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**SOLVABILITY OF DIRICHLET PROBLEMS FOR SEMILINEAR
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SOLVABILITY OF DIRICHLET PROBLEMS FOR SEMILINEAR ELLIPTIC EQUATIONS ON CERTAIN DOMAINS

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We demonstrate a method to solve Dirichlet problems for semilinear elliptic equations on certain domains by a combination of change of variables, variational method and super-sub-solutions method. We show that Dirichlet problems for a semilinear elliptic equation have a least one solution as long as a relationship between the growth rate of the nonlinear term and the size of the domain is satisfied. The result can be applied to semilinear elliptic equations with super-critical growth.

1. Introduction and Results.

Let Ω be a bounded domain in R^n , $n > 2$. We consider the Dirichlet problem for a semilinear elliptic equation

$$(D_0) \quad \begin{cases} -\Delta u = f(x, u) & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ is the standard Laplace operator, $f(x, u)$ is a local Hölder continuous function defined on $\bar{\Omega} \times R$.

Throughout the paper, we assume that:

(†) There are positive constants $M_1, M_2, q \geq 1$, such that

$$|f(x, t)| \leq M_1 + M_2|t|^q \quad \text{for all } x \in \bar{\Omega}, t \in R.$$

The main result of paper is

Theorem 1. *There is a constant $c(n, q)$ depending only on n and q , such that if we assume*

(1) (†);

(2) $|\Omega| \leq c(n, q) \left(M_2 M_1^{q-1}\right)^{-\frac{n}{2q}}$,

then (D_0) has at least one solution.

When $q < \frac{n+2}{n-2}$, a result similar to Theorem 1 was shown in [3]. The method used in [3] is the variational method. When $q > \frac{n+2}{n-2}$, a direct variational approach does not work. We shall use a combination of changes of variables, super- sub- solutions method and variational method to show the result.

As in [3], since the result requires the volume of the domain Ω to be dominated by something related to the nonlinear term, we need to distinguish the result from the triviality of using an implicit function theorem to get a similar result. Here are a few points. First of all, an implicit function theorem tells us that (D_0) has at least one solution when the size of the domain Ω is small, but usually one will not be able to get an explicit upper bound for the size of the domain as we do here. Secondly, in the case that M_2 is small relative to M_1 , the bound in Theorem 1 is not necessarily small at all. Lastly, the bound obtained in the result is invariant under the scaling of the domain (as explained in [3]).

When $f(x, 0) = 0$ on Ω , (D_0) has a trivial solution $u = 0$. And (1) and (2) in Theorem 1 are not enough to assure the existence of a nontrivial solution as indicated by the well known Pohozaev identity [5] for the case that $f(x, t) = |t|^{q-1}t$, $q > \frac{n+2}{n-2}$ and Ω is any ball (see [6] also). To get a non-trivial solution, additional conditions are needed. Let λ_1 be the first eigenvalue of $-\Delta$ on Ω with Dirichlet boundary conditions. Then we have

Theorem 2. *There is a constant $c(n, q)$ depending only on n and q , such that if*

$$(1) \quad (\dagger);$$

$$(2) \quad |\Omega| \leq c(n, q) \left(M_2 M_1^{q-1} \right)^{-\frac{2}{2q}};$$

$$(3) \quad \lim_{t \rightarrow 0^+} \frac{f(x, t)}{t} > \lambda_1 \quad \text{uniformly for } x \in \overline{\Omega},$$

then (D_0) has a positive solution.

Remark. Any function $f(x, t)$ will satisfy (3) in Theorem 2 if near $t = 0$, $t > 0$, $f(x, t)$ behaves like ct^β for some $c > 0$ and $\beta < 1$. Indeed, (3) assures that (D_0) has a family of very small positive subsolutions. And (3) can be replaced by any other conditions which assure the existence of small positive subsolutions for (D_0) .

The ideas of the proofs: since there is no restriction on q , one can not use the variational method directly to solve (D_0) . What we shall do is to combine a change of variable and the variational method to construct a pair of super- sub- solutions. For the purpose of illustration, we give a rough sketch of the proof of Theorem 1 here. Let $f^+(x, t) = \max\{f(x, t), 0\}$,

$f^-(x, t) = \min\{f(x, t), 0\}$. We look at a pair of quasilinear elliptic equations (α is a constant to be chosen).

$$(1) \quad \begin{cases} -\Delta u_1 = f^+(x, u_1) + \frac{\alpha - 1}{u_1} |\nabla u_1|^2 & \text{in } \Omega; \\ u_1 > 0 & \text{in } \Omega; \\ u_1 = 0 & \text{on } \partial\Omega; \end{cases}$$

and

$$(2) \quad \begin{cases} -\Delta u_2 = f^-(x, u_2) + \frac{\alpha - 1}{u_2} |\nabla u_2|^2 & \text{in } \Omega; \\ u_2 < 0 & \text{in } \Omega; \\ u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

If we can solve (1) and (2) for u_1 and u_2 , then $u_2 \leq u_1$, and we have a pair of super- sub- solutions. Thus (D_0) has a solution (for example, see Theorem 6.5 in [4]).

Usually it is not a good idea to solve a semilinear equation by looking at a quasilinear one. But here a change of variable will change the whole picture. For example if $q > \frac{n+2}{n-2}$, $\alpha > \frac{(q-1)(n-2)}{4}$, let $v = \frac{1}{\alpha} |u_1|^\alpha$ in (1), then v satisfies

$$\begin{cases} -\Delta v = f^+ \left(x, (\alpha|v|)^{\frac{1}{\alpha}} \right) (\alpha|v|)^{\frac{(\alpha-1)}{\alpha}} & \text{in } \Omega; \\ v > 0 & \text{in } \Omega; \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus the change of variable has transformed the quasilinear equation into semilinear one with sub- critical growth! Now we can use the variational method and the method used in [3] to get a super- solution u_1 . A sub- solution u_2 can be obtained similarly.

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2. Proofs.

2.1. Proof of Theorem 1. We may assume that $f(x, 0)$ is not identically zero, otherwise $u = 0$ is a trivial solution.

Step 1: Existence of a super- solution u_1 .

We may assume $f^+(x_1, 0) > 0$ for some $x_1 \in \Omega$, otherwise $u_1 = 0$ is a super- solution.

Let $\alpha \geq \max \left\{ \frac{(q-1)(n-2)}{4}, 1 \right\}$. The exact value of α will be determined later. Consider

$$(3) \quad \begin{cases} -\Delta u_1 = f^+(x, u_1) + \frac{\alpha-1}{u_1} |\nabla u_1|^2 & \text{in } \Omega; \\ u_1 > 0 & \text{in } \Omega; \\ u_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Change variable $v = \frac{1}{\alpha} |u_1|^\alpha$, then v satisfies

$$(4) \quad \begin{cases} -\Delta v = f^+ \left(x, (\alpha|v|)^{\frac{1}{\alpha}} \right) (\alpha|v|)^{\frac{(\alpha-1)}{\alpha}} & \text{in } \Omega; \\ v > 0 & \text{in } \Omega; \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

It is clear that every solution of (4) corresponds to a solution of (3).

Set $f_1(x, v) = f^+ \left(x, (\alpha|v|)^{\frac{1}{\alpha}} \right) |\alpha v|^{\frac{(\alpha-1)}{\alpha}}$. Then $f_1(x, v) \geq 0$ for all v and is Hölder continuous about v . (†) implies that for all v

$$(5) \quad 0 \leq f_1(x, v) \leq M_1 |\alpha v|^{\frac{(\alpha-1)}{\alpha}} + M_2 |\alpha v|^{\frac{(q+\alpha-1)}{\alpha}}.$$

Here we observe that $\frac{(q+\alpha-1)}{\alpha} < \frac{n+2}{n-2}$ if $\alpha > \frac{(q-1)(n-2)}{4}$. Thus $f_1(x, v)$ has subcritical growth if $\alpha > \frac{(q-1)(n-2)}{4}$.

Consider the functional

$$J_\alpha(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \int_\Omega F_1(x, v) dx, \quad v \in H_0^1(\Omega),$$

where $F_1(x, v) = \int_0^v f_1(x, s) ds$.

We shall show that $J_\alpha(v)$ has a nontrivial critical point for suitable choice of α (and under the assumption of Theorem 1). Then the regularity theory (see [1]) and the maximum principle imply that the non-trivial critical point is a positive solution to (4).

For any $v \in H_0^1(\Omega)$, from (5) we have

$$\int_\Omega F_1(x, v) dx \leq \frac{\alpha^{2-\frac{1}{\alpha}}}{2\alpha-1} M_1 \int_\Omega |v|^{\frac{2\alpha-1}{\alpha}} dx + \alpha^{\frac{q+2\alpha-1}{\alpha}} \frac{1}{q+2\alpha-1} M_2 \int_\Omega |v|^{\frac{q+2\alpha-1}{\alpha}} dx.$$

Then

$$\begin{aligned} J_\alpha(v) &\geq \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \frac{\alpha^{2-\frac{1}{\alpha}}}{2\alpha-1} M_1 \int_\Omega |v|^{\frac{2\alpha-1}{\alpha}} dx \\ &\quad - \alpha^{\frac{q+2\alpha-1}{\alpha}} \frac{1}{q+2\alpha-1} M_2 \int_\Omega |v|^{\frac{q+2\alpha-1}{\alpha}} dx. \end{aligned}$$

Let q_1, q_2 be defined by $\frac{1}{q_1} = \frac{2}{n} + \frac{1}{\alpha} \frac{n-2}{2n}$ and $\frac{1}{q_2} = \frac{2}{n} - \frac{(q-1)(n-2)}{\alpha 2n}$. Using Hölder inequality and Sobolev embedding inequality (see [8])

$$\left(\int_{\Omega} |v|^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{2n}} \leq S(n) \left(\int_{\Omega} |\nabla v|^2 dx \right)^{\frac{1}{2}} \quad v \in H_0^1(\Omega),$$

we have

$$\begin{aligned} J_{\alpha}(v) &\geq \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \frac{\alpha^{2-\frac{1}{\alpha}}}{2\alpha-1} S(n)^{\frac{2\alpha-1}{\alpha}} M_1 |\Omega|^{\frac{1}{q_1}} \left(\int_{\Omega} |\nabla v|^2 dx \right)^{\frac{2\alpha-1}{2\alpha}} \\ &\quad - \frac{1}{q+2\alpha-1} \alpha^{\frac{q+2\alpha-1}{\alpha}} S(n)^{\frac{q+2\alpha-1}{\alpha}} M_2 |\Omega|^{\frac{1}{q_2}} \left(\int_{\Omega} |\nabla v|^2 dx \right)^{\frac{q+2\alpha-1}{2\alpha}}. \end{aligned}$$

Denote $\left(\int_{\Omega} |\nabla v|^2 dx \right)^{\frac{1}{2}} = \rho$, we get

$$\begin{aligned} J_{\alpha}(v) &\geq \frac{1}{2} \rho^2 - \frac{\alpha^{2-\frac{1}{\alpha}}}{2\alpha-1} S(n)^{\frac{2\alpha-1}{\alpha}} M_1 |\Omega|^{\frac{1}{q_1}} \rho^{\frac{2\alpha-1}{\alpha}} \\ &\quad - \frac{1}{q+2\alpha-1} \alpha^{\frac{q+2\alpha-1}{\alpha}} S(n)^{\frac{q+2\alpha-1}{\alpha}} M_2 |\Omega|^{\frac{1}{q_2}} \rho^{\frac{q+2\alpha-1}{\alpha}} \\ &= \left(\frac{1}{2} - \frac{1}{q+2\alpha-1} \alpha^{\frac{q+2\alpha-1}{\alpha}} S(n)^{\frac{q+2\alpha-1}{\alpha}} M_2 |\Omega|^{\frac{1}{q_2}} \rho^{\frac{q-1}{\alpha}} \right) \rho^2 \\ &\quad - \frac{\alpha^{2-\frac{1}{\alpha}}}{2\alpha-1} S(n)^{\frac{2\alpha-1}{\alpha}} M_1 |\Omega|^{\frac{1}{q_1}} \rho^{\frac{2\alpha-1}{\alpha}}. \end{aligned}$$

Let ρ be defined by

$$(6) \quad \rho = \left(\frac{4}{q+2\alpha-1} \alpha^{\frac{q+2\alpha-1}{\alpha}} S(n)^{\frac{q+2\alpha-1}{\alpha}} |\Omega|^{\frac{1}{q_2}} \right)^{-\frac{\alpha}{q-1}} M_2^{-\frac{\alpha}{q-1}}.$$

Then

$$J_{\alpha}(v) \geq \frac{1}{4} \rho^2 - \frac{\alpha^{2-\frac{1}{\alpha}}}{2\alpha-1} S(n)^{\frac{2\alpha-1}{\alpha}} M_1 |\Omega|^{\frac{1}{q_1}} \rho^{\frac{2\alpha-1}{\alpha}}.$$

Thus if

$$(7) \quad \frac{1}{4} \rho^2 \geq \frac{\alpha^{2-\frac{1}{\alpha}}}{2\alpha-1} S(n)^{\frac{2\alpha-1}{\alpha}} M_1 |\Omega|^{\frac{1}{q_1}} \rho^{\frac{2\alpha-1}{\alpha}},$$

we shall have $J_{\alpha}(v) \geq 0$ on $\left(\int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}} = \rho$ with ρ determined by (6).

(7) is equivalent to

$$\frac{1}{4} \rho^{\frac{1}{\alpha}} \geq \frac{\alpha^{2-\frac{1}{\alpha}}}{2\alpha-1} S(n)^{\frac{2\alpha-1}{\alpha}} M_1 |\Omega|^{\frac{1}{q_1}}.$$

Combining this with (6) and definitions of q_1, q_2 , we have

$$(8) \quad |\Omega| \leq c(n, q, \alpha) \left(M_2 M_1^{q-1} \right)^{-\frac{n}{2q}},$$

for some constant $c(n, q, \alpha)$ depending only on n, q and α . And $c(n, q, \alpha)$ is continuous for $\alpha \geq 1$. Now we choose $\alpha = \frac{(q-1)(n-2)}{4} + 1$, denote $J_\alpha(v)$ by $J(v)$. Then there is a constant $c(n, q)$ depending only on q, n , such that if

$$(9) \quad |\Omega| \leq c(n, q) \left(M_2 M_1^{q-1} \right)^{-\frac{n}{2q}},$$

we have

$$J(v) \geq 0 \quad \text{for all } v \in H_0^1(\Omega) \quad \text{with } \|v\| = \rho \text{ given in (6).}$$

On the other hand, since $f^+(x_1, 0) > 0$ and $\alpha > 0$, we see that $f_1(x_1, v) \approx cv^{1-\frac{1}{\alpha}}$ for $v > 0$ small. Hence we can choose $v_1 \in H_0^1(\Omega)$ such that $\|v_1\| < \frac{1}{2}\rho$ and

$$(10) \quad J(v_1) < 0.$$

Now a standard argument in critical point theory (see [2] or [6]) implies that $J(v)$ has at least one nontrivial critical point v_2 (such that $J(v_2) < 0$).

Step 2: Existence of a sub- solution u_2 .

This part is almost identical to Step 1. We just sketch here.

We may assume $f^-(x_2, 0) < 0$ for some $x_2 \in \Omega$, otherwise $u_2 = 0$ is a sub-solution.

Let $\alpha \geq \max \left\{ \frac{(q-1)(n-2)}{4}, 1 \right\}$. The exact value of α will be determined later. Consider

$$(11) \quad \begin{cases} -\Delta u_2 = f^-(x, u_2) + \frac{\alpha-1}{u_2} |\nabla u_2|^2 & \text{in } \Omega; \\ u_2 < 0 & \text{in } \Omega; \\ u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Change variable $v = \frac{1}{\alpha} |u_2|^{\alpha-1} u_2$ in (11), then v satisfies

$$(12) \quad \begin{cases} -\Delta v = f^-\left(x, -(\alpha|v|)^{\frac{1}{\alpha}}\right) (\alpha|v|)^{\frac{(\alpha-1)}{\alpha}} & \text{in } \Omega; \\ v < 0 & \text{in } \Omega; \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

It is clear that every solution of (12) corresponds to a solution of (11).

Let $f_2(x, v) = f^+ \left(x, -(\alpha|v|)^{\frac{1}{\alpha}} \right) (\alpha|v|)^{\frac{(\alpha-1)}{\alpha}}$. Then $f_2(x, v) \leq 0$ for all v and is Hölder continuous about v . (†) implies that for all v

$$(13) \quad 0 \geq f_2(x, v) \geq -M_1 \alpha^{\frac{(\alpha-1)}{\alpha}} |v|^{\frac{\alpha-1}{\alpha}} - M_2 \alpha^{\frac{(\alpha-1)}{\alpha}} |v|^{\frac{(q+\alpha-1)}{\alpha}}.$$

Once again we notice that $\frac{q+\alpha-1}{\alpha} < \frac{n+2}{n-2}$ when $\alpha > \frac{(q-1)(n-2)}{4}$. Thus $f_2(x, v)$ has sub- critical growth in v if $\alpha > \frac{(q-1)(n-2)}{4}$.

Consider the functional

$$I_\alpha(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \int_\Omega F_2(x, v) dx, \quad v \in H_0^1(\Omega),$$

where $F_2(x, v) = \int_0^v f_2(x, s) ds$.

We shall show that $I_\alpha(v)$ has a nontrivial critical point for suitable value α (and under the assumptions of Theorem 1). Then the maximum principle implies that the non-trivial point is a negative solution of (12).

For $v \in H_0^1(\Omega)$, by (13), we have

$$\int_\Omega F_2(x, v) dx \leq \frac{\alpha^{2-\frac{1}{\alpha}}}{2\alpha-1} M_1 \int_\Omega |v|^{\frac{2\alpha-1}{\alpha}} dx + \frac{1}{q+2\alpha-1} \alpha^{\frac{q+2\alpha-1}{\alpha}} M_2 \int_\Omega |v|^{\frac{q+2\alpha-1}{\alpha}} dx.$$

Thus

$$\begin{aligned} I_\alpha(v) &\geq \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \frac{\alpha^{2-\frac{1}{\alpha}}}{2\alpha-1} M_1 \int_\Omega |v|^{\frac{2\alpha-1}{\alpha}} dx \\ &\quad - \frac{1}{q+2\alpha-1} \alpha^{\frac{q+2\alpha-1}{\alpha}} M_2 \int_\Omega |v|^{\frac{q+2\alpha-1}{\alpha}} dx. \end{aligned}$$

As we did in Step 1, we choose $\alpha = \frac{(q-1)(n-2)}{4} + 1$. Then there is a constant $c(n, q)$ depending only on q, n , such that if $|\Omega| \leq c(n, q) \left(M_2 M_1^{q-1} \right)^{-\frac{n}{2q}}$, (here $I_\alpha(v)$ is denoted by $I(v)$),

$$I(v) \geq 0 \quad \text{for all } v \in H_0^1(\Omega) \quad \text{with } \|v\| = \rho \text{ given in (6).}$$

Since $f^-(x_2, 0) < 0$ and $\alpha > 0$, we see that $f_2(x_2, v) \approx c|v|^{-\frac{1}{\alpha}} v$ for $v < 0$ small. Hence we can choose $v_3 \in H_0^1(\Omega)$ such that $\|v_3\| \leq \frac{1}{2}\rho$ and

$$I(v_3) < 0.$$

Thus $I(v)$ has at least one nontrivial critical point v_4 .

Step 3: Existence of at least one solution.

Since $u_2 \leq u_1$ is a pair of super- sub- solutions to (D_0) , (D_0) has a solution by Theorem 6.5 in [4]. □

Remark 1. From the proof we see that the choice of α is not unique. The choice of α will certainly have impact on the magnitude of the constant $c(n, q)$ in (9). Naturally one interesting question is for which value of α is the constant $c(n, q, \alpha)$ in (8) maximized. It is easy to check that the constant $c(n, q, \alpha)$ defined in (8) will tend to zero as $\alpha \rightarrow \infty$, so one might think that $c(n, q, \alpha)$ attains the maximum value when α is small. The smallest value that α can take is $\max \left\{ \frac{(q-1)(n-2)}{4}, 1 \right\}$ if $q \neq \frac{n+2}{n-2}$. And if $q = \frac{n+2}{n-2}$, then α can take any value arbitrary close to 1 (but greater than 1). It is not difficult to see that in any case the constant $c(n, q)$ in Theorem 1 can be obtained by choosing $\alpha = \max \left\{ \frac{(q-1)(n-2)}{4}, 1 \right\}$ in $c(n, q, \alpha)$.

The proof of Theorem 1 can be modified to obtain a more general version. Let $F(x, t) = \int_0^t f(x, s) ds$, $\Omega_1 = \{x | F(x, t) \neq 0 \text{ for some } t > 0\}$, $\Omega_2 = \{x | F(x, t) \neq 0 \text{ for some } t < 0\}$, We now impose the growth conditions on $f(x, t)$ and $F(x, t)$.

(F_+) There are positive constants $M_1, M_2, q_1 \geq 1$, such that

$$(14) \quad \limsup_{t \rightarrow +\infty} \frac{|f(x, t)|}{t^{q_1}} < +\infty,$$

and

$$(15) \quad |F(x, t)| \leq M_1|t| + M_2|t|^{q_1+1} \quad \text{for all } x \in \bar{\Omega}, t \geq 0.$$

(F_-) There are positive constants $m_1, m_2, q_2 \geq 1$, such that

$$(16) \quad \limsup_{t \rightarrow -\infty} \frac{|f(x, t)|}{|t|^{q_2}} < +\infty,$$

and

$$(17) \quad |F(x, t)| \leq m_1|t| + m_2|t|^{q_2+1} \quad \text{for all } x \in \bar{\Omega}, t \leq 0.$$

Then we have

Theorem 1*. *There are constants $c_1(n, q_1), c_2(n, q_2)$ depending only on q_1, q_2 and n , such that if we assume*

$$(1) \quad (F_+) \text{ and } |\Omega_1| \leq c_1(n, q_1) \left(M_2 M_1^{q_1-1} \right)^{-\frac{n}{2q_1}};$$

$$(2) \quad (F_-) \text{ and } |\Omega_2| \leq c_2(n, q_2) \left(m_2 m_1^{q_2-1} \right)^{-\frac{n}{2q_2}},$$

then (D_0) has a solution.

Proof. The proof here is more or less the same as that for Theorem 1. We only indicate the necessary changes here.

Once again, we may assume that $u = 0$ is not a solution, otherwise there is nothing to prove.

Let $\phi(t)$ be a smooth function defined by $\phi(t) = 0$ if $t < 1$, $\phi(t) = 1$ if $t > 2$, and $0 \leq \phi(t) \leq 1$ on $1 \leq t \leq 2$. For any small positive constant $0 < \delta < 1$, set $f_3(x, t) = f^+(x, t) + \phi(\frac{t}{\delta})f^-(x, t)$ if $t > 0$ and $f_3(x, t) = f^+(x, 0)$ if $t \leq 0$; $f_4(x, t) = f^-(x, t) + \phi(-\frac{t}{\delta})f^+(x, t)$ if $t < 0$ and $f_4(x, t) = f^-(x, 0)$ if $t \geq 0$. Then $f_3(x, t) = f(x, t)$ if $t \geq 2\delta$ and $f_4(x, t) = f(x, t)$ if $t \leq -2\delta$. Consider

$$(18) \quad \begin{cases} -\Delta u_1 = f_3(x, u_1) + \frac{\alpha - 1}{u_1} |\nabla u_1|^2 & \text{in } \Omega; \\ u_1 > 0 & \text{in } \Omega; \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$(19) \quad \begin{cases} -\Delta u_2 = f_4(x, u_2) + \frac{\alpha - 1}{u_2} |\nabla u_2|^2 & \text{in } \Omega; \\ u_2 < 0 & \text{in } \Omega; \\ u_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

It is clear that any solution of (18) is a super- solution of (D_0) and any solution of (19) is a sub- solution of (D_0) . Since $u_2 < u_1$ for any solutions u_2 and u_1 of (19) and (18) respectively, we only have to show that (18) and (19) have solutions.

Here we shall sketch the proof that (18) has a solution (under the assumption that $f^+(x, 0)$ is not identically zero, otherwise 0 is a super- solution). (The proof that (19) has a solution is similar.)

Change variable $v = \frac{1}{\alpha} |u_1|^\alpha$ in (18), then v satisfies

$$(20) \quad \begin{cases} -\Delta v = f_3 \left(x, (\alpha|v|)^{\frac{1}{\alpha}} \right) (\alpha|v|)^{\frac{(\alpha-1)}{\alpha}} & \text{in } \Omega; \\ v > 0 & \text{in } \Omega; \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Consider the functional

$$J_{\alpha,\delta}(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \int_{\Omega} F_3(x, v) dx, \quad v \in H_0^1(\Omega),$$

where $F_3(x, v) = \int_0^v f_3 \left(x, (\alpha|s|)^{\frac{1}{\alpha}} \right) (\alpha|s|)^{\frac{(\alpha-1)}{\alpha}} ds$.

Since $f_3(x, v) \geq 0$ when $v \leq \frac{\delta^\alpha}{\alpha}$, the maximum principle concludes that any non-trivial critical point of $J_{\alpha,\delta}(v)$ is a positive solution to (20).

Now let us show that $J_{\alpha,\delta}(v)$ has a non-trivial critical point for some small δ and $\alpha = \max \left\{ \frac{(q_1-1)(n-2)}{4}, 1 \right\}$.

For $v > 0$,

$$\begin{aligned}
F_3(x, v) &= \int_0^v f_3(x, (\alpha|s|)^{\frac{1}{\alpha}}) (\alpha|s|)^{\frac{(\alpha-1)}{\alpha}} ds \\
&= \int_0^v \left\{ f^+ \left(x, (\alpha|s|)^{\frac{1}{\alpha}} \right) + \phi \left(\frac{(\alpha|s|)^{\frac{1}{\alpha}}}{\delta} \right) f^- \left(x, (\alpha|s|)^{\frac{1}{\alpha}} \right) \right\} (\alpha|s|)^{\frac{(\alpha-1)}{\alpha}} ds \\
&= \int_0^v f \left(x, (\alpha|s|)^{\frac{1}{\alpha}} \right) (\alpha|s|)^{\frac{(\alpha-1)}{\alpha}} ds \\
&\quad + \int_0^v \left(\phi \left(\frac{(\alpha|s|)^{\frac{1}{\alpha}}}{\delta} \right) - 1 \right) f^- \left(x, (\alpha|s|)^{\frac{1}{\alpha}} \right) (\alpha|s|)^{\frac{(\alpha-1)}{\alpha}} ds \\
&\leq \int_0^v f \left(x, (\alpha|s|)^{\frac{1}{\alpha}} \right) (\alpha|s|)^{\frac{(\alpha-1)}{\alpha}} ds + c(f, n, q_1) \delta^\alpha \\
&= \int_0^{(\alpha v)^{\frac{1}{\alpha}}} f(x, z) z^{2(\alpha-1)} dz + c(f, n, q_1) \delta^\alpha \\
&= F \left(x, (\alpha v)^{\frac{1}{\alpha}} \right) (\alpha v)^{\frac{2(\alpha-1)}{\alpha}} \\
&\quad - 2(\alpha-1) \int_0^{(\alpha v)^{\frac{1}{\alpha}}} F(x, z) z^{2\alpha-3} dz + c(f, n, q_1) \delta^\alpha \\
&\leq F \left(x, (\alpha v)^{\frac{1}{\alpha}} \right) (\alpha v)^{\frac{2(\alpha-1)}{\alpha}} \\
&\quad + 2(\alpha-1) \int_0^{(\alpha v)^{\frac{1}{\alpha}}} |F(x, z)| z^{2\alpha-3} dz + c(f, n, q_1) \delta^\alpha \\
&\leq \left(M_1 (\alpha v)^{\frac{1}{\alpha}} + M_2 (\alpha v)^{\frac{q_1+1}{\alpha}} \right) (\alpha v)^{\frac{2\alpha-2}{\alpha}} \\
&\quad + 2(\alpha-1) \int_0^{(\alpha v)^{\frac{1}{\alpha}}} (M_1 z + M_2 z^{q_1+1}) z^{2\alpha-3} ds + c(f, n, q_1) \delta^\alpha \\
&= \frac{4\alpha-3}{2\alpha-1} M_1 (\alpha v)^{\frac{2\alpha-1}{\alpha}} + \frac{q_1+4\alpha-3}{q_1+2\alpha-1} M_2 (\alpha v)^{\frac{q_1+2\alpha-1}{\alpha}} + c(f, n, q_1) \delta^\alpha.
\end{aligned}$$

Now as we did in the proof of Theorem 1, it follows that there are constants $c(n, q_1)$ and ρ_1 depending only on n, q_1 , such that

$$\text{if } |\Omega_1| \leq c(n, q_1) \left(M_2 M_1^{q_1-1} \right)^{-\frac{n}{2q_1}},$$

$$J(v) \geq -c(f, n, q_1) \delta^\alpha \quad \text{for all } v \in H_0^1(\Omega) \text{ with } \|v\| = \rho_1.$$

On the other hand, $f^+(x_1, 0) \neq 0$ for some $x_1 \in \Omega$ implies that we can choose a v_5 independent of δ , such that $\|v_5\| < \frac{1}{2}\rho_1$ and $J(v_5) < 0$. Now if

we choose a $\delta > 0$ such that

$$-c(f, n, q_1)\delta^\alpha > J(v_5),$$

we see that $J(v)$ has a nontrivial critical point v_6 such that $\|v_6\| < \rho_1$ and $J(v_6) < J(v_5) < 0$. Thus there is a solution to (18).

The rest of the proof is clear. □

Remark 2. Since conditions (F_+) and (F_-) are imposed on $F(x, t)$, the behavior of $f(x, t)$ can be quite different. Furthermore the q_1 in (14) and (15) and the q_2 in (16) and (17)) can be two different numbers. That is, $f(x, t)$ and $F(x, t)$ can have different growth rates. If this is the case, the constant $c(n, q_1)$ will be changed accordingly. Finally if $q_1 < \frac{n+2}{n-2}$, we can take $\alpha = 1$ in the proof and replace $F(x, t)$ by $F^+(x, t) = \max\{F(x, t), 0\}$ in (14). Thus we have recovered the main result in [3].

When $f(x, 0) = 0$, (D_0) has a trivial solution $u = 0$. Then the main interest in this case is in non-trivial solutions. On the other hand, the conditions in Theorem 1 are not enough to assure a nontrivial solution. Indeed, if $f(x, t) = |t|^{q-1}t$ with $q > \frac{n+2}{n-2}$, the well known Pohozaev identity [5] concludes that (D_0) does not have any non-trivial solutions for any ball Ω . To get a nontrivial solution for (D_0) , we use an additional condition 3) in Theorem 2. Basically 3) in Theorem 2 assures that (D_0) has a very small positive sub- solution.

2.2. Proof of Theorem 2. Since $\lim_{t \rightarrow 0^+} \frac{f(x,t)}{t} > \lambda_1$, there is a $d > 0$, such that $f(x, t) > \lambda_1 t$ for $0 < t < d$. Then for any $0 < \delta < d$, $u_2 = \delta\varphi(x)$ is a sub- solution for (D_0) , where $\varphi(x)$ is the positive first eigenfunction of $-\Delta$ on Ω with Dirichlet boundary conditions and $\max_{\{x \in \Omega\}} \varphi(x) = 1$.

Now define

$$f^*(x, t) = \begin{cases} f(x, 0) & \text{if } t \leq 0; \\ f(x, t) & \text{if } t > 0. \end{cases}$$

Then $f^*(x, t)$ satisfies (†) with the same constants M_1 and M_2 as used by $f(x, t)$.

Consider

$$(P^*) \quad \begin{cases} -\Delta v = f^*(x, v) + \frac{\alpha - 1}{v} |\nabla u|^2 & \text{in } \Omega; \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

As we did in the Step 1 of the proof of Theorem 1, (P^*) has a positive solution $v > 0$ (under the assumptions (1) and (2) of Theorem 2, and we shall use $\lim_{t \rightarrow 0^+} \frac{f(x,t)}{t} > \lambda_1$ to find v_1 satisfying (10)). In particular v is a

super- solution for (D_0) . Since $f(x, t) > 0$ for $t > 0$ small, an application of maximum principle implies that $v(x) \geq \delta_1 \varphi(x)$ on Ω for some positive constant δ_1 .

Now fix a $0 < \delta < \delta_1$, then $u_2 = \delta \varphi(x) < v$, and u_2, v is a pair of super-sub- solutions. Therefore (D_0) has a positive solution. \square

Remark 3. If $f(x, t)$ is C^1 near $t = 0$ in Theorem 2, we see that (D_0) has two solutions $u_1 > 0$ and $u_2 < 0$.

Remark 4. It is straightforward to modify the method used here to obtain similar results for Dirichlet problems for a second order elliptic equations in divergent form

$$\begin{cases} -\frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) = f(x, u) & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

but now the constant $c(n, q)$ will depends on the dimension n , growth exponent q and the smallest eigenvalue of the positive matrix $(a_{ij}(x))$ on $\bar{\Omega}$.

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