Pacific Journal of Mathematics

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Volume 175 No. 2 October 1996

PACIFIC JOURNAL OF MATHEMATICS
Vol. 175, No. 2, 1996

# $L^P$-BOUNDS FOR HYPERSINGULAR INTEGRAL OPERATORS ALONG CURVES

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**It is known that the Hilbert transform along curves:**

$$\mathcal{H}_\Gamma f(x) = pv\int_{-\infty}^{\infty} f(x-\Gamma(t))\,\frac{dt}{t} \qquad (x \in \mathbf{R}^n)$$

**is bounded on $L^p$, $1 < p < \infty$, where $\Gamma(t)$ is an appropriate curve in $\mathbf{R}^n$. In particular, $\|\mathcal{H}_\Gamma f\|_p \leq C\|f\|_p$, $1 < p < \infty$, where $\Gamma(t) = (t, |t|^k \operatorname{sgn} t)$, $k \geq 2$, is a curve in $\mathbf{R}^2$.**
**It is easy to see that the *hypersingular integral operator***

$$\mathcal{T} f(x) = pv\int_{-1}^{1} f(x-\Gamma(t))\,\frac{dt}{t|t|^\alpha} \qquad (\alpha > 0),$$

**in which the singularity at the origin is worse than that in the Hilbert transform, is not bounded on $L^2(\mathbf{R}^2)$. To counter-balance this worsened singularity, we introduce an additional oscillation $e^{-2\pi i|t|^{-\beta}}$ and study the operator**

$$\mathcal{T}_{\alpha,\beta} f(x,y) = pv\int_{-1}^{1} f(x-t, y-\gamma(t))\,e^{-2\pi i|t|^{-\beta}}\,\frac{dt}{t|t|^\alpha} \qquad (\alpha, \beta > 0)$$

**along the curve $\Gamma(t) = (t, \gamma(t))$, where $\gamma(t) = |t|^k$ or $\gamma(t) = |t|^k \operatorname{sgn} t$, $k \geq 2$, in $\mathbf{R}^2$ and show that**

**(i)** $\|\mathcal{T}_{\alpha,\beta} f\|_2 \leq A_{\alpha,\beta}\|f\|_2$ **if and only if** $\beta \geq 3\alpha$;

**(ii)** $\|\mathcal{T}_{\alpha,\beta} f\|_p \leq B_{\alpha,\beta}\|f\|_p$ **whenever** $\beta > 3\alpha$, **and**

$$1 + \frac{3\alpha(\beta+1)}{\beta(\beta+1)+(\beta-3\alpha)} < p < \frac{\beta(\beta+1)+(\beta-3\alpha)}{3\alpha(\beta+1)} + 1.$$

## 1. Introduction.

In recent years, several mathematicians have studied the Hilbert transform along curves:

$$\mathcal{H}_\Gamma f(x) = pv\int_{-\infty}^{\infty} f(x-\Gamma(t))\,\frac{dt}{t} \qquad (x \in \mathbf{R}^n),$$

where $\Gamma(t)$ is an appropriate curve in $\mathbf{R}^n$. Fabes and Rivière were led to the study of $\mathcal{H}_\Gamma$ in their attempt to generalize the Method of Rotation of Calderon and Zygmund; for details see [**Fa, Ri**] and [**Wa2**].

Nagel, Rivière, Stein and Wainger, and several other mathematicians have studied the $L^p$-boundedness of $\mathcal{H}_\Gamma$ for a variety of curves $\Gamma$. A detailed survey of these results can be found in [**St, Wa**]; also see [**Wa1**]. Nagel, Rivière and Wainger proved in [**NRW1**] that $\mathcal{H}_\Gamma$ is a bounded operator on $L^p$, $1 < p < \infty$, when $\Gamma(t) = (|t|^{\alpha_1}\operatorname{sgn} t, \cdots, |t|^{\alpha_n}\operatorname{sgn} t)$, each $\alpha_k > 0$, is a curve in $\mathbf{R}^n$. In particular, $\|\mathcal{H}_\Gamma f\|_p \le C\|f\|_p$, $1 < p < \infty$, where $\Gamma(t) = (t, |t|^k \operatorname{sgn} t)$, $k \ge 2$, is a curve in $\mathbf{R}^2$. For more general curves see [**Na, Wa**], [**NVWW**], and [**Wa3**].

The kernel, $K(x) = \frac{1}{\pi x}$, of the Hilbert transform,

$$\mathcal{H}f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x-y)}{y}\, dy \qquad (x \in \mathbf{R}),$$

owing to its order of magnitude, is not integrable either at $0$ or $\infty$. It does, however, compensate for this deficiency by cancellation due to oscillation; this oscillatory property being reflected in the fact that its Fourier transform, $\hat{K}(x) = i \operatorname{sgn} x$, is bounded.

It is tempting to explore a situation where the order of magnitude of the singularity of $K$ at the origin is greater than that of $|x|^{-1}$, say of the order of $|x|^{-1-\alpha}$, $\alpha > 0$. It is reasonable to expect that some additional oscillation is required to compensate for this worsened singularity. This translates to the requirement that the Fourier transform of $K$, in addition to being bounded, have some decay at infinity; that is, $|\hat{K}(x)| \le C(1+|x|)^{-\beta}$ for some $\beta > 0$. For further discussion see Theorem 5 of [**St**].

Integral operators with strong singularities of the type described above, were studied by Hirschman in one dimension [**Hi**], Wainger in $k$-dimensions [**Wa**], Stein [**St**], Fefferman [**Fe**], and Fefferman and Stein [**Fe, St**].

It is not hard to see that the *hypersingular integral operator*

$$\mathcal{T}f(x) = pv\int_{-1}^{1} f(x - \Gamma(t))\,\frac{dt}{t|t|^\alpha} \qquad (\alpha > 0)$$

along $\Gamma(t) = (t, \gamma(t))$, where $\gamma(t) = |t|^k$ or $\gamma(t) = |t|^k \operatorname{sgn} t$, $k \ge 2$, is not bounded on $L^2(\mathbf{R}^2)$. The $L^2$-boundedness of this operator is equivalent to the *uniform* boundedness, in $\mathbf{R}^2$, of the *multiplier*

$$m(x,y) = pv \int_{-1}^{1} e^{-2\pi i[xt+y\gamma(t)]} \frac{dt}{t|t|^\alpha} \qquad (\alpha > 0).$$

It is easy to see that $|m(\frac{1}{4},0)| = \infty$ for $\alpha \geq 1$; for $0 < \alpha < 1$ and $x > 0$,

$$|m(x,0)| = 2\left| \int_0^1 \sin(2\pi xt)\frac{dt}{t^{1+\alpha}} \right| = 2\,(2\pi x)^\alpha \left| \int_0^{2\pi x} \sin s \frac{ds}{s^{1+\alpha}} \right| \to \infty$$

as $x \to \infty$.

One can ask if the worsened singularity at the origin can be counterbalanced by an oscillation. This leads us to the operator

$$\mathcal{T}_{\alpha,\beta} f(x,y) = pv \int_{-1}^{1} f(x-t, y-\gamma(t))\, e^{-2\pi i|t|^{-\beta}} \frac{dt}{t|t|^\alpha} \qquad (\alpha,\, \beta > 0)$$

along the curve $\Gamma(t) = (t, \gamma(t))$, $\gamma(t) = |t|^k$ or $\gamma(t) = |t|^k \operatorname{sgn} t$, $k \geq 2$, in $\mathbf{R}^2$.

Zielinski, in his thesis [**Zi**], studied the $L^2$-boundedness of $\mathcal{T}_{\alpha,\beta}$ along the parabola $\gamma(t) = (t, t^2)$, and proved that $\|\mathcal{T}_{\alpha,\beta} f\|_2 \leq C\|f\|_2 \iff \beta \geq 3\alpha$.

**1.1. Statement of the Main Result.** We state the main result of this paper as:

**Theorem 1.** *Suppose that* $\gamma(t) = |t|^k$ *or* $\gamma(t) = |t|^k \operatorname{sgn} t$, $k \geq 2$, *and*

$$\mathcal{T}_{\alpha,\beta} f(x,y) = pv \int_{-1}^{1} f(x-t, y-\gamma(t)) e^{-2\pi i|t|^{-\beta}} \frac{dt}{t|t|^\alpha} \qquad (\alpha,\, \beta > 0).$$

*Then*

(i) $\|\mathcal{T}_{\alpha,\beta} f\|_2 \leq A_{\alpha,\beta}\|f\|_2$ *if and only if* $\beta \geq 3\alpha$;

(ii) $\|\mathcal{T}_{\alpha,\beta} f\|_p \leq B_{\alpha,\beta}\|f\|_p$ *whenever* $\beta > 3\alpha$, *and*

$$1 + \frac{3\alpha\,(\beta+1)}{\beta\,(\beta+1) + (\beta - 3\alpha)} < p < \frac{\beta\,(\beta+1) + (\beta-3\alpha)}{3\alpha\,(\beta+1)} + 1.$$

Here $A_{\alpha,\beta}$ also depends on $k$, and $B_{\alpha,\beta}$ also depends on $p$.

**1.2. Outline of Proof.** In Section 2, we define an appropriate *one parameter family of dilations* $\{\delta_t\}_{t>0}$, and a corresponding distance function $\rho$, whose *homogeneity* with respect to $\delta_t$ is essential in proving the $L^2$ and $L^p$-boundedness of $\mathcal{T}_{\alpha,\beta}$.

In Section 3, we prove that $\mathcal{T}_{\alpha,\beta}$ is a bounded operator on $L^2$ if and only if $\beta \geq 3\alpha$. This is achieved by applying van der Corput's Lemma and its corollary to judiciously subdivided intervals, and the *asymptotics of oscillatory integrals.*

The $L^p$-boundedness, as stated in the second assertion of Theorem 1, is proven in Section 4. This is accomplished by showing that a certain *analytic family*, $\{\mathcal{T}_z^\epsilon\}$, of truncated operators is bounded on $L^2$ for an appropriate $\Re z > 0$; and it is bounded on $L^p$, $1 < p < \infty$, for an appropriate $\Re z < 0$; and that the bound in each case grows at most as fast as a polynomial in $|z|$. The result then follows by *analytic interpolation*.

## 2. Dilations and Homogeneity.

We define a *one parameter group of dilations* $\{\delta_t\}_{t>0}$, $\delta_t : \mathbf{R}^2 \to \mathbf{R}^2$, by $\delta_t = \mathrm{diag}[t^{1+\beta},\, t^{k+\beta}]$, with $A = \mathrm{diag}[1+\beta,\, k+\beta]$ and $a = \mathrm{trace}\, A = 2\beta+k+1$, and a corresponding *distance function* $\rho$ defined by: $\rho = \rho\,(x,y) = t$ such that

$$\left\|\delta_\rho^{-1}\,(x,y)\right\|^2 = \left(\frac{x}{\rho^{1+\beta}}\right)^2 + \left(\frac{y}{\rho^{k+\beta}}\right)^2 = 1$$

if $(x,y) \neq (0,0)$, and $\rho(0,0) = 0$. Then $\rho$ is *homogeneous* with respect to $\delta_t$: $\rho(\delta_t\, x) = t\,\rho(x), t > 0\,, x \in \mathbf{R}^2$; $\rho(x)$ is continuous and is in $C^\infty(\mathbf{R}^2 - 0)$; $\rho(x+y) \leq C[\rho(x)+\rho(y)]$, for some $C > 0$; and $\mathbf{R}^2$ can be coordinatized by the *polar-like* coordinates $\rho = \rho(x)$ and $u = \delta_\rho^{-1}\, x$, with $dx = \rho^{a-1}\, d\rho\,(Au,u)\, d\varsigma = \rho^{a-1}\, d\rho\, d\varphi$, where $d\varsigma$ is the linear measure on $\mathbf{S}^1$. For proofs of these assertions and additional properties of $\delta_t$ and $\rho$ see [**St**, **Wa**].

## 3. $L^2$-Boundedness.

The proof of sufficiency in the first assertion of Theorem 1 is accomplished as an easy consequence of Theorem 2, which we prove next. Our point of departure is the observation that

$$\widehat{(\mathcal{T}_{\alpha,\beta} f)}(x,y) = m_{\alpha,\beta}(x,y)\hat{f}(x,y) \qquad (f \in L^2),$$

where $\hat{}$ denotes the *Fourier transform*, and $m_{\alpha,\beta}(x,y)$ is the *multiplier* given by

$$m_{\alpha,\beta}(x,y) = pv \int_{-1}^{1} e^{-2\pi i[xt+y\gamma(t)+|t|^{-\beta}]} \frac{dt}{t|t|^{\alpha}} \qquad (\alpha, \beta > 0).$$

Thus, the boundedness of $\mathcal{T}_{\alpha,\beta}$ on $L^2$ is, by the Plancherel Theorem, equivalent to the *uniform boundedness*, in $x$ and $y$, of the multiplier $m_{\alpha,\beta}$ . So we first prove:

**Theorem 2.** *The multiplier* $m_{\alpha,\beta}(x,y)$ *is uniformly bounded in* $\mathbf{R}^2$ *for* $\beta \geq 3\alpha$. *More precisely:*

$$|m_{\alpha,\beta}(x,y)| \leq \begin{cases} C & \text{if } 0 \leq \rho \leq 1 \\ C\,\rho^{-\frac{\beta-3\alpha}{3}} & \text{if } \rho > 1 \end{cases}, \quad \beta \geq 3\alpha, \ (x,y) \in \mathbf{R}^2.$$

The proof of Theorem 2 depends mainly on the following:

**Lemma 3.1.** *Suppose that*

(i) $g$ *is real-valued and smooth for all* $t \in [a,b]$ , $0 < a < b$ ;

(ii) $|g^{(k)}(t)| \geq \rho > 0$ *for all* $t \in [a,b]$ *with* $k \geq 2$ ; *in addition,* $g'$ *is monotone on* $[a,b]$ *if* $k = 1$ ;

(iii) $z = \sigma + i\tau$ , $\sigma \geq 0$, $\tau \in \mathbf{R}$

(iv) $\alpha \geq 0$.

*Then,*

$$\left| \int_a^b e^{-2\pi i g(t)} \frac{dt}{t^{1+\alpha+z}} \right| \leq \frac{C(1+|z|)}{a^{1+\alpha+\sigma}} \rho^{-\frac{1}{k}} .$$

*Proof.* Let

$$G(t) = \int_a^t e^{-2\pi i g(s)}\, ds .$$

Then, by van der Corput's Lemma (see **[St3]**, Chapter VIII),

$$|G(t)| \leq C_k\, \rho^{-\frac{1}{k}}, \quad t \in [a,b].$$

Integrating by parts, we get

$$\left| \int_a^b e^{-2\pi i g(t)} \frac{dt}{t^{1+\alpha+z}} \right| = \left| \left[ \frac{G(t)}{t^{1+\alpha+z}} \right]_{t=a}^{t=b} - (1+\alpha+z) \int_a^b \frac{G(t)}{t^{2+\alpha+z}}\, dt \right|$$

$$\le C\,\rho^{-\frac{1}{k}}\left[\frac{1}{b^{1+\alpha+\sigma}}+\frac{1}{a^{1+\alpha+\sigma}}\right]$$
$$+\;C(1+\alpha+|z|)\,\rho^{-\frac{1}{k}}\int_a^b\frac{dt}{|t^{2+\alpha+z}|}$$
$$\le\frac{2C}{a^{1+\alpha+\sigma}}\,\rho^{-\frac{1}{k}}+\frac{C(1+\alpha+|z|)}{1+\alpha+\sigma}\left[\frac{1}{b^{1+\alpha+\sigma}}+\frac{1}{a^{1+\alpha+\sigma}}\right]\rho^{-\frac{1}{k}}$$
$$\le\frac{C(1+|z|)}{a^{1+\alpha+\sigma}}\,\rho^{-\frac{1}{k}}\;.$$

This completes the proof of Lemma 3.1. □

*Proof of Theorem* 2: We only need look at

$$m^+(x,y)=m^+_{\alpha,\beta}(x,y)=\int_0^1 e^{-2\pi i[xt+yt^k+t^{-\beta}]}\,\frac{dt}{t^{1+\alpha}}\,,$$

since the other half can be dealt with similarly.

Since $\rho(0,0)=0$ and $m(0,0)=0$; for $(x,y)\neq(0,0)$ but $x^2+y^2\le 1$, so that $0<\rho\le 1$, if we let

$$g(s)=xs+ys^k+s^{-\beta},$$

then

$$g'(s)=x+yk\,s^{k-1}-\;\beta\,s^{-(\beta+1)},$$

and so there exists a $T>0$ *independent* of $x$ and $y$ such that $g'(s)\le-\,\frac{\beta}{2}\,s^{-(\beta+1)}$ for $s\in(0,T]$. Then if we let

$$G(s)\;=\int_0^s e^{-2\pi i g(t)}\,dt\;,$$

we get $|\,G(s)\,|\le Cs^{\beta+1}$, by van der Corput's Lemma. Hence integrating by parts we get,

$$\left|\int_0^T e^{-2\pi i g(s)}\,\frac{ds}{s^{1+\alpha}}\right|\le\left|\left[\frac{G(s)}{s^{1+\alpha}}\right]_{s=0}^{s=T}\right|+\;(1+\alpha)\int_0^T\frac{|G(s)|}{s^{\alpha+2}}\,ds$$
$$\le C\left[\frac{s^{\beta+1}}{s^{\alpha+1}}\right]_{s=0}^{s=T}+\;C\,(1+\alpha)\int_0^T\frac{s^{\beta+1}}{s^{\alpha+2}}\,ds$$

$$= C\ \left[s^{\beta-\alpha}\right]_{s=0}^{s=T} + C\,(1+\alpha) \int_0^T s^{(\beta-\alpha)-1}\, ds\,.$$

Both of these exist if $\beta > \alpha$ . Thus,

$$\left| \int_0^T e^{-2\pi i g(s)}\ \frac{ds}{s^{1+\alpha}} \right| \le C.$$

For $s \in [T, 1]$ ,

$$\left| \int_T^1 e^{-2\pi i g(s)}\ \frac{ds}{s^{1+\alpha}} \right| \le \int_T^1 \frac{ds}{s^{1+\alpha}} = \frac{1}{\alpha}\left[\frac{1}{T^\alpha} - 1\right] \le C.$$

Thus $m^+(x,y)$ is uniformly bounbed when $0 \le \rho \le 1$. We now turn to the case when $\rho > 1$ . With $\rho = \rho(x,y)$ as defined above, the change of variable $t = s\,\rho^{-1}$ leads us to

$$m^+\,(x,y) = \rho^\alpha \int_0^\rho e^{-2\pi i[\frac{x}{\rho}s + \frac{y}{\rho^k}s^k + \rho^\beta s^{-\beta}]}\ \frac{ds}{s^{1+\alpha}}\,.$$

Thus, to prove the theorem, we need only show that

$$\left| \int_0^\rho e^{-2\pi i[\frac{x}{\rho}s + \frac{y}{\rho^k}s^k + \rho^\beta s^{-\beta}]}\ \frac{ds}{s^{1+\alpha}} \right| \le C\,\rho^{-\frac{\beta}{3}}$$

for all $(x,y) \in \mathbf{R}^2$. To this end, we show that the above integral is *uniformly bounded* in each of the four quadrants of $\mathbf{R}^2$.

*Note: For notational convenience, we shall write $x$ (resp. $y$) if $x$ (resp. $y$) is positive, and $-x$ (resp. $-y$) if $x$ (resp. $y$) is negative.*

*Case* I: $x < 0,\ y < 0$ .

Let

$$g(s) = -\,\frac{x}{\rho}\,s \;-\; \frac{y}{\rho^k}\,s^k \;+\; \rho^\beta s^{-\beta}.$$

Then,

$$g'(s) = -\,\frac{x}{\rho} \;-\; \frac{y}{\rho^k}\,k\,s^{k-1} \;-\; \rho^\beta\,\beta\,s^{-(\beta+1)},$$

and

$$g''(s) = -\frac{y}{\rho^k}\, k\,(k-1)\, s^{k-2} \;+\; \rho^\beta\, \beta\,(\beta+1)\, s^{-(\beta+2)}.$$

Let

$$G(s) = \int_0^s e^{-2\pi i g(t)}\, dt \,.$$

Since near $0$ we have $g'(s) \le -\rho^\beta\ \beta\ s^{-(\beta+1)}$, van der Corput's Lemma gives $|G(s)| \le C\,\rho^{-\beta}\, s^{\beta+1}$. Hence, integrating by parts as before, we get

$$\left| \int_0^1 e^{-2\pi i g(s)}\, \frac{ds}{s^{1+\alpha}} \right| \le C\,\rho^{-\beta} \qquad \text{for } \beta > \alpha.$$

To tackle the integral from 1 to $\rho$, we need to consider the following two cases:

$$\text{(i)}\ \frac{x}{\rho} \ge \frac{\rho^\beta}{2}\,; \qquad \text{(ii)}\ \frac{x}{\rho} \le \frac{\rho^\beta}{2}\,.$$

(i) $\frac{x}{\rho} \ge \frac{\rho^\beta}{2}$

This implies that $-\frac{x}{\rho} \le -\frac{\rho^\beta}{2}$. Thus $g'(s) \le -\frac{x}{\rho} \le -\frac{\rho^\beta}{2}$ on $[1,\rho]$, together with Lemma 3.1, yields

$$\left| \int_1^\rho e^{-2\pi i g(s)}\, \frac{ds}{s^{1+\alpha}} \right| \le C\,\rho^{-\beta}\,.$$

(ii) $\frac{x}{\rho} \le \frac{\rho^\beta}{2}$

By the definition of $\rho$, this implies that $-\frac{y}{\rho^k} \le -\frac{\rho^\beta}{2}$. Then,

$$\begin{aligned} g'(s) &= -\frac{x}{\rho} - \frac{y}{\rho^k}\, k\, s^{k-1} - \rho^\beta\, \beta\, s^{-(\beta+1)} \\ &\le -\frac{x}{\rho} - k\,\frac{\rho^\beta}{2} - \rho^\beta\, \beta\, s^{-(\beta+1)} \\ &\le -\frac{k}{2}\,\rho^\beta \quad \text{for} \quad s \in [1,\rho]. \end{aligned}$$

This, along with Lemma 3.1, gives

$$\left| \int_1^\rho e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C\, \rho^{-\beta} .$$

Hence, $|\, m^+(x,y)\,| \leq C\, \rho^{-(\beta-\alpha)}$ whenever $x, y < 0$ and $\beta > \alpha$. This completes *Case* I.

*Case* II: $x \geq 0,\ y \geq 0$.

In this case,

$$\begin{aligned} g(s) &= + \frac{x}{\rho}\, s + \frac{y}{\rho^k}\, s^k + \rho^\beta s^{-\beta} , \\ g'(s) &= + \frac{x}{\rho} + \frac{y}{\rho^k}\, k\, s^{k-1} - \rho^\beta\, \beta\, s^{-(\beta+1)} , \\ g''(s) &= + \frac{y}{\rho^k}\, k\,(k-1)\, s^{k-2} + \rho^\beta\, \beta\,(\beta+1)\, s^{-(\beta+2)} . \end{aligned}$$

In the vicinity of $0$, we have $g''(s) \geq C\rho^\beta s^{-(\beta+2)}$ ; and so

$$\left| \int_0^b e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C\rho^{-\frac{\beta}{2}} \quad \text{for } \beta > 2\alpha ,$$

using van der Corput's lemma, where $b$ can be chosen later.

Away from $0$, we have the following two cases:

$$\text{(i)}\ \frac{y}{\rho^k} \leq \frac{\rho^\beta}{2} ; \qquad \text{(ii)}\ \frac{y}{\rho^k} \geq \frac{\rho^\beta}{2} .$$

(i) $\frac{y}{\rho^k} \leq \frac{\rho^\beta}{2}$

This, and the definition of $\rho$ imply that $\frac{x}{\rho} \geq \frac{\rho^\beta}{2}$.

Then,

$$\begin{aligned} g'(s) &\geq \frac{\rho^\beta}{2} - \beta\, \rho^\beta\, s^{-(\beta+1)} \\ &\geq \frac{\rho^\beta}{2} - \frac{\rho^\beta}{4} \\ &\geq \frac{\rho^\beta}{4} \quad \text{whenever } s \geq (4\beta)^{\frac{1}{\beta+1}} . \end{aligned}$$

Note that $g'$ is increasing since $g'' > 0$ . Choosing $b = (4\beta)^{\frac{1}{\beta+1}}$ , and using Lemma 3.1, we get

$$\left| \int_b^\rho e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C\,\rho^{-\beta} \ .$$

(ii) $\frac{y}{\rho^k} \geq \frac{\rho^\beta}{2}$

Here, $g''(s) \geq C\,\frac{\rho^\beta}{2}$ . Choosing $b = 1$ , and using Lemma 3.1, we get

$$\left| \int_1^\rho e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C\,\rho^{-\frac{\beta}{2}} \ .$$

Thus, $|\, m^+(x, y)\,| \leq C\,\rho^{-(\frac{\beta}{2}-\alpha)}$ whenever $x, y \geq 0$ and $\beta > 2\alpha$ .

This completes *Case* II.

*Case* III : $x < 0$ , $y \geq 0$ .

Here,

$$\begin{aligned} g(s) &= -\frac{x}{\rho}\,s + \frac{y}{\rho^k}\,s^k + \rho^\beta s^{-\beta}\,, \\ g'(s) &= -\frac{x}{\rho} + \frac{y}{\rho^k}\,k\,s^{k-1} - \rho^\beta\,\beta\,s^{-(\beta+1)}\,, \\ g''(s) &= +\frac{y}{\rho^k}\,k\,(k-1)\,s^{k-2} + \rho^\beta\,\beta\,(\beta+1)\,s^{-(\beta+2)}. \end{aligned}$$

Close to $0$ , $g''(s) \geq \beta\,(\beta+1)\,\rho^\beta\,s^{-(\beta+2)}$ ; and so

$$\left| \int_0^b e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C\rho^{-\frac{\beta}{2}} \qquad \text{for } \beta > 2\alpha\,,$$

using van der Corput's Lemma , where $b$ can be chosen later.

Farther from $0$ , the following two cases need to be considered:

$$\text{(i)}\ \frac{y}{\rho^k} \geq \frac{\rho^{\frac{\beta}{3}}}{8k}\,; \qquad \text{(ii)}\ \frac{y}{\rho^k} \leq \frac{\rho^{\frac{\beta}{3}}}{8k}\,.$$

(i) $\frac{y}{\rho^k} \geq \frac{\rho^{\frac{\beta}{3}}}{8k}$

Here,

$$g''(s) \geq \frac{y}{\rho^k}\,k\,(k-1)\,s^{k-2}$$

$$\begin{aligned}&\geq \frac{(k-1)}{8}\,\rho^{\frac{\beta}{3}}\,s^{k-2}\\ &\geq C\rho^{\frac{\beta}{3}}\,\rho^{\frac{\beta}{3}}\\ &= C\rho^{\frac{2\beta}{3}} \qquad \text{whenever } s\in I=\left[\,\rho^{\frac{\beta}{3(k-2)}},\ \rho\,\right].\end{aligned}$$

Choosing $b=\rho^{\frac{\beta}{3(k-2)}}$ in the above, and using Lemma 3.1 we get

$$\left|\int_I e^{-2\pi i g(s)}\,\frac{ds}{s^{1+\alpha}}\right| \leq C\,\rho^{-\frac{\beta}{3(k-2)}}\,\rho^{-\frac{\beta}{3}} \leq C\,\rho^{-\frac{\beta}{3}}\,.$$

(ii) $\frac{y}{\rho^k} \leq \frac{\rho^{\frac{\beta}{3}}}{8k}$

This implies that $\frac{x}{\rho} \geq \frac{\rho^\beta}{2}$, and so

$$\begin{aligned}g'(s) &\leq -\frac{x}{\rho}+\frac{y}{\rho^k}\,k\,s^{(k-1)}\\ &\leq -\frac{\rho^\beta}{2}+\frac{\rho^{\frac{\beta}{3}}}{8}\,\rho^{\frac{2\beta}{3}}\\ &\leq -\frac{\rho^\beta}{4} \qquad \text{whenever } s\in\left[1,\ \rho^{\frac{2\beta}{3(k-1)}}\,\right].\end{aligned}$$

If $\beta \geq \frac{3(k-1)}{2}$, we are done using Lemma 3.1. If not, we need to subdivide further:

$$\text{(ii a)}\ \frac{y}{\rho^k} \leq \frac{\rho^{\beta-(k-1)}}{8k}\,; \qquad \text{(ii b)}\ \frac{\rho^{\beta-(k-1)}}{8k} \leq \frac{y}{\rho^k} \leq \frac{\rho^{\frac{\beta}{3}}}{8k}\,.$$

(ii a) $\frac{y}{\rho^k} \leq \frac{\rho^{\beta-(k-1)}}{8k}$; $\quad 0<\beta<\frac{3(k-1)}{2}$, $\quad s\in[\,1,\ \rho\,]$.

In this case,

$$\begin{aligned}g'(s) &\leq -\frac{\rho^\beta}{2}+\frac{\rho^{\beta-(k-1)}}{8}\,s^{(k-1)}\\ &\leq -\frac{\rho^\beta}{2}+\frac{\rho^\beta}{8}\\ &\leq -\frac{\rho^\beta}{4}\,.\end{aligned}$$

Thus, Lemma 3.1 gives

$$\left|\int_1^\rho e^{-2\pi i g(s)}\,\frac{ds}{s^{1+\alpha}}\right| \leq C\rho^{-\beta}\,.$$

(ii b) $\dfrac{\rho^{\beta-(k-1)}}{8k} \le \dfrac{y}{\rho^k} \le \dfrac{\rho^{\frac{\beta}{3}}}{8k}$; $\quad 0 < \beta < \dfrac{3(k-1)}{2}$, $\quad s \in [1, \rho]$.

There is a real number $j > 1$ such that $0 < j-1 \le \frac{2\beta}{3} \le j < k$. Then $\frac{\beta}{3} = \beta - \frac{2\beta}{3} \le \beta - (j-1) = \beta - [k - \{k - (j-1)\}]$.

Now, let

$$S_N = \sum_{n=0}^{N} \left(\frac{k-1}{k}\right)^n .$$

Then, $S_{m+1} = 1 + \left(\frac{k-1}{k}\right) S_m$; $m \ge 0$. We choose $N$ so that

$$S_N = \frac{1-\left(\frac{k-1}{k}\right)^{N+1}}{1-\left(\frac{k-1}{k}\right)} = k\left[1-\left(\frac{k-1}{k}\right)^{N+1}\right] \ge k - (j-1);$$

i.e., $(j-1) \ge k\left(\frac{k-1}{k}\right)^{N+1}$. This can be done since $\frac{k-1}{k} < 1$.

We now look at:

$$\frac{\rho^{\beta-[k-S_m]}}{8k} \le \frac{y}{\rho^k} \le \frac{\rho^{\beta-[k-S_{m+1}]}}{8k}; \quad m = 0, 1, 2, \dots, N-1 .$$

For $s \in I = \left[1, \rho^{1-\frac{S_m}{k}}\right]$, we have

$$\begin{aligned} g'(s) &= -\frac{x}{\rho} + \frac{y}{\rho^k} k\, s^{k-1} - \rho^\beta \beta\, s^{-(\beta+1)} \\ &\le -\frac{\rho^\beta}{2} + \frac{1}{8} \rho^{\beta-[k-S_{m+1}]} \rho^{k-1-\left(\frac{k-1}{k}\right)S_m} \\ &= -\frac{\rho^\beta}{2} + \frac{\rho^\beta}{8} \\ &\le -\frac{\rho^\beta}{4} . \end{aligned}$$

Hence,

$$\left| \int_I e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \le C \rho^{-\beta} \qquad \text{using Lemma 3.1.}$$

Next, for $s \in I = \left[\rho^{1-\frac{S_m}{k}}, \rho\right]$, we have

$$\begin{aligned} g''(s) &\ge k(k-1) \frac{y}{\rho^k} s^{k-2} \\ &\ge \frac{(k-1)}{8} \rho^{\beta-[k-S_m]} s^{k-2} \end{aligned}$$

$$\begin{aligned} &\geq \frac{(k-1)}{8}\,\rho^{\beta-[k-S_m]}\,\rho^{k-2-\left(\frac{k-2}{k}\right)S_m} \\ &= C\rho^{\beta-2+\frac{2}{k}S_m}. \end{aligned}$$

This, along with Lemma 3.1, gives

$$\left|\int_I e^{-2\pi i g(s)}\,\frac{ds}{s^{1+\alpha}}\right| \leq C\rho^{-\frac{\beta}{2}+1-\frac{S_m}{k}}\,\rho^{-1+\frac{S_m}{k}} = C\rho^{-\frac{\beta}{2}}.$$

Thus, $|\,m^+(x,y)\,|$ is uniformly bounded when $x, y < 0$ and $\beta \geq 3\alpha$.
This completes *Case* III.

*Case* IV: $x > 0$, $y < 0$.

Here,

$$\begin{aligned} g(s) &= +\frac{x}{\rho}\,s - \frac{y}{\rho^k}\,s^k + \rho^\beta s^{-\beta}, \\ g'(s) &= +\frac{x}{\rho} - \frac{y}{\rho^k}\,k\,s^{k-1} - \rho^\beta\,\beta\,s^{-(\beta+1)}, \\ g''(s) &= -\frac{y}{\rho^k}\,k\,(k-1)\,s^{k-2} + \rho^\beta\,\beta\,(\beta+1)\,s^{-(\beta+2)}, \\ g'''(s) &= -\frac{y}{\rho^k}\,k\,(k-1)\,(k-2)\,s^{k-3} - \rho^\beta\,\beta\,(\beta+1)\,(\beta+2)\,s^{-(\beta+3)}. \end{aligned}$$

We need to split as follows:

$$\text{(i) } \frac{y}{\rho^k} \geq C_1\,\rho^\beta\,; \qquad \text{(ii) } \frac{y}{\rho^k} \leq C_1\,\rho^\beta$$

where $0 < C_1 < 1$ is to be chosen appropriately at a later stage.

Note that, in the vicinage of $0$,

$$\begin{aligned} g'(s) &\leq \frac{x}{\rho} - \rho^\beta\,\beta\,s^{-(\beta+1)} \\ &\leq \rho^\beta - \rho^\beta\,\beta\,s^{-(\beta+1)} \\ &\leq -\frac{\beta}{2}\,\rho^\beta\,s^{-(\beta+1)} \end{aligned}$$

whenever $s \in I = \left(0,\ \left(\frac{\beta}{2}\right)^{\frac{1}{\beta+1}}\right]$.

Therefore,

$$\left|\int_I e^{-2\pi i g(s)}\,\frac{ds}{s^{1+\alpha}}\right| \leq C\,\rho^{-\beta} \quad \text{for } \beta > \alpha$$

using van der Corput's Lemma.

(i) $\frac{y}{\rho^k} \geq C_1 \rho^\beta$

Note that, $|g'''(s)| \geq \frac{y}{\rho^k} k(k-1)(k-2) s^{k-3} \geq C \rho^\beta$ whenever $s \in I = \left[ \left(\frac{\beta}{2}\right)^{\frac{1}{\beta+1}}, \rho \right]$. Thus,

$$\left| \int_I e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\frac{\beta}{3}}$$

using Lemma 3.1. This completes (i).

(ii) $\frac{y}{\rho^k} \leq C_1 \rho^\beta$

This needs to be split further:

$$\text{(ii a)} \frac{\rho^{\frac{2\beta}{3}}}{8k} \leq \frac{y}{\rho^k} \leq C_1 \rho^\beta ; \qquad \text{(ii b)} \frac{y}{\rho^k} \leq \frac{\rho^{\frac{2\beta}{3}}}{8k} .$$

(ii a) $\frac{\rho^{\frac{2\beta}{3}}}{8k} \leq \frac{y}{\rho^k} \leq C_1 \rho^\beta$

At $s_0 = \left[ \frac{\beta(\beta+1)}{k(k-1)y} \right]^{\frac{1}{\beta+k}} \rho$, we have $g''(s_0) = 0$. Since $g''' < 0$, $g'$ has a maximum at $s_0$.

Now,

$$\begin{aligned} g'(s_0) &= \frac{x}{\rho} - \frac{y}{\rho^k} k \left[ \frac{\beta(\beta+1)}{k(k-1)y} \right]^{\frac{k-1}{\beta+k}} \rho^{k-1} - \rho^{-1} \beta \left[ \frac{\beta(\beta+1)}{k(k-1)y} \right]^{-\frac{\beta+1}{\beta+k}} \\ &= \frac{x}{\rho} - C_{\beta,k} \frac{y^{\frac{\beta+1}{\beta+k}}}{\rho} , \end{aligned}$$

where $C_{\beta,k} = \left[ \frac{\beta(\beta+k)}{(k-1)} \right] \left[ \frac{k(k-1)}{\beta(\beta+1)} \right]^{\frac{\beta+1}{\beta+k}}$.

Now, choose $C_1$ so that $g'(s_0) \geq \frac{1}{2} \frac{x}{\rho}$. Next, choose $a < 1$ and $b > 1$ such that in the neighborhood $I_{a,b} = [\, a s_0 \,,\, b s_0 \,]$ of $s_0$, we have

$$g'(s) \geq \frac{x}{4\rho} \geq \frac{1}{4}(1 - C_1^2)^{\frac{1}{2}} \rho^\beta .$$

Then,

$$\left| \int_{I_{a,b}} e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\beta}$$

using van der Corput's Lemma on $[as_0, s_0]$ and $[s_0, bs_0]$.

Since $g'''(s) < 0$, $g''(s)$ is decreasing; and so on $I = \left[ \left(\frac{\beta}{2}\right)^{\frac{1}{\beta+1}}, as_0 \right]$, we have

$$g''(s) \geq g''(as_0) = -\frac{y}{\rho^k} k(k-1)(as_0)^{k-2} + \rho^\beta \beta(\beta+1)(as_0)^{-(\beta+2)}$$
$$= C_{a,\beta,k} \frac{y^{\frac{\beta+2}{\beta+k}}}{\rho^2} \geq C'_{a,\beta,k} \rho^{\frac{2\beta}{3}},$$

as a simple calculation shows.

Thus on $I$,

$$\left| \int_I e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\frac{\beta}{3}}$$

by Lemma 3.1.

Now, on $I = [bs_0, \rho]$,

$$g''(s) \leq g''(bs_0) \leq -C'_{b,\beta,k} \rho^{\frac{2\beta}{3}},$$

as before. Hence, once again,

$$\left| \int_I e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C \rho^{-\frac{\beta}{3}}$$

using Lemma 3.1. This completes (ii a).

(ii b) $\frac{y}{\rho^k} \leq \frac{\rho^{\frac{2\beta}{3}}}{8k}$

Have,

$$g'(s) = +\frac{x}{\rho} - \frac{y}{\rho^k} k s^{k-1} - \rho^\beta \beta s^{-(\beta+1)}$$
$$\geq \frac{\rho^\beta}{2} - \frac{\rho^{\frac{2\beta}{3}}}{8} s^{(k-1)} - \rho^\beta \beta s^{-(\beta+1)}$$
$$\geq \frac{\rho^\beta}{4} \quad \text{whenever } s \in I = \left[ (8\beta)^{\frac{1}{\beta+1}}, \rho^{\frac{\beta}{3(k-1)}} \right].$$

If $\beta \geq 3(k-1)$, we are done using Lemma 3.1. If not, we need to subdivide further:

$$\text{(ii b A)}\ \frac{y}{\rho^k} \leq \frac{\rho^{\beta-(k-1)}}{8k}; \qquad \text{(ii b B)}\ \frac{\rho^{\beta-(k-1)}}{8k} \leq \frac{y}{\rho^k} \leq \frac{\rho^{\frac{2\beta}{3}}}{8k}$$

with $0 < \beta < 3(k-1)$, and $s \in I = [1, \rho]$.

(ii b A) $\quad \frac{y}{\rho^k} \le \frac{\rho^{\beta-(k-1)}}{8k}\,; \quad 0 < \beta < 3(k-1)\,, \;\; s \in I \;=\; [\,1,\; \rho\,]$
Have,

$$\begin{aligned} g'(s) &= +\,\frac{x}{\rho} \;-\; \frac{y}{\rho^k}\, k\, s^{k-1} \;-\; \rho^{\beta}\, \beta\, s^{-(\beta+1)} \\ &\ge \frac{\rho^{\beta}}{2} \;-\; \frac{\rho^{\beta-(k-1)}}{8}\, \rho^{k-1} \;-\; \frac{\rho^{\beta}}{8} \\ &\ge \frac{\rho^{\beta}}{4} \quad \text{whenever} \;\; s \in I \;=\; \left[\, (8\beta)^{\frac{1}{\beta+1}} \;,\; \rho \,\right] . \end{aligned}$$

Using Lemma 3.1 once again, we are done.

(ii b B) $\quad \frac{\rho^{\beta-(k-1)}}{8k} \le \frac{y}{\rho^k} \le \frac{\rho^{\frac{2\beta}{3}}}{8k}\,; \quad 0 < \beta < 3(k-1)\,, \;\; s \in I \;=\; [\,1,\; \rho\,]$
We proceed here as in *Case* III (ii b) :
There is a real number $j > 1$ such that $0 < j - 1 \le \frac{\beta}{3} \le j < k$ .

Then $\frac{2\beta}{3} \;=\; \beta \;-\; \frac{\beta}{3} \;\le\; \beta \;-\; (j-1) \;=\; \beta \;-\; [\,k \;-\; \{k \;-\; (j \;-\; 1)\}]$ .

With $N$ , $S_N$ , and $S_m$ as in *Case* III (ii b) , for

$$\frac{\rho^{\beta-[k-S_m]}}{8k} \;\le\; \frac{y}{\rho^k} \;\le\; \frac{\rho^{\beta-[k-S_{m+1}]}}{8k}\,, \quad m \;=\; 0,1,2,\ldots,N-1\,,$$

and $s \in I \;=\; \left[\, (8\beta)^{\frac{1}{\beta+1}} \;,\; \rho^{1-\frac{S_m}{k}} \,\right]$ , we note that

$$\begin{aligned} g'(s) &\ge \frac{\rho^{\beta}}{2} \;-\; \frac{1}{8}\, \rho^{\beta-[k-S_{m+1}]}\, \rho^{k-1-\left(\frac{k-1}{k}\right)S_m} \;-\; \frac{\rho^{\beta}}{8} \\ &= \frac{\rho^{\beta}}{2} \;-\; \frac{\rho^{\beta}}{8} \;-\; \frac{\rho^{\beta}}{8} \;\ge\; \frac{\rho^{\beta}}{4}\,. \end{aligned}$$

Hence Lemma 3.1 implies that

$$\left| \int_I e^{-2\pi i g(s)}\, \frac{ds}{s^{1+\alpha}} \right| \;\le\; C\, \rho^{-\beta}\,.$$

For $s \in I \;=\; \left[\, \rho^{1-\frac{S_m}{k}} \;,\; \rho \,\right]$ , we use the fact that

$$\begin{aligned} |\, g'''(s)\,| &\ge k\,(k-1)\,(k-2)\, \frac{y}{\rho^k}\, s^{k-3} \\ &\ge C_k\, \rho^{\beta-[k-S_m]}\, \rho^{k-3-\frac{k-3}{k}S_m} \\ &\ge C_k\, \rho^{\beta-3+\frac{3}{k}S_m}\,. \end{aligned}$$

Lemma 3.1 now yields,

$$\left| \int_I e^{-2\pi i g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C\, \rho^{-\frac{\beta}{3}+1-\frac{S_m}{k}}\, \rho^{-1+\frac{S_m}{k}} = C\, \rho^{-\frac{\beta}{3}} .$$

On $\left[ \left(\frac{\beta}{2}\right)^{\frac{1}{\beta+1}} , \ (8\beta)^{\frac{1}{\beta+1}} \right]$ we use the fact that $|g'''(s)| \geq C'\, \rho^{\beta}$, and Lemma 3.1. This completes *Case* IV, and shows that $|m(x,y)| \leq C\, \rho^{-\frac{\beta}{3}+\alpha}$ ; i.e., the *multiplier* $m(x,y)$ is uniformly bounded in $\mathbf{R}^2$ whenever $\beta \geq 3\alpha$ . Thus the proof of Theorem 2 is complete. □

Plancherel's Theorem now shows that

$$\|\mathcal{T}_{\alpha,\beta} f\|_2 = \|\widehat{\mathcal{T}_{\alpha,\beta} f}\|_2 \leq A_{\alpha,\beta}\, \|\hat{f}\|_2 = A_{\alpha,\beta}\|f\|_2 \quad \text{for } \beta \geq 3\alpha.$$

**Theorem 3.** *Along the curve* $y = -C_{\beta,k}\, x^{\frac{\beta+k}{\beta+1}}$ $(x > 0)$,

$$\left| m\left(x, -C_{\beta,k}\, x^{\frac{\beta+k}{\beta+1}}\right) \right| \sim C\, \rho^{-\left[\frac{\beta}{3}-\alpha\right]} \quad \textit{as } \rho \longrightarrow \infty.$$

*Proof.* As before, it suffices to prove the above estimate for

$$m^+(x,y) = \int_0^1 e^{-2\pi i\left[xs + ys^k + s^{-\beta}\right]} \frac{ds}{s^{1+\alpha}} .$$

For $(x,y)$ on the above curve, write $x = C_{\beta,k}\, \tau^{\beta+1}$ and $y = -\tau^{\beta+k}$ $(\tau > 0)$. The change of variable $s \mapsto s\tau^{-1}$ yields

$$m^+\left(C_{\beta,k}\, \tau^{\beta+1}, -\tau^{\beta+k}\right) = \tau^{\alpha} \int_0^{\tau} e^{-2\pi i \tau^{\beta} g(s)} \frac{ds}{s^{1+\alpha}} ,$$

with $g(s) = \left[C_{\beta,k}\, s - s^k + s^{-\beta}\right]$. We split the above integral as

$$\int_0^{\tau} = \int_0^{a} + \int_a^{b} + \int_b^{\tau}$$

where $[a,b]$ is a small fixed interval centered at $s_0 = \left[\frac{\beta(\beta+1)}{k(k-1)}\right]^{\frac{1}{\beta+k}}$ . Then since $g'(s_0) = g''(s_0) = 0$, but $g'''(s_0) \neq 0$, we have

$$\tau^{\alpha} \left| \int_a^b e^{-2\pi i \tau^{\beta} g(s)} \frac{ds}{s^{1+\alpha}} \right| \leq C\, \tau^{\left(\alpha - \frac{\beta}{3}\right)} + O\left(\tau^{\left(\alpha - \frac{2\beta}{3}\right)}\right) \quad \text{as } \tau \longrightarrow \infty,$$

by a standard result on integral asymptotics; see [**St3**], Chapter VIII. Next, on $I_1 = \left(0, \left[\frac{\beta}{2C_{\beta,k}}\right]^{\frac{1}{\beta+1}}\right]$, we have

$$g'(s) \leq -\frac{\beta}{2}\, s^{-(\beta+1)} \leq -C_{\beta,k}\,.$$

Hence, by van der Corput's Lemma we get

$$\tau^{\alpha}\left|\int_{I_1} e^{-2\pi i \tau^{\beta} g(s)}\,\frac{ds}{s^{1+\alpha}}\right| \leq C\,\tau^{-(\beta-\alpha)}\,.$$

Since $g''' < 0$, $g''$ is decreasing on $I_2 = \left[\left[\frac{\beta}{2C_{\beta,k}}\right]^{\frac{1}{\beta+1}}, a\right]$. Thus

$$g''(s) \geq g''(a) = \left[-k(k-1)\,a^{k-2} + \beta(\beta+1)\,a^{-(\beta+2)}\right] = C > 0,$$

since $0 < a < s_0$ and $g''(s_0) = 0$. Hence,

$$\tau^{\alpha}\left|\int_{I_2} e^{-2\pi i \tau^{\beta} g(s)}\,\frac{ds}{s^{1+\alpha}}\right| \leq C\,\tau^{-\left[\frac{\beta}{2}-\alpha\right]}\,.$$

Since $g''' < 0$, as seen before, $g''$ is decreasing on $I_3 = [b, \tau]$. Then

$$g''(s) \leq g''(b) = \left[-k(k-1)\,b^{k-2} + \beta(\beta+1)\,b^{-(\beta+2)}\right] = -C < 0,$$

since $b > s_0$ and $g''(s_0) = 0$. Hence, by van der Corput's Lemma,

$$\tau^{\alpha}\left|\int_{I_3} e^{-2\pi i \tau^{\beta} g(s)}\,\frac{ds}{s^{1+\alpha}}\right| \leq C\,\tau^{-\left[\frac{\beta}{2}-\alpha\right]}\,.$$

Thus on $(0, a] \cup [b, \tau]$, $m^{+}(x, y)$ decays *faster* than required. This shows that

$$\left|m\left(C_{\beta,k}\,\tau^{\beta+1}, -\tau^{\beta+k}\right)\right| \sim C\,\tau^{-\left[\frac{\beta}{3}-\alpha\right]} \quad \text{as } \tau \longrightarrow \infty;$$

that is,

$$\left|m\left(x, -C_{\beta,k}\,x^{\frac{\beta+k}{\beta+1}}\right)\right| \sim C\,\rho^{-\left[\frac{\beta}{3}-\alpha\right]} \quad \text{as } \rho \longrightarrow \infty.$$

This completes the proof of Theorem 3. □

This shows that on the curve $y = -C_{\beta,k}\,x^{\frac{\beta+k}{\beta+1}}$ $(x > 0)$, the multiplier $m(x, y)$ becomes unbounded if $\beta < 3\alpha$; hence the bound $\beta \geq 3\alpha$ on $m(x, y)$ is *sharp*, and the first assertion of Theorem 1 is proved.

## 4. $L^p$-Boundedness.

To prove the second assertion of Theorem 1, we introduce an *analytic family* of truncated operators defined by

$$\widehat{(\mathcal{T}_z^\epsilon f)}(x,y) = \rho^z(x,y)\, m_z^\epsilon(x,y)\, \hat{f}(x,y) \qquad (f \in \mathcal{S}),$$

where

$$m_z^\epsilon(x,y) = \int_{\epsilon \le |t| \le 1} e^{-2\pi i[xt + y\gamma(t) + |t|^{-\beta}]}\, |t|^{-z} \frac{dt}{t|t|^\alpha}\,;\ \alpha > 0,\ \beta \ge 3\alpha, \ \text{ and } \epsilon > 0.$$

We note at the outset that $\mathcal{T}_0^0 = \mathcal{T}_{\alpha,\beta}$ is bounded on $L^2$. We need to prove that

$$\|\mathcal{T}_0^0 f\|_p \le C\|f\|_p \qquad (f \in L^p),$$

where $p$ is as in the statement of Theorem 1.

**Lemma 4.1.** *Let* $z = \sigma + i\tau$; $0 \le \sigma \le \frac{1}{2}\left[\frac{\beta}{3} - \alpha\right]$, $\tau \in \mathbf{R}$. *Then for simple* $f$

$$\|\mathcal{T}_z^\epsilon f\|_2 \le C(1+|z|)\, \|f\|_2 .$$

*Proof.* It suffices to show that for each $z$, $|m_z^\epsilon(x,y)|$ is uniformly bounded for $(x,y) \in \mathbf{R}^2$. The proof of this fact is very similar to that of Theorem 2 of Section 3, and shows that

$$|m_z^\epsilon(x,y)| \le \begin{cases} C(1+|z|) & \text{if } 0 \le \rho \le 1 \\ C(1+|z|)\, \rho^{-\frac{\beta}{3}+(\alpha+\sigma)} & \text{if } \rho > 1 \end{cases} \quad \text{for all } (x,y) \in \mathbf{R}^2.$$

Then for $\rho > 1$,

$$\begin{aligned} |\rho^z\, m_z^\epsilon(x,y)| &\le C(1+|z|)\, \rho^\sigma \rho^{-\frac{\beta}{3}+(\alpha+\sigma)} \\ &= C(1+|z|)\, \rho^{-(\frac{\beta}{3}-\alpha)+2\sigma} . \end{aligned}$$

For each $z$, this is uniformly bounded whenever $0 \le \sigma \le \frac{1}{2}\left[\frac{\beta}{3} - \alpha\right]$. The result now follows from the definition of $\mathcal{T}_z^\epsilon$ and the Plancherel theorem. This completes the proof of Lemma 4.1. □

To prove the $L^p$-boundedness of $T_z^\epsilon$, we need the following:

**Lemma 4.2.** *For* $-a < \Re z < 0$,

$$\rho^z(x,y) = \hat{h}_z(x,y)$$

*where*

(i) $h_z(x,y)$ *is a locally integrable function;*

(ii) $h_z \in C^\infty(\mathbf{R}^2 - 0)$*;*

(iii) $h_z(\delta_\lambda(x,y)) = \lambda^{-a-z} h_z(x,y),\ \lambda > 0,\ (x,y) \neq (0,0)$ *;*

(iv) *each derivative of* $h_z(x,y)$ *is bounded by a polynomial in* $|z|$ *if* $\rho(x,y) \geq 1$.

*Here* $a = (2\beta + k + 1) =$ *trace* $A$, *and the Fourier transform is to be taken in the sense of distributions.*

*Proof.* See [**St**, **Wa**]. □

**Remark 4.3.** If the line joining $x$ and $x - w$ avoids the origin, and $\frac{|w|}{|x|}$ is sufficiently small, then

$$\begin{aligned}
|h_z(x-w) - h_z(x)| &= \left| \int_0^1 \frac{d}{dt} h_z(x - tw)\, dt \right| \\
&= \left| -\int_0^1 \nabla h_z(x - tw) \cdot w\, dt \right| \\
&\leq |w| \int_0^1 |\nabla h_z(x - tw)|\, dt \\
&\leq C(z)\,|w| \qquad (4.3\text{-}1)
\end{aligned}$$

since the derivatives of $h_z$ are bounded by $C(z)$, by Lemma 4.2. This observation, and the homogeneity of $h_z$ with $\lambda = \rho(x)$ and $\|x\|$ sufficiently large, then imply that,

$$\begin{aligned}
&|h_z(x-w) - h_z(x)| \\
&= \left| h_z\left(\delta_{\rho(x)}\left(\delta^{-1}_{\rho(x)} x - \delta^{-1}_{\rho(x)} w\right)\right) - h_z\left(\delta_{\rho(x)}\left(\delta^{-1}_{\rho(x)} x\right)\right) \right| \\
&\leq C(z)\, \frac{\left|\delta^{-1}_{\rho(x)}\right|\, |w|}{\rho(x)^{(2\beta+k+1)+\sigma}} \qquad \text{by } (4.3-1) \\
&\leq C(z)\, \frac{|w|}{\rho(x)^{(2\beta+k+1)+\sigma+\beta+1}}\,. \qquad (4.3\text{-}2)
\end{aligned}$$

**Lemma 4.4.** *Suppose that*
(i) $\mathcal{T}_z^\epsilon f$ *is defined by*

$$\widehat{(\mathcal{T}_z^\epsilon f)}(x,y) = \rho^z(x,y)\, m_z^\epsilon(x,y)\, \hat{f}(x,y), \quad f \in \mathcal{S};$$

(ii) $z = \sigma + i\tau\,;\ -\alpha < \sigma \le -\alpha \left[\frac{\beta+1}{\beta+2}\right] < 0\,,\ \tau \in \mathbf{R}.$

*Then*

$$\|\mathcal{T}_z^\epsilon f\|_p \le C(z)\, \|f\|_p \qquad (1 < p < \infty),$$

*where, for fixed* $\alpha$ *and* $\beta$, $C(z)$ *grows at most as fast as a polynomial in* $|z|$.

*Proof.* By Lemma 4.2, for $f \in \mathcal{S}$, we see that

$$(\mathcal{T}_z^\epsilon f)(x) = (K_z * f)(x)\,, \tag{4.4-1}$$

where

$$K_z(x) = \int_{\epsilon \le |t| \le 1} h_z(x - \Gamma(t))\, |t|^{-z}\, e^{-2\pi i |t|^{-\beta}} \frac{dt}{t|t|^\alpha}$$

with $x \in \mathbf{R}^2$, and $\Gamma(t) = [t, \gamma(t)] \in \mathbf{R}^2$. It follows that (4.4-1) holds when $f$ is *simple*. Our aim now is to show that, for $x, y \in \mathbf{R}^2$,

$$\int_{\rho(x) > C\rho(y)} |\, K_z(x-y) - K_z(x)\,|\, dx \le C_1(z)\,, \tag{4.4-2}$$

where $C_1(z)$ has at most polynomial growth in $|z|$. Now $U_\alpha = \{x : \rho(x) < \alpha\}$ is a regular Vitali family; and proving (4.4-2) will prove our lemma by virtue of Theorem 4.1 of [**Ri**].

There are two cases to consider: $0 < \rho(y) \le 1$, and $\rho(y) \ge 1$.

*Case* I: $0 < \rho(y) \le 1$.

Since

$$\int_{\epsilon \le |t| \le 1} h_z(x)\, e^{-2\pi i |t|^{-\beta}}\, |t|^{-z} \frac{dt}{t|t|^\alpha} = 0\,;$$

$$K_z(x) = \int_{\epsilon \le |t| \le 1} [h_z(x - \Gamma(t)) - h_z(x)]\, e^{-2\pi i |t|^{-\beta}}\, |t|^{-z} \frac{dt}{t|t|^\alpha}\,.$$

The change of variable $t = s\,\rho(y)^{\beta+1}$ gives $dt = \rho(y)^{\beta+1}\, ds$, and

$$K_z(x) = \int_{\epsilon\, \rho(y)^{-(\beta+1)} \le |s| \le 1} \left[h_z(x - \Gamma(s\rho(y)^{\beta+1})) - h_z(x)\right] e^{-2\pi i |s\, \rho(y)^{\beta+1}|^{-\beta}}$$

$$\cdot \left|s\,\rho(y)^{\beta+1}\right|^{-z} \rho(y)^{-(\beta+1)\alpha} \frac{ds}{s|s|^{\alpha}}$$
$$+ \int_{1\le|s|\le\rho(y)^{-(\beta+1)}} \cdots$$
$$= K_z^1 + K_z^2 .$$

Now,

$$\int_{\rho(x)>C\rho(y)} |\, K_z^1(x)\,|\, dx$$
$$\le \int_{\rho(x)>C\rho(y)} \int_{\epsilon\,\rho(y)^{-(\beta+1)}\le|s|\le 1} \left|h_z\left(x - \Gamma\left(s\,\rho(y)^{\beta+1}\right)\right) - h_z(x)\right|$$
$$|s|^{-1-\alpha-\sigma}\,\rho(y)^{-(\beta+1)(\alpha+\sigma)}\,ds\,dx . \tag{4.4-3}$$

The change of variable $x = \delta_{\rho(y)}\,x'$ implies $dx = \rho(y)^{(2\beta+k+1)}\,dx'$; $\rho(x) = \rho(y)\,\rho(x')$; and that $\|x'\|$ is large. The right-hand side of (4.4-3) now becomes:

$$\int_{\rho(x')>C} \int_{\epsilon\,\rho(y)^{-(\beta+1)}\le|s|\le 1} \left|h_z\left(\delta_{\rho(y)}\left[x' - \delta^{-1}_{\rho(y)}\,\Gamma\left(s\,\rho(y)^{\beta+1}\right)\right]\right)\, h_z\left(\delta_{\rho(y)}\,x'\right)\right|$$
$$\cdot\, |s|^{-1-\alpha-\sigma}\,\rho(y)^{-(\beta+1)(\alpha+\sigma)}\,\rho(y)^{(2\beta+k+1)}\,ds\,dx'.$$

Now, using the homogeneity of $h_z$:

$$h_z\left(\delta_{\rho(y)}\,x\right) = \rho(y)^{-(2\beta+k+1)-z}\,h_z(x)\,,$$

and writing $x = x'$, this

$$= \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int_{\rho(x)>C} \int_{\epsilon\,\rho(y)^{-(\beta+1)}\le|s|\le 1}$$
$$\left|h_z\left(x - \left[s,\,\gamma(s)\,\rho(y)^{\beta(k-1)}\right]\right) - h_z(x)\right|\,|s|^{-1-\alpha-\sigma}\,ds\,dx$$
$$= \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int_{\rho(x)>C} \int_{\epsilon\,\rho(y)^{-(\beta+1)}\le|s|\le 1}$$
$$\left|h_z\left(\delta_{\rho(x)}\left(\delta^{-1}_{\rho(x)}\,x - \delta^{-1}_{\rho(x)}\left[s,\,\gamma(s)\,\rho(y)^{\beta(k-1)}\right]\right)\right) - h_z\left(\delta_{\rho(x)}\left(\delta^{-1}_{\rho(x)}\,x\right)\right)\right|$$
$$\cdot\, |s|^{-1-\alpha-\sigma}\,ds\,dx.$$

Note that $\left\|\delta^{-1}_{\rho(x)}\, x\right\| = 1$; and since $\rho(x)$ is large, $\|w\| = \left\|\delta^{-1}_{\rho(x)}\left[s,\, \gamma(s)\, \rho(y)^{\beta(k-1)}\right]\right\|$ is small. Fubini's theorem and (4.3-2) then imply that the above is

$$\leq C(z)\, \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int\limits_{\epsilon\, \rho(y)^{-(\beta+1)} \leq |s| \leq 1} |s|^{-1-\alpha-\sigma}\left[s^2 + s^{2k}\rho(y)^{2\beta(k-1)}\right]^{\frac{1}{2}} ds$$

$$\cdot \int\limits_{\rho(x)>C} \frac{dx}{\rho(x)^{(2\beta+k+1)+\sigma+\beta+1}}\,.$$

Changing to *polar-like* coordinates with $dx = \rho(x)^{(2\beta+k+1)-1}\, d\rho(x)\, d\varphi$, the above is

$$\leq C(z)\, \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int\limits_{\epsilon\, \rho(y)^{-(\beta+1)} \leq |s| \leq 1} |s|^{-\alpha-\sigma}\left[1 + s^{2k-2}\rho(y)^{2\beta(k-1)}\right]^{\frac{1}{2}} ds$$

$$\cdot \int\limits_{\mathbf{S}^1} d\varphi \int\limits_{\rho(x)>C} \frac{d\rho(x)}{\rho(x)^{\beta+\sigma+2}}.$$

For $\rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma}$ to be bounded, we need $-(\beta+1)(\alpha+\sigma)-\sigma \geq 0$; that is, $\sigma \leq -\alpha\left[\dfrac{\beta+1}{\beta+2}\right] < 0$. With $\sigma$ as in the preceding statement, $-\alpha-\sigma \geq -\dfrac{\alpha}{\beta+2} > -1$ since $\beta \geq 3\alpha$; and so $|s|^{-\alpha-\sigma}$ is integrable on $\epsilon\, \rho(y)^{-(\beta+1)} \leq |s| \leq 1$. For the $\rho$-integral to be bounded, we need $\beta + \sigma + 2 > 1$; that is, $\sigma > -(\beta+1)$. Thus, whenever $-(\beta+1) < -\alpha < \sigma \leq -\alpha\left[\dfrac{\beta+1}{\beta+2}\right]$, we have that $\displaystyle\int\limits_{\rho(x)>C\rho(y)} |K^1_z(x)|\, dx$ is bounded by $C(z)$.

Similarly, $\displaystyle\int\limits_{\rho(x)>C\rho(y)} |K^1_z(x-y)|\, dx \leq C(z)$ using the fact that $\rho(x+y) \leq C[\rho(x)+\rho(y)]$.

Next,

$$\int\limits_{\rho(x)>C\rho(y)} |\, K^2_z(x-y) - K^2_z(x)\,|\, dx$$

$$\leq \int\limits_{\rho(x)>C\rho(y)} \int\limits_{1\leq|s|\leq\rho(y)^{-(\beta+1)}}$$

$$\left|h_z\left(x-y-\Gamma\left(s\,\rho(y)^{\beta+1}\right)\right) - h_z\left(x-\Gamma\left(s\,\rho(y)^{\beta+1}\right)\right)\right|$$

$$\cdot\, |s|^{-1-\alpha-\sigma}\, \rho(y)^{-(\beta+1)(\alpha+\sigma)}\, ds\, dx\,. \tag{4.4-4}$$

Again, with $x = \delta_{\rho(y)}\, x'$ so that $\|x'\|$ is large, $dx = \rho(y)^{(2\beta+k+1)}\, dx'$, $\rho(x) = \rho(y)\,\rho(x')$, and using the homogeneity of $h_z$ with $\lambda = \rho(y)$, the right-hand side of (4.4-4) becomes

$$= \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int\limits_{\rho(x')>C} \int\limits_{1\le|s|\le\rho(y)^{-(\beta+1)}}$$
$$\left|h_z\left(x' - \delta_{\rho(y)}^{-1}\, y - \left[s, \gamma(s)\,\rho(y)^{\beta(k-1)}\right]\right) - h_z\left(x' - \left[s, \gamma(s)\,\rho(y)^{\beta(k-1)}\right]\right)\right|$$
$$\cdot\, |s|^{-1-\alpha-\sigma}\, ds\, dx'.$$

Writing $x = x'$ and $w = x - \left[s,\, \gamma(s)\,\rho(y)^{\beta(k-1)}\right]$, so that $dw = dx$ and $\|x\|$ is large, and using Fubini's theorem, this is

$$= \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int\limits_{1\le|s|\le\rho(y)^{-(\beta+1)}} |s|^{-1-\alpha-\sigma}\, ds$$
$$\cdot \left[\int\limits_{\rho(w)>C_2} + \int\limits_{\rho(w)\le C_2} \left|h_z\left(w - \delta_{\rho(y)}^{-1}\, y\right) - h_z(w)\right| dw\right]$$
$$= \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int\limits_{1\le|s|\le\rho(y)^{-(\beta+1)}} |s|^{-1-\alpha-\sigma}\, ds\, [I + II]\,;$$

where $C_2$ is a large constant. Now, using the homogeneity of $h_z$, and (4.3-2) we see that,

$$I = \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int\limits_{1\le|s|\le\rho(y)^{-(\beta+1)}} |s|^{-1-\alpha-\sigma}\, ds$$
$$\cdot \int\limits_{\rho(w)>C_2} \left|h_z\left(\delta_{\rho(w)}\left(\delta_{\rho(w)}^{-1} w - \delta_{\rho(w)}^{-1}\left(\delta_{\rho(y)}^{-1} y\right)\right)\right) - h_z\left(\delta_{\rho(w)}\left(\delta_{\rho(w)}^{-1} w\right)\right)\right| dw$$
$$\le C(z)\rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int\limits_{1\le|s|\le\rho(y)^{-(\beta+1)}} |s|^{-1-\alpha-\sigma}\, ds$$
$$\cdot \int\limits_{\rho(w)>C_2} \frac{\left|\delta_{\rho(y)}^{-1}\, y\right|}{\rho(w)^{(2\beta+k+1)+\sigma+\beta+1}}\, dw$$
$$\le C(z)\rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int\limits_{1\le|s|\le\rho(y)^{-(\beta+1)}} |s|^{-1-\alpha-\sigma}\, ds$$
$$\cdot \int\limits_{\mathbf{S}^1} d\varphi \int\limits_{\rho(w)>C_2} \frac{d\rho(w)}{\rho(w)^{\beta+\sigma+2}}\,.$$

For $-\alpha < \sigma \leq -\alpha \left[\frac{\beta+1}{\beta+2}\right] < 0$, we have $0 < \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \leq 1$, and $1+\alpha+\sigma > 1$; and so $|s|^{-1-\alpha-\sigma}$ is integrable on $|s| \geq 1$. The $\rho$-integral is bounded, since $\beta+\sigma+2 > 1$ whenever $\sigma > -\alpha > -(\beta+1)$. Hence, $I$ is bounded by $C(z)$.

Next,

$$II = \rho(y)^{-(\beta+1)(\alpha+\sigma)-\sigma} \int\limits_{1\leq|s|\leq\rho(y)^{-(\beta+1)}} |s|^{-1-\alpha-\sigma}\, ds \cdot \int\limits_{\rho(w)\leq C_2} \left| h_z\left(w - \delta_{\rho(y)}^{-1} y\right) - h_z(w)\right| dw .$$

The inner $w$-integral is bounded, since $h_z$ is locally integrable; the outer $s$-integral is bounded whenever $-\alpha < \sigma \leq -\alpha \left[\frac{\beta+1}{\beta+2}\right]$.

*Case* II: $\rho(y) > 1$.

Fubini's theorem, homogeneity of $h_z$, and (4.3-2) together imply that,

$$\begin{aligned}
&\int\limits_{\rho(x)>C\rho(y)} |K_z(x)|\, dx \\
&\leq \int\limits_{\epsilon\leq|t|\leq 1} |t|^{-1-\alpha-\sigma}\, dt \int\limits_{\rho(x)>C\rho(y)} |h_z(x-\Gamma(t)) - h_z(x)|\, dx \\
&= \int\limits_{\epsilon\leq|t|\leq 1} |t|^{-1-\alpha-\sigma}\, dt \\
&\quad \cdot \int\limits_{\rho(x)>C} \left| h_z\left(\delta_{\rho(x)}\left(\delta_{\rho(x)}^{-1} x - \delta_{\rho(x)}^{-1}\Gamma(t)\right)\right) - h_z\left(\delta_{\rho(x)}\left(\delta_{\rho(x)}^{-1} x\right)\right)\right| dx \\
&\leq C(z) \int\limits_{\epsilon\leq|t|\leq 1} |t|^{-1-\alpha-\sigma} |\Gamma(t)|\, dt \int\limits_{\rho(x)>C} \frac{dx}{\rho(x)^{(2\beta+k+1)+\sigma+\beta+1}} \\
&\leq C(z) \int\limits_{\epsilon\leq|t|\leq 1} |t|^{-\alpha-\sigma}\, dt \int\limits_{\mathbf{S}^1} d\varphi \int\limits_{\rho(x)>C} \frac{d\rho(x)}{\rho(x)^{\beta+\sigma+2}}
\end{aligned}$$

since $|\Gamma(t)| = (t^2+t^{2k})^{\frac{1}{2}} \leq \sqrt{2}\,|t|$ for $|t| \leq 1$. The last expression is bounded whenever $-\alpha < \sigma \leq -\alpha \left[\frac{\beta+1}{\beta+2}\right]$. Similarly, $\int\limits_{\rho(x)>C\rho(y)} |K_z(x-y)|\, dx$ is bounded by $C(z)$. This completes the proof of Lemma 4.4. □

This brings us to the final step of the proof of Theorem 1:

**4.1. Interpolation.** Lemma 4.1 shows that, for $f$ simple, $||\mathcal{T}_z^\epsilon f||_2 \leq C_1(z)||f||_2$ whenever $0 < \Re z \leq \frac{1}{2}\left[\frac{\beta}{3} - \alpha\right]$, $\beta > 3\alpha$; Lemma 4.4 shows that $||\mathcal{T}_z^\epsilon f||_p \leq C_2(z)||f||_p$, $1 < p < \infty$, whenever $-\alpha < \Re z \leq -\alpha\left[\frac{\beta+1}{\beta+2}\right] < 0$; each $C_i(z)$ $(i = 1, 2)$ grows at most as fast as a polynomial in $|z|$. It follows that $\{\mathcal{T}_z^\epsilon\}$ is an *admissible analytic family* for the *Stein analytic interpolation theorem* (see [**St**, **We**], page 205), defined for $z$ in the strip

$$S = \left\{ z \in \mathbf{C} : -\alpha\left[\frac{\beta+1}{\beta+2}\right] \leq \Re z \leq \frac{1}{2}\left[\frac{\beta}{3} - \alpha\right]\right\}.$$

Analytic interpolation and *duality* now imply that $\mathcal{T}_0^\epsilon = \mathcal{T}_{\alpha,\beta}^\epsilon$ is bounded on $L^p$ whenever

$$1 + \frac{3\alpha\,(\beta+1)}{\beta\,(\beta+1) + (\beta - 3\alpha)} < p < \frac{\beta\,(\beta+1) + (\beta - 3\alpha)}{3\alpha\,(\beta+1)} + 1,$$

for all *simple* $f$ on $\mathbf{R}^2$. An easy limiting argument shows that $||\mathcal{T}_{\alpha,\beta}^\epsilon f||_p \leq B_{\alpha,\beta}||f||_p$ for all $f \in \mathcal{S}$. The constant $B_{\alpha,\beta}$ is independent of $\epsilon$. Letting $\epsilon \to 0$, Fatou's lemma gives $||\mathcal{T}_{\alpha,\beta} f||_p \leq B_{\alpha,\beta}||f||_p$ for all $f \in \mathcal{S}$. Now, another limiting argument shows that the last inequality holds for all $f \in L^p$. This completes the proof of Theorem 1. □

**Acknowledgements.** I would like to express my deep gratitude to my *guru* Prof. Stephen Wainger for his invaluable guidance during the preparation of this paper. Under his supervision, portions of this paper comprised my *PhD* dissertation at the University of Wisconsin-Madison. I would also like to thank Prof. Alex Nagel, Prof. Dan Shea, and the referee for helpful suggestions.

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Received July 15, 1994 and revised December 7, 1995.

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# PACIFIC JOURNAL OF MATHEMATICS

Volume 175 No. 2 October 1996





0030-8730(1996)175:2;1-B