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ON A THEOREM OF KOCH

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We give a short proof of a slightly stronger version of a theorem of Koch: A complex quadratic field whose ideal class group contains a subgroup of type $(4, 4, 4)$ possesses an infinite unramified Galois pro-2 extension.

1. Koch's Theorem.

If K is a finite extension of \mathbb{Q} and p is a prime number, let $K^{(0)} = K$ and for $n \geq 1$ define $K^{(n)}$ to be the maximal abelian unramified p -extension of $K^{(n-1)}$. The smallest n such that $K^{(n)} = K^{(n+1)}$ is called the length of the p -class field tower of K ; if no such integer n exists, we say that K has infinite p -class field tower. By a group of type (m_1, \dots, m_t) we understand a group isomorphic to $\mathbb{Z}/m_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/m_t\mathbb{Z}$. The purpose of this note is to give a short proof of (a slightly strengthened version of) a theorem of Koch [4]:

Theorem 1. *If K is a complex quadratic field whose ideal class group contains a subgroup of type $(4, 4, 4)$, then the 2-class field tower of K is infinite.*

Koch's proof proceeds by showing that in a minimal presentation of the Galois group of the maximal unramified 2-extension of K by a free pro-2 group G , the relations lie deep in the Zassenhaus filtration of G . We replace this key ingredient of his proof, which can be thought of as the study of the quadratic unramified extensions of the genus field of K which are central over K [3, Satz 1], with a simple result from genus theory. Moreover, Koch's proof requires a generalization of the Vinberg/Gashütz sharpening of the Golod-Shafarevich theorem on the structure of pro- p groups [4, Satz 3]; for our proof, the original Vinberg/Gashütz inequality suffices (for an account of these inequalities, see, e.g., Koch's book [5]). Indeed, we will need only the following result (see Martinet [8]):

Theorem 2. *Suppose F is a totally real field of degree n , and E is a totally complex quadratic extension of F . Let t be the number of prime ideals of F which ramify in E . The 2-rank of the ideal class group of E is at least $t - 1$. If*

$$t \geq 3 + 2\sqrt{n+1},$$

then the 2-class field tower of E is infinite.

Corollary 3. *Suppose F is a totally real degree 4 extension of \mathbb{Q} . If two rational primes that split completely in F ramify in a complex quadratic field L , then $E = FL$ has an infinite 2-class field tower.*

Proof. With notation as in the theorem, we have $t \geq 8 \geq 3 + 2\sqrt{4+1}$. \square

Proof of Theorem 1. We know that at least four primes divide the discriminant D of K . If six or more primes divide D , then an application of Theorem 2 to K/\mathbb{Q} already yields the result. Assume first that exactly four primes divide D . By the criterion of Rédei-Reichardt [9] on the 4-rank of the class group of K , one knows that $D = -p_1 \cdot p_2 \cdot p_3 \cdot p_4$ where p_2, p_3, p_4 are odd primes satisfying $\left(\frac{p_i}{p_j}\right) = +1$ for $i, j > 1, i \neq j$, and one of the following is satisfied:

- (I) $p_1 = 4; p_j \equiv 1 \pmod{8}, j = 2, 3, 4.$
- (II) $p_1 = 8; p_j \equiv 1 \pmod{8}, j = 2, 3, 4.$
- (III) $p_1 = 8; p_2 \equiv 7 \pmod{8}; p_j \equiv 1 \pmod{8}, j = 3, 4.$
- (IV) $p_1 \equiv 3 \pmod{4}$ is an odd prime, $p_j \equiv 1 \pmod{4}, j = 2, 3, 4,$
and $\left(\frac{p_1}{p_j}\right) = +1$ for $j = 2, 3, 4.$

Incidentally, Koch's theorem was originally stated for case (IV) only. Let $F = \mathbb{Q}(\sqrt{p_3}, \sqrt{p_4})$ and $E = F(\sqrt{-p_1 \cdot p_2})$. In all cases, (p_2) and the unique rational prime divisor of (p_1) split completely in F . Hence, by Corollary 3, E has an infinite 2-class field tower. Since E/K is an unramified 2-extension, K has an infinite 2-class field tower as well. Now suppose exactly five primes p_1, \dots, p_5 divide the discriminant of K ; using the Rédei-Reichardt criterion [9], or its equivalent form due to Rédei [10], it is straightforward to check that for some $i, 1 \leq i \leq 5$, we have

$$p_i \equiv 1 \pmod{4}, \quad \left(\frac{p_i}{p_j}\right) = 1, j \neq i.$$

Now let $F = \mathbb{Q}(\sqrt{p_i}), E = K(\sqrt{p_i})$; E/F is a CM-extension with 8 ramified primes. By Theorem 2, E has an infinite 2-class field tower, and so does K . \square

2. Further Remarks.

Koch and Venkov [6] have proved that a complex quadratic field whose ideal class group has a subgroup of type (p, p, p) for some odd prime p has an

infinite p -class field tower. Therefore, a complex quadratic field possesses an infinite Hilbert class field tower whenever its ideal class group contains a subgroup of type (m, m, m) with $m \geq 3$. On the other hand, the field $\mathbb{Q}(\sqrt{-105})$, whose ideal class group is of type $(2, 2, 2)$, has a finite class field tower, since its root discriminant is just below the Odlyzko bound (see e.g. [8]). I am indebted to the referee for the above remark.

Note that the proof of Koch's theorem we have given relies only on the existence of two primes that split completely in a real biquadratic field. For instance, the primes 31, 89 split completely in $\mathbb{Q}(\sqrt{2}, \sqrt{5})$, hence $\mathbb{Q}(\sqrt{-2 \cdot 5 \cdot 31 \cdot 89})$ has an infinite 2-class field tower; its 2-ideal class group is of type $(4, 2, 2)$.

Taussky-Todd [12] proved that a number field with 2-ideal class group of type $(2, 2)$ has a finite 2-class field tower of length at most 2. It is natural to ask whether there are number fields with infinite 2-class field tower whose 2-class group is of type $(4, 2)$ or $(2, 2, 2)$ (simplest non-cyclic 2-groups after type $(2, 2)$). Using a minor variation on an idea first introduced by Schoof [11], we now show that there are complex quadratic fields with these properties. Consider, for example, $K = \mathbb{Q}(\sqrt{-5 \cdot 7 \cdot 41 \cdot 61})$, which has 2-ideal class group of type $(2, 2, 2)$. To show that this field has infinite 2-tower, let H_0 be the Hilbert class field of $K_0 = \mathbb{Q}(\sqrt{5 \cdot 41 \cdot 61})$, a real quadratic field with class number 16. Since 7 is inert in K_0 , it splits into 16 prime ideals in H_0 , all of which ramify in the CM extension $L = H_0(\sqrt{-7})$. Theorem 2 shows that L , an unramified 2-extension of K , has an infinite 2-class field tower, proving the claim. In fact, for any prime q satisfying $q \equiv 7 \pmod{5 \cdot 41 \cdot 61}$ (there are infinitely many such primes by Dirichlet's theorem), the same argument shows that $K_q = \mathbb{Q}(\sqrt{-5 \cdot 41 \cdot 61 \cdot q})$ has infinite 2-class field tower; furthermore, by Rédei-Reichardt, K_q has 2-class group of type $(2, 2, 2)$.

For the second example, let $K = \mathbb{Q}(\sqrt{-5 \cdot 11 \cdot 461})$; this field has 2-ideal class group of type $(4, 2)$. Observe that the rational prime ideal (11) splits into 16 prime ideals in H_0 , the Hilbert class field of the real quadratic field $K_0 = \mathbb{Q}(\sqrt{5 \cdot 461})$ with class number 16. Therefore, by the same argument as above, $L = H_0(\sqrt{-11})$, and thereby K , have infinite 2-class field tower. Let H be the 2-Hilbert class field of K . Benjamin [1] has shown that the 2-class field tower of a complex quadratic field E with 2-class group of type $(4, 2)$ has length at most 2 if the 2-Hilbert class field of E has elementary 2-class group ($E = \mathbb{Q}(\sqrt{-5 \cdot 13})$ is an example). Since K has infinite 2-tower, we conclude that H does not have elementary abelian 2-class group.

Finally, note that the 2-rank of the ideal class group of L is at least 15. Using Louboutin [7], we compute the 2-rank of the ideal class group of the biquadratic field $E = \mathbb{Q}(\sqrt{-11}, \sqrt{5 \cdot 461})$ to be 2. The arguments of [2] then show that the 2-rank of the ideal class group of L is 15, 16 or 17.

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