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## ON SPECTRA OF SIMPLE RANDOM WALKS ON ONE-RELATOR GROUPS

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 WITH AN APPENDIX BY PAUL JOLISSAINT

**For a one relator group  $\Gamma = \langle X : r \rangle$ , we study the spectra of the transition operators  $h_X$  and  $h_S$  associated with the simple random walks on the directed Cayley graph and ordinary Cayley graph of  $\Gamma$  respectively. We show that, generically (in the sense of Gromov), the spectral radius of  $h_X$  is  $(\#X)^{-1/2}$  (which implies that the semi-group generated by  $X$  is free). We give upper bounds on the spectral radii of  $h_X$  and  $h_S$ . Finally, for  $\Gamma$  the fundamental group of a closed Riemann surface of genus  $g \geq 2$  in its standard presentation, we show that the spectrum of  $h_S$  is an interval  $[-r, r]$ , with  $r \leq g^{-1}(2g - 1)^{1/2}$ . Techniques are operator-theoretic.**

### 1. Introduction.

Let  $\Gamma$  be a finitely generated group. Fix a finite, not necessarily symmetric generating subset  $X$ , and let  $S = X \cup X^{-1}$  be the symmetrization of  $X$ . With  $X$  and  $S$  are classically associated the usual Cayley graph  $G(\Gamma, S)$ , but also the Cayley digraph (or directed graph)  $G(\Gamma, X)$ , where the set of vertices is  $\Gamma$  and, for any  $\gamma \in \Gamma$  and  $s \in X$ , an oriented edge is drawn from  $\gamma$  to  $\gamma s$ . We denote by  $\#E$  the number of elements in the set  $E$ .

We consider the normalized adjacency operators, or transition operators,  $h_X$  and  $h_S$ ; these are operators of norm at most 1 on  $l^2(\Gamma)$ , defined by:

$$\begin{aligned} (h_X \xi)(x) &= \frac{1}{\#X} \sum_{s \in X} \xi(xs) \\ (h_S \xi)(x) &= \frac{1}{\#S} \sum_{s \in S} \xi(xs) \quad (\xi \in l^2(\Gamma), x \in \Gamma). \end{aligned}$$

Consider the nearest neighbour simple random walk on  $G(\Gamma, X)$  obtained by assigning probability  $1/(\#X)$  to each neighbour of a given vertex  $\gamma \in \Gamma$  (where a neighbour of  $\gamma$  is the extremity of an oriented edge with origin  $\gamma$ ); then, for any  $x, y \in \Gamma$ , the probability  $p^{(n)}(x, y)$  of a transition in  $n$  steps from  $x$  to  $y$  is given by  $\langle h_X^n \delta_x | \delta_y \rangle$  (where  $(\delta_x)_{x \in \Gamma}$  is the canonical basis of  $l^2(\Gamma)$ ); the

analogous probabilistic interpretation of  $h_S$  is classical. We denote by  $\text{Sp}(T)$  and  $r(T)$  the spectrum and spectral radius of a bounded operator  $T$  on a Hilbert space. That the spectra of  $h_X$  and  $h_S$  capture important information about the pairs  $(\Gamma, X)$  or  $(\Gamma, S)$  follows from the following results of Day and Kesten (see [Day], [Ke1], [Ke2]).

**Theorem 1.1.**

(a) *The following are equivalent:*

- (i)  $r(h_X) = 1$ ;
- (ii)  $1 \in \text{Sp}(h_X)$ ;
- (iii)  $\Gamma$  is amenable.

(b) *Assume  $\#X \geq 2$ ; then  $\frac{\sqrt{2(\#X) - 1}}{\#X} \leq r(h_S)$ , with equality if and only if  $\Gamma$  is isomorphic to the free group  $\mathbb{F}(X)$  on  $X$ ; in this case*

$$\text{Sp}(h_S) = \left[ -\frac{\sqrt{2(\#X) - 1}}{\#X}, \frac{\sqrt{2(\#X) - 1}}{\#X} \right].$$

In passing, we recall that, in the symmetric case,  $r(h_S)$  is the inverse of the radius of convergence of the Green kernel

$$G(x, y; z) = \sum_{n=0}^{\infty} p^{(n)}(x, y) z^n \quad (x, y \in \Gamma).$$

The qualitative study of the spectra of  $h_X$  and  $h_S$  was pursued in [HRV1] and [HRV2], where the following result was proved, with the exception of assertions concerning a symmetric  $X$ , that were obtained by Cartwright [Car] and Kesten [Ke1] respectively.

**Theorem 1.2.**

- (a) *Let  $\mathbb{T}$  be the group of complex numbers of modulus 1; fix  $z \in \mathbb{T}$ . If there exists a character  $\chi : \Gamma \rightarrow \mathbb{T}$  such that  $\chi(x) = z$  for any  $x \in X$ , then  $\text{Sp}(h_X)$  is invariant under multiplication by  $z$ . The converse is true if either  $\Gamma$  is amenable or  $X$  is symmetric (i.e.  $X = X^{-1}$ ).*
- (b) *Assume  $\#X \geq 2$ . Set  $\sigma(X) = \limsup_{k \rightarrow \infty} \|h_X^k\|_2^{1/k}$ , where  $h_X$  is now viewed as the normalized characteristic function of  $X$  and  $h_X^k$  denotes the  $k^{\text{th}}$  convolution power of  $h_X$ . Then*

$$\frac{1}{\sqrt{\#X}} \leq \sigma(X) \leq r(h_X)$$

*with  $\frac{1}{\sqrt{\#X}} = \sigma(X)$  if and only if  $X$  generates a free semi-group, and  $\sigma(X) = r(h_X)$  if either  $X$  is symmetric or  $\Gamma$  is hyperbolic in the sense of Gromov (but not in general).*

- (c) Let  $\Gamma$  be either the free group  $\mathbb{F}(X)$ , with  $\#X \geq 2$ , or the surface group  $\Gamma_g \cong \langle a_1, b_1, \dots, a_g, b_g : \prod_{i=1}^g [a_i, b_i] \rangle$  with  $X = a_1, b_1, \dots, a_g, b_g$  and  $g \geq 2$ ; then  $\text{Sp}(h_X) = \left\{ z \in \mathbb{C} : |z| \leq \frac{1}{\sqrt{\#X}} \right\}$ .

In the case of  $h_S$ , quantitative results on the spectrum were obtained mainly for virtually abelian groups (using Fourier analysis, as in [KeS]) or, at the other extreme, for virtually free groups or groups for which the Cayley graph is tree-like (using methods from combinatorics on trees, see e.g. [CS1], [CS2], [IoP], [KuS], [Mlo]). In the present paper, we deal with one-relator groups, i.e. groups of the form  $\Gamma = \langle X : r \rangle$  where  $r$ , the relator, is a cyclically reduced word in  $\mathbb{F}(X)$ . This class of groups contains the fundamental groups of all compact surfaces (even non-orientable ones), and one-relator groups share a number of interesting properties with surface groups (e.g., it follows from famous results of Lyndon [Lyn] and Stallings [Sta] that a torsion-free one-relator group which is not free must have cohomological dimension 2). To avoid degeneracies, we shall always assume  $\#X \geq 2$  and  $|r| \geq 3$ , i.e. the word length of  $r$  in  $\mathbb{F}(X)$  is at least 3.

Here is a summary of our results:

- (a) We propose the statistical result that “most” presentations  $\Gamma = \langle X : r \rangle$  give  $r(h_X) = \frac{1}{\sqrt{\#X}}$  (which implies in particular that the semi-group generated by  $X$  in  $\Gamma$  is free). More precisely, we prove that the ratio

$$\frac{\#\{\text{presentation } r \text{ with } r(h_X) = (\#X)^{-1/2} \text{ and } |r| = N\}}{\#\{\text{presentation } r \text{ with } |r| = N\}}$$

tends (exponentially fast) to 1 when  $N$  tends to  $+\infty$ . This is exactly the sense of genericity introduced by Gromov ([Gro], 0.2(A)), and studied further by Champetier [Ch2].

- (b) Let  $H_r$  (resp.  $H_l$ ) be the subgroup of  $\mathbb{F}(X)$  generated by all quotients  $xy^{-1}$ , with  $x, y \in X$  (resp.  $x^{-1}y$ , with  $x, y \in X$ ). Suppose that  $r$  is not in the union  $H_r \cup H_l$ ; then  $\|h_X\| = \frac{2\sqrt{\#X-1}}{\#X}$  and  $\max\{r(h_X), r(h_S)\} \leq \frac{2\sqrt{\#X-1}}{\#X}$ .

- (c) If  $r$  is in the exceptional set  $H_r \cup H_l$ , then  $r(h_X) = \frac{1}{\sqrt{\#X}}$  and  $\text{Sp}(h_X)$  is a union of concentric circles centered at 0; this is proved using Jolis-saint’s result from the Appendix.

- (d) Without restriction on  $r$  but with  $\#X \geq 4$ , we have

$$\max\{r(h_X), r(h_S)\} \leq \frac{2\sqrt{\#X - 2 + 1}}{\#X} < 1.$$

- (e) For the surface group  $\Gamma_g$  with  $g \geq 2$ , Peter Sarnak asked for the exact value of  $r(h_S)$  in terms of the genus  $g$ . We prove that  $\text{Sp}(h_S)$  is an interval  $[-r, r]$  with the non-trivial estimate  $r \leq \frac{\sqrt{2g - 1}}{g}$ .

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### 2. Rotational symmetries of spectra.

As is well-known, existence of homomorphisms is the easiest thing to check in the case of a finitely presented group. We exemplify this in the case of a one-relator group  $\Gamma = \langle X : r \rangle$ ; denote by  $\Sigma$  the sum of all exponents in  $r$  and fix  $z \in \mathbb{T}$ ; then:

- For  $z$  a primitive  $d$ -th root of 1, there exists a character  $\chi : \Gamma \rightarrow \mathbb{T}$  such that  $\chi(x) = z$  for any  $x \in X$  if and only if  $\Sigma \equiv 0 \pmod{d}$ ;
- for  $z$  not a root of 1, there exists a character  $\chi : \Gamma \rightarrow \mathbb{T}$  such that  $\chi(x) = z$  for any  $x \in X$  if and only if  $\Sigma = 0$ .

From this and Theorem 1.2 above, we immediately deduce:

**Proposition 2.1.**

- (a) If  $\Sigma \equiv 0 \pmod{d}$ , then  $\text{Sp}(h_X)$  is invariant under multiplication by  $\exp(2\pi i/d)$ ;
- (b) If  $\Sigma = 0$ , then  $\text{Sp}(h_X)$  is a union of concentric circles, centered at 0;
- (c)  $\text{Sp}(h_S)$  is symmetric with respect to 0 if and only if  $\Sigma$  is even.

### 3. Free semi-groups and small cancellation.

**Definition 3.1.** A word  $w \in \mathbb{F}(X)$  is **positive** if it involves only generators with positive exponents. Any non-empty reduced word  $r \in \mathbb{F}(X)$  can be written in a unique way as a product without cancellation, either  $r = w_1 w_2^{-1} w_3 \cdots w_n^{\pm 1}$  or  $r = w_1^{-1} w_2 w_3^{-1} \cdots w_n^{\pm 1}$ , where the  $w_i$ 's are positive words. We say that  $r$  **alternates enough** if  $n \geq 4$ , i.e. there are at least 3 changes of signs in the exponents of  $r$ .

**Lemma 3.2.** *Each of the following statements implies the next one:*

- (i)  $r(h_X) = \frac{1}{\sqrt{\#X}}$  ;

- (ii)  $X$  generates a free semi-group;
- (iii) the relator  $r$  alternates enough.

*Proof.* i)  $\Rightarrow$  ii) follows immediately from Theorem 1.2. To show ii)  $\Rightarrow$  iii), we assume that  $r$  does not alternate enough and prove that  $X$  does not generate a free semi-group. There are 3 cases to consider.

- (a)  $r$  has no change of signs in its exponents, i.e.  $r$  or  $r^{-1}$  is a positive word; then we have a positive word that represents the identity in  $\Gamma$ ;
- (b)  $r$  has exactly one change of sign, say  $r = w_1 w_2^{-1}$ , with  $w_1, w_2$  positive, distinct words; then  $w_1$  and  $w_2$  represent the same element in the semi-group generated by  $X$  in  $\Gamma$ ;
- (c)  $r$  has exactly two changes of sign, i.e.  $r$  or  $r^{-1}$  is of the form  $w_1 w_2^{-1} w_3$ , with  $w_1, w_2, w_3$  positive words; then by cyclically permuting we get  $w_3 w_1 w_2^{-1}$ , i.e. we are back to the preceding case.

□

*Example 3.3.* We give an example showing that the converse implication ii)  $\Rightarrow$  i) does not hold in general. It seems that this example was known to Y. Guivarc'h (private communication). Consider the one-relator group  $\Gamma = \langle y, z : yzy^{-1}z^{-1}yz^{-1} \rangle$ . We claim that, for  $X = \{y, z\}$ , we have  $r(h_X) = 1$  and  $X$  generates a free semi-group. To see it, set  $x = zy^{-1}$ ; in the generators  $x, y$ , the group  $\Gamma$  has the famous presentation

$$\Gamma = \langle x, y : yxy^{-1}x^{-2} \rangle$$

( $\Gamma$  is the first Baumslag-Solitar group).  $\Gamma$  is solvable, hence amenable, thus  $r(h_X) = 1$ . Let  $H$  be the subgroup of  $\Gamma$  generated by  $x$ . The relation

$$(*) \quad yxy^{-1} = x^2$$

exhibits  $\Gamma$  as an HNN-extension of  $H$  with respect to the monomorphism  $\Theta : H \rightarrow H$  such that  $x^k \rightarrow x^{2k}$ . Therefore,  $\Gamma$  acts on a tree  $T$ , whose construction we now recall (see [Ser], I.1.4, I.5.1). The homogeneous space  $\Gamma/H$  will be both the set of vertices and the set of edges of  $T$ : We define the extremity of the edge  $\gamma H$  as the vertex  $\gamma H$ , and the origin of  $\gamma H$  as the vertex  $\gamma y^{-1} H$ ; it follows from relation (\*) that this is well-defined. The resulting tree  $T$  is the homogeneous tree of degree 3 with, at each vertex, one incoming edge and two outgoing edges. We call descendant of order  $n$  of the vertex  $H$  any of the  $2^n$  vertices at distance  $n$  from  $H$  that can be reached from  $H$  by a positively oriented path. To prove that  $y$  and  $z$  generate a free semi-group in  $\Gamma$ , it suffices to prove the following

*Claim.* Any descendant of order  $n$  of  $H$  can be written as  $wH$ , where  $w$  is a positive word of length  $n$  in  $y$  and  $z$ .

Note that this writing is necessarily unique, since there are  $2^n$  positive words of length  $n$  in  $y$  and  $z$ . We prove the claim by induction over  $n$ , the case  $n = 0$  being obvious. So, let  $\gamma H$  be a descendant of order  $n + 1$  of  $H$ ; then  $\gamma y^{-1}H$  is a descendant of order  $n$  of  $H$ , so by the induction assumption we have  $\gamma y^{-1}H = wH$  for some positive word  $w$  of length  $n$  in  $y$  and  $z$ . Thus  $w = \gamma y^{-1}x^k$  for some  $k \in \mathbb{Z}$ . If  $k$  is even, we have  $w = \gamma x^{k/2}y^{-1}$ , i.e.  $\gamma H = wyH$ ; if  $k$  is odd, we have  $w = \gamma x^{(k+1)/2}y^{-1}x^{-1}$ , i.e.  $\gamma H = wzH$ . Both  $wy$  and  $wz$  are positive words of length  $n + 1$  in  $y$  and  $z$ .

*Example 3.4.* In Lemma 3.2, the converse implication iii)  $\Rightarrow$  ii) does not hold either. To see it, let  $n \geq 1$  be an integer, and consider the group  $\Gamma = \langle a, b : a(ab^{-1})^{n+1} \rangle$ . The relator  $r$  alternates enough. Set  $r' = (ab^{-1})^n a^2 b^{-1}$ , a cyclic permutation of  $r$ ; then  $r^{-1}ar'a^{-1} = bab^{-1}a^{-1}$ , so that  $X$  does not generate a free semi-group, since  $ab = ba$ . (Actually, this also shows that  $\Gamma$  is a quotient of  $\mathbb{Z}^2$ ; since the vector  $(n + 2, -n - 1)$  is primitive in  $\mathbb{Z}^2$ , one checks easily that  $\Gamma$  is isomorphic to  $\mathbb{Z}$ .)

This example typically displays absence of small cancellation, whose definition we recall now (see e.g. [LyS] for an extensive study).

**Definition 3.5.** Let  $\Gamma = \langle X : r \rangle$  be a one-relator group. Denote by  $R$  the set of words obtained by cyclic permutations of  $r$  and  $r^{-1}$ . A **piece** is a prefix  $u$  which is common to two distinct elements of  $R$  (by prefix we mean any not empty initial part of a word; a word is a prefix of itself). Fix  $\lambda \in ]0, 1[$ . We say that  $r$  satisfies **the small cancellation condition**  $C'(\lambda)$  if, for any piece  $u$ , one has:

$$|u| < \lambda|r|.$$

**Definition 3.6.** A one-relator group  $\Gamma = \langle X : r \rangle$  satisfies a **Dehn's algorithm** if, for any reduced word  $w \in \mathbb{F}(X)$  that represents 1 in  $\Gamma$ , there exists a prefix  $u$  of some word in  $R$  such that  $u$  is a subword of  $w$  and  $|u| > \frac{1}{2}|r|$ .

It is known that groups satisfying the small cancellation property  $C'(\lambda)$ , with  $\lambda \leq \frac{1}{6}$ , also satisfy a Dehn's algorithm (see [LyS], Theorem 4.4 of Chapter V; [Str], Theorem 25). On the other hand, by a result of Gromov, groups with a Dehn's algorithm are hyperbolic ([Gro], Theorem 2.3.D; see also [Str], Theorem 36 for a direct proof that  $C'(1/6)$ -groups are hyperbolic).

After these standard definitions, here is another one of our own.

Let  $\Gamma = \langle X : r \rangle$  be a one relator-group presentation. For any  $r' \in R$ , express  $r'$  as a reduced product in  $\mathbb{F}(X)$  either  $r' = w_1 w_2^{-1} w_3 \cdots w_n^{\pm 1}$  or  $r' = w_1^{-1} w_2 w_3^{-1} \cdots w_n^{\pm 1}$ , with the  $w_i$ 's positive words.

**Definition 3.7.** The presentation  $\Gamma = \langle X : r \rangle$  is **balanced** if one has  $|w_i| \leq \frac{|r|}{4}$  for  $i = 1, \dots, n$ .

Clearly, a balanced presentation alternates enough.

We are now in position to present a class of one-relator presentations for which the three conditions of Lemma 3.2 are equivalent.

**Lemma 3.8.** *Suppose that the presentation  $\Gamma = \langle X : r \rangle$  is balanced and satisfies a Dehn's algorithm (this latter assumption being verified if the presentation satisfies condition  $C'(1/6)$ ). Then  $r(h_X) = \frac{1}{\sqrt{\#X}}$ .*

*Proof.* We first show that  $X$  generates a free semi-group in  $\Gamma$ . Let  $N$  be the normal subgroup generated by  $r$  in  $\mathbb{F}(X)$  ( $N$  is the set of consequences of the relation  $r$ ). Fix  $w \in N$ , a reduced word. Thanks to the Dehn's algorithm, we find a subword  $u$  of  $w$  which is also a prefix of some  $r' \in R$ , with  $|u| > \frac{|r|}{2}$ . Because the presentation is balanced, we see that  $u$ , and a fortiori  $w$ , must contain at least 2 changes of sign in their exponents. Now, let  $v_1, v_2$  be distinct positive words. Since  $v_1 v_2^{-1}$  has exactly one change of sign in its exponents, we see that  $v_1 v_2^{-1}$  does not belong to  $N$ , i.e.  $v_1$  is distinct from  $v_2$  in  $\Gamma$ . This shows that the semi-group generated by  $X$  in  $\Gamma$  is free<sup>1</sup>. Since  $\Gamma$  is hyperbolic, Theorem 1.2 applies to give  $r(h_X) = \frac{1}{\sqrt{\#X}}$ .  $\square$

### Remarks.

- (1) It is stated in Theorem 3 of [New] (for a proof, see [LyS], Theorem 5.5 of Chapter IV) that a one-relator group with torsion satisfies a Dehn's algorithm, and hence is hyperbolic.
- (2) Lemma 3.8 and its proof naturally raise the question: Which one-relator groups are hyperbolic? It is conjectured that a one-relator group is hyperbolic if and only if every non-identity element has a cyclic centralizer (Conjecture 2 in [Juh]).

The following definition is basically due to Gromov ([Gro], 0.2(A)) and was made precise by Champetier ([Ch2]; this paper also contains many impressive results on "generic" properties).

**Definition 3.9.** Let  $\#X \geq 2$  be fixed. For any integer  $N \geq 1$ , denote by  $C(N)$  the number of cyclically reduced words of length  $N$  in  $\mathbb{F}(X)$ . Let  $(P)$  be a property of one-relator presentations. We say that  $(P)$  is **asymptotically almost sure** if the ratio

$$\frac{\#\{\text{presentation } \langle X : r \rangle \text{ with property } (P) \text{ and } |r| = N\}}{C(N)}$$

<sup>1</sup>On p. 100 of [HRV2], it was stated without proof that, in the surface group  $\Gamma_g$  ( $g \geq 2$ ) with the standard presentation,  $X$  generates a free semi-group. This provides a proof of this statement.

tends to 1 for  $N$  tending to  $+\infty$ .

**Lemma 3.10.** *Fix  $\lambda \in ]0, 1[$ . Condition  $C'(\lambda)$  is asymptotically almost sure.*

*Proof.* See [Ch2], Lemma 4.4. Note that the proof reveals in this case that, for  $N \rightarrow \infty$ , the convergence of the above ratio to 1 is exponentially fast.  $\square$

**Lemma 3.11.** *A one-relator presentation is asymptotically almost surely balanced.*

*Proof.* Set  $\#X = k$  for simplicity. First, notice that  $C(N)$  is not smaller than the number of reduced words of length  $N$  in  $\mathbb{F}(X)$  whose last letter is not the inverse of the first one, i.e.

$$(1) \quad C(N) \geq 2k(2k - 1)^{N-2}(2k - 2).$$

Now, we estimate the number  $B(N)$  of “bad” presentations, i.e those presentations  $\langle X : r \rangle$  such that there exists  $r' \in R$  beginning with a positive subword of length larger than  $N/4$ . Since there are at most  $2N$  elements in  $R$ , we have

$$B(N) \leq 2N \sum_{l=[N/4]+1}^N C(N, l)$$

where  $C(N, l)$  is the number of cyclically reduced words of length  $N$  beginning with a positive subword of length exactly  $l$ . Thus we certainly have:

$$(2) \quad B(N) \leq 2N \sum_{l=[N/4]+1}^N k^l(2k - 1)^{N-l}.$$

Dividing (2) by (1), we estimate the proportion of non-balanced presentations:

$$\begin{aligned} \frac{B(N)}{C(N)} &\leq \frac{N(2k - 1)^2}{2k(k - 1)} \sum_{l=[N/4]+1}^N k^l(2k - 1)^{-l} \\ &= \frac{N(2k - 1)^2}{2k(k - 1)} \frac{k^{[n/4]+1}(2k - 1)^{-[n/4]-1} - k^{N+1}(2k - 1)^{-N-1}}{1 - k(2k - 1)^{-1}}. \end{aligned}$$

Since  $k \geq 2$ , this ratio tends exponentially fast to 0 for  $N \rightarrow \infty$ .  $\square$

From this, we deduce:

**Theorem 3.12.** *Let  $\#X \geq 2$  be fixed. A presentation  $\Gamma = \langle X : r \rangle$  has asymptotically almost surely  $r(h_X) = \frac{1}{\sqrt{\#X}}$ .*

*Proof.* We combine Lemma 3.10 (with  $\lambda = 1/6$ ) and Lemma 3.11, and use the fact that the conjunction of two asymptotically almost sure properties is asymptotically almost sure. Thus, asymptotically almost surely, a one-relator presentation is balanced and satisfies  $C'(1/6)$ , so also satisfies  $r(h_X) = \frac{1}{\sqrt{\#X}}$ , by Lemma 3.8.  $\square$

**Remark.** Fix an integer  $k \geq 1$ . Let  $\Gamma_n = \langle X_n : r_n \rangle$  be a sequence of one-relator groups on  $k$  generators, with  $|r_n|$  tending to infinity for  $n \rightarrow \infty$ . Set  $S_n = X_n \cup X_n^{-1}$ . It was proved by Grigorchuk [Gri] (and recently reproved by Champetier [Ch1]) that, if all  $\Gamma_n$ 's satisfy the small cancellation condition  $C'(\lambda)$ , with  $\lambda < 1/6$ , then

$$\lim_{n \rightarrow \infty} r(h_{S_n}) = \frac{\sqrt{2(\#X) - 1}}{\#X}.$$

This corresponds to the intuitive idea that, as  $|r_n|$  becomes larger, the Cayley graph of  $\Gamma_n$  looks more and more like a tree.

#### 4. Estimates on norms and spectral radii.

First, we recall that, for any group  $\Gamma$ , the **right regular representation** is the representation  $\rho$  of  $\Gamma$  on  $l^2(\Gamma)$  defined by:

$$(\rho(g)\xi)(h) = \xi(hg) \quad (\xi \in l^2(\Gamma), g, h \in \Gamma).$$

If  $X$  is a finite generating subset of  $\Gamma$  and  $S = X \cup X^{-1}$ , our transition operators  $h_X, h_S$  may be expressed simply in terms of  $\rho$  as:

$$h_X = \frac{1}{\#X} \sum_{s \in X} \rho(s);$$

$$h_S = \frac{1}{\#S} \sum_{s \in S} \rho(s).$$

We shall need the following result of Akemann-Ostrand [AkO] (see also [Woe]).

**Lemma 4.1.** *Let  $x_1, x_2, \dots, x_n$  be elements of  $\Gamma$  that generate a free subgroup on  $n$  generators. Then*

$$\left\| \sum_{i=1}^n \rho(x_i) \right\| = 2\sqrt{n-1};$$

$$\left\| 1 + \sum_{i=1}^n \rho(x_i) \right\| = 2\sqrt{n}.$$

With this we may estimate the norm of  $h_X$ .

**Proposition 4.2.** *Let  $\Gamma = \langle X : r \rangle$  be a one-relator group, with  $\#X \geq 4$ ,  $|r| > 2$  and  $r$  cyclically reduced. Then*

$$\max\{r(h_X), r(h_S)\} \leq \|h_X\| \leq \frac{2\sqrt{\#X - 2} + 1}{\#X} < 1.$$

*Proof.* The inequality  $r(h_X) \leq \|h_X\|$  holds for any bounded operator. Now, since  $|r| > 2$  and  $r$  is cyclically reduced, the intersection  $X \cap X^{-1}$  is empty, so

$$r(h_S) = \|h_S\| = \left\| \frac{h_X + h_X^*}{2} \right\| \leq \|h_X\|,$$

where the first equality holds for any bounded self adjoint operator. This proves the first inequality in the statement. To prove the second, set  $X = \{x_1, \dots, x_k\}$ ; without loss of generality, we may assume that  $x_k$  appears in the relator  $r$ . Then

$$\|h_X\| \leq \frac{1}{k} \left( \left\| \sum_{i=1}^{k-1} \rho(x_i) \right\| + 1 \right).$$

Now, by Magnus' Freiheitssatz (see [LyS], Proposition 5.1 of Chapter II), the subgroup of  $\Gamma$  generated by  $x_1, \dots, x_{k-1}$  is free on  $k - 1$  generators, so Lemma 4.1 applies. Finally, the expression is  $1/k(2\sqrt{k-2} + 1)$  less than 1 provided  $k \geq 4$ . □

**Remark.** It was observed in Proposition 4 (iv) of [HRV2] that, for  $\#X = 2$ , one always has  $\|h_X\| = 1$ .

*Example 4.3.* For  $\#X = 3$ , Proposition 4.2 just gives the obvious bound  $\|h_X\| \leq 1$ . The following example shows that we cannot expect better in this case. Indeed, consider the group  $\Gamma = \langle a, b, c : [ac^{-1}, bc^{-1}] \rangle$ . Then, factoring out  $\rho(c)$  on the right, we get:

$$\|h_X\| = \frac{1}{3} \|\rho(ac^{-1}) + \rho(bc^{-1}) + 1\|.$$

Now,  $ac^{-1}$  and  $bc^{-1}$  commute and actually they generate a subgroup  $H$  isomorphic to  $\mathbb{Z}^2$ . Under Fourier transform  $C_r^*(H)$  is isometrically isomorphic to  $C(\mathbb{T}^2)$ , the  $C^*$ -algebra of continuous functions on the 2-torus

$\mathbb{T}^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2| = 1\}$ , the norm on  $C(\mathbb{T}^2)$  being the sup-norm. Thus

$$\|\rho(ac^{-1}) + \rho(bc^{-1}) + 1\| = \sup_{(z_1, z_2) \in \mathbb{T}^2} |z_1 + z_2 + 1| = 3.$$

We are now going to improve the bound of Proposition 4.2, and drop the assumption  $\#X \geq 4$ . For that, we shall have to exclude some presentations (like the one in Example 4.3). First we fix some notations. Let  $\Gamma = \langle X : r \rangle$  be a one-relator group; fix  $y \in X$ . We define a new generating subset  $X_y$  as

$$X_y = \{xy^{-1} : x \in X, x \neq y\} \cup \{y\};$$

note that  $\#X_y = \#X$ . Now, let  $r_y$  be the word  $r$  written in the alphabet  $X_y$ ; more precisely, if we set  $x' = xy^{-1}$  for  $x \in X, x \neq y$ , the word  $r_y$  is obtained from  $r$  by the change of variables (Nielsen transformation)

$$T_y : \begin{cases} x \rightarrow x'y & (x \neq y) \\ y \rightarrow y \end{cases}.$$

Then we define  $r'_y$  as the word obtained from  $r_y$  by cyclically reducing.

**Lemma 4.4.** *If  $r$  is a cyclically reduced word in  $\mathbb{F}(X)$ , then  $y$  is the only element that may disappear in  $r$  when  $T_y$  is applied. More precisely, if we denote by  $(\alpha_1, \dots, \alpha_l)$  the ordered set of elements in  $X \cup X^{-1} - \{y, y^{-1}\}$  appearing in  $r$  (i.e.  $r = y^{\nu_1} \alpha_1 y^{\nu_2} \alpha_2 y^{\nu_3} \dots y^{\nu_l} \alpha_l y^{\nu_{l+1}}$  where  $\nu_i \in \mathbb{Z}$ ), then the ordered set in  $r_y$  and hence in  $r'_y$  is  $(\alpha'_1, \dots, \alpha'_l)$ .*

*Proof.* We view  $T_y$  as an isomorphism from  $\mathbb{F}(X)$  to  $\mathbb{F}(X_y)$ . Note that  $T_y^{-1}$  is defined on the generators of  $\mathbb{F}(X_y)$  by  $T_y^{-1}(x') = xy^{-1}$  for  $x' \in X_y - \{y\}$  and  $T_y^{-1}(y) = y$ . Then for  $r = y^{\nu_1} \alpha_1 y^{\nu_2} \alpha_2 y^{\nu_3} \dots y^{\nu_l} \alpha_l y^{\nu_{l+1}}$ :

$$\begin{aligned} r_y &= T_y(r) = T_y(y^{\nu_1} \alpha_1 y^{\nu_2} \alpha_2 y^{\nu_3} \dots y^{\nu_l} \alpha_l y^{\nu_{l+1}}) \\ &= T_y(y^{\nu_1}) T_y(\alpha_1) T_y(y^{\nu_2}) T_y(\alpha_2) \dots T_y(y^{\nu_l}) T_y(\alpha_l) T_y(y^{\nu_{l+1}}) \\ &= y^{\mu_1} \alpha'_1 y^{\mu_2} \alpha'_2 y^{\mu_3} \dots y^{\mu_l} \alpha'_l y^{\mu_{l+1}}. \end{aligned}$$

Suppose that  $\alpha'_i$  and  $\alpha'_{i+1}$  cancel out in  $r_y$ . Then applying  $T_y^{-1}$ , we see that  $r$  cannot contain  $\alpha_i$  and  $\alpha_{i+1}$ , a contradiction. So the ordered set of  $r_y$  is exactly  $(\alpha'_1, \dots, \alpha'_l)$ .

As  $r$  is cyclically reduced, by the same argument, we conclude that we cannot cancel  $\alpha'_1$  and  $\alpha'_l$  by cyclic permutation of  $r_y$ . So  $(\alpha'_1, \dots, \alpha'_l)$  is also the ordered set of  $r'_y$ . That concludes the proof of 4.4.  $\square$

Recall from the introduction that we defined a subgroup  $H_r$  (resp.  $H_l$ ) of  $\mathbb{F}(X)$  as the subgroup generated by all right quotients  $xy^{-1}$ , with  $x, y \in X$

(resp. all left quotients  $x^{-1}y$ , with  $x, y \in X$ ); in passing, notice that  $H_r$  and  $H_l$  are free on  $(\#X) - 1$  generators.

**Lemma 4.5.** *For  $r$  cyclically reduced in  $\mathbb{F}(X)$ , the following are equivalent:*

- (i)  $r \in H_r \cup H_l$ ;
- (ii) For any  $y \in X$ , the letter  $y$  does not appear in  $r'_y$  ;
- (iii) There exists  $y \in X$  such that  $y$  does not appear in  $r'_y$ .

*Proof.* (i)  $\Rightarrow$  (ii) Suppose that  $r$  is in  $H_r$ ; write  $r = \prod_{i=1}^n a_i b_i^{-1}$  with  $a_i, b_i \in X$ .

We can have three kinds of factors  $a_i b_i^{-1}$ :

- (1)  $a, b \in X - \{y\}$ : then  $T_y(ab^{-1}) = a'yy^{-1}(b')^{-1} = a'(b')^{-1}$
- (2)  $T_y(ay^{-1}) = a'yy^{-1} = a'$
- (3)  $T_y(ya^{-1}) = y(a'y)^{-1} = (a')^{-1}$ .

Thus  $r_y$  does not contain  $y$  and so does  $r'_y$ .

If  $r$  is in  $H_l$  then the element  $s = yry^{-1}$  belongs to  $H_l$ . So  $y$  does not appear in  $s_y$ ,  $s_y$  is cyclically reduced (because  $r$  is cyclically reduced and Lemma 4.4) and  $s_y = r'_y$ .

(ii)  $\Rightarrow$  (iii) is obvious.

(iii)  $\Rightarrow$  (i) We assume that  $y$  does not appear in  $r'_y$  and analyse  $r$  in several steps. Write  $r = a_1 \cdots a_n$ , a word on  $X \cup X^{-1}$ , and look at the ordered subset  $(a_{i_1}, \dots, a_{i_l})$  of all letters in  $r$  from  $X \cup X^{-1} - \{y, y^{-1}\}$ . By Lemma 4.4, the corresponding ordered set in  $r'_y$  is  $(a'_{i_1}, \dots, a'_{i_l})$ . We will systematically use the following argument: Suppose that, after applying  $T_y$  to each letter of  $r$ , we find a word  $r_y$  in which  $y$  appears and there is no obvious cancellation to remove it: Then it is really impossible to remove  $y$ , because it would be necessary first to remove some  $a'_{i_j}$ , contradicting Lemma 4.4.

First step:  $r$  has no subword of the form  $ab$  or  $a^{-1}b^{-1}$ , with  $a, b \in X - \{y\}$ . Indeed, if this would be the case, by applying  $T_y$ , we would get either  $a'yb'$  or  $(a')^{-1}y^{-1}(b')^{-1}$  and  $y$  would appear in  $r'_y$ .

2nd Step:  $r$  does not contains  $y^2$  or  $y^{-2}$ . To see this, we suppose by contradiction that  $r$  contains such a subword, and show that  $r'_y$  contains  $y$  or  $y^{-1}$ . There are three cases to consider.

- (1)  $r$  contains  $a^{\pm 1}y^n b^{\pm 1}$  with  $n \geq 2$ , ( $a \neq y \neq b$ ). If the exponents of  $a$  and  $b$  are positive, then  $ay^n b$  becomes  $a'y^{n+1}b'$  after  $T_y$ . If the exponents of  $a$  and  $b$  are different,  $ay^n b^{-1}$  becomes  $a'yy^n y^{-1}(b')^{-1} = a'y^n (b')^{-1}$  and  $a^{-1}y^n b$  becomes  $(a')^{-1}y^n b'$ . Finally if the exponents of  $a$  and  $b$  are negative,  $a^{-1}y^n b^{-1}$  becomes  $(a')^{-1}y^{n-1}(b')^{-1}$ . The same can be done for  $n$  negative with  $|n| \geq 2$ .
- (2)  $r$  begins with  $y^n b^{\pm 1}$  ( $b \in X - \{y\}$ ). First assume  $n \geq 2$ . Since  $r$  is cyclically reduced,  $r$  ends with some letter  $a \in X \cup X^{-1} - \{y^{-1}\}$ . Then

$r_y = T_y(r)$  cannot end with  $y^{-1}$ , so there will be no cancellation when cyclically reducing  $r_y$  to get  $r'_y$ . On other hand, if  $r$  begins with  $y^n b$ , then  $r_y$  begin with  $y^n b'$ , and if  $r$  begins with  $y^n b^{-1}$ , then  $r_y$  begin with  $y^{n-1}(b')^{-1}$ . Since  $n \geq 2$ ,  $y$  appears in  $r'_y$ . The same can be done for  $n \leq -2$ .

(3) Similar arguments hold if  $r$  ends with  $b^{\pm 1}y^n$  ( $b \in X - \{y\}$ ,  $|n| \geq 2$ ).

Note that cases 2 and 3 also show that  $r$  cannot begin with  $yb$  or  $y^{-1}b^{-1}$ , neither end with  $by$  or  $b^{-1}y^{-1}$  ( $b \in X - \{y\}$ ).

3rd step: All exponents in  $r$  are equal to  $\pm 1$ , and exponents alternate in sign, i.e.  $r = a_1^{-1}a_2a_3^{-1} \cdots a_n^{\pm 1}$  or  $r = a_1a_2^{-1}a_3 \cdots a_n^{\pm 1}$ . Indeed, we have to show that  $r$  contains no subword of the form  $ab$  or  $a^{-1}b^{-1}$ , for  $a, b \in X$ . We already know that this holds if either  $a, b \in X - \{y\}$  (first step) or  $a = b = y$  (second step). It remains to show that  $r$  contains no subword of the form  $ay$  or  $yb$  ( $a, b \in X - \{y\}$ ), or an inverse of these. The remark at the end of the second step already shows that  $r$  cannot begin or end with such a subword. If  $ayb^{\pm 1}$  appears then we see that after applying  $T_y$ ,  $ayb^{\pm 1}$  becomes either  $a'yyb'y$  or  $a'yyy^{-1}(b')^{-1} = a'y(b')^{-1}$ . Similar arguments holds for  $yb, a^{-1}y^{-1}, y^{-1}b^{-1}$ .

Final step: To see that  $r$  is in  $H_r \cup H_l$ , we have to see that the exponent of  $a_n$  is the opposite of the exponent of  $a_1$ . By contradiction suppose that  $a_1$  and  $a_n$  have the same exponent  $+1$  (resp.  $-1$ ). Then  $r = a_1a_2^{-1}a_3 \cdots a_n$  becomes, after applying  $T_y$ ,  $a'_1(a'_2)^{-1}a'_3 \cdots a'_ny$  so  $y$  appear in  $r'_y$  (because  $a_1 \neq y^{-1}$ ). The argument for  $r = a_1^{-1}a_2a_3^{-1} \cdots a_n^{-1}$  is similar. That ends the proof. □

**Remark.** The group  $\Gamma = \langle X : r \rangle$  clearly also admits the presentation  $\Gamma = \langle X_y : r'_y \rangle$ . If  $r \in H_r \cup H_l$ , Lemma 4.5 reveals that  $\Gamma$  is the free product of  $\mathbb{Z} = \langle y \rangle$  with the one-relator group  $\Gamma_y = \langle X_y - y : r'_y \rangle$ . Now Shenitzer [She] has characterized those presentations  $\Gamma = \langle X : r \rangle$  such that  $\Gamma$  is isomorphic to the free product of  $\mathbb{Z}$  with another group; the criterion is that at least one generator of  $X$  must disappear from  $r$  by applying Nielsen transformations. Our Lemma 4.5 however does not seem to be a consequence of the result in [She] because we only consider very special Nielsen transformations, namely the  $T_y$ 's.

**Theorem 4.6.** *Let  $\Gamma = \langle X : r \rangle$  be a one-relator group, with  $\#X \geq 2$ ,  $|r| > 2$ ,  $r$  cyclically reduced and  $r \notin H_r \cup H_l$ . Then:*

$$\max\{r(h_x), r(h_s)\} \leq \|h_x\| = \frac{2\sqrt{\#X - 1}}{\#X}.$$

*Proof.* The inequality is proved as in Proposition 4.2. Fix  $y \in X$ ; then

$$\|h_X\| = \frac{1}{\#X} \left\| 1 + \sum_{x \in X - \{y\}} \rho(xy^{-1}) \right\| = \frac{1}{\#X} \left\| 1 + \sum_{x' \in X_y - \{y\}} \rho(x') \right\|.$$

Since  $r$  is not in  $H_r \cup H_l$ , it follows from Lemma 4.5 that  $y$  appears in  $r'_y$ ; again by Magnus' Freiheitssatz,  $X_y - \{y\}$  freely generates a free group on  $(\#X) - 1$  generators; Lemma 4.1 then applies to give the result.

Note that Example 4.3 above of a presentation with  $r \in H_r, \#X = 3$  and  $\|h_X\| = 1$  shows that the assumption on  $r$  in Theorem 4.6 cannot be dropped. We now discuss somewhat the ‘‘exceptional’’ presentations in  $H_r \cup H_l$ . We choose to work with  $H_r$  (the analogous results for  $H_l$  following by interchanging left and right). □

**Proposition 4.7.** *Let  $\Gamma = \langle X : r \rangle$  be a one-relator presentation, with  $r \in H_r$ . Then  $X$  generates a free semi-group in  $\Gamma$  and  $r(h_X) = \frac{1}{\sqrt{\#X}}$ . Moreover  $\text{Sp}(h_X)$  is a union of concentric circles centered at 0.*

*Proof.* Fix  $y \in X$ ; as mentioned in the remark following Lemma 4.5,  $\Gamma$  is the free product of  $\mathbb{Z} = \langle y \rangle$  with the one-relator group  $\Gamma_y = \langle X_y - y : r_y \rangle$ . We first prove that  $X$  generates a free semi-group. So, let  $w_1, w_2$  be two distinct positive words in  $\mathbb{F}(X)$ ; using the change of variables  $T_y$  together with the normal form for elements in a free product, we see that  $w_1$  and  $w_2$  define distinct elements of  $\Gamma$ . It follows from Theorem 1.2 that  $\sigma(X) = \frac{1}{\sqrt{\#X}} \leq r(h_X)$ . To prove the converse inequality  $r(h_X) \leq \sigma(X)$ , we appeal to Jolissaint’s result from the Appendix: there exists a constant  $C > 0$  such that, for any integer  $k \geq 0$ :

$$\|h_X^k\| \leq C(1+k)^3 \|h_X^k\|_2.$$

The desired inequality follows then straight from the definition of  $\sigma(X)$ . The final assertion follows from Proposition 2.1 by noticing that the sum of all exponents in  $r$  is 0. □

**Remarks.**

- (1) In the case of  $h_S$ , it would of course be desirable to find an upper bound on  $r(h_S)$  that depends on the relator  $r$  (for example on the length of  $r$ ); but we did not succeed in achieving that. Note that such a lower bound for  $r(h_S)$  was recently obtained by Paschke [**Pas**]: one has

$$r(h_S) \geq \min_{s>0} \left\{ \cosh(s) + (\#X - 1) Q \left( \frac{\cosh(|r|s) + 1}{\text{sh}(s) \text{sh}(|r|s)} \right) \right\}$$

where  $Q(t) = \frac{\sqrt{t^2 + 1} - 1}{t}$ .

- (2) Proposition 4.2 and Theorem 4.6 above show that one-relator groups are, in a certain sense, uniformly non-amenable: if we fix the number of generators, then the spectral radius of  $h_S$  is uniformly bounded away from 1. Since Proposition 4.2 and Theorem 4.6 provide upper bounds on  $r(h_S)$ , it is possible to deduce from them lower bounds on quantities that are known to depend on the width of the spectral gap, i.e. the quantity  $\epsilon = 1 - r(h_S)$ . One such quantity is the *Kazhdan constant* of the right regular representation  $\rho$  with respect to  $S$ , defined as

$$\kappa(\rho, S) = \inf_{\xi \in l^2(\Gamma), \|\xi\|=1} \max_{s \in S} \|\rho(s)\xi - \xi\|.$$

It is proved in Proposition I(6) of [HRV1] that  $\kappa(\rho, \Gamma) \geq \sqrt{2\epsilon}$ . Another such quantity is the isoperimetric constant of the Cayley graph  $G(\Gamma, S)$ , defined as

$$\iota(G(\Gamma, S)) = \inf \left\{ \frac{\#\partial A}{\#A} : A \text{ finite subset of } \Gamma \right\};$$

here  $\partial A$  is the boundary of  $A$ , i.e the set of edges of  $G(\Gamma, S)$  with one extremity in  $A$  and the other in  $\Gamma - A$ . It is well-known that

$$\iota(G(\Gamma, S)) \geq |S|\epsilon$$

(see e.g. Theorem 3.3 of [Moh] for a slightly better inequality).

### 5. Some computations of spectra.

We recall that the reduced  $C^*$ -algebra of the group  $\Gamma$ , denoted by  $C_r^*(\Gamma)$ , is the  $C^*$ -algebra generated by  $\rho(\Gamma)$ . If  $\Gamma$  is torsion free, a tantalizing conjecture of Kaplansky and Kadison states that  $C_r^*(\Gamma)$  has no idempotent except 0 and 1 (see [Val] for a survey). We explore here some consequences of this conjecture for one-relator groups.

**Proposition 5.1.** *Let  $\Gamma = \langle X : r \rangle$  be a torsion-free one-relator group satisfying the Kaplansky-Kadison conjecture. Denote by  $\Sigma$  the sum of all exponents in  $r$ . Then:*

- (i) *If  $\Sigma = 0$ , then  $\text{Sp}(h_X)$  is either a disk or an annulus centered at 0;*
- (ii) *If  $S$  is even, then  $\text{Sp}(h_S)$  is an interval symmetric with respect to 0.*

*Proof.* Any element in  $C_r^*(\Gamma)$  has a connected spectrum (otherwise, by holomorphic functional calculus, we construct non-trivial idempotents in  $C_r^*(\Gamma)$ ).

If  $\Sigma = 0$ , Proposition 2.1 says that  $\text{Sp}(h_X)$  is a union of concentric circles; by connectedness, it is either a disk or an annulus. Similarly, by connectedness  $\text{Sp}(h_S)$  must be an interval, symmetric with respect to 0 if  $\Sigma$  is even.  $\square$

**Corollary 5.2.** *Let  $\Gamma_g$  be the surface group  $\langle a_1, b_1, \dots, a_g, b_g : \prod_{i=1}^g [a_i, b_i] \rangle$  with  $X = \{a_1, b_1, \dots, a_g, b_g\}$  and  $g \geq 2$ ; then  $\text{Sp}(h_S) = [-r, r]$  with  $r \leq \frac{\sqrt{2g-1}}{g}$ .*

*Proof.* The fact that  $\Gamma_g$  satisfies the Kaplansky-Kadison conjecture was proved by Kasparov [Kas]. The result then follows by combining Proposition 5.1 with Theorem 4.6.

Of course, in the case of  $\Gamma_g$ , it is somewhat frustrating that we are able to compute explicitly the spectrum of  $h_X$  (see Theorem 1.2), but not the spectrum of its real part  $h_S$ .  $\square$

*Example 4.3 revisited.*

We consider again the group  $\Gamma = \langle a, b, c : [ac^{-1}, bc^{-1}] \rangle$  with  $X = \{a, b, c\}$ . We claim that  $\text{Sp}(h_X)$  is the disk  $\{z \in \mathbb{C} : |z| \leq \frac{1}{\sqrt{3}}\}$ . That the spectral radius is  $\frac{1}{\sqrt{3}}$  follows from Proposition 4.7. Now,  $\Gamma$  is the free product  $\mathbb{Z}^2 * \mathbb{Z}$ , so  $\Gamma$  satisfies the Kaplansky-Kadison conjecture as a corollary of a result of Rosenberg ([Ros], Proposition 2.10). By Proposition 5.1,  $\text{Sp}(h_X)$  is either a disk or an annulus centered at 0. To prove that it is a disk, we just have to prove that  $h_X$  is not invertible. But since  $h_X = \frac{1}{3}(\rho(ac^{-1}) + \rho(bc^{-1}) + 1)\rho(c)$ , it is enough to show that  $\rho(ac^{-1}) + \rho(bc^{-1}) + 1$  is not invertible. Since  $ac^{-1}$  and  $bc^{-1}$  generate a subgroup isomorphic to  $\mathbb{Z}^2$ , this follows from the fact that the function  $(z_1, z_2) \rightarrow z_1 + z_2 + 1$  is not invertible on the 2-torus.

### Appendix : An upper bound for the norms of powers of normalised adjacency operators.

*By Paul Jolissaint*

The aim of the present note is to prove a result needed in Proposition 3 of the previous article.

Let  $\Gamma$  be a group, let  $X$  be a finite subset of  $\Gamma$  containing 1, and let  $G$  be the free product  $\Gamma * \langle y \rangle$  of  $\Gamma$  with an infinite cyclic group generated by  $y$ .

Set also  $G_1 = \Gamma$ ,  $G_2 = \langle y \rangle$ ,  $X^* = X - \{1\}$  and  $G_j^* = G_j - \{1\}$  for  $j = 1, 2$ .

Recall that any element  $\omega$  of  $G$  can be uniquely written as a product  $\omega = \omega_1 \dots \omega_n$  with  $\omega_j \in G_{i_j}^*$ , and  $i_j \neq i_{j+1}$  for  $j \leq n - 1$ . The integer  $n$  is called the length of  $\omega$ , and we denote by  $\Lambda_n$  the set of words of length  $n$ .

The main result of this appendix is :

**Proposition 5.3.** *With the notation above, set*

$$h = \frac{1}{|X|} \sum_{x \in X} \rho(xy).$$

*Then there exists a positive constant  $C$  such that for every positive integer  $k$ , one has :*

$$\|h^k\| \leq C(1+k)^3 \|h^k\|_2,$$

*where  $\| \cdot \|$  in the left hand side is the operator norm and  $h^k$  is the  $k^{\text{th}}$  convolution power of  $h$ .*

$$\text{Let } \chi = \frac{1}{|X|} \sum_{x \in X} \delta_{xy}, \text{ so that } h = \rho(\chi).$$

It turns out that it will be more convenient to prove the inequality in Proposition 5.3 for  $\lambda(\chi)$  instead of  $h$ , where  $\lambda$  denotes the left regular representation of  $G$ . (As  $\lambda$  and  $\rho$  are equivalent representations, this will prove Proposition 5.3, as well.)

If  $k$  is a positive integer, it is easy to check by induction on  $k$  that the function  $\chi^k (= \chi * \dots * \chi, k \text{ times})$  is supported in  $\bigcup_{1 \leq j \leq 2k} \Lambda'_j$ , where  $\Lambda'_i$  is the set of reduced words  $\omega = \omega_1 \dots \omega_l$  such that

- (a) either  $\omega_j \in X^*$  or  $\omega_j = y^{\mu_j}$  with  $\mu_j > 0$  ;
- (b)  $\omega_l = y^{\mu_l}$  ;
- (c)  $\sum \mu_j \leq l$  .

Hence Proposition 5.3 follows immediately from the slightly more general:

**Proposition 5.4.** *There exists a positive constant  $C$  such that for any finitely supported function  $\varphi$  on  $G$  whose support lies in  $\bigcup_{1 \leq j \leq k} \Lambda'_j$  for some  $k$ , then*

$$\|\lambda(\varphi)\| \leq C(1+k)^3 \|\varphi\|_2.$$

The proof of Proposition 5.4 is similar to that of Theorem 2.2.2 of [Jol] and is based on an idea due to U. Haagerup in the case of the free group [Haa].

Using the same arguments as in the proof of Proposition 1.2.6 in [Jol], Proposition 5.4 is a consequence of the following result:

**Proposition 5.5.** *There exists a positive constant  $c$  such that for non-negative integers  $k, l$  and  $m$  satisfying:  $|k-l| \leq m \leq k+l$ , and for functions  $\varphi$  and  $\psi$  on  $G$  supported in  $\Lambda'_k$  and  $\Lambda_l$  respectively, one has*

$$\|(\varphi * \psi)\chi_{\Lambda_m}\|_2 \leq c(1+k) \|\varphi\|_2 \|\psi\|_2.$$

Let us recall the following result which is a special case of Lemma 2.2.1 of [Jol]:

**Lemma 5.6.** *Let  $k, l, m$  and  $q$  be non-negative integers such that  $m = k + l - q$ .*

*If  $\omega \in \Lambda_m$ , let  $\omega = \omega_1 \dots \omega_m$  be its reduced form.*

*Set also:  $E_{k,l}(\omega) = \{(u, v) \in \Lambda_k \times \Lambda_l \mid uv = \omega\}$ .*

*Then*

- (1) *If  $q = 2p$  is even, set  $u_\omega = \omega_1 \dots \omega_{k-p}$  and  $v_\omega = \omega_{k-p+1} \dots \omega_m$ ; then  $E_{k,l}(\omega)$  is the set of pairs  $(u, v) \in \Lambda_k \times \Lambda_l$  such that there exists  $a \in \Lambda_p$  with*

$$u = u_\omega a \text{ and } v = a^{-1} v_\omega.$$

- (2) *If  $q = 2p + 1$  is odd, set  $u_\omega = \omega_1 \dots \omega_{k-p-1}$  and  $v_\omega = \omega_{k-p+1} \dots \omega_m$ ; then  $E_{k,l}(\omega)$  is the set of pairs  $(u, v) \in \Lambda_k \times \Lambda_l$  such that there exists  $a \in \Lambda_p$  and  $b_1, b_2 \in \Lambda_1$  satisfying:*

$$b_1 b_2 = \omega_{k-p}, \quad u = u_\omega b_1 a \text{ and } v = a^{-1} b_2 v_\omega.$$

*Proof of Proposition 5.5.*

Set  $m = k + l - q$ ; then  $q$  is an integer such that  $0 \leq q \leq \min(k, l)$ . We divide the proof into two cases:

Case 1. Assume that  $q$  is even. Using the first part of the above lemma, one shows more generally that if  $\varphi$  is supported in  $\Lambda_k$  then:

$$\|(\varphi * \psi) \chi_{\Lambda_m}\|_2 \leq \|\varphi\|_2 \|\psi\|_2.$$

The proof is exactly the same as that of Lemma 1.3 in [Haa].

Case 2. Assume that  $q = 2p + 1$  is odd.

For every  $\omega \in \Lambda_m$ , set  $E'_{k,l}(\omega) = \{(u, v) \in \Lambda'_k \times \Lambda_l \mid uv = \omega\}$ . Let  $\omega \in \Lambda_m$  be such that  $E'_{k,l}(\omega) \neq \emptyset$  and let us write  $\omega$  as  $\omega = \omega_1 \dots \omega_{k-p-1} \omega_{k-p} \omega_{k-p+1} \dots \omega_m$  in its reduced form. Set  $u_\omega = \omega_1 \dots \omega_{k-p-1}$  and  $v_\omega = \omega_{k-p+1} \dots \omega_m$  as in the second part of the lemma. If  $(u, v) \in E'_{k,l}(\omega)$ , then  $u = u_1 \dots u_{k-1} y^{\mu_k}$  and  $v = v_1 \dots v_l$  for some  $u_i, v_i$  and  $\mu_k > 0$ .

Moreover there exists  $a \in \Lambda_p$  and  $b_1, b_2 \in \Lambda_1 = G_1^* \cup G_2^*$  such that  $u = u_\omega b_1 a, v = a^{-1} b_2 v_\omega$  and  $b_1 b_2 = \omega_{k-p}$ .

Then two cases may occur:

- (i)  $\omega_{k-p} \in G_1^*$ : Thus  $b_1, b_2 \in G_1^*$  and, since  $u \in \Lambda'_k$ , we must have  $b_1 \in X^*, a \in \Lambda'_p$  and  $a$  begins with  $y^\nu$  for some  $\nu > 0$ . (We will write:  $a \in \Lambda'_p, a_1 = y^{\nu_1}$ .)

- (ii)  $\omega_{k-p} \in G_2^*$ : Then  $b_1, b_2 \in G_2^*$  and  $a \in \Lambda'_p$  begins with some  $x \in X^*$ .

One has:

$$\begin{aligned}
 (1) \quad \|(\varphi * \psi)\chi_{\Lambda_m}\|_2^2 &= \sum_{\omega \in \Lambda_m} |(\varphi * \psi)(\omega)|^2 \\
 &= \sum_{\omega \in \Lambda_m} \left| \sum_{(u,v) \in E'_{k,i}(\omega)} \varphi(u)\psi(v) \right|^2 \\
 &= \sum_{\substack{\omega \in \Lambda_m \\ \omega_{k-p} \in G_1^*}} \left| \sum_{(u,v) \in E'_{k,i}(\omega)} \varphi(u)\psi(v) \right|^2 \\
 &\quad + \sum_{\substack{\omega \in \Lambda_m \\ \omega_{k-p} \in G_2^*}} \left| \sum_{(u,v) \in E'_{k,i}(\omega)} \varphi(u)\psi(v) \right|^2.
 \end{aligned}$$

Denote by  $\Sigma_1$  (resp.  $\Sigma_2$ ) the first (resp. the second) sum in the right handside of (1).

Let us estimate  $\Sigma_1$  first:

$$\begin{aligned}
 \Sigma_1 &= \sum_{\substack{\omega \in \Lambda_m \\ \omega_{k-p} \in G_1^*}} \left| \sum_{\substack{a \in \Lambda'_p \\ a_1 = y^v}} \sum_{x \in X^*} \varphi(u_\omega x a) \psi(a^{-1}(x^{-1}\omega_{k-p})v_\omega) \right|^2 \\
 &\leq \sum_{\substack{u \in \Lambda_{k-p-1} \\ u_{k-p-1} \in G_2^*}} \sum_{\substack{v \in \Lambda_{l-p-1} \\ v_1 \in G_2^*}} \sum_{\omega_{k-p} \in G_1^*} \left| \sum_{\substack{a \in \Lambda'_p \\ a_1 = y^v}} \sum_{x \in X^*} \varphi(x a) \psi(a^{-1}(x^{-1}\omega_{k-p})v) \right|^2 \\
 &\leq \sum_{\substack{u \dots \\ v \dots \\ \omega \dots}} \sum_{\omega \in G_1^*} \left( \sum_{\substack{a \in \Lambda'_{p+1} \\ a_1 \in X^*}} |\varphi(u a)| |\psi(a^{-1}\omega v)| \right)^2 \\
 &\leq \sum_{\substack{u \dots \\ v \dots \\ \omega \dots}} \left( \sum_{a \dots} |\varphi(u a)|^2 \right) \left( \sum_{a \dots} |\psi(a^{-1}\omega v)| \right)^2 \\
 &= \|\varphi\|_2^2 \sum_{\substack{v \in \Lambda_{l-p} \\ v_1 \in G_1^*}} \sum_{\substack{a \in \Lambda'_{p+1} \\ a_1 \in X^*}} |\psi(a^{-1}v)|^2.
 \end{aligned}$$

But

$$\sum_{\substack{v \in \Lambda_{l-p} \\ v_1 \in G_1^*}} \sum_{\substack{a \in \Lambda_{p+1}' \\ a_1 \in X^*}} |\psi(a^{-1}v)|^2 \leq (|X| - 1) \|\psi\|_2^2.$$

Let us finally estimate  $\Sigma_2$ :

We are going to use the fact that if  $f$  is a finitely supported function on  $G_2 (= \mathbb{Z})$ , then  $\|\lambda(f)\| \leq \frac{\pi}{\sqrt{3}} \|f\|_{2,1}$ , where  $\|f\|_{2,1}^2 = \sum_{x \in \mathbb{Z}} |f(x)|^2 (1 + |x|)^2$ . (See [Jol], Example 1.2.3.)

Then:

$$\Sigma_2 = \sum_{\substack{\omega \in \Lambda_m \\ \omega_{k-p} \in G_2^*}} \left| \sum_{\substack{a \in \Lambda_p' \\ a_1 \in X^*}} \lambda_{G_2}(\varphi_{u,\omega,a}) \psi_{v,\omega,a}(\omega_{k-p}) \right|^2,$$

where we set, for  $u \in \Lambda_{k-p-1}$ ,  $a \in \Lambda_p'$  and  $v \in \Lambda_{l-p-1}$ :  $\varphi_{u,a}(x) = \varphi(uxa)$  and  $\psi_{v,a}(x) = \psi(a^{-1}xv)$  for  $x \in G_2$ .

Hence

$$\begin{aligned} \Sigma_2 &\leq \sum_{\substack{u \in \Lambda_{k-p-1} \\ u_{k-p-1} \in X^*}} \sum_{\substack{v \in \Lambda_{l-p-1} \\ v_1 \in G_1^*}} \left\| \sum_{\substack{a \in \Lambda_p' \\ a_1 \in X^*}} \lambda_{G_2}(\varphi_{u,a}) \psi_{v,a} \right\|_2^2 \\ &\leq \sum_{\substack{u \dots \\ v \dots}} \left( \sum_{a \dots} \|\lambda_{G_2}(\varphi_{u,a})\|^2 \right) \left( \sum_{a \dots} \|\psi_{v,a}\|_2^2 \right) \\ &\leq \frac{\pi^2}{3} \left( \sum_{\substack{u \dots \\ a \dots}} \|\varphi_{u,a}\|_{2,1}^2 \right) \left( \sum_{\substack{v \dots \\ a \dots}} \|\psi_{v,a}\|_2^2 \right). \end{aligned}$$

But

$$\sum_{\substack{v \dots \\ a \dots}} \|\psi_{v,a}\|^2 = \|\psi\|_2^2 \quad \text{and}$$

$$\begin{aligned} \sum_{\substack{u \dots \\ a \dots}} \|\varphi_{u,a}\|_{2,1}^2 &= \sum_{u \dots} \sum_{a \dots} \sum_{x \in \{y^\nu; 1 \leq \nu \leq k\}} |\varphi(uxa)|^2 (1 + |x|)^2 \\ &\leq (1 + k)^2 \|\varphi\|_2^2. \end{aligned}$$

□

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**Added on proof:**

1) It has been pointed out to us by P. de la Harpe that, by combining Kesten's results [Ke1] with the Freiheitssatz, the upper bound on  $r(h_S) = \|h_S\|$  in Proposition 4.2 can be improved, to the effect that

$$r(h_S) \leq \frac{\sqrt{2(\#X) - 3} + 1}{\#X}.$$

In particular, for  $\#X \rightarrow \infty$ , one sees that  $r(h_S)$  behaves like  $\sqrt{\frac{2}{\#X}}$ .

2) The first author has recently extended the genericity result, Theorem 3.12, to all finitely presented groups; see P.-A. Cherix, *Generic result for the existence of a free semi-group*, Séminaire de théorie spectrale et géométrie, **13** (1994-95), Institut Fourier, Grenoble.

3) The Kaplansky-Kadison conjecture has now been proved for the class of torsion-free one-relator groups (see C. Béguin, H. Bettaieb & A. Valette, *The Baum-Connes conjecture for torsion-free one-relator groups*, preprint Neuchâtel, 1996). So this extra assumption can be dropped from Proposition 5.1.

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