exists $j_R \in \mathbb{N}$ and $r_0 (> 0)$ such that

$$(5.18) \quad \text{inj}_{(M_j, \tilde{g}_j)}(x) \geq r_0 > 0 \quad \text{for } x \in B_R(x_{a,j}; \tilde{g}_j), \quad j \geq j_R.$$

Theorem 5.3. *Let $B_r(x)$ be a metric ball of radius r in a Riemannian manifold (M, g) such that for $r' < r$, $\overline{B_{r'}(x)}$ is compact. Assume that on $B_r(x)$, $\omega \leq K_g \leq \kappa$ and $r \leq \frac{\pi}{\sqrt{\kappa}}$ (r arbitrary if $\kappa \leq 0$) for some constants ω, κ . Let B_r^ω be a geodesic ball of radius r in the simply-connected space form of constant curvature ω . Then, for positive constants r_0 and s with $r_0 + 2s \leq r$ and $r_0 \leq \frac{r}{4}$, the following inequality holds*

$$\text{inj}_{(M, g)}(x) \geq \frac{r_0}{2} \cdot \frac{1}{1 + \text{vol}(B_{r_0+s}^\omega) / \text{vol}(B_s(x))}.$$

It then follows from (5.11), (5.12), (5.18) and Theorem 3.5 that there exists a noncompact complete C^∞ pointed n -manifold $(N_a, x_{a,\infty})$ with $C^{1,\sigma}(0 <$

$\sigma < 1$) metric h_a such that, taking a subsequence if necessary, $((M_j, \tilde{g}_j), x_{a,j})$ converges to $((N_a, h_a), x_{a,\infty})$ in the pointed Hausdorff distance. Moreover, for each $r > 0$ there exists an into diffeomorphism $\Psi_j : B_r(x_{a,\infty}) (\subset N_a) \rightarrow M_j$ with $x_{a,j} \in \Psi_j(B_r(x_{a,\infty}))$ for j sufficiently large such that $(\Psi_j)^* \tilde{g}_j$ converges to h_a in the $C^{1,\sigma}$ topology on $B_r(x_{a,\infty})$ and that $\lim_{j \rightarrow \infty} \Psi_j^{-1}(x_{a,j}) = x_{a,\infty}$.

On the other hand, by using (5.10), (5.11), (5.17) and the Bishop comparison theorem [BC, Corollary 4, p. 245] in Theorem 3.7, then for any $s (> n)$ satisfying $1 - \frac{n}{s} > \sigma$ there exists $r_1 (> 0)$ such that

$$r_H(x) \geq r_1 > 0 \quad \text{for } x \in B_{\frac{1}{2}\sqrt{r_j \cdot \rho}}(x_{a,j}; \tilde{g}_j),$$

where $r_H(x)$ denotes the $L^{2,s}$ harmonic radius at $x \in M_j$. In terms of each harmonic coordinates in $\Psi_j^{-1} \left(B_{\frac{1}{2}\sqrt{r_j \cdot \rho}}(x_{a,j}; \tilde{g}_j) \right)$, by (5.11), (5.16), the L^p estimates and the Sobolev inequality (cf. [GT]), then the components $\tilde{h}(j)_{kl}$ of $\tilde{h}_j = (\Psi_j)^* \tilde{g}_j$, $\tilde{R}(j)_{kl}$ of $\text{Ric}_{\tilde{h}_j}$ and $\nabla_b \tilde{A}(j)_{ckl}$ of $\nabla A_{\tilde{h}_j}$ satisfy

$$(5.19) \quad \left\| \tilde{h}(j)_{kl} \right\|_{L^{2,s}} \leq C, \quad \left\| \tilde{h}(j)_{kl} \right\|_{C^{1,1-n/s}} \leq C,$$

$$(5.20) \quad \left\| \tilde{R}(j)_{kl} \right\|_{L^{2,q}} \leq C,$$

$$(5.21) \quad \left\| \nabla_b \tilde{A}(j)_{ckl} \right\|_{L^q} \leq C,$$

where C is independent of j . From (5.16) and (5.19)–(5.21), there exists a $L^{2,q}$ symmetric tensor P of type $(0,2)$ on N_a such that we obtain the following.

$$(5.22) \quad \tilde{h}(j)_{kl} \rightarrow (h_a)_{kl} \quad (j \rightarrow \infty)$$

in the $C^{1,\sigma}$ topology for $\sigma < 1 - \frac{n}{s}$ and weakly in the $L^{2,s}$ topology, where $(h_a)_{kl}$ denote the components of h_a .

$$(5.23) \quad \tilde{R}(j)_{kl} \rightarrow P_{kl} \quad (j \rightarrow \infty)$$

weakly in the $L^{2,q}$ topology.

$$(5.24) \quad \nabla_b \tilde{A}(j)_{ckl} \rightarrow 0 \quad (j \rightarrow \infty)$$

strongly in the L^q topology. It then follows from (5.22)–(5.24) that h_a is a weak $C^{1,1-n/s} \cap L^{2,s}$ solution and P a weak $L^{2,q}$ solution to the following

equations

$$(5.25) \quad (h_a)^{bc} \frac{\partial^2 (h_a)_{kl}}{\partial u^b \partial u^c} + Q_{kl}(\partial(h_a)) = -2P_{kl},$$

$$(5.26) \quad (h_a)^{bc} \frac{\partial^2 P_{kl}}{\partial u^b \partial u^c} + (\partial(h_a) * \partial P)_{kl} = [(\partial\partial(h_a) + \partial(h_a) * \partial(h_a) + P) * P]_{kl}.$$

Applying the elliptic regularity theory (cf. [Gi, GT]) to the equations (5.25) and (5.26), we obtain h_a is a C^∞ metric on N_a , and then $P = \text{Ric}_{h_a}$. From (5.10), (5.11), (5.14), (5.15) and (5.17), we also note that $S_{h_a} \equiv 0$ and

$$\begin{aligned} \sup_{N_a} |R_{h_a}| &= 1, \quad 0 < \int_{N_a} |R_{h_a}|^{n/2} dv_{h_a} \leq \Lambda_1, \\ \text{vol}(B_r(x); h_a) &\geq (5 \cdot 2^n (n-1))^{-n/2} (n-2)^{n/2} \mu_0^{n/2} r^n \end{aligned}$$

for $x \in N_a$ and $r > 0$.

From the definition of $A_{\tilde{g}_j}$ and Hölder’s inequality, we note

$$(5.27) \quad \begin{aligned} \int_{M_j} |A_{\tilde{g}_j}|^2 dv_{\tilde{g}_j} &\leq 2 \int_{M_j} |\text{Ric}_{\tilde{g}_j}| \cdot |\nabla A_{\tilde{g}_j}| dv_{\tilde{g}_j}, \\ &\leq 2 \left(\int_{M_j} |\text{Ric}_{\tilde{g}_j}|^{q/(q-1)} dv_{\tilde{g}_j} \right)^{(q-1)/q} \left(\int_{M_j} |\nabla A_{\tilde{g}_j}|^q dv_{\tilde{g}_j} \right)^{1/q}. \end{aligned}$$

By using Hölder’s inequality and $V_{g_j} = 1$ in $\int_{M_j} |A_{g_j}|^q dv_{g_j} \leq \Lambda_2$ and $q > \max\{1, \frac{n}{4}\}$, we may assume that

$$(5.28) \quad \frac{q}{q-1} \geq \frac{n}{2} \quad \text{when } n \leq 5.$$

From (5.11), (5.15), (5.16), (5.27) and (5.28), we then obtain for $n \leq 5$

$$(5.29) \quad \begin{aligned} \int_{M_j} |A_{\tilde{g}_j}|^2 dv_{\tilde{g}_j} &\leq 2 \left(\int_{M_j} |\text{Ric}_{\tilde{g}_j}|^{n/2} dv_{\tilde{g}_j} \right)^{(q-1)/q} \\ &\quad \cdot r_j^{-2+n/2q} \left(\int_{M_j} |\nabla A_{\tilde{g}_j}|^q dv_{\tilde{g}_j} \right)^{1/q} \\ &\leq 2r_j^{-2+n/2q} \Lambda_1^{(q-1)/q} \Lambda_2^{1/q} \longrightarrow 0 \quad (j \longrightarrow \infty). \end{aligned}$$

By (5.29), when $n \leq 5$ we have that $A_{h_a} \equiv 0$, i.e., h_a is of harmonic curvature. In particular, when $n = 3$ (N_a, h_a) is conformally flat.

The proof of (2°.6) follows from the lower semicontinuity of the curvature integral. This completes the proof of Theorem 1.1. \square

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Mogens L. Hansen and Richard V. Kadison , Banach algebras with unitary norms	535
Xin-hou Hua , Sharing values and a problem due to C.C. Yang	71
Jing-Song Huang , Harmonic analysis on compact polar homogeneous spaces	553
Min-Jei Huang , Commutators and invariant domains for Schrödinger propagators	83
Hisao Kato , Chaos of continuum-wise expansive homeomorphisms and dynamical properties of sensitive maps of graphs	93
Oliver Küchle , Some properties of Fano manifolds that are zeros of sections in homogeneous vector bundles over Grassmannians	117
Xin Li and Francisco Marcellan , On polynomials orthogonal with respect to Sobolev inner product on the unit circle	127
Steven Liedahl , Maximal subfields of $Q(i)$ -division rings	147
Alan L.T. Paterson , Virtual diagonals and n -amenability for Banach algebras	161
Claude Schochet , Rational Pontryagin classes, local representations, and K^G -theory	187
Sandra L. Shields , An equivalence relation for codimension one foliations of 3-manifolds	235
D. Siegel and E. O. Talvila , Uniqueness for the n -dimensional half space Dirichlet problem	571
Aleksander Simonič , A Construction of Lomonosov functions and applications to the invariant subspace problem	257
Endre Szabó , Complete intersection subvarieties of general hypersurfaces ..	271

PACIFIC JOURNAL OF MATHEMATICS

Volume 175 No. 2 October 1996

Mean-value characterization of pluriharmonic and separately harmonic functions	295
LEV ABRAMOVICH AĪZENBERG, CARLOS A. BERENSTEIN and L. WERTHEIM	
Convergence for Yamabe metrics of positive scalar curvature with integral bounds on curvature	307
KAZUO AKUTAGAWA	
Generalized modular symbols and relative Lie algebra cohomology	337
AVNER DOLNICK ASH and DAVID GINZBURG	
Convolution and limit theorems for conditionally free random variables	357
MAREK BOŹEJKO, MICHAEL LEINERT and ROLAND SPEICHER	
L^p -bounds for hypersingular integral operators along curves	389
SHARAD CHANDARANA	
On spectra of simple random walks on one-relator groups. With an appendix by Paul Jolissain	417
PIERRE-ALAIN CHERIX, ALAIN J. VALETTE and PAUL JOLISSAINT	
Every stationary polyhedral set in \mathbf{R}^n is area minimizing under diffeomorphisms	439
JAIGYOUNG CHOE	
Ramanujan's master theorem for symmetric cones	447
HONGMING DING, KENNETH I. GROSS and DONALD RICHARDS	
On norms of trigonometric polynomials on $SU(2)$	491
ANTHONY H. DOOLEY and SANJIV KUMAR GUPTA	
On the symmetric square. Unit elements	507
YUVAL ZVI FLICKER	
Stable constant mean curvature surfaces minimize area	527
KARSTEN GROSSE-BRAUCKMANN	
Banach algebras with unitary norms	535
MOGENS LEMVIG HANSEN and RICHARD VINCENT KADISON	
Harmonic analysis on compact polar homogeneous spaces	553
JING-SONG HUANG	
Uniqueness for the n -dimensional half space Dirichlet problem	571
DAVID SIEGEL and ERIK O. TALVILA	



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