GENERALIZED MODULAR SYMBOLS AND RELATIVE LIE ALGEBRA COHOMOLOGY

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In this paper we explore the limitations forced on the infinity type of a cohomological automorphic representation given the non-vanishing of an associated period over a generalized modular symbol. After some general remarks, we discuss the example of $GL(2n)$ over a totally real field.

Let $G$ be a reductive group defined over the number field $F$ and $\pi \approx \otimes_{v} \pi_{v}$ a cuspidal irreducible automorphic representation of $G(A)$, where $v$ runs over all the places of $F$ and $A$ denotes the adeles of $F$. Write $\omega$ for the central character of $\pi$. Let $G_{\infty} = \Pi G_{v}$ where $v$ runs over the archimedean places of $F$ and choose $K_{\infty}$ to be a compact subgroup of $G_{\infty}$ which contains the connected component of the identity of a maximal compact subgroup of $G_{\infty}$. Denote by $X$ the symmetric space $G_{\infty}/K_{\infty}Z_{\infty}^{0}$ where $Z$ is the center of $G$. We assume $X$ is non-compact.

Set $G_{f} = \Pi G_{v}$ where $v$ runs over the non-archimedean places of $F$ and choose a compact open subgroup $L$ of $G_{f}$. We let $\Gamma$ be the arithmetic subgroup of $G(F)$ defined to be the projection of $G(F) \cap G_{\infty}L$ into $G_{\infty}$. We assume $\Gamma/X$ is orientable. Let $\mathfrak{g} = \text{Lie} G_{\infty}/Z_{\infty}^{0}$ and $\tilde{K}_{\infty} = \text{image of } K_{\infty}$ in $G_{\infty}/Z_{\infty}^{0}$.

We recall the well-known isomorphism of cohomology groups $H^{*}_{\text{cusp}}(\Gamma/X, \mathbb{C}) \cong \otimes_{\omega} H^{*}(\mathfrak{g}, \tilde{K}_{\infty}; L^{2}_{\text{cusp}}(G(F)\backslash G(A), \omega))/L$. The latter contains $H^{*}(\mathfrak{g}, \tilde{K}_{\infty}; \pi_{\infty}) \otimes \pi_{f}^{L}$ as a summand (identifying $\pi$ with its image in $L^{2}_{\text{cusp}}(G(F)\backslash G(A), \omega)$ but taking care to remember that the isomorphism $\pi \approx \otimes_{v} \pi_{v}$ is an abstract one and doesn’t “take place” inside $L^{2}_{\text{cusp}}$). We let $d$ be a non-negative integer and choose $[\psi] \in H^{d}_{\text{cusp}}(\Gamma/X, \mathbb{C})$ where $\psi$ is a closed differential $d$-form on $\Gamma/X$ representing the cohomology class $[\psi]$ - we may even take $\psi$ harmonic. Under the isomorphism above, we suppose $\psi$ goes over to $\alpha \otimes \beta$, with $\alpha \in H^{d}(\mathfrak{g}, \tilde{K}_{\infty}; \pi_{\infty})$ and $\beta \in \pi_{f}^{L}$. Recall that $H^{d}(\mathfrak{g}, \tilde{K}_{\infty}; \pi_{\infty}) \approx \text{Hom}_{K_{\infty}}(\wedge^{d}\mathfrak{g}/\mathfrak{k}, \pi_{\infty})$ and we view $\alpha$ as such a homomorphism. (Here $\mathfrak{k} = \text{Lie } K_{\infty}$.)

Now let $H$ denote a reductive $F$-subgroup of $G$. We assume $H_{\infty}$ is connected and $H(A)$ satisfies strong approximation. Choose $e \in X$ fixed by $K_{\infty}$ and set $X_{H} = H(F_{\infty})e \subset X$. We assume $M = (H_{\infty} \cap \Gamma) \backslash X_{H}$ is orientable,
and we fix an orientation. Then the two propositions of Section 1 of \([AGR]\) imply that for some \(f\) in the space of \(\pi\)
\[
\int_H \psi = \int_{[Z(H) \cap H(F)]H(F)/H(H)} \omega^{-1}(h)f(h)dh.
\]
There is a canonical procedure for finding \(f\) given \(\psi\) or vice versa. Following the argument in Section 5.2 of \([AG]\), we take a basis \(Y_1, \ldots, Y_d\) of Lie \(H^0/(K_\infty \cap H^0)Z_{\infty}\) and set \(Y_M = Y = Y_1 \wedge \cdots \wedge Y_d\). Then up to a nonzero multiplicative constant we may take \(f = \alpha(Y)\beta\). In particular, if the integral doesn't vanish then \(\alpha(Y) \neq 0\), and of course \(d = \dim X_H = \dim M\).

We call \(f\) a cohomological vector for \(\pi\). We call such an integral a period (of the cuspform \(f\) or the cohomology class \([\psi]\)) over the (generalized) modular symbol \(M\). In our terminology, a modular symbol is an oriented locally finite cycle such as \(M\) arising as the projected orbit of a reductive group.

In \([AGR]\) it is shown that these integrals are absolutely convergent. Combining the topological methods of \([RS]\) with the deRham theorem, it is easy to construct modular symbols \(M\) that support non-vanishing periods. Here the reductive group \(H\) underlying \(M\) will be the fixed points in \(G\) of some finite group action.

The non-vanishing of periods seems to be connected with properties of \(\pi\) and its \(L\)-functions, e.g. whether \(\pi\) is a lift from some other group, or whether a certain \(L\)-function has a pole. This is being investigated by Jacquet, Rallis and others. See \([AG]\) for an example, and the references cited there.

On the local level, a non-vanishing period implies the existence of a nontrivial \(H_{\infty}\) invariant functional on \(\pi_\infty\), which should be related to whether \(\pi_\infty\) is a lift.

In this paper we begin to study the question: Does the non-vanishing of a period put a constraint on the isomorphism type of \(\pi_\infty\)? The case of \(GL(4)\) was studied already in \([AG]\) and there led to a proof of the non-vanishing of a \(p\)-adic \(L\)-function. This paper arose out of an attempt to extend those results to \(GL(2n)\) for \(n > 2\). We shall see that although many possibilities for \(\pi_\infty\) are ruled out by the nonvanishing of the period, already for \(GL(6)\) and \(GL(8)\) there are too many possibilities left to allow the use of the trick in Section 5 of \([AG]\) for \(n > 2\) to prove the non-vanishing of a certain archimedean integral and hence of the \(p\)-adic \(L\)-function.

In Section 1 we review the Vogan-Zuckerman classification of \(\pi_\infty\) with nontrivial \((g, K_\infty)\)-cohomology. In Section 2 we show how the nonvanishing period enters the picture and prove some propositions that can be used in practice to rule out certain \(\pi_\infty\)'s. In Section 3 we outline the example of \(GL(8)\) with remarks applying to \(GL(m)\) for various \(m\), notably \(m = 2, 4, 6\). In the appendix we give a heuristic connection between the existence of a
nontrivial $K_\infty^0 \cap H_\infty$-fixed vector in the cohomological $K$-type of $\pi_\infty$ and a nontrivial $H_\infty$-invariant continuous linear functional on $\pi_\infty$ in the case where $G = GL(2n)$ and $H = GL(n) \times GL(n)$.

We close this introduction by pointing out a comparison among the results in [A], [AGR], and this paper. In [A] the existence of a non-vanishing period for $\pi$ puts constraints on the local component $\pi_v$ of $\pi$ at a non-archimedean place, for local reasons. In this paper, we have similarly locally effected results at archimedean places. In [AGR], vanishing of certain periods was derived from global considerations.

1. Classification of representations with nontrivial $(g, K)$-cohomology.

For simplicity we assume in this section $G$ is a semi-simple, real, connected Lie group with finite center. Let $g = \text{Lie}(G) \otimes \mathbb{C}$ and $K \subset G$ a maximal compact subgroup. The modifications needed when $G$ is reductive or non-connected are most easily performed on an ad hoc basis. In [VZ] a finite list of irreducible admissible $(g, K)$-modules $\{\pi\}$ is given such that $H^*(g, K; \pi) \neq 0$ and it is shown that every irreducible unitary $G$-representation with nontrivial $(g, K)$-cohomology has its Harish-Chandra module isomorphic to some $\pi$ on the list. Later in [V] and [W] it was shown that each $\pi$ on the list is the Harish-Chandra module of a unitary $G$-representation. Hence the unitary nature of a $\pi_\infty$ arising from a cohomological cuspform places no restrictions on its isomorphism type. In [VZ] twisting $\pi$ by a finite dimensional representation is also allowed, but we are interested only in untwisted coefficients here. We summarize the properties of the classification that we will use. See [VZ] for complete details.

Let $\mathfrak{k} = \text{Lie}(K) \otimes \mathbb{C}$, $\theta$ be the corresponding Cartan involution, and $g = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition. A finite set $\{\mathfrak{q}\}$ of $\theta$-stable parabolic subalgebras of $g$ is defined. Write $\mathfrak{q} = \ell + \mathfrak{u}$, where $\ell$ is a Levi-factor and $\mathfrak{u}$ the radical of $\mathfrak{q}$. One chooses a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ which is contained in $\ell$ and let $\mu = \mu(q) = \text{irreducible representation of } K \text{ with highest weight } 2\rho(\mathfrak{u} \cap \mathfrak{p})$. Here $\rho(\mathfrak{u} \cap \mathfrak{p})$ is one-half the sum of the $\mathfrak{t}$ weights on $\mathfrak{u} \cap \mathfrak{p}$.

We shall call the isomorphism class of $\mu$ a cohomological $K$-type. It appears in $\wedge^*\mathfrak{p}$. There is a unique irreducible admissible $(g, K)$-module $A_\pi$ such that $H^*(g, K; A_\pi) = \text{Hom}_K(\wedge^*\mathfrak{p}, A_\pi) \neq 0$ and the only $K$-type shared by $\wedge^*\mathfrak{p}$ and $A_\pi$ is $\mu(q)$. Moreover for different $q$'s the $\mu(q)$'s (and hence the $A_q$'s) are distinct. Every irreducible admissible $(g, K)$-module $\pi$ with $H^*(g, K; \pi) \neq 0$ is isomorphic to one of the $A_q$'s.
2. Enter the nonvanishing period.

We maintain all the preceding notation.

Now suppose \( \pi_\infty \) is isomorphic to \( A_q \) for some \( q \) and that the period of a cohomological vector for \( \pi \) over \( H(A) \) doesn't vanish. In this case we shall say that \( \pi \) has a nontrivial \( H \)-period. Let \( d \) be the dimension of the corresponding modular symbol \( M \), with \( Y_M \in \wedge^d p \).

**Proposition 2.1.** Suppose \( \pi \) has a nontrivial \( H \)-period, and \( \pi_\infty \approx A_q \). Then

1. \( \mu(q) \) appears in \( \wedge^d p \);
2. \( \mu(q) \) contains a nontrivial vector invariant under \( H_\infty \cap K \);
3. The \( K \)-submodule of \( \wedge^d p \) generated by \( Y_M \) projected onto the \( \mu(q) \)-isotypic component of \( \wedge^d p \) is non-vanishing.

**Remark.** Although (1) and (2) immediately follow from (3) since \( Y_M \) is clearly \( H_\infty \cap K \)-invariant, we stated the three items in order of ease of checking in any given example.

**Proof.** As stated we need only prove (3). From the hypothesis, there exists \( \alpha \in \text{Hom}_K(\wedge^d p, A_q) \) such that \( \alpha(Y_M) \neq 0 \). Since \( \mu(p) \) is the only \( K \)-type shared by \( \wedge^d p \) and \( A_q \), (3) follows.

**Proposition 2.2.** Under the hypotheses of the previous proposition, suppose in addition there exists a connected noncompact semi-simple Lie group \( G_1 \) with Iwasawa decomposition \( G_1 = K_1 A_1 N_1 \) such that \( (\text{Lie} \ G_1) \otimes \mathbb{C} \) is isomorphic to \( (\text{Lie} \ K) \otimes \mathbb{C} \) by an isomorphism that takes \( (\text{Lie} \ K_1) \otimes \mathbb{C} \) onto \( \text{Lie}(H_\infty \cap K_1) \otimes \mathbb{C} \). Extend \( \text{Lie} \ A_1 \) to a maximal abelian subalgebra \( t_0 \) of \( g \), so that \( t_0 = t_0 \cap \text{Lie} \ K_1 \oplus \text{Lie} \ A_1 \) is a Cartan subalgebra of \( \text{Lie} \ G_1 \). Let \( \lambda \) be the highest weight of \( \mu(q) \) with respect to \( t_0 \). Then

1. \( \lambda \left( \sqrt{-1}(t_0 \cap \text{Lie} \ K_1) \right) = 0 \);
2. \( \frac{\langle \lambda, \alpha \rangle}{(\alpha, \alpha)} \in \mathbb{Z}^+ \) for all \( \alpha \in \Sigma^+ \)
where \( \Sigma^+ \) is the set of positive restricted roots on \( \text{Lie} \ A_1 \) with respect to the ordering induced by the choice of \( N_1 \).

**Proof.** This follows from Proposition 2.1 (2) and Helgason's criterion Theorem 4.1 p. 535 of [H] after complexifying the Lie algebras and taking the hypotheses into account.

In the following, with a view to our examples in the next section, we go back to the notation of Section 1 and allow \( G \) to be reductive and not necessarily connected. Thus \( g = \text{Lie} \ G^0_\infty / Z^0_\infty, \mathfrak{k} = \text{Lie} \ K_\infty \), etc.
The group of components $\tilde{K}_\infty/\tilde{K}_\infty^0$ acts on the set of cohomological $K$-types $\{\mu(q)\}$ in the obvious way. If $O$ is an orbit, there is an obvious way to make $\bigoplus_{\mu(q)\in O} A_q$ into an irreducible $(q, \tilde{K}_\infty)$-module. We will denote it by $B_q$ for any $q$ such that $\mu(q) \in O$. Every irreducible $(q, \tilde{K}_\infty)$-module with nontrivial cohomology is isomorphic to $B_q$ for some $q$.

Now let $K$ denote the algebraic $R$-group such that $K(R) = K_0$, so $\text{Lie } K(C) = t$. Given $q = I + u$, we have the Cartan subalgebra $t$ of $I$ contained in $t$ and we choose a Borel subalgebra $b = t + n$ of $t$ such that $u \subset n$. We use capital Roman letters to denote subgroups of $K(C)$ whose Lie-algebra equals the corresponding small Gothic letter. Thus $Q$ is a parabolic subgroup of $K = \tilde{K}(C)$ with Levi decomposition $Q = LU$. Also, $B$ is a Borel subgroup of $K$ with Levi decomposition $B = TN$. We let $H$ stand for $H(C)$.

We now make the following additional hypothesis. For an illustration of it, see Section 3.

**Hypothesis 2.3.** There exists a parabolic subgroup $P_0$ of $K$ with Levi decomposition $P_0 = L_0 U_0$ such that

(i) $P_0 \supset B$ and hence $U_0 \subset N$;

(ii) $T \subset L_0$;

(iii) $U_0$ contains a subgroup $W_0$ such that $\text{Lie } W_0 = \text{Lie } N \cap H$;

(iv) $L_0 \subset H$ and $L_0$ stabilizes $W_0$ under conjugation.

Now choose an order on $t^*$ so that $B$ corresponds to the positive roots $\Phi_+$ and for each $\alpha \in \Phi_+$ fix $u_\alpha : C \to U_\alpha \subset N$. Order the positive roots $\alpha_1, \ldots, \alpha_r$ and write $u_i = u_{\alpha_i}$. We assume the ordering chosen so that $u_1, \ldots, u_m$ generate $W_0$ and $u_{m+1}, \ldots, u_r$ generate $N \cap H$. Let $x_1, \ldots, x_r$ be indeterminates and view them as coordinates on $N$ by $x = (x_1, \ldots, x_r) = u_1(x_1) \ldots u_r(x_r) = u(x_1, \ldots, x_r) = u(x)$. Let $x' = (x_1, \ldots, x_m) = u_1(x_1) \ldots u_m(x_m) = u(x')$. We have an induced action of $L_0$ on $P \in C[x'] = C[x_1, \ldots, x_m] = C[W_0]$ by $(g \cdot P)(x') = P(g^{-1} \cdot x') = P(g^{-1}u(x')g)$.

Fix an irreducible $K$-module $V$ of $\wedge^d \mathfrak{g}/\mathfrak{k}$ with highest weight $\delta$ (all weights with respect to $t$) and let proj denote the $K$-equivariant projection onto $V$. Let $Y$ be a generator of the line $\wedge^d \text{Lie } H/\text{Lie } H \cap t$ in $\wedge^d \mathfrak{g}/\mathfrak{k}$. It has weight zero. For any $T$-module $M$ and weight $\lambda$ write $M_\lambda$ for the $\lambda$-isotypic component of $M$. For each weight $\mu$ in $V$ choose a $C$-basis $\{v_{\mu,i} : i = 1, \ldots, j_\mu\}$ of $V_\mu$. Since $V_\delta$ is one-dimensional we write $v_\delta$ in place of $v_{\delta,1}$.

**Lemma 2.4.** Define $\{P_{\mu,i}(x) \in C[x]\}$ by proj $u(x) \cdot Y = \Sigma_i P_{\mu,i}(x)v_{\mu,i}$.

(i) $P_{\mu,i}(x)$ is independent of $x_{m+1}, \ldots, x_r$ for all $\mu, i$.

(ii) $P_\delta$ is a maximal vector for $L_0 \cap B^\text{opp}$ of weight $-\delta$ and generates an
$L^0_0$-module contragredient to a quotient of $\text{Res}^{K}_{L_0^0} V$.

(iii) $V$ is contained in the $K$-span of $Y$ if and only if $P_\delta \neq 0$.

Proof. Since for $i > m$ $u_i(x_i) \in H \cap N$ and $H^0$ fixes $Y$, statement (i) is true. To prove (ii) reindex the $\{v_{\mu,i}\}$ as $\{v_k\}$. Let $g \in L_0^0 \subset H$, so $gY = Y$. Then

$$\text{proj } u(g \cdot x')Y = \text{proj } gu(x')g^{-1}gY = \Sigma P_k(x')gv_k.$$ 

On the other hand

$$\text{proj } u(g \cdot x')Y = \Sigma P_k(g \cdot x')v_k = \Sigma g^{-1}P_k(x')v_k.$$ 

Comparing the right hand sides, we see that the matrix representation of $g$ on the span of $\{P_k\}$ is a quotient of the contragredient $V^*$ of $V$. Thus $P_\delta$ generates an $L^0_0$-module isomorphic to a quotient of $\text{Res}^{K}_{L_0^0} V^*$.

View $\delta$ as a character on $T$ and extend it to $B$ by making it trivial on $N$. If $g \in B^{\text{opp}}$, since $v_\delta$ is a maximal vector in $V$, we have $gv_k$ has no $v_\delta$-component unless $v_k = v_\delta$ and then $gv_\delta = \delta(g)v_\delta$. Comparing the right hand sides again we get

$$\delta(g)P_\delta = g^{-1}P_\delta.$$ 

Statement (iii) follows from Lemma 5.5.1 of [AG] except we use $V$ in place of the whole isotypic component of type $\delta$. □

Lemma 2.5. If $\delta$ is the cohomological $K$-type of $B_\delta$ and $Q = LU$ is the Levi decomposition, then as $L \cap K$-module, $V = V_\delta \oplus (\sum_{\mu \neq \delta} V_\mu)$.

Proof. From the proof of Proposition 3.6 of [VZ] we see that $L \cap K$ fixes the line $V_\delta$. Therefore, if $\alpha$ is any root of $L \cap K$, the $\alpha$-string of weights of $V$ in $\delta + \mathbb{Z}\alpha$ is just $\{\delta\}$. (Incidentally this proves that $< \delta, \alpha > = 0$ in the notation of Section 21.3 of [Hu].) So if $\mu$ is a weight of $V$, $\mu \neq \delta$, the $\alpha$-string through $\mu$ can’t reach to $\delta$. Hence $\sum_{\mu \neq \delta} V_\mu$ is also $L \cap K$-invariant. □

Lemma 2.6. Suppose $s \in L \cap L_0$ such that $s \cdot P_\delta = aP_\delta$, $s \cdot Y = bY$, $s \cdot v_\delta = cv_\delta$ with $a, b, c \in \mathbb{C}$. Then if $abc \neq 1$, $V$ is not contained in the $K$-span of $Y$ in $\wedge^d_{\mathfrak{g}/\mathfrak{k}}$.

Proof. As in the proof of (ii) of Lemma 2.4 we obtain $b\Sigma P_k sv_k = \Sigma s^{-1} \cdot P_k v_k$. From Lemma 2.5 we can equate the terms involving $v_\delta$ to get $bP_\delta sv_\delta = s^{-1}P_\delta v_\delta$ or $abcP_\delta = P_\delta$. If $abc \neq 1, P_\delta = 0$ and the conclusion follows from (iii) of Lemma 2.4. □
3. Examples: $GL(2n)$.

In this section we apply the foregoing to the example whose interest stems from [AG]. We refer the reader to the introduction of that paper for motivation.

We let $G = GL(2n)/\mathbb{Q}$ for $n \geq 1$. Choose $K_\infty = O(2n, \mathbb{R})$ and $H = GL(n) \times GL(n)$. Although $H$ doesn’t satisfy all the hypotheses made in Section 1, in this particular example all the conclusions there and in Section 2 remain true, as comparison with Section 5 of [AG] will show.

We found in [AG] that for $n = 2$, the nonvanishing of the $H$-period determined $\pi_\infty$ uniquely up to isomorphism. The same is easily seen to be the case for $n = 1$. Here we will investigate $n = 3$ and $n = 4$.

Of particular interest in the following calculations is the invariant theory that comes in.

We will present the $GL(8)$ case in detail and summarize our results for the $GL(6)$ case. The methods in both cases are basically the same, but since $GL(6)$ is smaller than $GL(8)$, less variety appears.

3.1. Case of $GL(N)$. First we present the list of irreducible $(\mathfrak{g}, K)$-modules $\pi$ with non-trivial cohomology. We thank J.S. Li for providing us with this, which may be derived either from Speh’s original article [S] or from the general theory of Vogan and Zuckermann [VZ].

In this subsection, let $G = GL(N, \mathbb{R}), K = O(N), \mathfrak{g} = \text{Lie } G, \mathfrak{k} = \text{Lie } K, \mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition. Let $\epsilon_j, 1 \leq j \leq [N/2]$ be the usual basis for the dual of a Cartan subalgebra of $\mathfrak{k}$.

Let $r_1, \ldots, r_k$ be positive integers with $m = r_1 + \cdots + r_k \leq N/2$. We allow the case $k = 0$. There corresponds a $\theta$-stable parabolic subalgebra $q = \mathfrak{q} + u$ whose corresponding Levi subgroup is

$$L = GL(r_1, \mathbb{C}) \times \cdots \times GL(r_k, \mathbb{C}) \times GL(N - 2m, \mathbb{R}).$$

In the notation of Section 2 the $(\mathfrak{g}, K)$-module $B_q$ is irreducible, unitarizable and $H^*(\mathfrak{g}, K; B_q) \neq 0$. Any such $\pi$ is isomorphic to $B_q$ for some $q = q(r_1, \ldots, r_k)$ arising this way.

Set $m_s = r_1 + \cdots + r_s, 1 \leq s \leq k$. Then the cohomological $K$-type of $B_q$ has highest weight

$$2\rho(u \cap p) = \sum_{1 \leq s \leq k} \sum_{m_{s-1} \leq i \leq m_s} (N + 1 - m_{s-1} - m_s)\epsilon_i.$$

This is the unique $K$-type of $A(q)$ that occurs in $\wedge^*(\mathfrak{g}/\mathfrak{k})$.

Let $P$ be the standard parabolic subgroup of $G$ with Levi component

$$M = GL(2r_1, \mathbb{R}) \times \cdots \times GL(2r_k, \mathbb{R}) \times GL(N - 2m, \mathbb{R}).$$
Let $\pi_s$ be the Speh representation of $GL(2r_s, \mathbb{R})$ which is the Langlands quotient of

$$\operatorname{Ind} \left( \sigma_s | \text{det} \right)^{\frac{m_s-1}{2}} \otimes \cdots \otimes \sigma_s | \text{det} \right)^{\frac{-m_s+1}{2}}$$

where $\sigma_s$ is the discrete series representation of $GL(2, \mathbb{R})$ given by $\sigma_s = \pi(\mu_s, -\mu_s)$ with $\mu_s = \frac{1}{2}(N - m_s - 1 - m_s)$. We also let $1$ denote the trivial representation of $GL(N - 2m, \mathbb{R})$. Then $B_q \approx \operatorname{Ind}^G_p(\pi_1 \otimes \cdots \otimes \pi_k \otimes 1)$.

For $N = 6$ and $8$ we record this information in tabular form. The case $N = 4$ is already treated in [AG]. We give each representation an identifying number for later reference.

### Table for $GL(6)$

<table>
<thead>
<tr>
<th>$#$</th>
<th>$k$</th>
<th>$r_1, \ldots, r_k$</th>
<th>$m_1, \ldots, m_k$</th>
<th>$\delta(\epsilon \text{- basis})$</th>
<th>$\delta(f \text{- basis})$</th>
</tr>
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<tbody>
<tr>
<td>1</td>
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<td>$-$</td>
<td>$-$</td>
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<td>$0, 0, 0$</td>
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<td>3</td>
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<td>2, 3</td>
<td></td>
<td>$5, 5, 2$</td>
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</tr>
<tr>
<td>8</td>
<td>3</td>
<td>1, 1, 1</td>
<td>1, 2, 3</td>
<td>$6, 4, 2$</td>
<td>$6, 2, 2$</td>
</tr>
</tbody>
</table>

In these tables, $\delta$ refers to the cohomological $K$-type. The $\epsilon$-basis was defined above; $(a, \ldots, b, c)$ stands for $ae_1 \cdots + b\epsilon_{n-1} + c\epsilon_n, N = 2n$. The $f$-basis refers to the parametrization of $K$-types in terms of fundamental weights; draw the Dynkin diagram so that the all but two of the nodes lie along a horizontal line, and the outer automorphism switches the two nodes on the far right; then $(a, \ldots, b, c)$ in that basis stands for $a$ times the leftmost weight plus ... plus $b$ times the upper rightmost weight plus $c$ times the lower rightmost weight.

The last entry in each table is the unique representation on the list which could occur as the infinity type of a global cuspidal representation on $GL(N)/\mathbb{Q}$.

If $\pi$ is isomorphic to $B_q$ for the $q$ from the $i$-th line on the list, write $\pi = \pi_i = \pi_\delta$ where $\delta$ is the corresponding cohomological $K$-type.

### 3.2. Case of $GL(8)$

Resume of notations: $G_\infty = GL(8, \mathbb{R}), K_\infty = O(8), H_\infty = GL(4, \mathbb{R}) \times GL(4, \mathbb{R}), g_\infty = \mathfrak{t}_\infty \oplus p_\infty$ where $p_\infty$ can be viewed as $8 \times 8$ symmetric matrices. Let $0p_\infty$ denote the traceless matrices in $p_\infty$. Identify $g_\infty/\mathfrak{t}_\infty$ with $p_\infty$ and let $Y \in \Lambda^{19}p_\infty$ be the wedge of a fixed basis of $\text{Lie } H_\infty \cap 0p_\infty$. 
Table for $GL(8)$

<table>
<thead>
<tr>
<th>n</th>
<th>k</th>
<th>$r_1,\ldots,r_k$</th>
<th>$m_1,\ldots,m_k$</th>
<th>$\delta(\epsilon$- basis)</th>
<th>$\delta(f$- basis)</th>
</tr>
</thead>
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<td>$-$</td>
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<td>2</td>
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<td>0700</td>
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<td>0066</td>
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<td>2,4</td>
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<td>0326</td>
</tr>
<tr>
<td>16</td>
<td>4</td>
<td>1,1,1,1</td>
<td>1,2,3,4</td>
<td>8642</td>
<td>2226</td>
</tr>
</tbody>
</table>

**Theorem.**

(i) If $\alpha \in \text{Hom}_{K_{\infty}}(\Lambda^3 p_{\infty}, \pi)$ and $\alpha(Y) \neq 0$ then $\pi$ is type 8, 11 or 16.

(ii) Conversely, if $\pi$ is one of those three types, there exists $\alpha$ such that $\alpha(Y) \neq 0$.

**Proof.** We do part (i) by eliminating possibilities.

Because $Y$ is invariant under $SO(4) \times SO(4)$ we can apply Proposition 2.2: $\pi_\delta$ contains a nontrivial $K^0_{\infty} \cap H_{\infty}$-fixed vector if and only if

$$
\frac{(\delta|\beta)}{(\beta|\beta)} \in \mathbb{Z} \quad \text{for all roots } \beta \text{ of } \mathfrak{t}_{\infty}.
$$

In the $\epsilon$-basis we have $(\epsilon_i|\epsilon_j) = \delta_{ij}$. Since $(\beta|\beta) = 2$ for all $\beta$, the criterion becomes $(\delta|\beta) \in 2\mathbb{Z}$. Write $\delta = \Sigma \epsilon_i \epsilon_i$. Each $\beta$ has the form $\epsilon_i \pm \epsilon_j$ for $i \neq j$.

Thus $(\delta|\beta) \in 2\mathbb{Z} \iff$ all $\epsilon_i$'s have same parity $\iff$ either all $r_i$'s have same parity or $m_k < n$ and all $r_i$'s are odd. This eliminates types 3, 7, 9, 13, 14, 15.

The other cases require a more detailed analysis. It will be convenient to complexify and work with a split version of $K_{\infty}$. We let $K = O(2n, \mathbb{C})$, $p = p_{\infty} \otimes \mathbb{C}$, $\delta p = \delta p_{\infty} \otimes \mathbb{C}$. However we have to keep track of $H_{\infty}$ when we do this. Let $\theta$ be the standard Cartan involution $g \rightarrow ^tg^{-1}$ and $\sigma$ be the involution $(I_n - I_n)$ so that $H$ is the fixed-points of $\sigma$. Then we conjugate $\theta$
and σ by the same complex $2n \times 2n$ matrix to get a split form of $K$ and the new $H$. Let

$$J_m = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \in GL(m),$$

$$J = \begin{pmatrix} 0 & J_n \\ J_n & 0 \end{pmatrix} \in GL(2n),$$

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} J_n & I_n \\ -iI_n & iJ_n \end{pmatrix}.$$

Then $AJ^tA = I$ and $A^{-1}\sigma A = AdJ$, so conjugation by $A$ takes $O(2n)$ to $O(J)$ and $\sigma$ to $AdJ$. We can then conjugate further by $g \in O(J)$ such that $gJg^{-1} = \left( \begin{pmatrix} I_{n/2} \\ -iI_{n/2} \end{pmatrix} \right) = \xi$ assuming $n$ is even. (The odd $n$ case is a little more complicated – see the section on $GL(6)$.)

From now on, assume $n$ even and set $K = O(J)(\mathbb{C}), H = Ad(\xi)$-fixed points in $GL(2n, \mathbb{C})$:

$$H = \left\{ \begin{pmatrix} * & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix} \begin{pmatrix} n/2 \\ n/2 \\ n/2 \end{pmatrix} \right\}.$$

If $X \in M_{2n}$, let $X_T$ denote the transpose of $X$ about the non-main diagonal. Then

$$K = \{ g \in GL(2n, \mathbb{C}) | g_T^{-1} = g \},$$

$$\mathfrak{p} = \{ X \in M_{2n}(\mathbb{C}) | X_T = X \},$$

$$Y = \text{generator of } \wedge^{top} (\mathfrak{p} \cap \text{Lie } H).$$

Now define some groups that will satisfy Hypothesis 2.3. First let

$$B = \{ \text{upper triangular matrices in } K \},$$

$$N = \{ \text{unipotent matrices in } B \},$$

$$T = \{ \text{diagonal matrices in } B \}.$$
\[ P_0 = L_0U_0 \text{ with } U_0 = R_u(P_0) \text{ and } \]
\[ L_0 = \left\{ \begin{pmatrix} * & * \\ * & * \end{pmatrix} \right\} \cap K. \]

Finally set
\[ W_0 = \exp \left( \left\{ \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix} \right\} \cap K \right). \]

We make the choices prescribed after Hypothesis 2.3 so that we are in a position to apply the rest of Section 2.

Now set \( n = 4 \). Fix a cohomological \( K \)-type \( \delta = (\delta_1, \delta_2, \delta_3, \delta_4) \) in the \( \epsilon \)-coordinates. Put coordinates on \( T \) and \( W_0 \) as follows:

\[
t = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4^{-1} \\ d_3^{-1} \\ d_2^{-1} \\ d_1^{-1} \end{pmatrix} \in T
\]

\[
w(X, Y) = w = \begin{pmatrix} 1 & 0 & X_1 & X_2 & X_3 & X_4 & * & * \\ 0 & 1 & Y_1 & Y_2 & Y_3 & Y_4 & * & * \\ 1 & 0 & 0 & 0 & -Y_4 & -X_4 & & \\ 1 & 0 & 0 & -Y_3 & -X_3 & & & \\ 1 & 0 & -Y_2 & -X_2 & & & & \\ 1 & -Y_1 & -X_1 & & & & & \\ 1 & 0 & & & & & & \\ 0 & 1 & & & & & & \end{pmatrix} \in W_0
\]

so \( \epsilon_i(t) = d_i \). If
\[
\ell = \begin{pmatrix} A \\ B \\ A_T^{-1} \end{pmatrix} \in L_0
\]
then \( \ell \) acts by conjugation on \( W_0 \) via \( M \to AMB^{-1} \) where

\[
M = \begin{pmatrix}
X_1 & X_2 & X_3 & X_4 \\
Y_1 & Y_2 & Y_3 & Y_4
\end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}.
\]

Lemma 3.4. The space of polynomials \( P(X,Y) \) fixed under the induced action by \( L_0 \cap R_u(B^{opp}) \) is the \( \mathbb{C} \)-span of the 6 polynomials \( P_1, \ldots, P_6 \) in the table. Each of these is an eigenpolynomial for the action of \( T \). The right hand column of the table gives the character \( \chi_i \) such that \( P_i(t \cdot w) = \chi_i(t) P_i(w) \).

Table of semi-invariants for \( L_0 \cap B^{opp} \) in \( \text{Sym}^\ast(W_0) \):

<table>
<thead>
<tr>
<th>( P_i )</th>
<th>( \chi_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_1 = X_4 )</td>
<td>( \chi_1 = d_1 d_3 )</td>
</tr>
<tr>
<td>( P_2 = X_2 Y_4 - Y_2 X_4 )</td>
<td>( \chi_2 = d_1 d_2 d_3 d_4^{-1} )</td>
</tr>
<tr>
<td>( P_3 = X_3 Y_4 - Y_3 X_4 )</td>
<td>( \chi_3 = d_1 d_2 d_4 )</td>
</tr>
<tr>
<td>( P_4 = X_1 X_4 + X_2 X_3 )</td>
<td>( \chi_4 = d_1^2 )</td>
</tr>
<tr>
<td>( P_5 = Y_1 X_4^2 - X_1 X_4 Y_4 + X_2 Y_3 X_4 + Y_2 X_3 X_4 - 2 X_2 X_3 Y_4 )</td>
<td>( \chi_5 = d_1^2 d_3 )</td>
</tr>
<tr>
<td>( P_6 = \det(MJ_4^t M) )</td>
<td>( \chi_6 = d_1^2 d_2 )</td>
</tr>
</tbody>
</table>

Proof. It is easily checked each \( P_i \) is semi-invariant with the designated character. To show these span the space of semi-invariants one can use a result from [P-SR]. The local unramified computation in that paper induces a decomposition of the symmetric algebra of \( GL(2, \mathbb{C})^3 \approx GL(2, \mathbb{C}) \times GO(4, \mathbb{C}) \). Using this decomposition one gets the desired assertion.

Now consider types 2, 4, 6, 12. They all have \( \delta_4 = 0 \). Writing \( P_\delta = \Pi P_i^{\delta_4} \), as we may by Lemma 2.4 (iii), we see that necessarily \( e_2 = e_3 \), since \( \delta = \Pi \chi_i^{\delta_4} \).

Set

\[
s = \begin{pmatrix}
I_3 \\
0 & 1 & 0 \\
1 & 0 & I_3
\end{pmatrix} \in K.
\]

Then \( s \) induces the permutation \((23)\) on the indices of \( X \) and \( Y \). Since \( s(P_i) = P_i \) for \( i \neq 2, 3 \) and \( s(P_2) = P_3, s(P_3) = P_2 \), we have \( sP_\delta = P_\delta \) in the case where \( \delta_4 = 0 \).

Also, \( s \) acts on \( {}^0p \) by conjugation and preserves \( H \), hence \( Y \). It’s easy to see \( sY = -Y \).
Now let $V$ be an irreducible $K$-submodule of $\Lambda^1\mathfrak{p}$ of type $\delta$, with highest weight vector $v_\delta$. Since $\delta_4 = 0$, $s$ preserves $\delta$ and hence $sv_\delta = \pm v_\delta$.

**Lemma 3.5.** If $\delta$ is type 2, 4, 6, or 12 then $sv_\delta = v_\delta$.

**Proof.** By the proof of Theorem 3.3 p. 64 of [VZ], if $q = \ell + u$ corresponds to type $\delta$, then $v_\delta = \alpha \wedge \beta$ for some $\beta \in \Lambda^R(u \cap \mathfrak{p})$ and some $\alpha \in (\Lambda^{19} - R\ell \cap \mathfrak{p})^{\text{tr}}$, where $R = \dim u \cap \mathfrak{p}$. Now $(\Lambda^s \ell \cap \mathfrak{p})^{\text{tr}}$ is isomorphic to the space of $L^0$-invariant differential forms on the symmetric space for $L^0$, which is in turn isomorphic to the cohomology of the compact dual. The latter is explicitly computed in [B].

We need only consider $V$ contained in the $K$-span of $Y$, hence contained in $^0\mathfrak{p}$. So we may assume $\alpha \in (\Lambda^{d-R} \ell \cap ^0\mathfrak{p})^{\text{tr}}$. Of course $\beta \in \Lambda^R(u \cap ^0\mathfrak{p}) = \Lambda^R(u \cap \mathfrak{p})$.

A case by case calculation based on [B] now shows that in the cases under consideration $s\alpha = (-1)^m\alpha$ and $s\beta = (-1)^m\beta$. Hence $sv_\delta = v_\delta$.

We omit the details, but sketch out one case as an example. Consider type 6. Then $k = 2, (r_1, r_2) = (1, 1), m_k = 2, R = 12$. In this case, $L \cong \prod_{i=1}^k GL(r_i, \mathbb{C}) \times GL(8 - 2m_k, \mathbb{R}) \approx \mathbb{C}^\times \times \mathbb{C}^\times \times GL(4, \mathbb{R})$ and

$$L(\mathbb{C}) \cong \left\{ \begin{pmatrix} t_1 & t_2 \\ t_3 \\ t_4 \end{pmatrix} \begin{pmatrix} g \\ g \in GL(4, \mathbb{R}) \end{pmatrix} \mid t_1 \ldots t_4 \in \mathbb{C}^\times \right\}$$

and $s$ acts on $L$ as conjugation by

$$\begin{pmatrix} I_3 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The compact dual symmetric space for $L$ is $\prod_{i=1}^k U(r_i) \times U(8 - 2m_k)/SO(8 - 2m_k)$. Since we consider only the traceless matrices in $\ell \cap ^0\mathfrak{p}$ we have that $(\Lambda^{d-R} \ell \cap ^0\mathfrak{p})^{\text{tr}}$ is isomorphic to the cohomology of

$$Y = \prod_{i=1}^k U(r_i) \times SU(8 - 2m_k)/SO(8 - 2m_k).$$

In our case $Y = U(1) \times U(1) \times SU(4)/SO(4)$ and $s$ acts nontrivially only on the last factor, and there as conjugation by an element of determinant $-1$ in $O(4)$. 
By [B] we know that $H^*(SU(4)/SO(4)) \approx E[x_4, x_5]$ where $E$ stands for the exterior algebra generated by generators $x_i$ in deg $i$. Also $s$ acts on $x_i$ as multiplication by $(-1)^{i+1}$. We also know that $H^*(U(1)) = E[y_1]$.

So $H^*(Y) \approx E[y_1, y'_1, x_4, x_5]$ and $sy_1 = y_1, sy'_1 = y'_1, sx_4 = -x_4, sx_5 = x_5$. Now $\alpha$ corresponds to an element in $H^{19-R}(Y) = H^7(Y)$, so the only possibility is $y_1 \land y'_1 \land x_5$. It follows that the $K$-type $\delta_5$ appears with multiplicity one in $\Lambda^{19-0}p$, and that $s\alpha = \alpha$. (The only case among 2, 4, 6, 12 with more than one linearly independent choice of $\alpha$ is case 12, with multiplicity two. One simply checks that for all possible $\alpha$, $s\alpha = (-1)^{m_\alpha}\alpha$.)

Next $\beta$ is the wedge of 12 vectors in $u \cap p$, indicated schematically as

$$\beta = \Lambda^{\text{top}} \begin{pmatrix} 0 & a & b & c & d & e & f & g \\ 0 & h & i & j & k & l & f \\ 0 & 0 & 0 & 0 & k & e \\ 0 & 0 & 0 & 0 & j & d \\ 0 & 0 & 0 & 0 & i & c \\ 0 & 0 & 0 & 0 & h & b \\ 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$  

Now $s$ switches $c$ and $d$, and $i$ and $j$. Hence $s$ acts as $+1$ on $\beta$. So $s\alpha = \alpha, s\beta = \beta$ and $sv_5 = v_5$.

So by Lemma 2.6, since $sP_5 = P_5, sv_5 = v_5$ and $sY = -Y$ for types 2, 4, 6, 12, they can’t occur in the $K$-span of $Y$.

Finally, using $v_5 = \alpha \land \beta$ and [B] again, one sees that types 1, 5 and 10 can’t occur in $\Lambda^{19-0}p$.

To prove (ii) we must exhibit $v_9$ in the $K$-span of $Y$ for $\delta$ of type 8, 11 and 16. First we treat cases 8 and 11. Setting $\delta = \Pi X_i^{e_i}$ we find that we must have $P_{5_8} = c_8P_3^4P_4^2$ and $P_{5_{11}} = c_{11}P_3^4P_2^2$ where $c_8$ and $c_{11}$ are constants.

Let’s treat case 11; case 8 is similar. In the notation of Section 2, we have after specialization

$$\text{proj } w \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \cdot Y = c_{11}P_3^4P_2^2 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} v_5 + \text{lower-weight-terms}.$$  

Set

$$w_0 = w \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$  

Thus $\text{proj } w_0 \cdot Y = c_{11}v_5 + \ell.\text{w.t.}$, and we must show $c_{11} \neq 0$.

Now $Y$ is a wedge of 19 vectors in the 35-dimensional space $0p$. Even with computer aided symbolic algebra it is not feasible just to ask for $w_0 \cdot Y$ and pick out the $v_5$-term.
Instead, we write $v_δ = \alpha \wedge \beta$ as above. Writing $Y$ as the wedge of 19 vectors taken from a basis of $^0p$ which includes a basis of $u \cap p$, we then apply $w_0$ to $Y$. We see that if, as we remove the parentheses and expand terms, we are to get a term of the form (something)$\wedge \beta$ then certain choices are forced. For a schematic example, if $Y = a \wedge b \wedge c \wedge \ldots$ and $w_0Y = (w_0a) \wedge (w_0b) \wedge (w_0c) \wedge \ldots = (d + e + f) \wedge (g + h) \wedge (e + j + k + \ell) \wedge \ldots$ and if $d$ is a basis vector appearing in the pure wedge $\beta$, and if $d$ doesn’t appear in the other 18 terms, then we must keep $d$ from the first term and discard $e$ and $f$. Now if $e$ is also in $\beta$ and appears only in the terms shown, we can’t get $e$ from the first term any more, so we must get it from the third term and discard $j + k + \ell$.

In this way, we can actually write down the exact formula $w_0Y = \psi \wedge \beta + \text{other-weight-terms}$ for an explicit $\psi \in \Lambda^{60}p$. Moreover $\psi$ is a weight zero wedge of vectors from $\tilde{\ell} \cap \mathfrak{p}$ where $\tilde{\ell}$ is the Lie-subalgebra of $\mathfrak{q}$

$$\tilde{\ell} = \left\{ \begin{pmatrix} X_1 \\ 0 \\ X_2 \end{pmatrix} \right\}.$$  

It follows that $\text{proj } w_0Y = c_{11}v_δ + \ell.w.t.$ and $c_{11} \neq 0$ only if the projection of $\psi$ to $(\Lambda^6(\ell \cap \mathfrak{p}))^{\ell.w.t}$ is nonzero.

Computing this projection of $\psi$ is a problem in $GL(3)$. For convenience we apply the Hodge $*$ operator and work in $\Lambda^3$. To see if our explicit form has a nonzero projection to the $\tilde{\ell} \cap \mathfrak{p}$-invariants we look instead (by duality) to see if it fails to lie in the linear span $C$ of vectors of the form $(gv - v), g \in GL(3)$. We compute $C$ and find that $\ast \psi$ is not in $C$.

The proof of (ii) in case 16 is similar but easier because we don’t have to worry about invariant theory in $GL(3)$. We do have to pick judiciously an element $w \in W_0$ such that $\text{proj } wY = c_{16}v_δ + \ell.w.t.$ In fact, we let

$$w = w \begin{pmatrix} x_1 & x_2 & x_3 & 0 \\ y_1 & y_2 & 0 & y_4 \end{pmatrix}$$

and compute:

$$\text{proj } wY = cf(x, y)v_δ + \ell.w.t.$$  

for some $c \neq 0$ and

$$f(x, y) = y_1x_1 + y_3x_3 - \frac{1}{2}(x_3y_2 + y_4x_1) \left( x_1 + \frac{y_2y_3}{y_4} \right).$$  

Clearly $f(x, y) \neq 0$ for some choice of $x, y$. Again this computation is performed completely by hand. \qed
3.3. Case of $GL(6)$. Here $H$ does not have as simple a form as in the $GL(2n)$ cases with $n$ even. We may take

$$\text{Lie } H \cap p = \begin{pmatrix}
0 & d & e & -f & 0 \\
-e & f & 0 & 0 & 0 \\
i & j & g & h & -e b \\
i & -j & h & g & e b \\
0 & k & -j & j & d \\
0 & 0 & i & i & 0 & a
\end{pmatrix},$$

$$\text{Lie } H \cap \mathfrak{t} = \begin{pmatrix}
(t_1 & 0 & y & y & 0 & 0 \\
0 & t_2 & x & -x & 0 & 0 \\
y' & x' & t_3 & 0 & x & -y \\
y' & -x' & 0 & -t_3 & -x & -y \\
0 & 0 & x' & -x' - t_2 & 0 \\
0 & 0 & -y' & -y' & 0 & t_1
\end{pmatrix},$$

$$s = \begin{pmatrix}
I_2 & 0 & 1 \\
0 & 1 & 0 \\
I_2 & \end{pmatrix}.$$

Types 3, 6, 7 are ruled out by Proposition 2.2. As in the $GL(8)$ case we use Lemma 2.6 to rule out types 1, 2 and 5. The invariant theory for finding $P_\delta$ reduces to finding weights, since $L_0$ in the $GL(6)$ case is a torus. We get $sP_\delta = P_\delta$ in these cases. A twist occurs for $GL(6)$ because now $sY = Y$.

However computation of $\ell \cap k$ invariants in $\Lambda^* \ell \cap \mathfrak{o}_p$ using [B] gives that $s\alpha = (-1)^m \alpha + 1 \alpha$ in these three cases. We also see that $s\beta = (-1)^m \beta$ so that $s\nu_\beta = -\nu_\beta$.

We rule in types 4 and 8 by explicit computations similar to the $GL(8)$ case. Thus we prove:

**Theorem.**

(i) If $\alpha \in \text{Hom}_{K_{\infty}}(\Lambda^{1+} p_{\infty}, \pi)$ and $\alpha(Y) \neq 0$ then $\pi$ is type 4 or 8.

(ii) Conversely if $\pi$ is one of these two types, there exists $\alpha$ such that $\alpha(Y) \neq 0$.

**Appendix.** Periods and Liftings.

Several relationships between the existence of a nonzero period for an automorphic representation $\pi$ and the fact that $\pi$ is a lift from another group (in the sense of “Langlands’ philosophy”) are known, and more are conjectured. In particular, if $\pi$ is a cuspidal irreducible automorphic representation for $GL(2n)/F$ it is conjectured that $\pi$ has a nonzero period over $GL(n) \times GL(n)$ if and only if $\pi$ is a lift from $GO(2n + 1)$ (cf. the introduction to [AG]).
We can rephrase this locally at a place $v$ in terms of $L$-groups by conjecturing that an irreducible admissible representation $\pi_v$ of $GL(2n, F_v)$ possesses a $GL(n, F_v) \times GL(n, F_v)$-invariant continuous functional if and only if the $L$-parameter classifying $\pi_v$ factors through the symplectic group.

In this appendix we prove the following proposition which is a heuristic analog of this conjecture in the "geometric" setting for $v$ a real place:

**Proposition.** Let $\pi$ be an irreducible admissible representation for $GL(2n, \mathbb{R})$ with nontrivial $(g, K)$-cohomology, and let $V_\delta$ be a representative of its cohomological $K$-type ($K = O(2n, \mathbb{R})$). Then $V_\delta$ contains a vector invariant under $SO(n) \times SO(n)$ if and only if the $L$-parameter corresponding to $\pi$

$$\Phi : W_\mathbb{R} \rightarrow GL(2n, \mathbb{C})$$

factors through $GSp(2n, \mathbb{C})$.

**Remark.** The connection with a nonvanishing period for $H = GL(n) \times GL(n)$ is given by Proposition 2.1.

**Proof.** Suppose $\pi$ is given by the data $(r_1, \ldots, r_k)$ as in Section 3.1. As in the proof of the theorem in Section 3.2, we apply Proposition 2.2 to show that $V_\delta$ contains an $SO(n) \times SO(n)$-invariant if and only (i) all the $r_s$ have the same parity and (ii) if $m_k < n$ then that parity is odd. So we must show that $\Phi$ factors through $GSp(2n, \mathbb{C})$ if and only if (i) and (ii) hold.

From the description of $\pi$ as a Langlands’ quotient in Section 3.1 it is easy to write down $\Phi$ (or more precisely a representative for $\Phi$, which is only determined up to choice of a basis in $GL(2n, \mathbb{C})$).

Recall that $W_\mathbb{R} = \mathbb{C}^\times \cup j\mathbb{C}^\times$ with $j^2 = -1$ and $jzj^{-1} = \bar{z}$ for any $z \in \mathbb{C}^\times$. Let $a(z) = z/|z|$ and $t(z) = z\bar{z}$. For any integers $M, r$ with $r > 0$ let $A(M, r)$ denote the $2r \times 2r$ matrix:

$$A(M, r) = \text{diag} \left( a^M t^{r-1 \over 2}, a^{-M} t^{r-1 \over 2}, a^M t^{r-3 \over 2}, a^{-M} t^{r-3 \over 2}, \ldots, a^M t^{1-r \over 2}, a^{-M} t^{1-r \over 2} \right).$$

Also let $I(M, r)$ denote the $2r \times 2r$ matrix

$$I(M, r) = \begin{pmatrix} 0 & I_r \\ (-I_r)^M & 0 \end{pmatrix}.$$

For $s = 1, \ldots, k$, let $m_s = r_1 + \cdots + r_s$ and $M_s = (2n - m_{s-1} - m_s)$. Also $r_0 = 2n - 2m_k$. Recall that $r_s > 0$ for all $s$ and $r_1 + \cdots + r_k \leq n$. Hence $M_1 > M_2 > \cdots > M_k > 0$.

Then we can give $\Phi$ in block diagonal form by

$$\Phi(z) = \text{diag} \left( A(M_1, r_1), \ldots, A(M_k, r_k), A(0, r_0) \right);$$
\[ \Phi(j) = \text{diag}(I(M_1, r_1), \ldots, I(M_k, r_k), I_{2r_0}). \]

Now suppose \( \Phi \) factors through \( \text{GSp}(2n, \mathbb{C}) \) up to conjugacy. That means there exists a skew symmetric \( 2n \times 2n \) matrix \( J \) and a character \( \lambda \) of \( W_\mathbb{R} \) such that for any \( w \in W_\mathbb{R} \),

\[ ^t\Phi(w)J\Phi(w) = \lambda(w)J. \]

Applying this to \( \Phi(z) \), which has determinant 1, we first see that \( \lambda(z)^{2n} = 1 \) and then (by taking a generic \( z \)) that \( J_{ij} = 0 \) except for the entries of \( J \) along the non-main diagonal of each block. In other words \( J = \text{diag}(J_1, \ldots, J_k, J_0) \) with

\[
J_i = \begin{pmatrix} 0 & B_i \\ -B_i & 0 \end{pmatrix} \quad \text{where} \quad B_i = \begin{pmatrix} & \ast \\ \ast & \end{pmatrix} \quad (r \times r).
\]

Now apply the same formula to \( \Phi(j) \). Since

\[
^tI(M_s, r_s)J_sI(M_s, r_s) = (-1)^{m_s+1}J_s
\]

for \( s = 1, \ldots, k \) and \( I_{2r_0}J_0I_{2r_0} = J_0 \), we see that \( \lambda(j) = (-1)^{M_s+1} \) for all \( s \) and further that \( \lambda(j) = 1 \) if \( r_0 \neq 0 \), i.e. if \( m_k < n \). Since \( r_s \equiv M_s \pmod{2} \) for all \( s \), we are finished.

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