ON NORMS OF TRIGONOMETRIC POLYNOMIALS ON SU(2)

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A conjecture about the $L^4$-norms of trigonometric polynomials on $SU(2)$ is discussed and some partial results are proved.

1. Introduction.

If $G$ is a compact abelian group, an elementary argument shows that $M_p(G) = M_q(G)$ where $M_p(G)$ denotes the space of $L^p$-multipliers on $G$ and $p$ and $q$ are conjugate indices. Oberlin [7] found a nonabelian totally disconnected compact group $G$ for which $M_p(G) \neq M_q(G)$. Herz [4] conjectured that inequality holds for all those infinite nonabelian compact groups $G$ whose degrees of the irreducible representations are unbounded. However, for compact connected groups, the situation is still unresolved, even for $SU(2)$.

The present paper arose from an attempt to study the Herz conjecture for $SU(2)$. In his unpublished M.Sc. thesis [8], S. Roberts formulated a conjecture for $SU(2)$ which, if proved, would settle the Herz conjecture for all compact connected groups. We believe that Robert’s conjecture is interesting in its own right as it makes a rather delicate statement connecting the $L^p$-norms of noncentral trigonometric polynomials with the growth of the Clesh-Gordon coefficients.

We have pursued this interesting conjecture and make some partial progress towards settling it. Our results open the way to a detailed study of some new aspects of $L^p$ analysis on compact Lie groups.

In Section 2, we establish our notation. We state the conjecture in Section 3 and prove some partial results (Theorem 3.2). In Section 4, we show the relevance of the conjecture to Herz’s conjecture.

2. Notation and remarks.

2.1. Irreducible representations of $SU(2)$. We summarise some notation and definitions from [6] concerning the irreducible representation of $SU(2)$.
Let $n$ be a rational number of the form $k/2$, where $k \in \mathbb{N}$ and $H_n$ be the space of homogeneous polynomials on $\mathbb{C}^2$ of degree $2n$; i.e. of functions of the form

\begin{equation}
    f(z_1, z_2) = \sum_{i=-n}^{+n} a_i z_1^{n-i} z_2^{n+i}.
\end{equation}

Let $(f, g)$ be the inner product on $H_n$ given by the formula

\begin{equation}
    \left( \sum_{i=-n}^{+n} a_i z_1^{n-i} z_2^{n+i}, \sum_{j=-n}^{+n} b_j z_1^{n-j} z_2^{n+j} \right) = \sum_{i=-n}^{+n} (n-i)!(n+i)!a_ib_i.
\end{equation}

Let $U(H_n)$ denote the set of unitary operators on $H_n$ with respect to the inner product (2.2). The mapping $T_n : SU(2) \rightarrow U(H_n)$ given by

\begin{equation}
    \left( T_n \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} f \right)(z_1, z_2) = f(\alpha z_1 - \beta z_2, \beta z_1 + \bar{\alpha} z_2)
\end{equation}

is an irreducible representation of $SU(2)$ and in fact the set $\{T_n : n = 0, 1/2, \ldots\}$ forms a complete set of inequivalent irreducible representations of $SU(2)$.

To each operator $T_n(x), x \in SU(2)$, there corresponds a unitary matrix (relative to the natural orthonormal basis of $H_n$) whose elements will be denoted by $t^n_{jk} (-n \leq j, k \leq n)$. These matrix elements are continuous functions on $SU(2)$. We shall be estimating their norms as convolution operators on $L^p$.

There are many results on the $L^p$ multiplier norms of central trigonometric polynomials - see for example [2], or the more recent optimal results of Sogge on Riesz kernels on arbitrary compact manifolds (c.f. also [9], [10]). However, the $t^n_{jk}$'s considered here are non-central.

A word about the geometric significance of the matrix coefficients $t^n_{jk}$ is in order. By the Peter-Weyl theorem, $L^2(G)$ decomposes as a direct sum of the irreducible representations of $G$, each one occurring with multiplicity equal to its dimension. These isotypic components represent the eigenspaces of the Laplace-Beltrami operator, and convolution by $(2n+1)\chi_n$, where $\chi_n$ is the character of the $n$th irreducible representation, is the projection onto this space.

For each $j (-n \leq j \leq n)$, the functions $\{t^n_{jk} : -n \leq k \leq n\}$ span one of the above copies of the representation space of degree $2n+1$. Convolution
on the left by \((2n+1)\delta_{j,j}\) is a projection of \(L^2(G)\) onto this copy. Convolution by the function \((2n+1)\delta_{j,j}\) are the natural isometries between the various copies of the \(n\)th irreducible representation inside the isotypic component.

### 2.2. Expressions of products of functions \(\delta_{j,j}\): The tensor product of any two nontrivial irreducible unitary representations of \(SU(2)\) is always reducible. If one decomposes such a tensor product into its irreducible components, then the coefficients which appear in the decomposition are known as the Clebsch-Gordan coefficients.

In the case of \(SU(2)\) the Clebsch-Gordan coefficients \(C(n_1, n_2, n_3, j_1, j_2, j_3)\) make their appearance in this way in the formula

\[
\sum_{m=|n_1-n_2|}^{n_1+n_2} C(n_1, n_2, m, j_1, k_1, k_1 + j_1) \cdot C(n_1, n_2, m, j_2, k_2, k_2 + j_2)\delta_{j_1+k_1,j_2+k_2}.
\]

While the Clebsch-Gordan coefficients are, in general, very complicated [8], there are simple formulas for them in certain situations. Two such cases are given below; they will be of interest in Section 3.

\[
C(n, n, 2n, j, j + k) = \left(\frac{(2n + j + k)!(2n + j - k)!2n!2n!}{(n - j)!(n + j)!(n - k)!(n + k)!4n!}\right)^{1/2}
\]

\[
C(n, n, 2m, j, j, 2j) = (-1)^{n-m} \left(\frac{(4m + 1)(2j + 2m)!2m - 2m!2m - 2j!}{(2n + 2m + 1)!}\right)^{1/2} \times \frac{(m + n)!}{(m + j)!(m - j)!(n - m)!}
\]

if \(n \geq m \geq |j|\) and 0 otherwise

\[
C(n, n, 2m + 1, j, j, 2j) = 0.
\]

We denote by \(C_{ik}^{2m}\) the Clebsch-Gordan coefficient \(C(n, n, 2m, i, k, i + k)\) where \(n\) will considered be fixed throughout the argument.

In Section 4, we use the following convolution identity ([5], 27.20)

\[
t_{jk}^{n} * t_{pq}^{m} = \frac{1}{2n+1} \delta_{nm} \delta_{kp} t_{jq}^{n}.
\]

By \(A_n \approx B_n\), \(n > 1\) we mean that there exist positive constants \(\alpha, \beta\) such that

\[
\beta B_n \leq A_n \leq \alpha B_n, \quad \forall n \geq 1.
\]

The same symbol \(C\) may denote two different constants in two different lines.
3. The conjecture.

In this section we state the conjecture and prove some partial results.

**Conjecture.** Denote by $z^{(n)} \in \mathbb{C}^{2n+1}$ the vector with components $\{z_i^{(n)}\}_{i=-n}^{+n}$. Let

$$A_n = \frac{1}{n^{1/8}} \sup_{\sum_{i=-n}^{+n} |z_i^{(n)}| = 1} \frac{\left\| \sum_{i=-n}^{+n} z_i^{(n)} t_{0i}^{(n)} \right\|_4}{\left\| \sum_{i=-n}^{+n} z_i^{(n)} t_{ni}^{(n)} \right\|_4}.$$ 

Then $A_n \to 0$ as $n \to \infty$.

**Remark 3.1.** For the motivation of the conjecture, see Section 4.

We will prove the following theorem which is a weaker version of the conjecture:

**Theorem 3.2.**

(A) Let $z^{(n)} \in \mathbb{C}^{2n+1}$ and $z_i^{(n)} \geq 0$, $\forall i = -n, \ldots, n$. Define $F_n(z^{(n)}) = \{i | z_i^{(n)} \neq 0\}$. Suppose that $\sum_{i=-n}^{+n} |z_i^{(n)}| = 1$.

If

$$\left| F_n(z^{(n)}) \right| \leq \frac{Cn^{1/3}}{(\log n)^{2/3}} \quad \forall n \geq 2,$$

then

$$\frac{1}{n^{1/8}} \left\| \sum_{i=-n}^{+n} z_i^{(n)} t_{0i}^{(n)} \right\|_4 \leq \frac{C}{(\log n)^{1/4}}.$$ 

(B) Let $\{p_n\}_{n=1}^\infty$ be a sequence of natural numbers such that $p_n \geq 2 \forall n$. Define $j_n = \left[ \frac{\log n}{\log p_n} \right]$. Then

$$\frac{1}{n^{1/8}} \sup_{\sum_{i=-n}^{+n} |z_i^{(n)}| = 1} \left\| \sum_{i=-n}^{+n} z_i^{(n)} t_{jip_n}^{(n)} \right\|_4 \leq C \frac{\log n}{n^{1/4}}.$$ 

To prove Theorem 3.2, we first obtain an expression for $\left\| \sum_{i=-n}^{+n} z_i t_{ji}^{(n)} \right\|_4$ in terms of Clebsch-Gordan coefficients. Let $z \in \mathbb{C}^{2n+1}$. 


Set
\[ \varphi_j^n(z) = \sum_{i=-n}^{+n} z_i t_{ji}^n \]
\[ (\varphi_j^n(z))^2 = \left( \sum_{i=-n}^{+n} z_i t_{ji}^n \right) \left( \sum_{k=-n}^{+n} z_k t_{jk}^n \right) \]
\[ = \sum_{-n \leq i, k \leq n} z_i z_k t_{ji}^n t_{jk}^n \]
\[ = \sum_{-n \leq i, k \leq n} z_i z_k \left( \sum_{m=0}^{2n} C(n, n, j, j, 2j) C(n, n, i, k, i + k) t_{2j}^{2m} \right). \]

Using (2.6)-(2.7), we get
\[ (\varphi_j^n(z))^2 = \sum_{-n \leq i, k \leq n} z_i z_k \sum_{m=|j|}^{n} C(n, n, 2m, j, j, 2j) C(n, n, 2m, i, k, i + k) t_{2j}^{2m} \]
\[ = \sum_{m=|j|}^{n} C_{jj}^{2m} \sum_{r=-2m}^{+2m} \left( \sum_{i+k=r, -n \leq i, k \leq n} z_i z_k C_{ik}^{2m} \right) t_{2j}^{2m} r. \]

Since \( \{ \sqrt{2n + 1} \ t_{ij}^n \}_{-n \leq i, j \leq n} \) is an orthonormal set in \( L^2(SU(2)) \), we get
\[ \left\| (\varphi_j^n(z))^2 \right\|^2 = \sum_{m=|j|}^{n} \frac{C_{jj}^{2m}}{(4m + 1)} \sum_{r=-2m}^{+2m} \left( \sum_{i+k=r, -n \leq i, k \leq n} z_i z_k C_{ik}^{2m} \right)^2. \]

In particular,
\[ \| \varphi_0^n(z) \|_4 = \left[ \sum_{m=-n}^{n} \frac{(C_{00}^{2m})^2}{4m + 1} \sum_{r=-2m}^{+2m} \left( \sum_{i+k=r, -n \leq i, k \leq n} z_i z_k C_{ik}^{2m} \right)^2 \right]^{1/4} \]
\[ \| \varphi_n^n(z) \|_4 = \left[ \frac{(C_{nn}^{2n})^2}{(4n + 1)} \sum_{r=-2n}^{+2n} \left( \sum_{i+k=r, -n \leq i, k \leq n} z_i z_k C_{ik}^{2n} \right)^2 \right]^{1/4} \]
\[ = \left[ \frac{1}{(4n + 1)} \sum_{r=-2n}^{+2n} \left( \sum_{i+k=r, -n \leq i, k \leq n} z_i z_k C_{ik}^{2n} \right)^2 \right]^{1/4} \]

as \( C_{nn}^{2n} = 1. \)
Next we prove two Lemmas.

**Lemma 3.5.** There exist constants $C_1, C_2 > 0$ satisfying: $(n \geq 2)$

1. $\frac{C_2}{n^{1/4}} \leq |C_{00}^{2n}| \leq \frac{C_2}{n^{1/4}}$.
2. $\frac{C_2}{\sqrt{2n+1}} \leq |C_{00}^{00}| \leq \frac{C_1}{\sqrt{2n+1}}$.
3. Let $0 \leq |j| \leq n - 1$. Then

$$\frac{C_2 n^{1/4}}{(n + j)^{1/4}(n - j)^{1/4}} \leq |C_{jj}^{2n}| \leq \frac{C_1 n^{1/4}}{(n + j)^{1/4}(n - j)^{1/4}}.$$  

4. Let $|j| + 1 \leq m \leq n - 1$. Then

$$\frac{C_2 \sqrt{m}}{(m + n)^{1/4}(m + j)^{1/4}(m - j)^{1/4}(n - m)^{1/4}} \leq |C_{jj}^{2m}| \leq \frac{C_1 \sqrt{m}}{(m + n)^{1/4}(m + j)^{1/4}(m - j)^{1/4}(n - m)^{1/4}}.$$  

5. Let $1 \leq j \leq n - 1$. Then

$$\frac{C_2 j^{1/4}}{(n + j)^{1/4}(n - j)^{1/4}} \leq |C_{jj}^{2j}| \leq \frac{C_1 j^{1/4}}{(n + j)^{1/4}(n - j)^{1/4}}.$$  

6. Let $1 \leq j \leq n - 1$. Then

$$\frac{C_2}{(n + j)^{1/4}(n - j)^{1/4}} \leq |C_{00}^{2j}| \leq \frac{C_1}{(n + j)^{1/4}(n - j)^{1/4}}.$$  

**Proof.** The easy proof using the following inequality

$$e^{7/8} \leq \frac{n!}{(n/e)^n n^{1/2}} \leq e$$  
for $n = 1, 2, 3, ...$

is left to the reader.

□

**Lemma 3.6.** Let $n \geq 2$ be a natural number and $-n \leq i \leq n$. Then there exists a positive constant $C$ such that

$$\frac{1}{n^{1/8}} \frac{\| t_{0i}^n \|_4}{\| t_{n,i}^n \|_4} \leq C \left( \frac{\log n}{n} \right)^{1/4}. \tag{3.7}$$

Also for every $\epsilon$, $0 < \epsilon < 1$, there exists a $C_\epsilon > 0$ such that for $0 \leq |i| \leq n\epsilon$, we have

$$C_\epsilon \left( \frac{\log n}{n} \right)^{1/4} \leq \frac{1}{n^{1/8}} \frac{\| t_{0i}^n \|_4}{\| t_{n,i}^n \|_4} \leq C \left( \frac{\log n}{n} \right)^{1/4}. \tag{3.8}$$
Proof. Using (3.3)-(3.4), we get

\[
(3.9) \| t^n_{0,i} \|_4 = \left[ \sum_{m=0}^{n} \frac{(C^{2m}_{00})^2}{(4m+1)} (C^{2m}_{ii})^2 \right]^{1/4} = \left[ \sum_{m=|i|}^{n} \frac{(C^{2m}_{00})^2 (C^{2m}_{|i||i|})^2}{4m+1} \right]^{1/4}
\]

as \( C^{2m}_{ii} = C^{2m}_{|i||i|} \) and \( C^{2m}_{|i||i|} = 0 \) for \( m < |i| \) and

\[
(3.10) \| t^n_{n,i} \|_4 = \frac{\sqrt{C^{2n}_{|i||i|}}}{(4n+1)^{1/4}}.
\]

From (3.9)-(3.10) we see that \( \| t^n_{0,i} \|_4 = \| t^n_{0,-i} \|_4 \) and \( \| t^n_{n,i} \|_4 = \| t^n_{n,-i} \|_4 \).

Therefore we assume that \( 0 < i < n \). We divide the rest of the proof in four steps:

**Step 1.** \( i = 0 \)

\[
\frac{1}{n^{1/8}} \| t^n_{0,0} \|_4 = \left[ \sum_{m=0}^{n} \frac{(C^{2m}_{00})^4}{4m+1} \sqrt{n} \right]^{1/4} \approx \left[ \frac{1}{n} + \sum_{m=1}^{n-1} \frac{1}{(n+m)(n-m)(4m+1)} \right]^{1/4} \approx \left[ \frac{1}{n} + \sum_{m=1}^{n-1} \frac{1}{m(n-m)} \right]^{1/4} \approx \left( \log n \right)^{1/4}.
\]

**Step 2.** \( i = n \)

\[
\frac{1}{n^{1/8}} \| t^n_{0,n} \|_4 = \left[ \frac{4n+1}{\sqrt{n}} \frac{(C^{2n}_{00})^2}{(4n+1)} \right]^{1/4} \approx \frac{1}{n^{1/4}}.
\]

**Step 3.** \( 1 \leq i \leq n - 1 \)
\[ \frac{1}{n^{1/8}} \| e_{n,4} \| = \left[ \frac{4n+1}{\sqrt{n}} \sum_{m=0}^{n} \frac{(C_{0,0}^{2m})^2 (C_{1,1}^{2m})^2}{(4m+1) (C_{1,1}^{2n})^2} \right]^{1/4} \]

\[ = \left[ \frac{4n+1}{\sqrt{n}} \left\{ \frac{(C_{0,0}^{2i})^2 (C_{1,1}^{2i})^2}{(4i+1) (C_{1,1}^{2n})^2} + \sum_{m=i+1}^{n-1} \frac{(C_{0,0}^{2m})^2 (C_{1,1}^{2m})^2}{(4m+1) (C_{1,1}^{2n})^2} + \frac{(C_{0,0}^{2n})^2}{(4n+1)} \right\} \right]^{1/2} \]

\[ = \left[ \frac{4n+1}{\sqrt{n}} \left\{ A_n + Z_n + B_n \right\} \right]^{1/4} \quad \text{(say)} \]

\[ A_n = \frac{(C_{0,0}^{2i})^2 (C_{1,1}^{2i})^2}{(4i+1) (C_{1,1}^{2n})^2} \]

\[ \approx \frac{1}{(n+i)^{1/2} (n-i)^{1/2}} \left\{ i^{1/2} \right\} 1 \left( \frac{n+i}{n-i} \right)^{1/2} \left( \frac{n+i}{n-i} \right)^{1/2} (4i+1) n^{1/2} \]

\[ \approx \frac{1}{n \sqrt{n-i \sqrt{i}}} \]

Hence \( A_n \leq \frac{C}{n^{3/2}}. \)

\[ B_n = \frac{(C_{0,0}^{2n})^2}{(4n+1)} \approx \frac{1}{n^{3/2}} \]

\[ Z_n = \sum_{m=i+1}^{n-1} \frac{(C_{0,0}^{2m})^2 (C_{1,1}^{2m})^2}{(4m+1) (C_{1,1}^{2n})^2} \]

\[ \approx \sum_{m=i+1}^{n-1} \frac{m(n+i)^{1/2} (n-i)^{1/2}}{(n+m)^{1/2} (n-m)^{1/2} (n+m)^{1/2} (n-m)^{1/2} (m+i)^{1/2} (m-i)^{1/2} (4m+1) n^{1/2}} \]

\[ \approx \frac{1}{n} \sum_{m=i+1}^{n-1} \frac{(n-i)^{1/2}}{(n-m)(m+i)^{1/2} (m-i)^{1/2}}. \]

Now we further divide Step 3 into three cases:

a) \( i = n - 1. \) Then

\[ A_n \leq \frac{C}{n^{3/2}}, \quad B_n \approx \frac{1}{n^{3/2}}, \quad Z_n = 0. \]

Therefore \[ \frac{1}{n^{1/8}} \| e_{n,4} \| \approx \frac{1}{n^{1/4}}. \]
b) \(1 \leq i \leq \frac{n}{2}\). Then

\[
Z_n \leq \frac{C}{\sqrt{n}} \sum_{m=i+1}^{n-1} \frac{1}{(n-m)(m-i)}
\]

\[
= \frac{C}{\sqrt{n}} \sum_{m=i+1}^{n-1} \left[ \frac{1}{n-m} + \frac{1}{m-i} \right] \frac{1}{(n-i)}
\]

\[
= \frac{2C}{\sqrt{n}} \left[ \sum_{j=1}^{n-i-1} \frac{1}{j} \right] \frac{1}{(n-i)}
\]

\[
\leq \frac{4C}{n^{3/2}} [1 + \log(n - i - 1)] \leq \frac{8C (\log n)}{n^{3/2}}.
\]

Therefore \(\frac{1}{n^{1/8}} \| T^{\ast}_{n,n} \|_4 \leq C \left( \frac{\log n}{n} \right)^{1/4}\).

c) \(\frac{n}{2} \leq i \leq n-2\). Then

\[
Z_n \leq \frac{C}{n^{3/2}} \sum_{m=i+1}^{n-1} \frac{\sqrt{n-i}}{(n-m)\sqrt{m-i}}
\]

\[
= \frac{C}{n^{3/2}\sqrt{n-i}} \sum_{m=i+1}^{n-1} \left[ \frac{\sqrt{m-i}}{(n-m)} + \frac{1}{\sqrt{m-i}} \right]
\]

\[
\leq \frac{C}{n^{3/2}} \left[ \sum_{m=i+1}^{n-1} \frac{1}{(n-m)} + \frac{1}{\sqrt{n-i}} \sum_{m=i+1}^{n-1} \frac{1}{\sqrt{m-i}} \right]
\]

\[
\leq \frac{C \log n}{n^{3/2}}.
\]

Therefore \(\frac{1}{n^{1/8}} \| T^{\ast}_{i,n} \|_4 \leq C \left( \frac{\log n}{n} \right)^{1/4}\).

Step 4. Let \(0 < \epsilon < 1\) and \(0 \leq i \leq n\epsilon\). In this case,

\[
Z_n \approx \frac{1}{\sqrt{n}} \sum_{l=i+1}^{n-1} \frac{1}{(n-l)(l+i)^{1/2}(l-i)^{1/2}}.
\]
Therefore

\[
Z_n \geq \frac{1}{\sqrt{n}} \sum_{i=1}^{n-1} \frac{1}{(n-l)(l+i)}
\]

\[
= \frac{1}{\sqrt{n(n+i)}} \sum_{i=1}^{n-1} \left[ \frac{1}{(n-l)} + \frac{1}{(l+i)} \right]
\]

\[
\geq \frac{1}{n^{3/2}} \sum_{j=1}^{n-i-1} \frac{1}{j} \geq \frac{\log(n-i)}{n^{3/2}}
\]

\[
\geq C_{\epsilon} \frac{\log n}{n^{3/2}}.
\]

Hence

\[
\frac{1}{n^{1/8}} \left\| t_{n,0}^{-1} t_{n}^{n,1} \right\|_4 \geq C_{\epsilon} \left( \frac{\log n}{n} \right)^{1/4}.
\]

Now by Step 1 and Step 3 we have

\[
\frac{1}{n^{1/8}} \left\| t_{n,0}^{-1} t_{n}^{n,1} \right\|_4 \leq C \left( \frac{\log n}{n} \right)^{1/4}
\]

for all \(0 \leq i \leq n - 1\).

Therefore

\[
C_{\epsilon} \left( \frac{\log n}{n} \right)^{1/4} \leq \frac{1}{n^{1/8}} \left\| t_{n,0}^{-1} t_{n}^{n,1} \right\|_4 \leq C \left( \frac{\log n}{n} \right)^{1/4}.
\]

Proof of Theorem 3.2.

(A) Let \( z^{(n)} \) be as in the hypothesis of Theorem 3.2(A).

Consider

\[
\left\| \sum_{i=-n}^{n} z_{i}^{(n)} t_{n,i} \right\|_4 = \left[ \frac{1}{(4n+1)} \sum_{r=-2n}^{2n} \sum_{i+k=r}^{2n} \left| z_{i}^{(n)} z_{k}^{(n)} C_{i,k}^{2n} \right|^2 \right]^{1/4}
\]

\[
\geq \left[ \frac{1}{(4n+1)} \sum_{i=-n}^{n} \left( C_{i,i}^{2n} \right)^2 \left| z_{i}^{(n)} \right|^4 \right]^{1/4} \quad \text{as } C_{i,k}^{2n} \geq 0, \forall i, k
\]

\[
= \left( \sum_{i=-n}^{n} \left| z_{i}^{(n)} \right|^4 \right)^{1/4} \left\| t_{n,0}^{-1} t_{n}^{n,1} \right\|_4^{1/4}.
\]
Now

\[
\frac{1}{n^{1/8}} \left\| \sum_{i=-n}^{+n} z_i^{(n)} t_{ni} \right\|_4 \leq \frac{1}{n^{1/8}} \left( \frac{\sum_{i=-n}^{+n} z_i^{(n)} \left\| t_{ni} \right\|_4}{\left( \sum_{i=-n}^{+n} \left( z_i^{(n)} \right)^4 \left\| t_{ni} \right\|_4^4 \right)^{1/4}} \right)
\]

\[
\leq \frac{1}{n^{1/8}} \left[ \left( \sum_{i=-n}^{+n} \left( z_i^{(n)} \right)^4 \left\| t_{ni} \right\|_4^4 \right)^{1/4} \right] F_n \left( z^{(n)} \right)^{3/4}
\]

\[
\leq C \left( \frac{\log n}{n} \right)^{1/4} \left( \frac{n^{1/3}}{\left( \log n \right)^{2/3}} \right)^{3/4} \leq \frac{C}{\left( \log n \right)^{1/4}},
\]

(by Lemma 3.6).

This completes the proof of part (A).

(B) Consider

\[
\left\| \sum_{i=0}^j z_i^{(n)} t_{n p_n^i} \right\|_4 \leq \left[ \frac{1}{(4n + 1)} \sum_{r=0}^{+2n} \sum_{p_n^i + p_n^k = r \atop 0 \leq i, k \leq j_n} \sum_{i=0}^{j_n} z_i^{(n)} z_k^{(n)} C_{p_n^i p_n^k}^{2n} \right]^{2-1/4}
\]

\[
\geq \left[ \frac{1}{(4n + 1)} \sum_{i=0}^{j_n} \left| z_i^{(n)} \right|^4 \left( C_{p_n^i p_n^l}^{2n} \right)^2 \right]^{1/4}
\]

as

\[
\sum_{p_n^i + p_n^k = r} \left| z_i^{(n)} z_k^{(n)} C_{p_n^i p_n^k}^{2n} \right|^2 = \left| z_i^{(n)} \right|^4 \left( C_{p_n^i p_n^l}^{2n} \right)^2 \text{ if } r = 2p_n^l.
\]

Therefore

\[
\left\| \sum_{i=0}^j z_i^{(n)} t_{n p_n^i} \right\|_4 \geq \left[ \sum_{i=0}^{j_n} \left| z_i^{(n)} \right|^4 \left\| t_{n p_n^i} \right\|_4^4 \right]^{1/4}.
\]
Hence

\[ \frac{1}{n^{1/8}} \left\| \sum_{i=0}^{j_n} z_i^{(n)} t_{0,p_n}^n \right\|_4 \leq \frac{1}{n^{1/8}} \frac{\sum_{i=0}^{j_n} z_i^{(n)} \left\| t_{0,p_n}^n \right\|_4}{\sum_{i=0}^{j_n} z_i^{(n)} \left\| t_{n,p_n}^n \right\|_4}^{1/4} \]

\[ \leq \frac{1}{n^{1/8}} \left[ \sum_{i=0}^{j_n} z_i^{(n)} \left\| t_{0,p_n}^n \right\|_4 \right]^{1/4} \frac{j_n^{3/4}}{\sum_{i=0}^{j_n} z_i^{(n)} \left\| t_{n,p_n}^n \right\|_4^{1/4}} \]

(by Hölder's inequality),

\[ \leq C \left( \frac{\log n}{n} \right)^{1/4} (\log n)^{3/4} \]

\[ = C \frac{\log n}{n^{1/4}}. \]

This completes the proof of the Theorem.

\[ \square \]

**Remark 3.11.** The following inequality can be proved by using the ideas of the proof of Theorem 3.2(B):

\[ \frac{1}{n^{1/8}} \left\| z_1 t_{0,p}^n + z_2 t_{0,q}^n \right\|_4 \leq C \left( \frac{\log n}{n} \right)^{1/4} \]

for \(-n \leq p, q \leq n\).

### 4.

Let \( G \) be a compact group and let \( \Gamma \) be the dual object of \( G \), the set of equivalence classes of irreducible unitary representations of \( G \). For each \( \sigma \in \Gamma \), select a representation \( U_{\sigma} \in \sigma \), let \( H_\sigma \) be the Hilbert space on which \( U_{\sigma} \) acts, and let \( d_\sigma \) be the dimension of \( H_\sigma \). Let \( B(H_\sigma) \) denote the space of linear operators on \( H_\sigma \) and \( C(\Gamma) \) denote the space \( \prod_{\sigma \in \Gamma} B(H_\sigma) \).

**Definition.** Fix \( p \in [1, \infty] \). Let \( m \) be an element of \( C(\Gamma) \), so that for each \( \sigma \), \( m(\sigma) \in B(H_\sigma) \). The function \( m \) is a (left) multiplier of \( L^p(= L^p(G)) \) if for each \( f \in L^p \), the series

\[ \sum_{\sigma \in \Gamma} d_\sigma \text{tr} \left[ m(\sigma) \hat{f}(\sigma) U_\sigma(x) \right] \]
is the Fourier series of some function $L_m f \in L^p$. The collection of all such $m$ is denoted by $M_p(G)$ or simply $M_p$.

For each $m \in M_p$, the map $f \to L_m f$ defines a bounded linear operator on $L^p$, an operator which commutes with left translations by the elements of $G$. We regard $M_p$ as a Banach space under the operator norm.

When $G$ is abelian, an easy argument shows that if $\frac{1}{p} + \frac{1}{q} = 1$, then $M_p = M_q$. It is known that for $1 < p < 2$, $M_p \neq M_q$ ($\frac{1}{p} + \frac{1}{q} = 1$) for many nonabelian groups $G$ (see [1, 2, 3, 4, 6]).

For connected compact non-abelian group $G$ and for $1 < p < 2$, it is an open problem whether $M_p = M_q$, $\frac{1}{p} + \frac{1}{q} = 1$.

S.G. Roberts has shown in [8] that if the conjecture is true then $M_p(G) \neq M_q(G)$ for every connected compact non-abelian group and $1 < p < 2$, $\frac{1}{p} + \frac{1}{q} = 1$.

We give an easy proof that if the conjecture is true then $M_p(SU(2)) \neq M_q(SU(2))$ for $1 < p < 2$, $\frac{1}{p} + \frac{1}{q} = 1$. This proof is essentially due to Roberts [8], but has never to the best of our knowledge been published. We present it here for completeness. The result will follow if we show that

$$\frac{\|t^n_{0n}\|_p}{\|t^n_{0n}\|_q} \to \infty \text{ as } n \to \infty$$

where $\|t^n_{0n}\|_p$ denotes the norm of $t^n_{0n}$ as an element in $M_p$.

To prove (4.1), we use the following norm estimates for $t^n_{0n}$ and $t^n_{n}$ which are easy to establish (see [8]).

$$\|t^n_{0n}\|_p \approx \frac{1}{n^{1/4+1/2p}}, \quad \|t^n_{n}\|_p = \frac{1}{(np + 1)^{1/p}}$$

$$\|t^n_{0n}\|_1 = \|t^n_{0n}\|_1 \approx \frac{1}{n^{3/4}}, \quad \|t^n_{0n}\|_2 = \frac{1}{2n + 1}.$$ 

Now by Riesz convexity theorem, we get

$$\|t^n_{0n}\|_p \leq \|t^n_{0n}\|_1^\alpha \|t^n_{0n}\|_2^{1-\alpha}$$

where

$$\alpha = \frac{2 - p}{p}.$$ 

Hence

$$\|t^n_{0n}\|_p \leq \frac{C}{n^{(5/4) - (1/2p)}}.$$
Also
\[ \|t^n_{0n}\|_p \geq \frac{\|t^n_{0n} * t^n_{n}\|_p}{\|t^n_{0n}\|_p} = \frac{1}{(2n+1)} \frac{\|t^n_{0n}\|_p}{\|t^n_{n}\|_p} \geq C \frac{n}{n^{(5/4)-(1/2p)}}. \]

Therefore (4.1) is true if
\[(4.2) \quad n^{(5/4)-(1/2p)} \|t^n_{0n}\|_q \to 0 \text{ as } n \to \infty.\]

A routine argument using Riesz convexity theorem shows that (4.2) is true if
\[(4.3) \quad n^{-7/8} \|t^n_{0n}\|_4 \to 0 \text{ as } n \to \infty.\]

Now
\[ \|t^n_{0n}\|_4 = \sup_{f \in L^n, f \neq 0} \frac{\|t^n_{0n} * f\|_4}{\|f\|_4} = \sup_{f \in L^n, f \neq 0} \frac{\|t^n_{0n} * t^n_{n} * f\|_4}{\|f\|_4} (2n + 1) \]

and
\[ (2n + 1) \|t^n_{n} * f\|_4 \leq (2n + 1) \|t^n_{n}\|_1 \|f\|_4 \]
\[ = \frac{2}{(n+1)} \|f\|_4 \leq 2 \|f\|_4. \]

So
\[ \|t^n_{0n}\|_4 \leq 2 \sup_{f \in L^n, f \neq 0} \frac{\|t^n_{0n} * t^n_{n} * f\|_4}{\|t^n_{n} * f\|_4} \]
\[ = \frac{2}{(2n + 1)} \sup_{\sum_{i=-n}^{n} |z_i^{(n)}| \neq 0} \frac{\|\sum_{i=-n}^{n} z_i^{(n)} t^n_{0i}\|_4}{\|\sum_{i=-n}^{n} z_i^{(n)} t^n_{ni}\|_4}. \]

Therefore (4.3) is true if the conjecture is true. Hence (4.1) is true if the conjecture is true.

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