$L^p$-BOUNDS FOR HYPERSINGULAR INTEGRAL OPERATORS ALONG CURVES

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It is known that the Hilbert transform along curves:

$$H_{\Gamma}f(x) = \text{pv} \int_{-\infty}^{\infty} f(x - \Gamma(t)) \frac{dt}{t} \quad (x \in \mathbb{R}^n)$$

is bounded on $L^p$, $1 < p < \infty$, where $\Gamma(t)$ is an appropriate curve in $\mathbb{R}^n$. In particular, $\|H_{\Gamma}f\|_p \leq C\|f\|_p$, $1 < p < \infty$, where $\Gamma(t) = (t, |t|^k \text{sgn} t)$, $k \geq 2$, is a curve in $\mathbb{R}^2$.

It is easy to see that the hypersingular integral operator

$$Tf(x) = \text{pv} \int_{-1}^{1} f(x - \Gamma(t)) \frac{dt}{t|t|^\alpha} \quad (\alpha > 0),$$

in which the singularity at the origin is worse than that in the Hilbert transform, is not bounded on $L^2(\mathbb{R}^2)$. To counterbalance this worsened singularity, we introduce an additional oscillation $e^{-2\pi i|t|^{-\beta}}$ and study the operator

$$T_{\alpha, \beta}f(x, y) = \text{pv} \int_{-1}^{1} f(x - t, y - \gamma(t)) e^{-2\pi i|t|^{-\beta}} \frac{dt}{t|t|^\alpha} \quad (\alpha, \beta > 0)$$

along the curve $\Gamma(t) = (t, \gamma(t))$, where $\gamma(t) = |t|^k$ or $\gamma(t) = |t|^k \text{sgn} t$, $k \geq 2$, in $\mathbb{R}^2$ and show that

(i) $\|T_{\alpha, \beta}f\|_2 \leq A_{\alpha, \beta}\|f\|_2$ if and only if $\beta \geq 3\alpha$;
(ii) $\|T_{\alpha, \beta}f\|_p \leq B_{\alpha, \beta}\|f\|_p$ whenever $\beta > 3\alpha$, and

$$1 + \frac{3\alpha(\beta + 1)}{\beta(\beta + 1) + (\beta - 3\alpha)} < p < \frac{\beta(\beta + 1) + (\beta - 3\alpha)}{3\alpha(\beta + 1)} + 1.$$

1. Introduction.

In recent years, several mathematicians have studied the Hilbert transform along curves:

$$H_{\Gamma}f(x) = \text{pv} \int_{-\infty}^{\infty} f(x - \Gamma(t)) \frac{dt}{t} \quad (x \in \mathbb{R}^n),$$
where \( \Gamma(t) \) is an appropriate curve in \( \mathbb{R}^n \). Fabes and Rivièrè were led to the study of \( \mathcal{H}_\Gamma \) in their attempt to generalize the Method of Rotation of Calderon and Zygmund; for details see [Fa, Ri] and [Wa2].

Nagel, Rivièrè, Stein and Wainger, and several other mathematicians have studied the \( L^p \)-boundedness of \( \mathcal{H}_\Gamma \) for a variety of curves \( \Gamma \). A detailed survey of these results can be found in [St, Wa]; also see [Wa1]. Nagel, Rivièrè and Wainger proved in [NRW1] that \( \mathcal{H}_\Gamma \) is a bounded operator on \( L^p \), \( 1 < p < \infty \), when \( \Gamma(t) = (|t|^{\alpha_1} \text{sgn} t, \cdots, |t|^{\alpha_n} \text{sgn} t) \), each \( \alpha_k > 0 \), is a curve in \( \mathbb{R}^n \). In particular, \( \|\mathcal{H}_\Gamma f\|_p \leq C\|f\|_p \), \( 1 < p < \infty \), where \( \Gamma(t) = (t, |t|^k \text{sgn} t) \), \( k \geq 2 \), is a curve in \( \mathbb{R}^2 \). For more general curves see [Na, Wa], [NVWW], and [Wa3].

The kernel, \( K(x) = \frac{1}{\pi x} \), of the Hilbert transform,

\[
\mathcal{H}f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x-y)}{y} dy \quad (x \in \mathbb{R}),
\]

owing to its order of magnitude, is not integrable either at 0 or \( \infty \). It does, however, compensate for this deficiency by cancellation due to oscillation; this oscillatory property being reflected in the fact that its Fourier transform, \( \hat{K}(x) = i \text{sgn} x \), is bounded.

It is tempting to explore a situation where the order of magnitude of the singularity of \( K \) at the origin is greater than that of \( |x|^{-1} \), say of the order of \( |x|^{-1-\alpha} \), \( \alpha > 0 \). It is reasonable to expect that some additional oscillation is required to compensate for this worsened singularity. This translates to the requirement that the Fourier transform of \( K \), in addition to being bounded, have some decay at infinity; that is, \( |\hat{K}(x)| \leq C(1 + |x|)^{-\beta} \) for some \( \beta > 0 \). For further discussion see Theorem 5 of [St].

Integral operators with strong singularities of the type described above, were studied by Hirschman in one dimension [Hi], Wainger in \( k \)-dimensions [Wa], Stein [St], Fefferman [Fe], and Fefferman and Stein [Fe, St].

It is not hard to see that the **hypersingular integral operator**

\[
Tf(x) = pv \int_{-1}^{1} f(x - \Gamma(t)) \frac{dt}{t|t|^\alpha} \quad (\alpha > 0)
\]

along \( \Gamma(t) = (t, \gamma(t)) \), where \( \gamma(t) = |t|^k \) or \( \gamma(t) = |t|^k \text{sgn} t, k \geq 2 \), is not bounded on \( L^2(\mathbb{R}^2) \). The \( L^2 \)-boundedness of this operator is equivalent to
the uniform boundedness, in $\mathbb{R}^2$, of the multiplier

$$m(x, y) = \text{pv} \int_{-1}^{1} e^{-2\pi i [xt + y\gamma(t)]} \frac{dt}{t|t|^\alpha} \quad (\alpha > 0).$$

It is easy to see that $|m(\frac{1}{4}, 0)| = \infty$ for $\alpha \geq 1$; for $0 < \alpha < 1$ and $x > 0$,

$$|m(x, 0)| = 2 \left| \int_0^1 \sin(2\pi xt) \frac{dt}{t^{1+\alpha}} \right| = 2 (2\pi x)^\alpha \left| \int_0^{2\pi x} \sin s \frac{ds}{s^{1+\alpha}} \right| \to \infty$$
as $x \to \infty$.

One can ask if the worsened singularity at the origin can be counterbalanced by an oscillation. This leads us to the operator

$$T_{\alpha, \beta} f(x, y) = \text{pv} \int_{-1}^{1} f(x - t, y - \gamma(t)) e^{-2\pi i t |t|^{-\beta}} \frac{dt}{t|t|^\alpha} \quad (\alpha, \beta > 0)$$
along the curve $\Gamma(t) = (t, \gamma(t))$, $\gamma(t) = |t|^k$ or $\gamma(t) = |t|^k \text{sgn} t$, $k \geq 2$, in $\mathbb{R}^2$.

Zielinski, in his thesis [Zi], studied the $L^2$-boundedness of $T_{\alpha, \beta}$ along the parabola $\gamma(t) = (t, t^2)$, and proved that $\|T_{\alpha, \beta} f\|_2 \leq C \|f\|_2 \iff \beta \geq 3\alpha$.

1.1. Statement of the Main Result. We state the main result of this paper as:

**Theorem 1.** Suppose that $\gamma(t) = |t|^k$ or $\gamma(t) = |t|^k \text{sgn} t$, $k \geq 2$, and

$$T_{\alpha, \beta} f(x, y) = \text{pv} \int_{-1}^{1} f(x - t, y - \gamma(t)) e^{-2\pi i t |t|^{-\beta}} \frac{dt}{t|t|^\alpha} \quad (\alpha, \beta > 0).$$

Then

(i) $\|T_{\alpha, \beta} f\|_2 \leq A_{\alpha, \beta} \|f\|_2$ if and only if $\beta \geq 3\alpha$;

(ii) $\|T_{\alpha, \beta} f\|_p \leq B_{\alpha, \beta} \|f\|_p$ whenever $\beta > 3\alpha$, and

$$1 + \frac{3\alpha (\beta + 1)}{\beta (\beta + 1) + (\beta - 3\alpha)} < p < \frac{\beta (\beta + 1) + (\beta - 3\alpha)}{3\alpha (\beta + 1)} + 1.$$

Here $A_{\alpha, \beta}$ also depends on $k$, and $B_{\alpha, \beta}$ also depends on $p$. 
1.2. Outline of Proof. In Section 2, we define an appropriate one parameter family of dilations \{δ_t\}_{t>0}, and a corresponding distance function ρ, whose homogeneity with respect to δ_t is essential in proving the \(L^2\) and \(L^p\)-boundedness of \(\mathcal{T}_{\alpha,\beta}\).

In Section 3, we prove that \(\mathcal{T}_{\alpha,\beta}\) is a bounded operator on \(L^2\) if and only if \(\beta \geq 3\alpha\). This is achieved by applying van der Corput’s Lemma and its corollary to judiciously subdivided intervals, and the asymptotics of oscillatory integrals.

The \(L^p\)-boundedness, as stated in the second assertion of Theorem 1, is proven in Section 4. This is accomplished by showing that a certain analytic family, \{\mathcal{T}_z\}, of truncated operators is bounded on \(L^2\) for an appropriate \(\Re z > 0\); and it is bounded on \(L^p\), \(1 < p < \infty\), for an appropriate \(\Re z < 0\); and that the bound in each case grows at most as fast as a polynomial in \(|z|\). The result then follows by analytic interpolation.

2. Dilations and Homogeneity.

We define a one parameter group of dilations \{δ_t\}_{t>0}, \(δ_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2\), by \(δ_t = \text{diag}[t^{1+\beta}, t^{k+\beta}]\), with \(A = \text{diag}[1+\beta, k+\beta]\) and \(a = \text{trace } A = 2\beta + k + 1\), and a corresponding distance function ρ defined by: \(ρ = ρ(x, y) = t\) such that

\[
\left\| δ_ρ^{-1}(x, y) \right\|^2 = \left( \frac{x}{ρ^{1+β}} \right)^2 + \left( \frac{y}{ρ^{k+β}} \right)^2 = 1
\]

if \((x, y) \neq (0, 0)\), and \(ρ(0, 0) = 0\). Then ρ is homogeneous with respect to \(δ_t\): \(ρ(δ_t x) = t ρ(x), t > 0, x ∈ \mathbb{R}^2\); \(ρ(x)\) is continuous and is in \(C^∞(\mathbb{R}^2 - 0)\); \(ρ(x + y) ≤ C[ρ(x) + ρ(y)]\), for some \(C > 0\); and \(\mathbb{R}^2\) can be coordinatized by the polar-like coordinates \(ρ = ρ(x)\) and \(u = δ_ρ^{-1}x\), with \(dx = ρ^{a-1} dρ (Au, u) dσ = ρ^{a-1} dρ dφ\), where \(dσ\) is the linear measure on \(S^1\). For proofs of these assertions and additional properties of \(δ_t\) and \(ρ\) see [St, Wa].

3. \(L^2\)-Boundedness.

The proof of sufficiency in the first assertion of Theorem 1 is accomplished as an easy consequence of Theorem 2, which we prove next. Our point of departure is the observation that

\[
(\mathcal{T}_{α,β} f)(x, y) = m_{α,β}(x, y) \hat{f}(x, y) \quad (f ∈ L^2),
\]
where \( \Lambda \) denotes the Fourier transform, and \( m_{\alpha,\beta}(x, y) \) is the multiplier given by
\[
m_{\alpha,\beta}(x, y) = pv \int_{-1}^{1} e^{-2\pi i [xt + y^2(t) + t^{-\alpha}]} \frac{dt}{t^{|t|^\alpha}} \quad (\alpha, \beta > 0).
\]

Thus, the boundedness of \( T_{\alpha,\beta} \) on \( L^2 \) is, by the Plancherel Theorem, equivalent to the uniform boundedness, in \( x \) and \( y \), of the multiplier \( m_{\alpha,\beta} \). So we first prove:

**Theorem 2.** The multiplier \( m_{\alpha,\beta}(x, y) \) is uniformly bounded in \( \mathbb{R}^2 \) for \( \beta \geq 3\alpha \). More precisely:
\[
|m_{\alpha,\beta}(x, y)| \leq \begin{cases} C \\ C \rho^{-\frac{\beta - 3\alpha}{2}} \end{cases} \quad \text{if } 0 \leq \rho \leq 1, \quad \beta \geq 3\alpha, \ (x, y) \in \mathbb{R}^2.
\]

The proof of Theorem 2 depends mainly on the following:

**Lemma 3.1.** Suppose that
(i) \( g \) is real-valued and smooth for all \( t \in [a, b], \ 0 < a < b \);
(ii) \( |g^{(k)}(t)| \geq \rho > 0 \) for all \( t \in [a, b] \) with \( k \geq 2 \); in addition, \( g' \) is monotone on \([a, b]\) if \( k = 1 \);
(iii) \( z = \sigma + i\tau, \ \sigma \geq 0, \ \tau \in \mathbb{R} \)
(iv) \( \alpha \geq 0 \).

Then,
\[
\left| \int_{a}^{b} e^{-2\pi i g(t)} \frac{dt}{t^{1+\alpha+z}} \right| \leq \frac{C(1 + |z|)}{a^{1+\alpha+\sigma}} \rho^{-\frac{k}{2}}.
\]

**Proof.** Let
\[
G(t) = \int_{a}^{t} e^{-2\pi i g(s)} \, ds.
\]
Then, by van der Corput’s Lemma (see [St3], Chapter VIII),
\[
|G(t)| \leq C_k \rho^{-\frac{k}{2}}, \quad t \in [a, b].
\]

Integrating by parts, we get
\[
\left| \int_{a}^{b} e^{-2\pi i g(t)} \frac{dt}{t^{1+\alpha+z}} \right| = \left| \left[ \frac{G(t)}{t^{1+\alpha+z}} \right]_{t=a}^{t=b} - (1 + \alpha + z) \int_{a}^{b} \frac{G(t)}{t^{2+\alpha+z}} \, dt \right|
\]
This completes the proof of Lemma 3.1.

Proof of Theorem 2: We only need look at

\[ m^+(x, y) = m^+_{\alpha, \beta}(x, y) = \int_0^1 e^{-2\pi i \theta [xt + yt^\alpha + t^\beta]} \frac{dt}{t^{1+\alpha}}, \]

since the other half can be dealt with similarly.

Since \( p(0, 0) = 0 \) and \( m(0, 0) = 0 \); for \( (x, y) \neq (0, 0) \) but \( x^2 + y^2 \leq 1 \), so that \( 0 < \rho \leq 1 \), if we let

\[ g(s) = xs + ys^\alpha + s^{-\beta}, \]

then

\[ g'(s) = x + yk s^{k-1} - \beta s^{-(\beta+1)}, \]

and so there exists a \( T > 0 \) independent of \( x \) and \( y \) such that \( g'(s) \leq -\beta s^{-(\beta+1)} \) for \( s \in (0, T) \). Then if we let

\[ G(s) = \int_0^s e^{-2\pi i g(t)} \, dt, \]

we get \( |G(s)| \leq C s^{\beta+1} \), by van der Corput's Lemma. Hence integrating by parts we get,

\[
\left| \int_0^T e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq \left| \left[ \frac{G(s)}{s^{1+\alpha}} \right]_{s=0}^{s=T} \right| + (1 + \alpha) \int_0^T \frac{|G(s)|}{s^{\alpha+2}} \, ds \\
\leq C \left[ \frac{s^{\beta+1}}{s^{\alpha+1}} \right]_{s=0}^{s=T} + C (1 + \alpha) \int_0^T \frac{s^{\beta+1}}{s^{\alpha+2}} \, ds.
\]
Both of these exist if \( \beta > \alpha \). Thus,

\[
\left| \int_0^T e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C.
\]

For \( s \in [T, 1] \),

\[
\left| \int_T^1 e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq \int_T^1 \frac{ds}{s^{1+\alpha}} = \frac{1}{\alpha} \left[ \frac{1}{T^\alpha} - 1 \right] \leq C.
\]

Thus \( m^+(x, y) \) is uniformly bounded when \( 0 \leq \rho \leq 1 \). We now turn to the case when \( \rho > 1 \). With \( \rho = \rho(x, y) \) as defined above, the change of variable \( t = s \rho^{-1} \) leads us to

\[
m^+(x, y) = \rho^\alpha \int_0^\rho e^{-2\pi i \left[ \frac{\xi s^\rho - y}{\rho^k s^k + \rho^\beta s^{-\beta}} \right]} \frac{ds}{s^{1+\alpha}}.
\]

Thus, to prove the theorem, we need only show that

\[
\left| \int_0^\rho e^{-2\pi i \left[ \frac{\xi s^\rho - y}{\rho^k s^k + \rho^\beta s^{-\beta}} \right]} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\frac{x}{2}}
\]

for all \( (x, y) \in \mathbb{R}^2 \). To this end, we show that the above integral is uniformly bounded in each of the four quadrants of \( \mathbb{R}^2 \).

**Note:** For notational convenience, we shall write \( x \) (resp. \( y \)) if \( x \) (resp. \( y \)) is positive, and \(-x\) (resp. \(-y\)) if \( x \) (resp. \( y \)) is negative.

**Case I:** \( x < 0, y < 0 \).

Let

\[
g(s) = -\frac{x}{\rho} s - \frac{y}{\rho^k} s^k + \rho^\beta s^{-\beta}.
\]

Then,

\[
g'(s) = -\frac{x}{\rho} - \frac{y}{\rho^k} k s^{k-1} - \rho^\beta \beta s^{-(\beta+1)}.
\]
and

\[ g''(s) = \frac{y}{\rho^k} k (k - 1) s^{k-2} + \rho^\beta \beta (\beta + 1) s^{-(\beta+2)}. \]

Let

\[ G(s) = \int_0^s e^{-2\pi i g(t)} \, dt. \]

Since near 0 we have \( g'(s) \leq -\rho^\beta \beta s^{-(\beta+1)} \), van der Corput’s Lemma gives \( |G(s)| \leq C \rho^{-\beta} s^{\beta+1} \). Hence, integrating by parts as before, we get

\[ \left| \int_0^1 e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\beta} \quad \text{for } \beta > \alpha. \]

To tackle the integral from 1 to \( \rho \), we need to consider the following two cases:

(i) \( \frac{x}{\rho} \geq \frac{\rho^\beta}{2} \); (ii) \( \frac{x}{\rho} \leq \frac{\rho^\beta}{2} \).

(i) \( \frac{x}{\rho} \geq \frac{\rho^\beta}{2} \)

This implies that \( -\frac{x}{\rho} \leq -\frac{\rho^\beta}{2} \). Thus \( g'(s) \leq -\frac{x}{\rho} \leq -\frac{\rho^\beta}{2} \) on \([1, \rho]\), together with Lemma 3.1, yields

\[ \left| \int_1^\rho e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\beta}. \]

(ii) \( \frac{x}{\rho} \leq \frac{\rho^\beta}{2} \)

By the definition of \( \rho \), this implies that \( -\frac{y}{\rho^k} \leq -\frac{\rho^\beta}{2} \). Then,

\[ g'(s) = -\frac{x}{\rho} - \frac{y}{\rho^k} k s^{k-1} - \rho^\beta \beta s^{-(\beta+1)} \leq -\frac{x}{\rho} - k \frac{\rho^\beta}{2} - \rho^\beta \beta s^{-(\beta+1)} \leq -\frac{k}{2} \rho^\beta \quad \text{for } s \in [1, \rho]. \]
This, along with Lemma 3.1, gives

$$\left| \int_{1}^{\rho} e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\beta} .$$

Hence, \( |m^+(x,y)| \leq C \rho^{-(\beta-\alpha)} \) whenever \( x, y < 0 \) and \( \beta > \alpha \). This completes Case I.

**Case II:** \( x \geq 0, \ y \geq 0 \).

In this case,

\[
g(s) = \frac{x}{\rho} \ s + \frac{y}{\rho^k} \ s^k + \rho^2 s^{-\beta},
\]

\[
g'(s) = \frac{x}{\rho} + \frac{y}{\rho^k} \ k \ s^{k-1} - \rho^\beta \ s^{-(\beta+1)},
\]

\[
g''(s) = \frac{y}{\rho^k} \ k (k-1) \ s^{k-2} + \rho^\beta \ (\beta + 1) \ s^{-(\beta+2)}.
\]

In the vicinity of 0, we have \( g''(s) \geq C \rho^\beta s^{-(\beta+2)} \); and so

\[
\left| \int_{0}^{b} e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\frac{\beta}{2}} \text{ for } \beta > 2\alpha,
\]

using van der Corput’s lemma, where \( b \) can be chosen later.

Away from 0, we have the following two cases:

(i) \( \frac{y}{\rho^k} \leq \frac{\rho^\beta}{2} \); \quad (ii) \( \frac{y}{\rho^k} \geq \frac{\rho^\beta}{2} \).

(i) \( \frac{y}{\rho^k} \leq \frac{\rho^\beta}{2} \)

This, and the definition of \( \rho \) imply that \( \frac{x}{\rho} \geq \frac{\rho^\beta}{2} \).

Then,

\[
g'(s) \geq \frac{\rho^\beta}{2} - \beta \rho^\beta s^{-(\beta+1)} \geq \frac{\rho^\beta}{2} - \frac{\rho^\beta}{4} \geq \frac{\rho^\beta}{4} \ \text{ whenever } s \geq (4\beta)^{\frac{1}{\beta+1}}.
\]
Note that \( g' \) is increasing since \( g'' > 0 \). Choosing \( b = \left(4/3\right)^{\frac{1}{\beta}} \), and using Lemma 3.1, we get
\[
\left| \int_b^\rho e^{-2\pi ig(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\beta}.
\]

(ii) \( \frac{y}{\rho^k} \geq \frac{\rho^\beta}{2} \)

Here, \( g''(s) \geq C \frac{\rho^\beta}{2} \). Choosing \( b = 1 \), and using Lemma 3.1, we get
\[
\left| \int_1^\rho e^{-2\pi ig(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\frac{\beta}{2}}.
\]

Thus, \( |m^+(x,y)| \leq C \rho^{-\left(\frac{\beta}{2} - \alpha\right)} \) whenever \( x, y \geq 0 \) and \( \beta > 2\alpha \).

This completes Case II.

Case III: \( x < 0 \), \( y \geq 0 \).

Here,
\[
g(s) = -\frac{x}{\rho} s + \frac{y}{\rho^k} s^k + \rho^\beta s^{-\beta},
\]
\[
g'(s) = -\frac{x}{\rho} + \frac{y}{\rho^k} k s^{k-1} - \rho^\beta \beta s^{-(\beta+1)},
\]
\[
g''(s) = + \frac{y}{\rho^k} k (k-1) s^{k-2} + \rho^\beta \beta (\beta + 1) s^{-(\beta+2)}.
\]

Close to \( 0 \), \( g''(s) \geq \beta (\beta + 1) \rho^\beta s^{-(\beta+2)} \); and so
\[
\left| \int_0^b e^{-2\pi ig(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\frac{\beta}{2}} \quad \text{for} \quad \beta > 2\alpha,
\]

using van der Corput’s Lemma, where \( b \) can be chosen later.

Farther from \( 0 \), the following two cases need to be considered:

(i) \( \frac{y}{\rho^k} \geq \frac{\rho^\beta}{8k} \); (ii) \( \frac{y}{\rho^k} \leq \frac{\rho^\beta}{8k} \).

(i) \( \frac{y}{\rho^k} \geq \frac{\rho^\beta}{8k} \)

Here,
\[
g''(s) \geq \frac{y}{\rho^k} k (k-1) s^{k-2}.
\]
Choosing $b = \rho^{\frac{k}{3(k-2)}}$ in the above, and using Lemma 3.1 we get
\[
\left| \int_I e^{-2\pi ig(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\frac{\beta}{3(k-2)}} \rho^{-\frac{\beta}{8}} \leq C \rho^{-\frac{\beta}{8}}.
\]

(ii) $\frac{y}{\rho^k} \leq \frac{\rho^{\frac{k}{8}}}{8k}$

This implies that $\frac{x}{\rho} \geq \frac{\rho^\beta}{2}$, and so
\[
g'(s) \leq -\frac{x}{\rho} + \frac{y}{\rho^k} k s^{(k-1)}
\leq -\frac{\rho^\beta}{2} + \frac{\rho^{\frac{k}{8}}}{8} \rho^{\frac{2k}{8}}
\leq -\frac{\rho^\beta}{4} \quad \text{whenever} \ s \in \begin{bmatrix} 1, \rho^{\frac{2k}{3(k-2)}} \end{bmatrix}.
\]

If $\beta \geq \frac{3(k-1)}{2}$, we are done using Lemma 3.1. If not, we need to subdivide further:

(ii a) $\frac{y}{\rho^k} \leq \frac{\rho^\beta-(k-1)}{8k}$; \hspace{1cm} (ii b) $\frac{\rho^\beta-(k-1)}{8k} \leq \frac{y}{\rho^k} \leq \frac{\rho^{\frac{k}{8}}}{8k}$.

(ii) a) $\frac{y}{\rho^k} \leq \frac{\rho^\beta-(k-1)}{8k}; \quad 0 < \beta < \frac{3(k-1)}{2}, \quad s \in [1, \rho].$

In this case,
\[
g'(s) \leq -\frac{\rho^\beta}{2} + \frac{\rho^\beta-(k-1)}{8} s^{(k-1)}
\leq -\frac{\rho^\beta}{2} + \frac{\rho^\beta}{8}
\leq -\frac{\rho^\beta}{4}.
\]

Thus, Lemma 3.1 gives
\[
\left| \int_1^\rho e^{-2\pi ig(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\beta}.
\]
There is a real number \( j > 1 \) such that \( 0 < j - 1 < \frac{2\beta}{3} \leq j < k \). Then
\[
\frac{\beta}{3} = \beta - \frac{2\beta}{3} \leq \beta - (j - 1) = \beta - [k - \{k - (j - 1)\}] .
\]

Now, let
\[
S_N = \sum_{n=0}^{N} \left(\frac{k-1}{k}\right)^n .
\]

Then, \( S_{m+1} = 1 + \left(\frac{k-1}{k}\right) S_m \); \( m \geq 0 \). We choose \( N \) so that
\[
S_N = 1 - \left(\frac{k-1}{k}\right)^{N+1} = k \left[ 1 - \left(\frac{k-1}{k}\right)^{N+1} \right] \geq k - (j - 1) ;
\]
i.e., \( (j - 1) \geq k \left(\frac{k-1}{k}\right)^{N+1} \). This can be done since \( \frac{k-1}{k} < 1 \).

We now look at:
\[
\frac{\rho^{\beta-[k-S_m]}}{8k} \leq \frac{y}{\rho^k} \leq \frac{\rho^{\beta-[k-S_{m+1}]}}{8k} ; \quad m = 0, 1, 2, \ldots, N - 1 .
\]

For \( s \in I = \left[ 1, \rho^{1-\frac{S_m}{k}} \right] \), we have
\[
g'(s) = -\frac{x}{\rho} + \frac{y}{\rho^k} k s^{k-1} - \rho^\beta \beta s^{-(\beta+1)} \\
\leq -\frac{\rho^\beta}{2} + \frac{1}{8} \rho^{\beta-[k-S_{m+1}]} \rho^{k-1-(\frac{k-1}{k})} S_m \\
= -\frac{\rho^\beta}{2} + \frac{\rho^\beta}{8} \\
\leq -\frac{\rho^\beta}{4} .
\]

Hence,
\[
\left| \int_I e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\beta} \quad \text{using Lemma 3.1.}
\]

Next, for \( s \in I = \left[ \rho^{1-\frac{S_m}{k}}, \rho \right] \), we have
\[
g''(s) \geq k (k-1) \frac{y}{\rho^k} s^{k-2} \\
\geq \frac{(k-1)}{8} \rho^{\beta-[k-S_m]} s^{k-2}
\]
This, along with Lemma 3.1, gives

\[
\left| \int_{I} e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\frac{\alpha}{2} + \frac{\beta}{4} - \frac{S_{m}}{k} \rho^{-1+\frac{S_{m}}{k}} = C \rho^{-\frac{\alpha}{2}}.
\]

Thus, \( |m^+(x, y)| \) is uniformly bounded when \( x, y < 0 \) and \( \beta \geq 3\alpha \).

This completes Case III.

Case IV: \( x > 0, y < 0 \).

Here,

\[
g(s) = \frac{x}{\rho} s - \frac{y}{\rho^k} s^k + \rho^\beta s^{-\beta},
\]

\[
g'(s) = \frac{x}{\rho} - \frac{y}{\rho^k} k s^{k-1} - \rho^\beta \beta s^{-(\beta+1)},
\]

\[
g''(s) = -\frac{y}{\rho^k} k (k-1) s^{k-2} + \rho^\beta (\beta+1) s^{-(\beta+2)},
\]

\[
g'''(s) = -\frac{y}{\rho^k} k (k-1) (k-2) s^{k-3} - \rho^\beta (\beta+1) (\beta+2) s^{-(\beta+3)}.
\]

We need to split as follows:

\[
(i) \quad \frac{y}{\rho^k} \geq C_1 \rho^\beta; \quad \text{ (ii) } \frac{y}{\rho^k} \leq C_1 \rho^\beta
\]

where \( 0 < C_1 < 1 \) is to be chosen appropriately at a later stage.

Note that, in the vicinage of \( 0 \),

\[
g'(s) \leq \frac{x}{\rho} - \rho^\beta \beta s^{-(\beta+1)}
\]

\[
\leq \rho^\beta - \rho^\beta \beta s^{-(\beta+1)}
\]

\[
\leq -\frac{\beta}{2} \rho^\beta s^{-(\beta+1)}
\]

whenever \( s \in I = \left[ 0, \left( \frac{\beta}{2} \right) \pi^{\frac{1}{\beta}} \right] \).

Therefore,

\[
\left| \int_{I} e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\beta} \quad \text{for } \beta > \alpha
\]
using van der Corput’s Lemma.

(i) \( \frac{y}{\rho^k} \geq C_1 \rho^\beta \)

Note that, \( |g'''(s)| \geq \frac{y}{\rho^k} k(k-1)(k-2) s^{k-3} \geq C \rho^\beta \) whenever \( s \in I = \left[ \left( \frac{\beta}{2} \right)^{\frac{1}{k+1}}, \rho \right] \). Thus,

\[
\left| \int_I e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\frac{\alpha}{5}}
\]

using Lemma 3.1. This completes (i).

(ii) \( \frac{y}{\rho^k} \leq C_1 \rho^\beta \)

This needs to be split further:

(ii a) \( \frac{y}{\rho^k} \leq C_1 \rho^\beta \); (ii b) \( \frac{y}{\rho^k} \leq \frac{\rho^{2\beta}}{8k} \).

(ii a) \( \frac{\rho^{2\beta}}{8k} \leq \frac{y}{\rho^k} \leq C_1 \rho^\beta \)

At \( s_0 = \left[ \frac{\beta (\beta + 1)}{k (k-1) y} \right]^{\frac{1}{k+1}} \rho \), we have \( g''(s_0) = 0 \). Since \( g'' < 0 \), \( g' \) has a maximum at \( s_0 \).

Now,

\[
g'(s_0) = \frac{x}{\rho} - \frac{y}{\rho^2} k \left[ \frac{\beta (\beta + 1)}{k (k-1) y} \right]^{\frac{k-1}{k+1}} \rho^{k-1} - \rho^{-1} \beta \left[ \frac{\beta (\beta + 1)}{k (k-1) y} \right]^{\frac{k+1}{k+1}}
\]

\[
= \frac{x}{\rho} - C_{\beta,k} \frac{y^{\frac{k+1}{k+1}}}{\rho},
\]

where \( C_{\beta,k} = \left[ \frac{\beta (\beta + k)}{k (k-1)} \right] \left[ \frac{k (k-1)}{\beta (\beta + 1)} \right]^{\frac{k+1}{k+1}} \).

Now, choose \( C_1 \) so that \( g'(s_0) \geq \frac{1}{2} \frac{x}{\rho} \). Next, choose \( a < 1 \) and \( b > 1 \) such that in the neighborhood \( I_{a,b} = [a s_0, b s_0] \) of \( s_0 \), we have

\( g'(s) \geq \frac{x}{4\rho} \geq \frac{1}{4} (1 - C_1^2) \frac{1}{2} \rho^\beta \).

Then,

\[
\left| \int_{I_{a,b}} e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\beta}
\]
using van der Corput’s Lemma on \([a s_0, s_0]\) and \([s_0, bs_0]\).

Since \(g''(s) < 0\), \(g''(s)\) is decreasing; and so on \(I = \left[\left(\frac{\beta}{2}\right)^{\frac{1}{k+1}}, a s_0\right]\), we have

\[
g''(s) \geq g''(a s_0) = -\frac{y}{\rho^k} k (k - 1) (a s_0)^{k-2} + \rho^\beta \beta (\beta + 1) (a s_0)^{-(\beta+2)}
\]

\[= C_{a,\beta,k} \frac{y^{\frac{2\beta+2}{k+2}}}{\rho^2} \geq C'_{a,\beta,k} \rho^{\frac{2\beta}{k+1}},\]

as a simple calculation shows.

Thus on \(I\),

\[
\left| \int_I e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\frac{\beta}{2}}
\]

by Lemma 3.1.

Now, on \(I = [bs_0, \rho]\),

\[
g''(s) \leq g''(b s_0) \leq -C'_{b,\beta,k} \rho^{\frac{2\beta}{k+1}},
\]

as before. Hence, once again,

\[
\left| \int_I e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\frac{\beta}{2}}
\]

using Lemma 3.1. This completes (ii a).

(ii b) \(\frac{y}{\rho^k} \leq \frac{\rho^{\frac{2\beta}{k+1}}}{8k}\)

Have,

\[
g'(s) = \frac{x}{\rho} - \frac{y}{\rho^k} k s^{k-1} - \rho^\beta \beta s^{-(\beta+1)}
\]

\[\geq \frac{\rho^\beta}{2} - \frac{\rho^{\frac{2\beta}{k+1}}}{8} s^{(k-1)} - \rho^\beta \beta s^{-(\beta+1)}
\]

\[\geq \frac{\rho^\beta}{4} \text{ whenever } s \in I = \left[\left(\frac{\beta}{2}\right)^{\frac{1}{k+1}}, \rho^{\frac{\beta}{3(k-1)}}\right].\]

If \(\beta \geq 3(k-1)\), we are done using Lemma 3.1. If not, we need to subdivide further:

(ii b A) \(\frac{y}{\rho^k} \leq \frac{\rho^{\beta-(k-1)}}{8k}\); (ii b B) \(\frac{\rho^{\beta-(k-1)}}{8k} \leq \frac{y}{\rho^k} \leq \frac{\rho^{\frac{2\beta}{k+1}}}{8k}\)

with \(0 < \beta < 3(k-1)\), and \(s \in I = [1, \rho]\).
Using Lemma 3.1 once again, we are done.

(ii b A) \[ \frac{y}{\rho^k} \leq \frac{\rho^{\beta-(k-1)}}{8k} ; \quad 0 < \beta < 3(k-1), \ s \in I = [1, \rho] \]

Have,

\[
g'(s) = \frac{x}{\rho} - \frac{y}{\rho^k} k s^{k-1} - \rho^\beta s^{-(\beta+1)}
\geq \frac{\rho^\beta}{2} - \frac{\rho^{\beta-(k-1)}}{8} \rho^{k-1} - \frac{\rho^\beta}{8}
\geq \frac{\rho^\beta}{4} \quad \text{whenever } s \in I = \left[ (8\beta)^{-1}, \rho \right].
\]

(ii b B) \[ \frac{\rho^{\beta-(k-1)}}{8k} \leq \frac{y}{\rho^k} \leq \frac{3\beta}{8k} ; \quad 0 < \beta < 3(k-1), \ s \in I = [1, \rho] \]

We proceed here as in Case III (ii b):

There is a real number \( j > 1 \) such that \( 0 < j - 1 \leq \frac{\beta}{3} \leq j < k \).

Then \( \frac{2\beta}{3} = \beta - \frac{\beta}{3} \leq \beta - (j - 1) = \beta - [k - (k - (j - 1))] \).

With \( N, S_N, \) and \( S_m \) as in Case III (ii b), for

\[ \frac{\rho^{\beta-[k-S_m]}}{8k} \leq \frac{y}{\rho^k} \leq \frac{\rho^{\beta-[k-S_{m+1}]}}{8k} , \ m = 0,1,2,\ldots,N-1 , \]

and \( s \in I = \left[ (8\beta)^{-1}, \rho^{1-S_m} \right] \), we note that

\[
g'(s) \geq \frac{\rho^\beta}{2} - \frac{1}{8} \rho^{\beta-[k-S_{m+1}]} \rho^{k-1-(\frac{k-1}{k})S_m} - \frac{\rho^\beta}{8}
= \frac{\rho^\beta}{2} - \frac{\rho^\beta}{8} - \frac{\rho^\beta}{8} \geq \frac{\rho^\beta}{4} .
\]

Hence Lemma 3.1 implies that

\[
\int_I e^{-2\pi ig(s)} \frac{ds}{s^{1+\alpha}} \leq C \rho^{-\beta} .
\]

For \( s \in I = \left[ \rho^{1-S_m}, \rho \right] \), we use the fact that

\[
|g''(s)| \geq k(k-1)(k-2) \frac{y}{\rho^k} s^{k-3}
\geq C_k \rho^{\beta-[k-S_m]} \rho^{k-3-(\frac{k-1}{k})S_m}
\geq C_k \rho^{\beta-3+\frac{k}{k}}S_m .
\]
Lemma 3.1 now yields,
\[ \left| \int e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\frac{\beta}{2}+1-\frac{\beta}{2}+\frac{1}{\alpha}} \rho^{-1+\frac{2\beta}{\alpha}} = C \rho^{-\frac{\beta}{2}}. \]

On \[ \left[ \left( \frac{\beta}{2} \right)^{\frac{2}{\beta+1}}, (8\beta)^{\frac{2}{\beta+1}} \right] \] we use the fact that \(|g''(s)| \geq C' \rho^\beta\), and Lemma 3.1. This completes Case IV, and shows that \(|m(x,y)| \leq C \rho^{-\frac{\beta}{2}+\alpha}\); i.e., the multiplier \(m(x,y)\) is uniformly bounded in \(\mathbb{R}^2\) whenever \(\beta \geq 3\alpha\). Thus the proof of Theorem 2 is complete. \(\Box\)

Plancherel's Theorem now shows that
\[ ||T_{a,\beta}f||_2 = ||T_{a,\beta}f||_2 \leq A_{a,\beta} ||f||_2 = A_{a,\beta} ||f||_2 \] for \(\beta \geq 3\alpha\).

**Theorem 3.** Along the curve \(y = -C_{\beta,k} \frac{s^{\beta+1}}{s^{\beta+1}} (x > 0)\),
\[ |m(x, -C_{\beta,k} \frac{s^{\beta+1}}{s^{\beta+1}})| \sim C \rho^{-\frac{\beta}{2}-\alpha} \quad \text{as} \quad \rho \to \infty. \]

**Proof.** As before, it suffices to prove the above estimate for
\[ m^+(x, y) = \int_0^1 e^{-2\pi i [xs+y^\beta+s^{-\beta}]} \frac{ds}{s^{1+\alpha}}. \]

For \((x, y)\) on the above curve, write \(x = C_{\beta,k} \tau^{\beta+1}\) and \(y = -\tau^{\beta+k} \quad (\tau > 0)\).

The change of variable \(s \mapsto s\tau^{-1}\) yields
\[ m^+ (C_{\beta,k} \tau^{\beta+1}, -\tau^{\beta+k}) = \tau^\alpha \int_0^\tau e^{-2\pi i \tau^\beta g(s)} \frac{ds}{s^{1+\alpha}}, \]
with \(g(s) = [C_{\beta,k} s - s^k + s^{-\beta}]\). We split the above integral as
\[ \int_0^\tau = \int_a^b + \int_b^\tau \]
where \([a, b]\) is a small fixed interval centered at \(s_0 = \left[ \frac{\beta(\beta+1)}{k(k-1)} \right]^{\frac{1}{\beta+k}}\). Then since \(g'(s_0) = g''(s_0) = 0\), but \(g'''(s_0) \neq 0\), we have
\[ \tau^\alpha \left| \int_a^b e^{-2\pi i \tau^\beta g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \tau^{(\alpha-\frac{\beta}{2})} + O \left( \tau^{(\alpha-\frac{2\beta}{2})} \right) \quad \text{as} \quad \tau \to \infty, \]
by a standard result on integral asymptotics; see [St3], Chapter VIII. Next, on 
$I_1 = \left(0, \left[\frac{\beta}{2C_{\beta,k}}\right]^{\frac{1}{\beta+1}}\right)$, we have
\[ g'(s) \leq -\frac{\beta}{2} s^{-(\beta+1)} \leq -C_{\beta,k}. \]
Hence, by van der Corput’s Lemma we get
\[ \tau^\alpha \left| \int_{I_1} e^{-2\pi i r^\theta g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \tau^{-(\beta-\alpha)}. \]

Since $g''' < 0$, $g''$ is decreasing on $I_2 = \left(\left[\frac{\beta}{2C_{\beta,k}}\right]^{\frac{1}{\beta+1}}, a\right)$. Thus
\[ g''(s) \geq g''(a) = \left[-k(k-1) a^{k-2} + \beta(\beta+1) a^{-(\beta+2)}\right] = C > 0, \]
since $0 < a < s_0$ and $g''(s_0) = 0$. Hence,
\[ \tau^\alpha \left| \int_{I_2} e^{-2\pi i r^\theta g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \tau^{-\left[\frac{\theta}{2}-\alpha\right]}. \]

Since $g''' < 0$, as seen before, $g''$ is decreasing on $I_3 = [b, \tau)$. Then
\[ g''(s) \leq g''(b) = \left[-k(k-1) b^{k-2} + \beta(\beta+1) b^{-(\beta+2)}\right] = -C < 0, \]
since $b > s_0$ and $g''(s_0) = 0$. Hence, by van der Corput’s Lemma,
\[ \tau^\alpha \left| \int_{I_3} e^{-2\pi i r^\theta g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \tau^{-\left[\frac{\theta}{2}-\alpha\right]}. \]

Thus on $(0, a) \cup [b, \tau]$, $m^+(x,y)$ decays faster than required. This shows that
\[ \left| m\left(C_{\beta,k} \tau^{\beta+1}, -\tau^{\beta+k}\right) \right| \sim C \tau^{-\left[\frac{\theta}{2}-\alpha\right]} \quad \text{as} \quad \tau \to \infty; \]
that is,
\[ \left| m\left(x, -C_{\beta,k} x^{\frac{\theta+k}{\theta+1}}\right) \right| \sim C \rho^{-\left[\frac{\theta}{2}-\alpha\right]} \quad \text{as} \quad \rho \to \infty. \]
This completes the proof of Theorem 3. 

This shows that on the curve $y = -C_{\beta,k} x^{\frac{\theta+k}{\theta+1}}$ ($x > 0$), the multiplier $m(x,y)$ becomes unbounded if $\beta < 3\alpha$; hence the bound $\beta \geq 3\alpha$ on $m(x,y)$ is sharp, and the first assertion of Theorem 1 is proved.
4. $L^p$-Boundedness.

To prove the second assertion of Theorem 1, we introduce an analytic family of truncated operators defined by

$$(T_z^e f)(x,y) = \rho^z(x,y) m_z^e(x,y) \hat{f}(x,y) \quad (f \in \mathcal{S}),$$

where

$$m_z^e(x,y) = \int_{|t| \leq 1} e^{-2\pi i [\sigma t + y_0(t) + |t|^{-\beta}]} \frac{dt}{t|t|^\alpha} ; \quad \alpha > 0, \beta \geq 3\alpha, \text{ and } \epsilon > 0.$$  

We note at the outset that $T_0^0 = T_{\alpha,\beta}$ is bounded on $L^2$. We need to prove that

$$\|T_0^0 f\|_p \leq C\|f\|_p \quad (f \in L^p),$$

where $p$ is as in the statement of Theorem 1.

**Lemma 4.1.** Let $z = \sigma + i\tau$; $0 \leq \sigma \leq \frac{1}{2} \left[ \frac{\beta}{3} - \alpha \right]$, $\tau \in \mathbb{R}$. Then for simple $f$

$$\|T_z^e f\|_2 \leq C(1 + |z|) \|f\|_2.$$

**Proof.** It suffices to show that for each $z$, $|m_z^e(x,y)|$ is uniformly bounded for $(x,y) \in \mathbb{R}^2$. The proof of this fact is very similar to that of Theorem 2 of Section 3, and shows that

$$|m_z^e(x,y)| \leq \begin{cases} C(1 + |z|) & \text{if } 0 \leq \rho \leq 1 \\ C(1 + |z|) \rho^{-\frac{\epsilon}{3} + (\alpha + \sigma)} & \text{if } \rho > 1 \end{cases}$$

for all $(x,y) \in \mathbb{R}^2$.

Then for $\rho > 1$,

$$|\rho^z m_z^e(x,y)| \leq C(1 + |z|) \rho^{\sigma} \rho^{-\frac{\epsilon}{3} + (\alpha + \sigma)} = C(1 + |z|) \rho^{-(\frac{\epsilon}{3} - \alpha) + 2\sigma}.$$  

For each $z$, this is uniformly bounded whenever $0 \leq \sigma \leq \frac{1}{2} \left[ \frac{\beta}{3} - \alpha \right]$. The result now follows from the definition of $T_z^e$ and the Plancherel theorem. This completes the proof of Lemma 4.1. $\square$

To prove the $L^p$-boundedness of $T_z^e$, we need the following:
Lemma 4.2. For $-a < \Re z < 0$,

$$\rho^z(x, y) = h_z(x, y)$$

where

(i) $h_z(x, y)$ is a locally integrable function;
(ii) $h_z \in C\infty(R^2 - 0)$;
(iii) $h_z(\delta_\lambda (x, y)) = \lambda^{-a-z} h_z(x, y)$, $\lambda > 0$, $(x, y) \neq (0, 0)$;
(iv) each derivative of $h_z(x, y)$ is bounded by a polynomial in $|z|$ if $\rho(x, y) \geq 1$.

Here $a = (2\beta + k + 1) = \text{trace } A$, and the Fourier transform is to be taken in the sense of distributions.

Proof. See [St, Wa].

Remark 4.3. If the line joining $x$ and $x - w$ avoids the origin, and $|w|/|x|$ is sufficiently small, then

$$|h_z(x - w) - h_z(x)| = \left| \int_0^1 \frac{d}{dt} h_z(x - tw) \right| dt$$

$$= \left| -\int_0^1 \nabla h_z(x - tw) \cdot w \right| dt$$

$$\leq |w| \int_0^1 |\nabla h_z(x - tw)| dt$$

(4.3-1)

$$\leq C(z) |w|$$

since the derivatives of $h_z$ are bounded by $C(z)$, by Lemma 4.2. This observation, and the homogeneity of $h_z$ with $\lambda = \rho(x)$ and $||x||$ sufficiently large, then imply that,

$$|h_z(x - w) - h_z(x)|$$

$$= \left| h_z(\delta_{\rho(x)}(\delta^{-1}_{\rho(x)} x - \delta^{-1}_{\rho(x)} w)) - h_z(\delta_{\rho(x)}(\delta^{-1}_{\rho(x)} x)) \right|$$

$$\leq C(z) \frac{||w||}{\rho(x)^{2\beta+k+1}+\sigma}$$

by (4.3 - 1)

(4.3-2)

$$\leq C(z) \frac{||w||}{\rho(x)^{(2\beta+k+1)+\sigma+\beta+1}}.$$
Lemma 4.4. Suppose that
(i) \( T_z f \) is defined by
\[
(T_z f)(x, y) = \rho^z(x, y) m_z(x, y) \tilde{f}(x, y), \quad f \in \mathcal{S};
\]
(ii) \( z = \sigma + i\tau; \quad -\alpha < \sigma \leq -\alpha \left[ \frac{\beta + 1}{\beta + 2} \right] < 0, \quad \tau \in \mathbb{R}. \)
Then
\[
\|T_z f\|_p \leq C(z) \|f\|_p \quad (1 < p < \infty),
\]
where, for fixed \( \alpha \) and \( \beta \), \( C(z) \) grows at most as fast as a polynomial in \( |z| \).

Proof. By Lemma 4.2, for \( f \in \mathcal{S} \), we see that
\[
(T_z f)(x) = (K_z * f)(x),
\]
where
\[
K_z(x) = \int_{\epsilon \leq |t| \leq 1} h_z(x - \Gamma(t)) |t|^{-\sigma} e^{-2\pi i |t|^{-\beta}} \frac{dt}{t|t|^\alpha}
\]
with \( x \in \mathbb{R}^2 \), and \( \Gamma(t) = [t, \gamma(t)] \in \mathbb{R}^2 \). It follows that (4.4-1) holds when \( f \) is simple. Our aim now is to show that, for \( x, y \in \mathbb{R}^2 \),
\[
\int_{\rho(x) > C\rho(y)} |K_z(x - y) - K_z(x)| \, dx \leq C_1(z),
\]
where \( C_1(z) \) has at most polynomial growth in \( |z| \). Now \( U_\alpha = \{x: \rho(x) < \alpha\} \) is a regular Vitali family; and proving (4.4-2) will prove our lemma by virtue of Theorem 4.1 of [Ri].

There are two cases to consider: \( 0 < \rho(y) \leq 1 \), and \( \rho(y) \geq 1 \).

Case I: \( 0 < \rho(y) \leq 1 \).

Since
\[
\int_{\epsilon \leq |t| \leq 1} h_z(x) e^{-2\pi i |t|^{-\beta}} |t|^{-\sigma} \frac{dt}{t|t|^\alpha} = 0;
\]

\[
K_z(x) = \int_{\epsilon \leq |t| \leq 1} [h_z(x - \Gamma(t)) - h_z(x)] e^{-2\pi i |t|^{-\beta}} |t|^{-\sigma} \frac{dt}{t|t|^\alpha}.
\]
The change of variable \( t = s \rho(y)^{\beta + 1} \) gives \( dt = \rho(y)^{\beta + 1} ds \), and
\[
K_z(x) = \int_{\epsilon \rho(y)^{-(\beta + 1)} \leq |s| \leq 1} [h_z(x - \Gamma(s \rho(y)^{\beta + 1})) - h_z(x)] e^{-2\pi i |s| \rho(y)^{\beta + 1}} |s|^{-\sigma} \frac{ds}{s|s|^\alpha}.
\]
\[
\cdot \left| s \rho(y)^{\beta+1} \right|^{-2} \rho(y)^{-(\beta+1)\alpha} \frac{ds}{s|s|^\alpha} \\
+ \int_{1 \leq |s| \leq \rho(y)^{-(\beta+1)}} \ldots \\
= K_1^1 + K_2^2.
\]

Now,

\[
\int_{\rho(x) > C \rho(y)} \left| K_1^1(x) \right| dx \\
\leq \int_{\rho(x) > C \rho(y)} \int_{\epsilon \rho(y)^{-(\beta+1)} \leq |s| \leq 1} \left| h_z \left( x - \Gamma \left( s \rho(y)^{\beta+1} \right) \right) - h_z(x) \right| ds dx.
\]

(4.4-3)

\[
|s|^{-1-\alpha-\sigma} \rho(y)^{-(\beta+1)(\alpha+\sigma)} ds dx.
\]

The change of variable \( x = \delta_{\rho(y)} x' \) implies \( dx = \rho(y)^{(2\beta+k+1)} dx' \); \( \rho(x) = \rho(y) \rho(x') \); and that \( \|x'\| \) is large. The right-hand side of (4.4-3) now becomes:

\[
\int_{\rho(x') > C} \int_{\epsilon \rho(y)^{-(\beta+1)} \leq |s| \leq 1} \left| h_z \left( \delta_{\rho(y)} \left[ x' - \delta_{\rho(y)}^{-1} \Gamma \left( s \rho(y)^{\beta+1} \right) \right] \right) - h_z(x') \right| \cdot |s|^{-1-\alpha-\sigma} \rho(y)^{-(\beta+1)(\alpha+\sigma)} \rho(y)^{(2\beta+k+1)} ds dx'.
\]

Now, using the homogeneity of \( h_z \):

\[
h_z \left( \delta_{\rho(y)} x \right) = \rho(y)^{-(2\beta+k+1)-2} h_z(x),
\]

and writing \( x = x' \), this

\[
= \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int_{\rho(x) > C} \int_{\epsilon \rho(y)^{-(\beta+1)} \leq |s| \leq 1} \left| h_z \left( x - \left[ s, \gamma(s) \rho(y)^{(k-1)} \right] \right) - h_z(x) \right| |s|^{-1-\alpha-\sigma} ds dx
\]

\[
= \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int_{\rho(x) > C} \int_{\epsilon \rho(y)^{-(\beta+1)} \leq |s| \leq 1} \left| h_z \left( \delta_{\rho(x)} \left( \delta_{\rho(x)}^{-1} x - \delta_{\rho(x)}^{-1} \left[ s, \gamma(s) \rho(y)^{(k-1)} \right] \right) \right) - h_z \left( \delta_{\rho(x)} \left( \delta_{\rho(x)}^{-1} x \right) \right) \right| |s|^{-1-\alpha-\sigma} ds dx.
\]
Note that \( \left\| \delta_{\rho(x)}^{-1} x \right\| = 1 \); and since \( \rho(x) \) is large, 
\[ \|w\| = \left\| \delta_{\rho(x)}^{-1} \left[ s, \gamma(s) \rho(y)^{\beta(k-1)} \right] \right\| \] is small. Fubini’s theorem and (4.3-2) then imply that the above is

\[ \leq C(z) \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int_{\epsilon \rho(y)^{-(\beta+1)} \leq |s| \leq 1} |s|^{-1-\alpha-\sigma} \left[ s^2 + s^2 \rho(y)^{2\beta(k-1)} \right]^{1/2} ds \]

\[ \cdot \int_{\rho(x) > C} \frac{\rho(x)^{2\beta+k+1} \rho(x)^{\beta+\sigma+1}}{\rho(x)^{(2\beta+k+1)+\sigma+\beta+1}}. \]

Changing to polar-like coordinates with \( dx = \rho(x)^{(2\beta+k+1)\frac{1}{2}} \rho(x) d\phi \), the above is

\[ \leq C(z) \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int_{\epsilon \rho(y)^{-(\beta+1)} \leq |s| \leq 1} |s|^{-\alpha-\sigma} \left[ 1 + s^{2k-2} \rho(y)^{2\beta(k-1)} \right]^{1/2} ds \]

\[ \cdot \int_{S^1} d\phi \int_{\rho(x) > C} \frac{d\rho(x)}{\rho(x)^{\beta+\sigma+2}}. \]

For \( \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \) to be bounded, we need \(- (\beta + 1)(\alpha + \sigma) - \sigma \geq 0\); that is, \( \sigma \leq -\alpha \left[ \frac{\beta + 1}{\beta + 2} \right] < 0 \). With \( \sigma \) as in the preceding statement, \(-\alpha - \sigma \geq -\alpha \left[ \frac{\beta + 1}{\beta + 2} \right] > -1 \) since \( \beta \geq 3\alpha \); and so \( |s|^{-\alpha-\sigma} \) is integrable on \( \epsilon \rho(y)^{-(\beta+1)} \leq |s| \leq 1 \). For the \( \rho \)-integral to be bounded, we need \( \beta + \sigma + 2 > 1 \); that is, \( \sigma > -(\beta + 1) \). Thus, whenever \(- (\beta + 1) < -\alpha < \sigma \leq -\alpha \left[ \frac{\beta + 1}{\beta + 2} \right] \), we have that \( \int_{\rho(x) > C \rho(y)} |K_1(x) - y| dx \) is bounded by \( C(z) \).

Similarly, \( \int_{\rho(x) > C \rho(y)} |K_1^2(x) - y| dx \leq C(z) \) using the fact that \( \rho(x + y) \leq C[\rho(x) + \rho(y)] \).

Next,

\[ \int_{\rho(x) > C \rho(y)} |K_2^2(x) - y - K_2^2(x) - y| dx \]

\[ \leq \int_{\rho(x) > C \rho(y)} \int_{1 \leq |s| \leq \rho(y)^{-(\beta+1)}} |h_z \left( x - y - \Gamma \left( s \rho(y)^{\beta+1} \right) \right) - h_z \left( x - \Gamma \left( s \rho(y)^{\beta+1} \right) \right)| \]

\[ \cdot |s|^{-1-\alpha-\sigma} \rho(y)^{-(\beta+1)(\alpha+\sigma)} ds dx. \]
Again, with \( x = \delta_{\rho(y)} x' \) so that \( \|x'\| \) is large, \( dx = \rho(y)^{(2\beta+k+1)} dx' \), \( \rho(x) = \rho(y) \rho(x') \), and using the homogeneity of \( h_z \) with \( \lambda = \rho(y) \), the right-hand side of (4.4-4) becomes

\[
= \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int_{\rho(x') > C} \int_{1 \leq |s| \leq \rho(y)^{-(\beta+1)}} \left\{ h_z \left( x' - \delta_{\rho(y)}^{-1} y - \left[ s, \gamma(s) \rho(y)^{\beta(k-1)} \right] \right) - h_z \left( x' - \left[ s, \gamma(s) \rho(y)^{\beta(k-1)} \right] \right) \right\} \cdot |s|^{-1-\alpha-\sigma} \, ds \, dx'.
\]

Writing \( x = x' \) and \( w = x - \left[ s, \gamma(s) \rho(y)^{\beta(k-1)} \right] \), so that \( dw = dx \) and \( \|x\| \) is large, and using Fubini's theorem, this is

\[
= \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int_{1 \leq |s| \leq \rho(y)^{-(\beta+1)}} |s|^{-1-\alpha-\sigma} \, ds \\
\cdot \left\{ \int_{\rho(w) > C_2} h_z \left( w - \delta_{\rho(y)}^{-1} y \right) - h_z (w) \right\} \left( w \right) \, dw \\
= \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int_{1 \leq |s| \leq \rho(y)^{-(\beta+1)}} |s|^{-1-\alpha-\sigma} \, ds \left[ I + II \right];
\]

where \( C_2 \) is a large constant. Now, using the homogeneity of \( h_z \) and (4.3-2) we see that,

\[
I = \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int_{1 \leq |s| \leq \rho(y)^{-(\beta+1)}} |s|^{-1-\alpha-\sigma} \, ds \\
\cdot \int_{\rho(w) > C_2} \left| h_z \left( \delta_{\rho(w)} \left( \delta_{\rho(w)}^{-1} w \right) \right) - h_z \left( \delta_{\rho(w)} \left( \delta_{\rho(w)}^{-1} w \right) \right) \right| \, dw \\
\leq C(z) \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int_{1 \leq |s| \leq \rho(y)^{-(\beta+1)}} |s|^{-1-\alpha-\sigma} \, ds \\
\cdot \int_{\rho(w) > C_2} \frac{\left| \delta_{\rho(y)}^{-1} w \right|}{\rho(w)^{(2\beta+k+1)+\alpha+\beta+1}} \, dw \\
\leq C(z) \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int_{1 \leq |s| \leq \rho(y)^{-(\beta+1)}} |s|^{-1-\alpha-\sigma} \, ds \\
\cdot \int_{S^1} \frac{d\varphi}{\rho(w)^{\beta+\alpha+2}} \\
\int_{\rho(w) > C_2} \frac{d\rho(w)}{\rho(w)^{(2\beta+k+1)+\alpha+\beta+1}}.
\]
For $-\alpha < \sigma \leq -\alpha \left[\frac{\beta+1}{\beta+2}\right] < 0$, we have $0 < \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \leq 1$, and $1 + \alpha + \sigma > 1$; and so $|s|^{-1-\alpha-\sigma}$ is integrable on $|s| \geq 1$. The $p$-integral is bounded, since $\beta + \sigma + 2 > 1$ whenever $\sigma > -\alpha > -(\beta + 1)$. Hence, $I$ is bounded by $C(z)$.

Next,

$$II = \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int_{1 \leq |s| \leq \rho(y)^{-(\beta+1)}} |s|^{-1-\alpha-\sigma} \, ds \cdot \int_{\rho(w) \leq C_2} |h_z(w - \delta_{\rho(y)} y) - h_z(w)| \, dw.$$

The inner $w$-integral is bounded, since $h_z$ is locally integrable; the outer $s$-integral is bounded whenever $-\alpha < \sigma \leq -\alpha \left[\frac{\beta+1}{\beta+2}\right]$.

**Case II**: $\rho(y) > 1$.

Fubini's theorem, homogeneity of $h_z$, and (4.3-2) together imply that,

$$\int_{\rho(x) > C \rho(y)} |K_z(x)| \, dx \leq \int_{\varepsilon \leq |t| \leq 1} |t|^{-1-\alpha-\sigma} \, dt \int_{\rho(x) > C \rho(y)} |h_z(x - \Gamma(t)) - h_z(x)| \, dx$$

$$= \int_{\varepsilon \leq |t| \leq 1} |t|^{-1-\alpha-\sigma} \, dt \cdot \int_{\rho(x) > C} \left| h_z \left( \delta_{\rho(x)} \left( \delta_{\rho(x)}^{-1} x - \delta_{\rho(x)}^{-1} \Gamma(t) \right) \right) - h_z \left( \delta_{\rho(x)} \left( \delta_{\rho(x)}^{-1} x \right) \right) \right| \, dx$$

$$\leq C(z) \int_{\varepsilon \leq |t| \leq 1} |t|^{-1-\alpha-\sigma} |\Gamma(t)| \, dt \int_{\rho(x) > C} \frac{dx}{\rho(x)^{2\beta+k+1} + \sigma + \beta + 1}$$

$$\leq C(z) \int_{\varepsilon \leq |t| \leq 1} |t|^{-1-\alpha-\sigma} \int_{S^1} \int_{\rho(x) > C} \frac{d\varphi}{\rho(x)^{\beta+\sigma+2}} \frac{d\rho(x)}{\rho(x)^{2\beta+k+1} + \sigma + \beta + 1}$$

since $|\Gamma(t)| = (t^2 + t^{2k})^{\frac{1}{2}} \leq \sqrt{2} |t|$ for $|t| \leq 1$. The last expression is bounded whenever $-\alpha < \sigma \leq -\alpha \left[\frac{\beta+1}{\beta+2}\right]$. Similarly, $\int_{\rho(x) > C \rho(y)} |K_z(x - y)| \, dx$ is bounded by $C(z)$. This completes the proof of Lemma 4.4.

This brings us to the final step of the proof of Theorem 1:
4.1. Interpolation. Lemma 4.1 shows that, for $f$ simple, $\|T_z f\|_2 \leq C_1(z)\|f\|_2$ whenever $0 < \Re z \leq \frac{1}{2}\left[\frac{\beta}{3} - \alpha\right]$, $\beta > 3\alpha$; Lemma 4.4 shows that $\|T_z f\|_p \leq C_2(z)\|f\|_p$, $1 < p < \infty$, whenever $-\alpha < \Re z \leq -\alpha\left[\frac{\beta + 1}{\beta + 2}\right] < 0$; each $C_i(z)$ grows at most as fast as a polynomial in $|z|$. It follows that $\{T_z^\epsilon\}$ is an admissible analytic family for the Stein analytic interpolation theorem (see [St, We], page 205), defined for $z$ in the strip

$$S = \left\{ z \in \mathbb{C} : -\alpha\left[\frac{\beta + 1}{\beta + 2}\right] \leq \Re z \leq \frac{1}{2}\left[\frac{\beta}{3} - \alpha\right]\right\}.$$  

Analytic interpolation and duality now imply that $T_0^\epsilon = T_{\alpha,\beta}^\epsilon$ is bounded on $L^p$ whenever

$$1 + \frac{3\alpha(\beta + 1)}{\beta(\beta + 1) + (\beta - 3\alpha)} < p < \frac{\beta(\beta + 1) + (\beta - 3\alpha)}{3\alpha(\beta + 1)} + 1,$$

for all simple $f$ on $\mathbb{R}^2$. An easy limiting argument shows that $\|T_{\alpha,\beta}^\epsilon f\|_p \leq B_{\alpha,\beta}\|f\|_p$ for all $f \in S$. The constant $B_{\alpha,\beta}$ is independent of $\epsilon$. Letting $\epsilon \to 0$, Fatou’s lemma gives $\|T_{\alpha,\beta}^\epsilon f\|_p \leq B_{\alpha,\beta}\|f\|_p$ for all $f \in S$. Now, another limiting argument shows that the last inequality holds for all $f \in L^p$. This completes the proof of Theorem 1.

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