ON NORMS OF TRIGONOMETRIC POLYNOMIALS ON $SU(2)$

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A conjecture about the $L^4$-norms of trigonometric polynomials on $SU(2)$ is discussed and some partial results are proved.

1. Introduction.

If $G$ is a compact abelian group, an elementary argument shows that $M_p(G) = M_q(G)$ where $M_p(G)$ denotes the space of $L^p$-multipliers on $G$ and $p$ and $q$ are conjugate indices. Oberlin [7] found a nonabelian totally disconnected compact group $G$ for which $M_p(G) \neq M_q(G)$. Herz [4] conjectured that inequality holds for all those infinite nonabelian compact groups $G$ whose degrees of the irreducible representations are unbounded. However, for compact connected groups, the situation is still unresolved, even for $SU(2)$.

The present paper arose from an attempt to study the Herz conjecture for $SU(2)$. In his unpublished M.Sc. thesis [8], S. Roberts formulated a conjecture for $SU(2)$ which, if proved, would settle the Herz conjecture for all compact connected groups. We believe that Robert's conjecture is interesting in its own right as it makes a rather delicate statement connecting the $L^p$-norms of noncentral trigonometric polynomials with the growth of the Clesh-Gordon coefficients.

We have pursued this interesting conjecture and make some partial progress towards settling it. Our results open the way to a detailed study of some new aspects of $L^p$ analysis on compact Lie groups.

In Section 2, we establish our notation. We state the conjecture in Section 3 and prove some partial results (Theorem 3.2). In Section 4, we show the relevance of the conjecture to Herz's conjecture.

2. Notation and remarks.

2.1. Irreducible representations of $SU(2)$. We summarise some notation and definitions from [6] concerning the irreducible representation of $SU(2)$. 

491
Let \( n \) be a rational number of the form \( k/2 \), where \( k \in \mathbb{N} \) and \( H_n \) be the space of homogeneous polynomials on \( \mathbb{C}^2 \) of degree \( 2n \); i.e. of functions of the form

\[
(2.1) \quad f(z_1, z_2) = \sum_{i=-n}^{+n} a_i z_1^{n-i} z_2^{n+i}.
\]

Let \((f, g)\) be the inner product on \( H_n \) given by the formula

\[
(2.2) \quad \left( \sum_{i=-n}^{+n} a_i z_1^{n-i} z_2^{n+i}, \sum_{j=-n}^{+n} b_j z_1^{n-j} z_2^{n+j} \right) = \sum_{i=-n}^{+n} (n-i)!(n+i)!a_i b_i.
\]

Let \( U(H_n) \) denote the set of unitary operators on \( H_n \) with respect to the inner product (2.2). The mapping \( T_n : SU(2) \to U(H_n) \) given by

\[
(2.3) \quad \left( T_n \left( \begin{array}{cc} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{array} \right) f \right)(z_1, z_2) = f(\alpha z_1 - \bar{\beta} z_2, \beta z_1 + \bar{\alpha} z_2)
\]

is an irreducible representation of \( SU(2) \) and in fact the set \( \{T_n : n = 0, 1/2, \ldots\} \) forms a complete set of inequivalent irreducible representations of \( SU(2) \).

To each operator \( T_n(x) \), \( x \in SU(2) \), there corresponds a unitary matrix (relative to the natural orthonormal basis of \( H_n \)) whose elements will be denoted by \( t_{jk}^{n}(= -n \leq j, k \leq n) \). These matrix elements are continuous functions on \( SU(2) \). We shall be estimating their norms as convolution operators on \( L^p \).

There are many results on the \( L^p \) multiplier norms of central trigonometric polynomials - see for example [2], or the more recent optimal results of Sogge on Riesz kernels on arbitrary compact manifolds (c.f. also [9], [10]). However, the \( t_{jk}^n \)'s considered here are non-central.

A word about the geometric significance of the matrix coefficients \( t_{jk}^n \) is in order. By the Peter-Weyl theorem, \( L^2(G) \) decomposes as a direct sum of the irreducible representations of \( G \), each one occurring with multiplicity equal to its dimension. These isotypic components represent the eigenspaces of the Laplace-Beltrami operator, and convolution by \((2n + 1)\chi_n\), where \( \chi_n \) is the character of the \( n \)th irreducible representation, is the projection onto this space.

For each \( j \) \((= -n \leq j \leq n)\), the functions \( \{t_{jk}^n : -n \leq k \leq n\} \) span one of the above copies of the representation space of degree \( 2n + 1 \). Convolution
on the left by $(2n + 1) t_{i,j}^n$ is a projection of $L^2(G)$ onto this copy. Convolution
by the function $(2n + 1) t_{i,j}^n$ are the natural isometries between the various
copies of the $n$th irreducible representation inside the isotypic component.

2.2. Expressions of products of functions $t_{i,j}^n$: The tensor product of
any two nontrivial irreducible unitary representations of $SU(2)$ is always reducible. If one decomposes such a tensor product into its irreducible com-
ponents, then the coefficients which appear in the decomposition are known
as the Clebsch-Gordan coefficients.

In the case of $SU(2)$ the Clebsch-Gordan coefficients $C(n_1, n_2, n_3, j_1, j_2, j_3)$
make their appearance in this way in the formula

$$t_{i,j_1}^{n_1} t_{k,l_1}^{n_2} = \sum_{m=|n_1-n_2|}^{n_1+n_2} C(n_1, n_2, m, j_1, k_1, k_1+j_1)
\cdot C(n_1, n_2, m, j_2, k_2, k_2+j_2) t_{i+k_1,j_1+k_2}^{n_1}.$$

While the Clebsch-Gordan coefficients are, in general, very complicated [8],
there are simple formulas for them in certain situations. Two such cases are
given below. They will be of interest in Section 3.

$$C(n, n, 2n, j, k, j+k) = \left(\frac{(2n + j + k)! (2n - j - k)! 2n! 2n!}{(n - j)! (n + j)! (n - k)! (n + k)! 4n!}\right)^{1/2}$$

$$C(n, n, 2m, j, j, 2j) = (-1)^{n-m} \left(\frac{(4m + 1) (2j + 2m)! 2n - 2m! 2m - 2j!}{(2n + 2m + 1)!}\right)^{1/2}
\times \frac{(m + n)!}{(m + j)! (m - j)! (n - m)!}$$

if $n \geq m \geq |j|$ and 0 otherwise

$$C(n, n, 2m + 1, j, j, 2j) = 0.$$

We denote by $C_{i,k}^{2m}$ the Clebsch-Gordan coefficient $C(n, n, 2m, i, k, i + k)$
where $n$ will considered be fixed throughout the argument.

In Section 4, we use the following convolution identity ([5], 27.20)

$$t_{i,k}^{n} * t_{p,q}^{m} = \frac{1}{2n+1} \delta_{nm}\delta_{kp} t_{i,j}^{n}.$$

By $A_n \approx B_n$, $n > 1$ we mean that there exist positive constants $\alpha, \beta$ such
that

$$\beta B_n \leq A_n \leq \alpha B_n, \ \forall n \geq 1.$$

The same symbol $C$ may denote two different constants in two different lines.
3. The conjecture.

In this section we state the conjecture and prove some partial results.

**Conjecture.** Denote by \( z^{(n)} \in \mathbb{C}^{2n+1} \) the vector with components \( \{ z_i^{(n)} \}_{i=-n}^{n} \).

Let

\[
A_n = \frac{1}{n^{1/8}} \sup_{\sum_{i=-n}^{n} |z_i^{(n)}|^2 = 1} \left\| \frac{\sum_{i=-n}^{n} z_i^{(n)} t_{0i}^{n}}{\sum_{i=-n}^{n} z_i^{(n)} t_{ni}^{n}} \right\|_4.
\]

Then \( A_n \to 0 \) as \( n \to \infty \).

**Remark 3.1.** For the motivation of the conjecture, see Section 4.

We will prove the following theorem which is a weaker version of the conjecture:

**Theorem 3.2.**

(A) Let \( z^{(n)} \in \mathbb{C}^{2n+1} \) and \( z_i^{(n)} \geq 0, \quad \forall i = -n, \ldots, n \). Define \( F_n(z^{(n)}) = \{ i | z_i^{(n)} \neq 0 \} \). Suppose that \( \sum_{i=-n}^{n} |z_i^{(n)}| = 1 \).

If

\[
\left| F_n(z^{(n)}) \right| \leq \frac{C n^{1/3}}{(\log n)^{2/3}} \quad \forall n \geq 2,
\]

then

\[
\frac{1}{n^{1/8}} \left\| \sum_{i=-n}^{n} z_i^{(n)} t_{0i}^{n} \right\|_4 \leq \frac{C}{(\log n)^{1/4}}.
\]

(B) Let \( \{ p_n \}_{n=1}^{\infty} \) be a sequence of natural numbers such that \( p_n \geq 2 \forall n \).

Define \( j_n = \left\lfloor \frac{\log n}{\log p_n} \right\rfloor \). Then

\[
\frac{1}{n^{1/8}} \sup_{\sum_{i=-n}^{n} |z_i^{(n)}|^2 = 1} \left\| \sum_{i=-n}^{n} z_i^{(n)} t_{0p_i^{n}}^{n} \right\|_4 \leq \frac{C \log n}{n^{1/4}}.
\]

To prove Theorem 3.2, we first obtain an expression for \( \left\| \sum_{i=-n}^{n} z_i^{(n)} t_{j_i}^{n} \right\|_4 \) in terms of Clebsch-Gordan coefficients. Let \( z \in \mathbb{C}^{2n+1} \).
Set
\[ \varphi^n_j(z) = \sum_{i=-n}^{+n} z_i t^n_{ji} \]
\[ (\varphi^n_j(z))^2 = \left( \sum_{i=-n}^{+n} z_i t^n_{ji} \right) \left( \sum_{k=-n}^{+n} z_k t^n_{jk} \right) \]
\[ = \sum_{-n \leq i, k \leq n} z_i z_k t^n_{ji} t^n_{jk} \]
\[ = \sum_{-n \leq i, k \leq n} z_i z_k \left( \sum_{m=0}^{2n} C(n, n, m, j, j, 2j) C(n, n, i, i, k + k) t^m_{2j, i+k} \right). \]

Using (2.6)-(2.7), we get
\[ (\varphi^n_j(z))^2 = \sum_{-n \leq i, k \leq n} z_i z_k \sum_{m=|j|}^{n} C(n, n, 2m, j, j, 2j) C(n, n, 2m, i, i, k + k) t^m_{2j, i+k} \]
\[ = \sum_{m=|j|}^{n} C^2_{jj} \sum_{r=-2m}^{+2m} \left( \sum_{i+k=r} z_i z_k C^2_{ik} \right) t^m_{2j, r}. \]

Since \( \left\{ \sqrt{2n+1} t^n_{ij} \right\}_{-n \leq i, j \leq n} \) \( n=0, 1/2, ... \) is an orthonormal set in \( L^2(SU(2)) \), we get
\[ \left\| (\varphi^n_j(z))^2 \right\|_2^2 = \sum_{m=|j|}^{n} \frac{(C^2_{jj})^2}{(4m+1)} \sum_{r=-2m}^{+2m} \left( \sum_{i+k=r} z_i z_k C^2_{ik} \right)^2. \]

In particular,
\[ (3.3) \left\| \varphi^n_0(z) \right\|_4 = \left[ \sum_{m=-n}^{n} \frac{(C^2_{00})^2}{4m+1} \sum_{r=-2m}^{+2m} \left( \sum_{i+k=r} z_i z_k C^2_{ik} \right)^2 \right]^{1/4} \]
\[ (3.4) \left\| \varphi^n_n(z) \right\|_4 = \left[ \frac{(C^2_{nn})^2}{(4n+1)} \sum_{r=-2n}^{+2n} \left( \sum_{i+k=r} z_i z_k C^2_{ik} \right)^2 \right]^{1/4} \]
\[ = \left[ \frac{1}{(4n+1)} \sum_{r=-2n}^{+2n} \left( \sum_{i+k=r} z_i z_k C^2_{ik} \right)^2 \right]^{1/4} \]
\[ \text{as } C^2_{nn} = 1. \]
Next we prove two Lemmas.

**Lemma 3.5.** There exist constants $C_1, C_2 > 0$ satisfying: $(n \geq 2)$

(i) $\frac{C_2}{n^{1/4}} \leq |C_{00}^{2n}| \leq \frac{C_1}{n^{1/4}}$.

(ii) $\frac{C_2}{\sqrt{2n+1}} \leq |C_{00}^0| \leq \frac{C_1}{\sqrt{2n+1}}$.

(iii) Let $0 \leq |j| \leq n - 1$. Then

$$\frac{C_2n^{1/4}}{(n + j)^{1/4}(n - j)^{1/4}} \leq |C_{jj}^{2n}| \leq \frac{C_1n^{1/4}}{(n + j)^{1/4}(n - j)^{1/4}}.$$ 

(iv) Let $|j| + 1 \leq m \leq n - 1$. Then

$$\frac{C_2\sqrt{m}}{(m + n)^{1/4}(m + j)^{1/4}(m - j)^{1/4}(n - m)^{1/4}} \leq |C_{jj}^{2m}|.$$ 

$$|C_{jj}^{2m}| \leq \frac{C_1\sqrt{m}}{(m + n)^{1/4}(m + j)^{1/4}(m - j)^{1/4}(n - m)^{1/4}}.$$ 

(v) Let $1 \leq j \leq n - 1$. Then

$$\frac{C_2j^{1/4}}{(n + j)^{1/4}(n - j)^{1/4}} \leq |C_{jj}^{2j}| \leq \frac{C_1j^{1/4}}{(n + j)^{1/4}(n - j)^{1/4}}.$$ 

(vi) Let $1 \leq j \leq n - 1$. Then

$$\frac{C_2}{(n + j)^{1/4}(n - j)^{1/4}} \leq |C_{00}^{2j}| \leq \frac{C_1}{(n + j)^{1/4}(n - j)^{1/4}}.$$ 

**Proof.** The easy proof using the following inequality

$$e^{7/8} \leq \frac{n!}{(n/e)^n n^{1/2}} \leq e$$

for $n = 1, 2, 3, ...$ is left to the reader. \qed

**Lemma 3.6.** Let $n \geq 2$ be a natural number and $-n \leq i \leq n$. Then there exists a positive constant $C$ such that

$$\frac{1}{n^{1/8}} \|t_{i}^n\|_4 \leq C \left( \frac{\log n}{n} \right)^{1/4}.$$ 

Also for every $\epsilon$, $0 < \epsilon < 1$, there exists a $C_\epsilon > 0$ such that for $0 \leq |i| \leq n\epsilon$, we have

$$C_\epsilon \left( \frac{\log n}{n} \right)^{1/4} \leq \frac{1}{n^{1/8}} \|t_{i}^n\|_4 \leq C \left( \frac{\log n}{n} \right)^{1/4}.$$
Proof. Using (3.3)-(3.4), we get

\[(3.9) \quad \|t_{0i}\|_4 = \left[ \sum_{m=0}^{n} \frac{(C_{00}^{2m})^2}{(4m+1)} (C_{2m}^{2i})^2 \right]^{1/4} = \left[ \sum_{m=|i|}^{n} \frac{(C_{00}^{2m})^2 (C_{2m}^{2i})^2}{4m+1} \right]^{1/4}\]

as \(C_{2m}^{2i} = C_{|i||i|}^{2m}\) and \(C_{|i||i|}^{2m} = 0\) for \(m < |i|\) and

\[(3.10) \quad \|t_{ni}\|_4 = \frac{\sqrt{C_{|i||i|}^{2n}}}{(4n+1)^{1/4}}.\]

From (3.9)-(3.10) we see that \(\|t_{0i}\|_4 = \|t_{0n-i}\|_4\) and \(\|t_{ni}\|_4 = \|t_{n-i}\|_4\).

Therefore we assume that \(0 < i < n\). We divide the rest of the proof in four steps:

Step 1. \(i = 0\)

\[\frac{1}{n^{1/8}} \|t_{00}\|_4 = \left[ \sum_{m=0}^{n} \frac{(C_{00}^{2m})^4}{4m+1} \left( \frac{C_{00}^{2m}}{\sqrt{n}} \right) \right]^{1/4}\]

\[\approx \left[ \frac{1}{n} + \sum_{m=1}^{n-1} \frac{1}{(n+m)(n-m)(4m+1)} \frac{(4n+1)}{\sqrt{n}} + \frac{1}{n} \right]^{1/4}\]

\[\approx \left[ \frac{1}{n} + \sum_{m=1}^{n-1} \frac{1}{m(n-m)} \right]^{1/4}\]

\[\approx \left[ \frac{1}{n} + \sum_{m=1}^{n-1} \frac{1}{n} \left( \frac{1}{m} + \frac{1}{n-m} \right) \right]^{1/4}\]

\[\approx \left[ \frac{1}{n} + \frac{1}{n} \sum_{m=1}^{n-1} \frac{1}{n} \right]^{1/4}\]

\[\approx \left( \frac{\log n}{n} \right)^{1/4}.
\]

Step 2. \(i = n\)

\[\frac{1}{n^{1/8}} \|t_{nn}\|_4 = \left[ \frac{4n+1}{\sqrt{n}} \left( \frac{C_{00}^{2n}}{4n+1} \right) \right]^{1/4}\]

\[\approx \frac{1}{n^{1/4}}.
\]

Step 3. \(1 \leq i \leq n - 1\)
\[
\frac{1}{n^{1/8}} \| e_{n}^{n} \|_{4} = \left[ \frac{4n + 1}{\sqrt{n}} \sum_{m=0}^{n} \left( \frac{C_{0}^{2m}}{(4m + 1) (C_{i}^{2m})^2} \right) \right]^{1/4}
\]

\[
= \left[ \frac{4n + 1}{\sqrt{n}} \left\{ \frac{C_{0}^{2i}}{(4i + 1) (C_{i}^{2i})^2} + \sum_{m=i+1}^{n-1} \frac{(C_{0}^{2m})^2 (C_{i}^{2m})^2}{(4m + 1) (C_{i}^{2m})^2} + \frac{(C_{0}^{2n})^2}{(4n + 1)} \right\} \right]^{2}
\]

\[
= \left[ \frac{4n + 1}{\sqrt{n}} \left\{ A_{n} + Z_{n} + B_{n} \right\} \right]^{1/4} \quad \text{(say)}
\]

\[
A_{n} = \frac{(C_{0}^{2i})^2 (C_{i}^{2i})^2}{(4i + 1) (C_{i}^{2n})^2}
\]

\[
\approx \frac{1}{(n + i)^{1/2} (n - i)^{1/2}} \frac{1}{(n + i)^{1/2} (n - i)^{1/2} (4i + 1)} \left( \frac{(n + i)^{1/2} (n - i)^{1/2}}{n^{1/2}} \right)
\]

\[
\approx \frac{1}{n \sqrt{n - i} \sqrt{i}}.
\]

Hence \( A_{n} \leq \frac{C}{n^{2/3}} \).

\[
B_{n} = \frac{(C_{0}^{2n})^2}{(4n + 1)} \approx \frac{1}{n^{3/2}}
\]

\[
Z_{n} = \sum_{m=i+1}^{n-1} \frac{(C_{0}^{2m})^2 (C_{i}^{2m})^2}{(4m + 1) (C_{i}^{2m})^2}
\]

\[
\approx \sum_{m=i+1}^{n-1} \frac{m(n+i)^{1/2}(n-i)^{1/2}}{(n+m)^{1/2}(n-m)^{1/2}(n+m)^{1/2}(n-m)^{1/2}(m+i)^{1/2}(m-i)^{1/2}(4m+1)n^{1/2}}
\]

\[
\approx \frac{1}{n} \sum_{m=i+1}^{n-1} \frac{(n-i)^{1/2}}{(n-m)(m+i)^{1/2}(m-i)^{1/2}}.
\]

Now we further divide Step 3 into three cases:

a) \( i = n - 1 \). Then

\[
A_{n} \leq \frac{C}{n^{3/2}}, \quad B_{n} \approx \frac{1}{n^{3/2}}, \quad Z_{n} = 0.
\]

Therefore \( \frac{1}{n^{1/8}} \| e_{n}^{n} \|_{4} \approx \frac{1}{n^{1/4}} \).
b) $1 \leq i \leq \frac{n}{2}$. Then

$$Z_n \leq \frac{C}{\sqrt{n}} \sum_{m=i+1}^{n-1} \frac{1}{(n-m)(m-i)}$$

$$= \frac{C}{\sqrt{n}} \sum_{m=i+1}^{n-1} \left[ \frac{1}{n-m} + \frac{1}{m-i} \right] \frac{1}{(n-i)}$$

$$= 2C \sqrt{n} \left[ \sum_{j=1}^{n-i-1} \frac{1}{j} \right] \frac{1}{(n-i)}$$

$$\leq \frac{4C}{n^{3/2}} \left[ 1 + \log(n-i-1) \right] \leq \frac{8C \left( \log n \right)}{n^{3/2}}.$$

Therefore $\frac{1}{n^{1/8}} \left\| T_{n+1}^n \right\|_{n+1}^{1/4} \leq C \left( \frac{\log n}{n} \right)^{1/4}$.

c) $\frac{n}{2} \leq i \leq n-2$. Then

$$Z_n \leq \frac{C}{n^{3/2}} \sum_{m=i+1}^{n-1} \frac{\sqrt{n-i}}{(n-m)\sqrt{m-i}}$$

$$= \frac{C}{n^{3/2} \sqrt{n-i}} \sum_{m=i+1}^{n-1} \left[ \frac{\sqrt{m-i}}{(n-m)} + \frac{1}{\sqrt{m-i}} \right]$$

$$\leq \frac{C}{n^{3/2}} \left[ \sum_{m=i+1}^{n-1} \frac{1}{(n-m)} + \frac{1}{\sqrt{n-i}} \sum_{m=i+1}^{n-1} \frac{1}{\sqrt{m-i}} \right]$$

$$\leq \frac{C \log n}{n^{3/2}}.$$

Therefore $\frac{1}{n^{1/8}} \left\| T_{n+1}^n \right\|_{n+1}^{1/4} \leq C \left( \frac{\log n}{n} \right)^{1/4}$.

Step 4. Let $0 < \epsilon < 1$ and $0 \leq i \leq n\epsilon$. In this case,

$$Z_n \approx \frac{1}{\sqrt{n}} \sum_{l=i+1}^{n-1} \frac{1}{(n-l)(l+i)^{1/2}(l-i)^{1/2}}.$$
Therefore

\[ Z_n \geq \frac{1}{\sqrt{n}} \sum_{l=i+1}^{n-1} \frac{1}{(n-l)(l+i)} \]

\[ = \frac{1}{\sqrt{n}(n+i)} \sum_{l=i+1}^{n-1} \left[ \frac{1}{(n-l)} + \frac{1}{(l+i)} \right] \]

\[ \geq \frac{1}{n^{3/2}} \sum_{j=1}^{n-i-1} \frac{1}{j} \geq \frac{\log(n-i)}{n^{3/2}} \]

\[ \geq C_\epsilon \frac{\log n}{n^{3/2}}. \]

Hence \( \frac{1}{n^{1/8}} \left\| t_n^{1/4} \right\| \geq C_\epsilon \left( \frac{\log n}{n} \right)^{1/4} \).

Now by Step 1 and Step 3 we have

\[ \frac{1}{n^{1/8}} \left\| t_n^{1/4} \right\| \leq C \left( \frac{\log n}{n} \right)^{1/4} \text{ for all } 0 \leq i \leq n - 1. \]

Therefore \( C_\epsilon \left( \frac{\log n}{n} \right)^{1/4} \leq \frac{1}{n^{1/8}} \left\| t_n^{1/4} \right\| \leq C \left( \frac{\log n}{n} \right)^{1/4}. \)

Proof of Theorem 3.2.

(A) Let \( z^{(n)} \) be as in the hypothesis of Theorem 3.2(A). Consider

\[ \left\| \sum_{i=-n}^{n} z_i^{(n)} t_n^{1/4} \right\|_4 \]

\[ = \left[ \frac{1}{(4n+1)} \sum_{r=-2n}^{+2n} \left( \sum_{i+k=r, -n \leq i, k \leq n} \left| z_i^{(n)} z_k^{(n)} C_{ik}^{2n} \right| \right) \right]^{2n-1/4} \]

\[ \geq \left[ \frac{1}{(4n+1)} \sum_{i=-n}^{+n} \left( C_{ii}^{2n} \right)^{2} \left| z_i^{(n)} \right|^4 \right]^{1/4} \text{ as } C_{ik}^{2n} \geq 0, \forall i, k \]

\[ = \left[ \sum_{i=-n}^{+n} \left( z_i^{(n)} \right)^4 \left\| t_n^{1/4} \right\|^4 \right]^{1/4}. \]
Now

\[
\frac{1}{n^{1/8}} \left\| \sum_{i=-n}^{+n} z_i^{(n)} n^{n} n_i \right\|_4 \leq \frac{1}{n^{1/8}} \left( \sum_{i=-n}^{+n} \left( z_i^{(n)} \right)^4 \left\| t_{0i}^n \right\|_4 \right)^{1/4} \\
\leq \frac{1}{n^{1/8}} \left( \sum_{i=-n}^{+n} \left( z_i^{(n)} \right)^4 \left\| t_{n}^n \right\|_4 \right)^{1/4} \left| F_n \left( z_i^{(n)} \right) \right|^{3/4} \\
\leq C \left( \frac{\log n}{n} \right)^{1/4} \left( \frac{n^{1/3}}{(\log n)^{2/3}} \right)^{3/4} \leq \frac{C}{(\log n)^{1/4}}.
\]

(by Lemma 3.6).

This completes the proof of part (A).

(B) Consider

\[
\sum_{i=0}^{j_a} \left\| \sum_{r=0}^{+2n} \sum_{p^*_k, p^*_k = r}^n z_i^{(n)} z_k^{(n)} C_{p^*_k, p^*_k}^{2n} \right\|_4^2 = \left( \frac{1}{4n+1} \right) \left[ \sum_{i=0}^{j_a} \left( z_i^{(n)} \right)^4 \left( C_{p^*_k, p^*_k}^{2n} \right)^2 \right]^{1/4} as
\]

\[
\left\| \sum_{p^*_k + p^*_k = r} z_i^{(n)} z_k^{(n)} C_{p^*_k, p^*_k}^{2n} \right\|_4^2 = \left| z_i^{(n)} \right|^4 \left( C_{p^*_k, p^*_k}^{2n} \right)^2 \text{ if } r = 2p^*_k.
\]

Therefore

\[
\left\| \sum_{i=0}^{j_a} z_i^{(n)} n_i \right\|_4 \geq \left( \sum_{i=0}^{j_a} \left| z_i^{(n)} \right|^4 \left\| t_{n}^n \right\|_4 \right)^{1/4}.
\]
Hence
\[ \frac{1}{n^{1/8}} \left\| \sum_{i=0}^{j_n} z_i^{(n)} t_0 n p_k \right\|_4 \leq \frac{1}{n^{1/8}} \left( \sum_{i=0}^{j_n} \left| z_i^{(n)} \right|^4 \right)^{1/4} \left( \| t_0 n p_k \|_4 \right)^{1/4} \]
(by Hölder's inequality),
\[ \leq C \left( \frac{\log n}{n} \right)^{1/4} (\log n)^{3/4} \]
\[ = C \frac{\log n}{n^{1/4}}. \]

This completes the proof of the Theorem. \( \square \)

**Remark 3.11.** The following inequality can be proved by using the ideas of the proof of Theorem 3.2(B):
\[ \frac{1}{n^{1/8}} \left\| z_1 t_0 p_n + z_2 t_0 q_n \right\|_4 \leq C \left( \frac{\log n}{n} \right)^{1/4} \]
for \(-n \leq p, q \leq n\).

4.

Let \( G \) be a compact group and let \( \Gamma \) be the dual object of \( G \), the set of equivalence classes of irreducible unitary representations of \( G \). For each \( \sigma \in \Gamma \), select a representation \( U_\sigma \in \sigma \), let \( H_\sigma \) be the Hilbert space on which \( U_\sigma \) acts, and let \( d_\sigma \) be the dimension of \( H_\sigma \). Let \( B(H_\sigma) \) denote the space of linear operators on \( H_\sigma \) and \( C(\Gamma) \) denote the space \( \prod_{\sigma \in \Gamma} B(H_\sigma) \).

**Definition.** Fix \( p \in [1, \infty] \). Let \( m \) be an element of \( C(\Gamma) \), so that for each \( \sigma \), \( m(\sigma) \in B(H_\sigma) \). The function \( m \) is a (left) multiplier of \( L^p (= L^p (G)) \) if for each \( f \in L^p \), the series
\[ \sum_{\sigma \in \Gamma} d_\sigma \text{ tr} \left[ m(\sigma) \hat{f}(\sigma) U_\sigma(x) \right] \]
is the Fourier series of some function $L_m f \in L^p$. The collection of all such $m$ is denoted by $M_p(G)$ or simply $M_p$.

For each $m \in M_p$, the map $f \to L_m f$ defines a bounded linear operator on $L^p$, an operator which commutes with left translations by the elements of $G$. We regard $M_p$ as a Banach space under the operator norm.

When $G$ is abelian, an easy argument shows that if $\frac{1}{p} + \frac{1}{q} = 1$, then $M_p = M_q$. It is known that for $1 < p < 2$, $M_p \neq M_q \left( \frac{1}{p} + \frac{1}{q} = 1 \right)$ for many nonabelian groups $G$ (see [1, 2, 3, 4, 6]).

For connected compact non-abelian group $G$ and for $1 < p < 2$, it is an open problem whether $M_p = M_q$, $\frac{1}{p} + \frac{1}{q} = 1$.

S.G. Roberts has shown in [8] that if the conjecture is true then $M_p(G) \neq M_q(G)$ for every connected compact non-abelian group and $1 < p < 2$, $\frac{1}{p} + \frac{1}{q} = 1$.

We give an easy proof that if the conjecture is true then $M_p(SU(2)) \neq M_q(SU(2))$ for $1 < p < 2$, $\frac{1}{p} + \frac{1}{q} = 1$. This proof is essentially due to Roberts [8], but has never to the best of our knowledge been published. We present it here for completeness. The result will follow if we show that

\[(4.1) \quad \frac{\|t^n_{0,n}\|_p}{\|t^n_{0,n}\|_q} \to \infty \text{ as } n \to \infty\]

where $\|t^n_{0,n}\|_p$ denotes the norm of $t^n_{0,n}$ as an element in $M_p$.

To prove (4.1), we use the following norm estimates for $t^n_{0,n}$ and $t^n_{n,n}$ which are easy to establish (see [8]).

$$\|t^n_{0,n}\|_p \approx \frac{1}{n^{1/4 + 1/2p}} \quad \|t^n_{n,n}\|_p = \frac{1}{(np + 1)^{1/p}}$$

$$\|t^n_{0,n}\|_1 = \|t^n_{0,n}\|_2 \approx \frac{1}{n^{3/4}} \quad \|t^n_{0,n}\|_2 = \frac{1}{2n + 1}.$$

Now by Riesz convexity theorem, we get

$$\|t^n_{0,n}\|_p \leq \|t^n_{0,n}\|_1^{\alpha} \|t^n_{0,n}\|_2^{1-\alpha}$$

where

$$\alpha = \frac{2 - p}{p}.$$

Hence

$$\|t^n_{0,n}\|_p \leq \frac{C}{n^{(5/4) - (1/2p)}}.$$
Also

\[ \|t_0^n\|_p \geq \frac{\|t_0^n \ast t_{nn}^n\|_p}{\|t_0^n\|_p} = \frac{1}{(2n + 1)} \|t_0^n\|_p \geq \frac{C}{n^{(5/4)-(1/2p)}}. \]

Therefore (4.1) is true if

\[ n^{(5/4)-(1/2p)} \|t_0^n\|_q \to 0 \text{ as } n \to \infty. \]

A routine argument using Riesz convexity theorem shows that (4.2) is true if

\[ n^{7/8} \|t_0^n\|_q \to 0 \text{ as } n \to \infty. \]

Now

\[ \|t_0^n\|_q = \sup_{f \in L^q \setminus \{0\}} \frac{\|t_0^n \ast f\|_4}{\|f\|_4} = \sup_{f \in L^q \setminus \{0\}} \frac{\|t_0^n \ast t_{nn}^n \ast f\|_4}{\|f\|_4} (2n + 1) \]

and

\[ (2n + 1) \|t_{nn}^n \ast f\|_4 \leq (2n + 1) \|t_{nn}^n\|_1 \|f\|_4 \]

\[ = \frac{(2n + 1)}{(n + 1)} \|f\|_4 \leq 2 \|f\|_4. \]

So

\[ \|t_0^n\|_q \leq 2 \sup_{f \in L^q \setminus \{0\}} \frac{\|t_0^n \ast t_{nn}^n \ast f\|_4}{\|t_{nn}^n \ast f\|_4} \]

\[ = \frac{2}{(2n + 1)} \sup_{\|z_i\|_q \neq 0} \left\| \sum_{i=-n}^{+n} z_i^{(n)} t_0^n i \right\|_4. \]

Therefore (4.3) is true if the conjecture is true. Hence (4.1) is true if the conjecture is true.

References


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<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mogens L. Hansen and Richard V. Kadison</td>
<td>Banach algebras with unitary norms</td>
<td>535</td>
</tr>
<tr>
<td>Xin-hou Hua</td>
<td>Sharing values and a problem due to C.C. Yang</td>
<td>71</td>
</tr>
<tr>
<td>Jing-Song Huang</td>
<td>Harmonic analysis on compact polar homogeneous spaces</td>
<td>553</td>
</tr>
<tr>
<td>Min-Jei Huang</td>
<td>Commutators and invariant domains for Schrödinger propagators</td>
<td>83</td>
</tr>
<tr>
<td>Hisao Kato</td>
<td>Chaos of continuum-wise expansive homeomorphisms and dynamical properties of sensitive maps of graphs</td>
<td>93</td>
</tr>
<tr>
<td>Oliver Küchle</td>
<td>Some properties of Fano manifolds that are zeros of sections in homogeneous vector bundles over Grassmannians</td>
<td>117</td>
</tr>
<tr>
<td>Xin Li and Francisco Marcellan</td>
<td>On polynomials orthogonal with respect to Sobolev inner product on the unit circle</td>
<td>127</td>
</tr>
<tr>
<td>Steven Liedahl</td>
<td>Maximal subfields of $Q(i)$-division rings</td>
<td>147</td>
</tr>
<tr>
<td>Alan L.T. Paterson</td>
<td>Virtual diagonals and $n$-amenability for Banach algebras</td>
<td>161</td>
</tr>
<tr>
<td>Claude Schochet</td>
<td>Rational Pontryagin classes, local representations, and $K^G$-theory</td>
<td>187</td>
</tr>
<tr>
<td>Sandra L. Shields</td>
<td>An equivalence relation for codimension one foliations of 3-manifolds</td>
<td>235</td>
</tr>
<tr>
<td>D. Siegel and E. O. Talvila</td>
<td>Uniqueness for the $n$-dimensional half space Dirichlet problem</td>
<td>571</td>
</tr>
<tr>
<td>Aleksander Simonić</td>
<td>A Construction of Lomonosov functions and applications to the invariant subspace problem</td>
<td>257</td>
</tr>
<tr>
<td>Endre Szabó</td>
<td>Complete intersection subvarieties of general hypersurfaces</td>
<td>271</td>
</tr>
</tbody>
</table>
Mean-value characterization of pluriharmonic and separately harmonic functions
Lev Abramovich Aizenberg, Carlos A. Berenstein and L. Wertheim

Convergence for Yamabe metrics of positive scalar curvature with integral bounds on curvature
Kazuaki Kutagawa

Generalized modular symbols and relative Lie algebra cohomology
Avner D. Ash and David Ginzburg

Convolution and limit theorems for conditionally free random variables
Marek Bożejko, Michael Leinert and Roland Speicher

L_p-bounds for hypersingular integral operators along curves
Sharad Chandarana

On spectra of simple random walks on one-relator groups. With an appendix by Paul Jolissain
Pierre-Alain Cherix, Alain J. Valette and Paul Jolissaint

Every stationary polyhedral set in \( \mathbb{R}^n \) is area minimizing under diffeomorphisms
Jaigyoung Choe

Ramanujan’s master theorem for symmetric cones
Hongming Ding, Kenneth I. Gross and Donald Richards

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Anthony H. Dooley and Sanjiv Kumar Gupta

On the symmetric square. Unit elements
Yuval Zvi Flicker

Stable constant mean curvature surfaces minimize area
Karsten Grosse-Brauckmann

Banach algebras with unitary norms
Mogens Lemvig Hansen and Richard Vincent Kadison

Harmonic analysis on compact polar homogeneous spaces
Jing-Song Huang

Uniqueness for the \( n \)-dimensional half space Dirichlet problem
David Siegel and Erik O. Talvila