

*Pacific  
Journal of  
Mathematics*

**EVERY STATIONARY POLYHEDRAL SET IN  $\mathbb{R}^n$  IS AREA  
MINIMIZING UNDER DIFFEOMORPHISMS**

JAIGYOUNG CHOE

## EVERY STATIONARY POLYHEDRAL SET IN $R^n$ IS AREA MINIMIZING UNDER DIFFEOMORPHISMS

JAIGYOUNG CHOE

**It is shown that every stationary polyhedral set in the Euclidean space is area minimizing under diffeomorphisms leaving the boundary fixed. Similar theorems are also proved for crystals and immiscible fluids.**

There are infinitely many minimal cones in  $R^3$ . Of these only three are area minimizing under Lipschitz maps: the plane; the three half planes meeting along their common boundary line at an angle of 120 degrees; the cone over the one-skeleton of the regular tetrahedron (see [T]). A recent work of Lawlor and Morgan [LM] has produced a generalization to higher dimensional cones in  $R^n$ . Namely, the hypercone over the  $(n - 2)$ -skeleton of the regular simplex in  $R^n$  has the least area among all surfaces separating the  $(n - 1)$ -dimensional faces of the simplex. Consequently this cone is area minimizing under Lipschitz maps. Moreover, Brakke has proved that the hypercone over the  $(n - 2)$ -skeleton of the cube in  $R^n$  is area minimizing under Lipschitz maps when and only when  $n \geq 4$  [B].

In this paper we prove that every stationary polyhedral set in  $R^n$  is area minimizing under *diffeomorphisms* leaving the boundary fixed. Therefore if we only consider competing surfaces of diffeomorphic images of a minimal cone  $C$  in  $R^3$ ,  $C$  has the least area. Hence in  $R^3$  all minimal cones are *stable*.

We wish to thank Frank Morgan for helpful comments on the extension of the main theorem.

### 1. Terminology.

An  $m$ -dimensional set  $C \subset R^n$  is said to be *polyhedral* if there exist  $m$ -dimensional planes  $\{\Pi_i\}_{i \in I}$  in  $R^n$  such that  $C \subset \bigcup_{i \in I} \Pi_i$ . Each  $m$ -dimensional set  $F_i = C \cap \Pi_i$  is called a *face* of  $C$ . The *singular* set  $S$  of  $C$  is the largest  $(m - 1)$ -dimensional subset of  $C$  which lies inside  $\bigcup_{i \neq j} (\Pi_i \cap \Pi_j)$ . A *singular edge* of  $C$  is the  $(m - 1)$ -dimensional subset  $E$  of  $S$  defined by  $E = C \cap (\Pi_i \cap \Pi_j)$  for each pair  $i, j \in I$ . So each singular edge of  $C$  is the intersection of two faces of  $C$ . For each face  $F_i$  of  $C$  the *boundary edge*  $B_{\bar{\gamma}}$  of  $C$  in  $F_i$  is the closure of  $\partial F_i \sim S$ . The union of all boundary edges of  $C$  is called the *boundary*  $\partial C$  of  $C$ .

A polyhedral set  $C \subset R^n$  is said to be *area minimizing under diffeomorphisms* (*Lipschitz maps*, respectively) if

$$\text{Volume}(C) \leq \text{Volume}(\varphi(C))$$

for any diffeomorphism (Lipschitz map, respectively)  $\varphi$  of  $R^n$  leaving the boundary of  $C$  fixed.

A polyhedral set  $C \subset R^n$  is *stationary* if

$$\frac{d}{dt} \text{Volume}(\phi_t(C)) \Big|_{t=0} = 0$$

for any 1-parameter family of diffeomorphisms  $\{\phi_t\}_{-1 < t < 1}$  of  $R^n$  with  $\phi_0 = \text{id}$  and leaving  $\partial C$  fixed. A stationary polyhedral set  $C$  is said to be *stable* if

$$\frac{d^2}{dt^2} \text{Volume}(\phi_t(C)) \Big|_{t=0} \geq 0$$

for any  $\{\phi_t\}$  as above.

## 2. Main theorem.

**Theorem 1.** *Every  $m$ -dimensional stationary polyhedral set  $C$  in  $R^n$  is area minimizing under diffeomorphisms.*

*Proof.* Let  $C \subset R^n$  be an  $m$ -dimensional stationary polyhedral set with faces  $\{F_i\}_{i \in I}$ , boundary edges  $\{B_i\}_{i \in I}$ , and singular edges  $\{E_j\}_{j \in J}$ . Then  $\bigcup_{i \in I} B_i$  is the boundary of  $C$ ,  $\bigcup_{j \in J} E_j$  is the singular set  $S$  of  $C$ , and  $\bigcup_{i \in I} F_i$  becomes  $C$  itself.

In order to simplify computations we shall frequently classify the singular edges of  $C$  in terms of the faces. For this purpose let us double-index  $\{E_j\}_{j \in J}$  using two indices: Given a face  $F_i$ , define a reindexed set  $\{E_{i1}, E_{i2}, \dots, E_{im_i}\}$  to be the set of all singular edges  $E_j$  such that  $E_j \subset \partial F_i$ . Hence

$$\partial F_i = B_i \cup E_{i1} \cup \dots \cup E_{im_i}.$$

Similarly we classify the faces of  $C$  in terms of the singular edges: Given a singular edge  $E_j$ , define

$$\{F_{j1}, F_{j2}, \dots, F_{jn_j}\} = \{F_i : \partial F_i \supset E_j\}.$$

Then

$$E_j = \bigcap_{k=1}^{n_j} \partial F_{jk}.$$

Since a face contains at least two singular edges, each face is double-indexed at least two times. Likewise each singular edge is double-indexed at least three times because a singular edge is part of the boundary of at least three faces.

Let  $\varphi$  be a diffeomorphism of  $R^n$  leaving  $\partial C$  fixed. Since  $\varphi$  is homotopic to the identity map in  $R^n$ , the singular set  $S$  is *homologous* to  $\varphi(S)$ . More precisely, let  $\varphi_t$  be the homotopy from the identity to  $\varphi$ . For each singular edge  $E_j$ , there exists an  $m$ -dimensional smooth submanifold  $G_j$  of  $R^n$  such that  $G_j$  is the set swept out by  $\varphi_t(E_j)$ . Clearly  $\partial G_j \supset E_j \cup \varphi(E_j)$ .  $\{G_j\}_{j \in J}$  can be double-indexed in the same manner as  $\{E_j\}$ : We write  $G_j = G_{ik}$  if  $E_j = E_{ik}$ . Similarly, given a face  $F_i$ , there exists an  $(m + 1)$ -dimensional smooth submanifold  $D_i$  swept out by  $\varphi_t(F_i)$  such that

$$\partial D_i = F_i \cup \varphi(F_i) \cup G_{i1} \cup \dots \cup G_{im_i}.$$

Now let us equip submanifolds  $F_i$ ,  $\varphi(F_i)$ ,  $G_{ik}$ , and  $D_i$  with appropriate orientations in such a way that

$$(1) \quad \partial D_i = F_i - \varphi(F_i) + G_{i1} + \dots + G_{im_i}.$$

One can find a coordinate system  $\{x_1, \dots, x_n\}$  in  $R^n$  such that the coordinate frame fields  $\partial/\partial x_1, \dots, \partial/\partial x_n$  are orthonormal and  $\partial/\partial x_1, \dots, \partial/\partial x_m$  are parallel to  $F_i$ . Define

$$\omega_i = dx_1 \wedge \dots \wedge dx_m.$$

Then  $d\omega_i = 0$ . Reordering  $\{x_1, \dots, x_m\}$  according to the orientation of  $F_i$ , if necessary, one sees that

$$(2) \quad \int_{F_i} \omega_i = \text{Volume}(F_i).$$

Then

$$(3) \quad 0 = \int_{D_i} d\omega_i = \int_{\partial D_i} \omega_i = \text{Volume}(F_i) - \int_{\varphi(F_i)} \omega_i + \sum_{k=1}^{m_i} \int_{G_{i,k}} \omega_i.$$

Let  $\xi$  be the volume form of  $\varphi(F_i)$  and  $\xi^*$  the  $m$ -vector on  $\varphi(F_i)$  with  $\xi(\xi^*) = 1$ . Note that  $\omega_i(\xi^*) \leq 1$  and

$$\int_{\varphi(F_i)} \omega_i = \int_{\varphi(F_i)} \omega_i(\xi^*) \xi \leq \text{Volume}(\varphi(F_i)).$$

Then summing up (3) for all  $i \in I$ , we have

$$\text{Volume}(C) = \sum_{i \in I} \text{Volume}(F_i) \leq \text{Volume}(\varphi(C)) - \sum_{i \in I} \sum_{k=1}^{m_i} \int_{G_{i,k}} \omega_i.$$

Here let us double-index  $\{\omega_i\}_{i \in I}$  such that  $\omega_i = \omega_{jk}$  if  $F_i = F_{jk}$ . Then rearranging the summation in terms of the singular edges gives

$$(4) \quad \text{Volume}(C) \leq \text{Volume}(\varphi(C)) - \sum_{j \in J} \sum_{k=1}^{n_j} \int_{G_j} \omega_{jk}.$$

In the integral in (4), however, the orientation of  $G_j$  is ambiguous since it depends on the orientation of  $F_{jk}$  for each  $k$  subject to (1). But since  $\int_{G_j} \omega_{jk} = \int_{-G_j} -\omega_{jk}$ , one can fix the orientation of  $G_j$  by taking the negative of  $\omega_{jk}$  if necessary. Then through (1) the orientation of  $G_j$  determines that of  $F_{jk}$ , which in turn determines  $\omega_{jk}$  through (2). Now  $m$ -forms  $\omega_{j1}, \dots, \omega_{jn_j}$  can be expressed more explicitly as follows. Let  $\nu_k$  be a unit constant vector field in  $R^n$  parallel to  $F_{jk}$  and perpendicular to  $E_j$ . Assume further that  $\nu_k \mid F_{jk}$  points inward along  $E_j$ . Define  $\theta_k$  to be the 1-form in  $R^n$  dual to  $\nu_k$ , i.e.,  $\theta_k(v) = \nu_k \cdot v$  for any vector field  $v$ . Let  $\eta_j$  be a volume form of  $E_j$  for an appropriate orientation of  $E_j$ . Then one can easily check that

$$\omega_{jk} = \eta_j \wedge \theta_k.$$

The stationarity of  $C$  states that

$$\sum_{k=1}^{n_j} \nu_k = 0 \quad \text{and} \quad \sum_{k=1}^{n_j} \theta_k = 0.$$

Therefore

$$\sum_{k=1}^{n_j} \omega_{jk} = 0.$$

Thus it follows from (4) that

$$\text{Volume}(C) \leq \text{Volume}(\varphi(C)).$$

□

**Corollary 1.** *Every stationary cone in  $R^3$  is area minimizing under diffeomorphisms.*

*Proof.* All stationary cones in  $R^3$  are polyhedral. □

**Corollary 2.** *Every  $m$ -dimensional stationary polyhedral set in  $R^n$  is stable.*

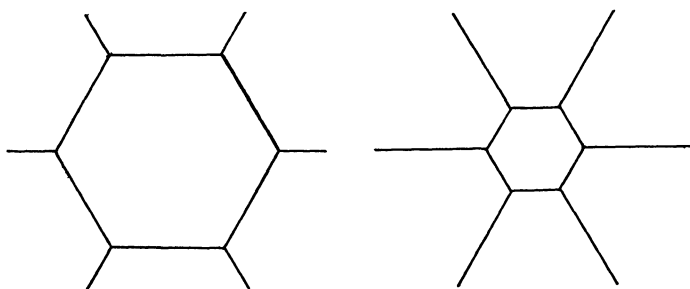
**Remarks.** i) In fact, the set of competing surfaces in Theorem 1 can be enlarged from diffeomorphic images  $\varphi(C)$  of  $C$  to the surfaces *homologous*

to  $C$ : If  $\tilde{C} = \bigcup_{i \in I} \tilde{F}_i$ ,  $\tilde{C} \supset \partial C$ , and if each  $\tilde{F}_i$  satisfies  $\partial D_i = F_i \cup \tilde{F}_i \cup G_{i1} \cup \dots \cup G_{im}$ , then  $\text{Volume}(\tilde{C}) \geq \text{Volume}(C)$ .

ii) There are nonpolyhedral stationary cones that are unstable: e.g. the cones over  $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2})$  in  $R^4$  and over  $S^2(1/\sqrt{2}) \times S^2(1/\sqrt{2})$  in  $R^6$  [S].

iii) B. White has also shown that stationary polyhedral cones are always stable.

iv) It should be mentioned that not every stationary polyhedral set is a unique minimizer. Figure below illustrates two diffeomorphic 1-dimensional stationary polyhedral sets of equal length. However, if we assume that each face of the  $m$ -dimensional stationary polyhedral set  $C$  has nonempty intersection with the boundary of  $C$ , then  $\text{Volume}(\varphi(C)) = \text{Volume}(C)$  for a diffeomorphism  $\varphi$  leaving  $\partial C$  fixed if and only if  $\varphi(C) = (C)$ . This is because  $\text{Volume}(\varphi(C)) = \text{Volume}(C)$  if and only if  $\omega_i(\xi^*) = 1$  on  $\varphi(F_i)$  for all  $i \in I$  if and only if  $\varphi(F_i) = F_i$  or  $\varphi(C) = C$ . But is there an  $m$ -dimensional stationary polyhedral set which is not a unique minimizer? Indeed it seems to be an interesting problem to find an  $m$ -dimensional ( $m \geq 2$ ) stationary polyhedral set  $C$  which has an interior face, i.e., a face disjoint from the boundary of  $C$  (like the edges of the hexagons in the 1-dimensional polyhedral sets of figure below)



Two diffeomorphic stationary sets of equal length.

### 3. Extensions to crystals and immiscible fluids.

Crystals tend to minimize the surface energy which is given by an integral  $\int_S \Psi(n)$  in which the weighting of area depends on the unit normal  $n$  at each point. Immiscible fluids try to minimize the total interface energy. This energy is proportional to area, but the constant of proportionality depends on a pair of fluids separated by the interface. In this section we extend Theorem 1 to the stationary polyhedral hypersurfaces (interfaces) of crystals and immiscible fluids.

**Definition.** A norm  $\Psi$  in  $R^n$  is a homogeneous convex function on  $R^n$ , positive except at 0. The dual norm  $\Psi^*$  is defined by

$$\Psi^*(w) = \sup\{w \cdot v : \Psi(v) = 1\}.$$

It follows immediately that

$$v \cdot w \leq \Psi(v)\Psi^*(w).$$

If equality holds, we say that  $w$  is dual to  $v$ . One can easily see that  $w$  is dual to a  $\Psi$ -unit vector  $v$  when  $w$  is an outward-pointing normal to the unit  $\Psi$ -ball at  $v$ .

For a hypersurface  $S$  in  $R^n$  with a unit normal  $n$ , the energy  $\Psi(S)$  of  $S$  associated with the norm  $\Psi$  is defined by

$$\Psi(S) = \int_S \Psi(n).$$

**Theorem 2.** Let  $\Psi$  be a norm in  $R^n$ , and let  $C$  be an  $(n-1)$ -dimensional polyhedral set in  $R^n$  which is stationary with respect to the  $\Psi$ -energy. Then  $C$  is energy minimizing under diffeomorphisms, i.e., for any diffeomorphism  $\varphi$  leaving  $\partial C$  fixed,

$$\Psi(C) \leq \Psi(\varphi(C)).$$

*Proof.* For the faces and edges of  $C$  and their “swept-out” sets, we use the same notations  $F_i, F_{jk}, E_j, D_i, G_j, G_{ik}$  as used in the proof of Theorem 1. Also we employ the same double-indexing convention as used there. Let  $n_i$  be a unit normal to  $F_i$ . Extend  $n_i$  to a constant unit vector field  $n_i$  in  $R^n$ . Let  $n_i^*$  be the  $\Psi^*$ -unit vector field dual to  $n_i$ , that is,

$$v \cdot n_i^* \leq \Psi(v)$$

with equality for  $v = n_i$ . Let  $dV$  be the volume form of  $R^n$  and define

$$\omega_i = n_i^* \lrcorner dV.$$

Then  $d\omega_i = 0$ . Hence

$$\begin{aligned} \Psi(F_i) &= \int_{F_i} \Psi(n_i) = \int_{F_i} n_i \cdot n_i^* = \int_{F_i} \omega_i \\ &= \int_{\varphi(F_i)} \omega_i - \sum_{k=1}^{m_i} \int_{G_{i,k}} \omega_i = \int_{\varphi(F_i)} \nu \cdot n_i^* - \sum_{k=1}^{m_i} \int_{G_{i,k}} \omega_i \end{aligned}$$

where  $\nu$  is a unit normal to  $\varphi(F_i)$ . Therefore

$$(5) \quad \Psi(F_i) \leq \int_{\varphi(F_i)} \Psi(\nu) - \sum_{k=1}^{m_i} \int_{G_{i,k}} \omega_i = \Psi(\varphi(F_i)) - \sum_{k=1}^{m_i} \int_{G_{i,k}} \omega_i.$$

Adding up (5) for all  $i$  gives

$$(6) \quad \Psi(C) \leq \Psi(\varphi(C)) - \sum_{i \in I} \sum_{k=1}^{m_i} \int_{G_{i,k}} \omega_i.$$

Now, for a singular edge  $E_j$ , the faces  $F_{j_1}, F_{j_2}, \dots, F_{j_{n_j}}$  are assumed to be indexed in the order they appear around  $E_j$ . Also the unit normals  $n_{jk}$  to  $F_{jk}$  are chosen in such a way that  $n_{jk}$  points from  $F_{jk}$  to  $F_{j(k+1)}$  (to  $F_{j_1}$  if  $k = n_j$ ). Then the stationarity of  $C$  implies that

$$\sum_{k=1}^{n_j} n_{jk}^* = 0.$$

(See [LM, Theorem 4.2].) Hence we have

$$\sum_{k=1}^{n_j} \omega_{jk} = 0.$$

Therefore the last term in (6) vanishes by the same reason as in the proof of Theorem 1. □

**Definition.** Let  $S$  be a union of hypersurfaces  $S_i$  of  $R^n$  and  $a_i$  the multiplicity constant (interface energy) of  $S_i$ . Given a diffeomorphism  $\varphi$  of  $R^n$ , define the *total interface energy*  $M(\varphi(S))$  of  $\varphi(S)$  by

$$M(\varphi(S)) = \sum_i a_i \text{Volume}(\varphi(S_i)).$$

**Theorem 3.** *Given an  $(n - 1)$ -dimensional polyhedral set  $C$  in  $R^n$  with faces  $F_i$  of multiplicity  $a_i$ , suppose  $C$  is stationary with respect to the total interface energy. Then  $C$  is energy minimizing in its diffeomorphism class.*

*Proof.* Employing the same notations as in the proof of Theorem 2, we define an  $(n - 1)$ -form  $\omega_i$  in  $R^n$  by

$$\omega_i = a_i n_i \lrcorner dV.$$



Then  $d\omega_i = 0$  and so

$$\begin{aligned} M(F_i) &= \int_{F_i} a_i = \int_{F_i} \omega_i = \int_{\varphi(F_i)} \omega_i - \sum_{k=1}^{m_i} \int_{G_{i,k}} \omega_i \\ &= \int_{\varphi(F_i)} a_i n_i \cdot \nu - \sum_{k=1}^{m_i} \int_{G_{i,k}} \omega_i \leq M(\varphi(F_i)) - \sum_{k=1}^{m_i} \int_{G_{i,k}} \omega_i. \end{aligned}$$

Hence

$$M(C) \leq M(\varphi(C)) - \sum_{i \in I} \sum_{k=1}^{m_i} \int_{G_{i,k}} \omega_i.$$

Since  $C$  is stationary, we have

$$\sum_{k=1}^{n_j} a_{jk} n_{jk} = 0,$$

where  $a_{jk}$  is the multiplicity of the face  $F_{jk}$ , and  $n_{jk}$  the unit normal to  $F_{jk}$ . Hence

$$\sum_{k=1}^{n_j} \omega_{jk} = 0$$

and we get

$$M(C) \leq M(\varphi(C)),$$

for any diffeomorphism  $\varphi$  leaving  $\partial C$  fixed. □

## References

- [B] K. Brakke, *Minimal cones on hypercubes*, J. Geom. Anal., **1** (1991), 329–338.
- [LM] G. Lawlor and F. Morgan, *Paired calibrations applied to soapfilms, immiscible fluids, and surfaces or networks minimizing other norms*, Pacific J. Math., **166** (1994), 55–83.
- [S] J. Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math., **88** (1968), 62–105.
- [T] J.E. Taylor, *The Structure of singularities in soap-bubble-like and soap-film-like minimal surfaces*, Ann. of Math., **103** (1976), 489–539.

Received December 7, 1993. Supported in part by KOSEF (921-0100-001-2), GARC.

POSTECH  
 POHANG, 790-600, SOUTH KOREA  
*E-mail address:* jchoe@vision.postech.ac.kr

<b>Mogens L. Hansen and Richard V. Kadison</b> , Banach algebras with unitary norms .....	535
<b>Xin-hou Hua</b> , Sharing values and a problem due to C.C. Yang .....	71
<b>Jing-Song Huang</b> , Harmonic analysis on compact polar homogeneous spaces .....	553
<b>Min-Jei Huang</b> , Commutators and invariant domains for Schrödinger propagators .....	83
<b>Hisao Kato</b> , Chaos of continuum-wise expansive homeomorphisms and dynamical properties of sensitive maps of graphs .....	93
<b>Oliver Küchle</b> , Some properties of Fano manifolds that are zeros of sections in homogeneous vector bundles over Grassmannians .....	117
<b>Xin Li and Francisco Marcellan</b> , On polynomials orthogonal with respect to Sobolev inner product on the unit circle .....	127
<b>Steven Liedahl</b> , Maximal subfields of $Q(i)$ -division rings .....	147
<b>Alan L.T. Paterson</b> , Virtual diagonals and $n$ -amenability for Banach algebras .....	161
<b>Claude Schochet</b> , Rational Pontryagin classes, local representations, and $K^G$ -theory .....	187
<b>Sandra L. Shields</b> , An equivalence relation for codimension one foliations of 3-manifolds .....	235
<b>D. Siegel and E. O. Talvila</b> , Uniqueness for the $n$ -dimensional half space Dirichlet problem .....	571
<b>Aleksander Simonič</b> , A Construction of Lomonosov functions and applications to the invariant subspace problem .....	257
<b>Endre Szabó</b> , Complete intersection subvarieties of general hypersurfaces ..	271

# PACIFIC JOURNAL OF MATHEMATICS

Volume 175 No. 2 October 1996

---

Mean-value characterization of pluriharmonic and separately harmonic functions	295
LEV ABRAMOVICH AĪZENBERG, CARLOS A. BERENSTEIN and L. WERTHEIM	
Convergence for Yamabe metrics of positive scalar curvature with integral bounds on curvature	307
KAZUO AKUTAGAWA	
Generalized modular symbols and relative Lie algebra cohomology	337
AVNER DOLNICK ASH and DAVID GINZBURG	
Convolution and limit theorems for conditionally free random variables	357
MAREK BOŹEJKO, MICHAEL LEINERT and ROLAND SPEICHER	
$L^p$ -bounds for hypersingular integral operators along curves	389
SHARAD CHANDARANA	
On spectra of simple random walks on one-relator groups. With an appendix by Paul Jolissain	417
PIERRE-ALAIN CHERIX, ALAIN J. VALETTE and PAUL JOLISSAINT	
Every stationary polyhedral set in $\mathbf{R}^n$ is area minimizing under diffeomorphisms	439
JAIGYOUNG CHOE	
Ramanujan's master theorem for symmetric cones	447
HONGMING DING, KENNETH I. GROSS and DONALD RICHARDS	
On norms of trigonometric polynomials on $SU(2)$	491
ANTHONY H. DOOLEY and SANJIV KUMAR GUPTA	
On the symmetric square. Unit elements	507
YUVAL ZVI FLICKER	
Stable constant mean curvature surfaces minimize area	527
KARSTEN GROSSE-BRAUCKMANN	
Banach algebras with unitary norms	535
MOGENS LEMVIG HANSEN and RICHARD VINCENT KADISON	
Harmonic analysis on compact polar homogeneous spaces	553
JING-SONG HUANG	
Uniqueness for the $n$ -dimensional half space Dirichlet problem	571
DAVID SIEGEL and ERIK O. TALVILA	



0030-8730(1996)175:2;1-B