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TOHRU NAKASHIMA AND YUICHIRO TAKEDA

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Let $\overline{F}$ be a Hermitian vector bundle on an arithmetic variety X over $\mathbb{Z}$. We prove an inequality between the L^2-norm of an element in $H^1(X, F^\vee)$ and arithmetic Chern classes of $\overline{F}$ under certain stability condition. This is a higher dimensional analogue of a result of C. Soulé for Hermitian line bundles on arithmetic surfaces. We observe that our result is related to a conjectural inequality of Miyaoka-Yau type.

Introduction.

In a recent paper [**S**], C. Soulé obtained an analogue for arithmetic surfaces of Kodaira-Ramanujam vanishing theorem. Let K be a number field and $\mathcal{O}_K$ its ring of integers. Let X be an arithmetic surface over $\mathcal{O}_K$ with the smooth, geometrically irreducible generic fiber and $\overline{L} = (L, h)$ a Hermitian line bundle on X. We denote by $\hat{c}_1(\overline{L})$ its arithmetic first Chern class. The main result in *loc.cit.* states that if $\overline{L}$ satisfies certain positivity assumption, then there exist explicit constants A, B such that for any non-torsion element $e \in H^1(X, L^\vee)$, we have

$$\hat{c}_1(\overline{L})^2 \leq A \log \|e\| + B.$$

Here $\|e\|$ denotes the supremum of the L^2-norm $\|\sigma(e)\|_{L^2}$ when σ runs over all infinite places of K.

In this paper we prove a similar inequality for Chern classes of certain Hermitian vector bundles on higher dimensional arithmetic varieties over $\mathbb{Z}$. For this purpose, we need the assumption that these bundles are stable with respect to an arithmetically ample line bundle. Unfortunately we have to put further rather restrictive assumptions on their first Chern classes (see §1. for a precise statement).

There are some applications of our main result. First we obtain an arithmetic vanishing of the first cohomology groups $H^1(X, F^\vee)$: It is possible to bound the number of elements of L^2-norm less than or equal to one by certain constants. Secondly we prove an inequality for Chern classes of arithmetic varieties with arithmetically ample relative canonical bundle $\overline{\omega}_{X/\mathbb{Z}}$, which can be considered as an arithmetic analogue of Miyaoka-Yau inequality ([**Y**]).

Our proof follows the method of Soulé. Let $\overline{F}$ be a Hermitian vector bundle on an arithmetic variety X. Every non-torsion element $e \in H^1(X, F^\vee)$ corresponds to a non-split extension

$$0 \to \mathcal{O}_X \to E \to F \to 0.$$

To obtain an estimate of L^2-norm of the induced class e_∞, we put a special Einstein-Hermitian metric on the holomorphic bundle E_∞ on the infinite fiber X_∞. Since our assumption on F_∞ ensures the stability of E_∞, the desired inequality follows from an estimate for Donaldson functionals and an inequality of Bogomolov-Gieseker type for semistable bundles on arithmetic varieties which is due to A. Moriwaki ([**M**]).

1. Statement of the main result.

Let $f : X \to \operatorname{Spec} \mathbb{Z}$ be an arithmetic variety of dimension $d+1$. Namely X is a regular integral scheme and f is flat, projective of pure relative dimension d with the smooth generic fiber $X_\mathbb{Q}$. For an integer $p \geq 0$, we denote by $\widehat{CH}^p(X)$ the arithmetic Chow group of codimension p. In [**GS1**] (see also [**SABK**]), H. Gillet and C. Soulé showed that there exist an intersection product

$$\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \to \widehat{CH}^{p+q}(X)_{\mathbb{Z}} \otimes \mathbb{Q}$$

and a direct image map

$$\widehat{\deg} : \widehat{CH}^{d+1}(X) \to \widehat{CH}^1(\operatorname{Spec} \mathbb{Z}) = \mathbb{R}.$$

Let $\overline{E} = (E, h)$ be a Hermitian vector bundle of rank n on X. For $p \geq 0$, we denote by $\hat{c}_p(\overline{E})$ the p-th arithmetic Chern class of $\overline{E}$ ([**GS2**]). For $x \in \widehat{CH}^2(X)$ and a Hermitian line bundle $\overline{H}$, we write simply $x \cdot \hat{c}_1(\overline{H})^{d-1}$ instead of $\widehat{\deg}(x \cdot \hat{c}_1(\overline{H})^{d-1})$.

Let $f : X \to \operatorname{Spec} \mathbb{Z}$ be an arithmetic variety of dimension $d+1 \geq 2$ with the smooth, geometrically irreducible generic fiber $X_\mathbb{Q}$. If E is a vector bundle on X, we denote by E_∞ the associated holomorphic bundle on the infinite fiber X_∞. Let $\overline{H} = (H, k)$ be a Hermitian line bundle on X. $\overline{H}$ is said to be *arithmetically ample* if the following conditions are satisfied:

(1) H is f-ample;

(2) The first Chern form $c_1(H_\infty, k)$ is a Kähler form on X_∞;

(3) For every irreducible horizontal subvariety Y, we have $\hat{c}_1(\overline{H}_{|Y})^{\dim Y} > 0$.

Assume that X_∞ is equipped with a Kähler metric g which is invariant under complex conjugation and let ω_g be the associated Kähler form. Let

$\overline{F} = (F, h^F)$ be a Hermitian vector bundle on X. Using g and h^F, we can define the L^2-norm $||\cdot||_{L^2}$ on the cohomology group $H^1(X, F^\vee)$.

The main result of this paper is the following

Theorem 1.1. *Let $f : X \to \operatorname{Spec}\mathbb{Z}$ be an arithmetic variety of dimension $d+1 \geq 2$ with the smooth, geometrically irreducible generic fiber and $\overline{H} = (H, k)$ an arithmetically ample Hermitian line bundle on X with $c_1(H,k) = \omega_g$. Let $\overline{F} = (F, h^F)$ be a rank r Hermitian vector bundle on X and $m := c_1(F_\infty)\cdot c_1(H_\infty)^{d-1}$. Assume that F_∞ is μ-stable with respect to H_∞ and one of the follwing conditions is satisfied.*

(1) $m = 1$;

(2) $\operatorname{Num}(X_\infty) \cong \mathbb{Z}[H_\infty]$.

Here $\operatorname{Num}(X_\infty)$ *denotes the numerical equivalence class group. Then for every non-torsion element $e \in H^1(X, F^\vee)$, the following inequality holds*

$$(r\hat{c}_1(\overline{F})^2 - 2(r+1)\hat{c}_2(\overline{F}))\cdot \hat{c}_1(\overline{H})^{d-1} \leq m\left(2\log\|e\|_{L^2} + 1 + \log\frac{(r+1)(d-1)!}{m}\right).$$

The significance of arithmetic ampleness in the above theorem stems from the following result of Moriwaki, which is an arithmetic analogue of the usual Bogomolov-Gieseker inequality in the geometric case.

Proposition 1.2 ([M]). *Let $f : X \to \operatorname{Spec}\mathbb{Z}$ be an arithmetic variety of dimension $d+1 \geq 2$ and $\overline{H} = (H,k)$ an arithmetically ample Hermitian line bundle on X. Let $\overline{E} = (E, h)$ be a Hermitian vector bundle of rank r on X. If E_∞ is μ-semistable with respect to H_∞, then we have*

$$\left(\hat{c}_2(\overline{E}) - \frac{r-1}{2r}\hat{c}_1(\overline{E})^2\right)\cdot \hat{c}_1(\overline{H})^{d-1} \geq 0.$$

2. Preliminaries on Bott-Chern classes.

In this section we prove results concerning the Bott-Chern secondary classes, which will be needed for the proof of Theorem 1.1 in the next section. Throughout this section X will be a projective algebraic manifold over $\mathbb{C}$ of dimension d and g a Kähler metric on X which is invariant under complex conjugation. The Kähler form associated to g is defined by

$$\omega_g = \frac{i}{2\pi}\sum_{\alpha,\beta} g\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}\right) dz_\alpha \wedge d\bar{z}_\beta.$$

We fix an ample line bundle H on X equipped with a Hermitian metric k such that the first Chern form $c_1(H,k) =: \mu_H$ is equal to ω_g.

Let

$$\mathcal{E} : 0 \to S \to E \to Q \to 0$$

be an exact sequence of holomorphic vector bundles on X. Assume that the bundles S, E, Q are equipped with Hermitian metrics h', h, h'' respectively and let $\overline{\mathcal{E}} = (\mathcal{E}, h', h, h'')$. Let ϕ be any symmetric power series in r variables. We denote by $\tilde{\phi}(\overline{\mathcal{E}})$ the Bott-Chern secondary characteristic class of $\overline{\mathcal{E}}$ ([**GS2**]). In the case when $S \to E$ is the identity map and $Q = 0$, we write $\tilde{\phi}(\overline{\mathcal{E}}) = \tilde{\phi}(E, h', h)$.

Lemma 2.1. *Let E be a holomorphic vector bundle of* $\mathrm{rk}(E) = r$ *on X and h a Hermitian metric on E. For any positive real number $s > 0$, we have*

$$\tilde{c}_1(E, h, sh) = r\log(s), \tag{2.1.1}$$

$$\tilde{c}_2(E, h, sh) = (r-1)\log(s)c_1(E, h). \tag{2.1.2}$$

Proof. (2.1.1) follows immediately from [**GS2**, (1.2.5.1)]. Assume that we are given two Hermitian metrics h, k on E. Let $\{h_t\}$, $0 \le t \le 1$ be a C^∞ path of Hermitian metrics on E such that $h_0 = k$, $h_1 = h$ and let R_t be the curvature of h_t. By [**BGS**, Cor. 1.30], we have

$$\widetilde{ch}_2(E, k, h) = \frac{i}{2\pi}\int_0^1 \mathrm{tr}\,(h_t^{-1}\partial_t h_t \cdot R_t)\,dt.$$

If we set $k = sh$ and $h_t = s^{1-t}h$, we have $R_t = R_1$ for all t. Then the above formula yields

$$\begin{aligned}\widetilde{ch}_2(E, sh, h) &= -\frac{i}{2\pi}\int_0^1 \mathrm{tr}\,(\log(s)R_1)\,dt \\ &= -\frac{i\log(s)}{2\pi}\mathrm{tr}(R_1) = -\log(s)c_1(E,h).\end{aligned}$$

Therefore, replacing h by $s^{-1}h$, we have $\widetilde{ch}_2(E, h, sh) = \log(s)c_1(E,h)$ for every $s > 0$. On the other hand, by (2.1.1) we obtain

$$\begin{aligned}&\widetilde{ch}_2(E, h, sh) \\ &\quad = \frac{1}{2}\{c_1(E,h)\cdot\tilde{c}_1(E,h,sh) + \tilde{c}_1(E,h,th)\cdot c_1(E,sh)\} - \tilde{c}_2(E,h,sh) \\ &\quad = r\log(s)c_1(E,h) - \tilde{c}_2(E,h,sh).\end{aligned}$$

Hence (2.1.2) follows. □

Let F be a holomorphic vector bundle of $\operatorname{rk} F = r$ on X with a Hermitian metric h^F. Let E be a bundle which sits in the extension

$$\mathcal{E} : 0 \to \mathcal{O}_X \to E \to F \to 0.$$

Assume that $\mathcal{O}_X$ is equipped with the trivial metric τ and that there exists a Hermitian metric h on E such that the canonical isomorphim

$$\det(E) \tilde{\to} \det(F) \tag{2.2}$$

becomes an isometry. Let $\mathcal{M}$ be the set of such Hermitian merics on E. We denote the Bott-Chern form $\tilde{\phi}(\overline{\mathcal{E}})$ for $\overline{\mathcal{E}} = (\mathcal{E}, \tau, h, h^F)$ by $\tilde{\phi}(\mathcal{E}, h, h^F)$.

Lemma 2.2. *For every positive real number $t > 0$ and $h \in \mathcal{M}$, we have*

$$\tilde{c}_2(\mathcal{E}, th, t^{\frac{r+1}{r}} h^F) = \tilde{c}_2(\mathcal{E}, h, h^F) + \frac{\log t}{r} c_1(F, h^F).$$

Proof. We put $s := t^{\frac{r+1}{r}}$. If we replace the metric h^F by sh^F and h by th, (2.2) is still an isometry. Hence we have

$$\tilde{c}_1(\mathcal{E}, h, h^F) = \tilde{c}_1(\mathcal{E}, th, sh^F) = 0,$$
$$c_1(E, h) = c_1(F, h^F).$$

It follows from [**GS2**, Prop. 1.3.4] and Lemma 2.1 that

$$\begin{aligned} &\tilde{c}_2(\mathcal{E}, h, h^F) - \tilde{c}_2(\mathcal{E}, th, sh^F) \\ &= -\tilde{c}_2(E, h, th) + \tilde{c}_2(F, h^F, sh^F) \\ &= -r \log(t) c_1(E, h) + (r-1) \log(s) c_1(F, h^F) \\ &= -\frac{\log(t)}{r} c_1(F, h^F) \end{aligned}$$

as desired. □

Assume that an element $h \in \mathcal{M}$ gives a smooth splitting of $\mathcal{E}$: We have an isometry of C^∞ Hermitian bundles $(E, h) \cong (\mathcal{O}_X, \tau) \oplus (F, h^F)$. Then we can write the Cauchy-Riemann operator $\overline{\partial}_E$ on E as follows

$$\overline{\partial}_E = \begin{pmatrix} \overline{\partial}_{\mathcal{O}_X} & \alpha \\ 0 & \overline{\partial}_F \end{pmatrix}$$

where $\alpha \in A^{0,1}(\operatorname{Hom}(F, \mathcal{O}_X))$ is the closed form corresponding to the extension class $e \in H^1(X, F^\vee)$.

As in [**S**], we introduce a functional on $\mathcal{M}$ which is defined for $h \in \mathcal{M}$ as follows

$$\Phi(h, h^F) = -\int_X \tilde{c}_2(\mathcal{E}, h, h^F) \cdot \frac{\mu_H^{d-1}}{d!}.$$

Lemma 2.3. *Let α, $\Phi(h, h^F)$ be as above. Then we have*

$$\Phi(h, h^F) = \frac{1}{d}\int_X \|\alpha\|^2 \frac{\mu_H^d}{d!}.$$

Proof. By [**D**, 10.1] we have

$$\tilde{c}_2(\mathcal{E}, h, h^F) = \frac{1}{2\pi i}\operatorname{tr}(\alpha^*\alpha)$$

where $\alpha^* \in A^{1,0}(\mathrm{Hom}(\mathcal{O}_X, F))$ denotes the conjugate of α. Let $\theta^1, \dots, \theta^d$ be a local unitary frame of the holomorphic cotangent bundle of X. Using this, we write locally $\mu_H = \frac{i}{2\pi}\sum_{j=1}^d \theta^j \wedge \overline{\theta}^j$, $\alpha = \sum_{j=1}^d \alpha_j \overline{\theta}^j$ and $\alpha^* = \sum_{j=1}^d \alpha_j^* \theta^j$. Then

$$\begin{aligned}
\operatorname{tr}(\alpha^*\alpha) \cdot \frac{\mu_H^{d-1}}{(d-1)!} &= \sum_{j=1}^d \alpha_j^*\alpha_j \theta^j \wedge \overline{\theta}^j \cdot \frac{\mu_H^{d-1}}{(d-1)!} \\
&= \sum_{j=1}^d \alpha_j^*\alpha_j \left(-2\pi i \frac{\mu_H^d}{d!}\right) \\
&= -2\pi i \|\alpha\|^2 \frac{\mu_H^d}{d!}.
\end{aligned}$$

Integrating the above forms over X, the claim follows. □

Proposition 2.4. *Assume that $m = c_1(F).c_1(H)^{d-1} > 0$ and that E admits a Hermitian metric h in $\mathcal{M}$ which is Einstein-Hermitian with respect to μ_H. Then*

$$\Phi(h^E, h^F) \leq \frac{m}{(r+1)d!}\left(2\log\|e\|_{L^2} + 1 + \log\frac{(r+1)(d-1)!}{m}\right).$$

Proof. For fixed $g \in \mathcal{M}$, let

$$\Psi(g, h) = \int_X \left(\widetilde{ch}_2(E, g, h) + \lambda\tilde{c}_1(E, g, h)\mu_H\right) \cdot \frac{\mu_H^{d-1}}{d!}$$

be the Donaldson functional. Here λ is a constant independent of g, h. If h is in $\mathcal{M}$, we have $\tilde{c}_1(E, g, h) = 0$. Hence

$$\Phi(h, h^F) = \Psi(g, h) - \int_X \tilde{c}_2(\mathcal{E}, g, h^F) \cdot \frac{\mu_H^{d-1}}{d!}$$

since we have

$$\begin{aligned}\widetilde{ch}_2(E,g,h) &= \widetilde{ch}_2(\mathcal{E},h,h^F) - \widetilde{ch}_2(\mathcal{E},g,h^F)\\ &= -\tilde{c}_2(\mathcal{E},h,h^F) + \tilde{c}_2(\mathcal{E},g,h^F).\end{aligned}$$

Thus $\Phi(h,h^F)$ takes the absolute minimum at h^E since so does $\Psi(g,h)$ for fixed g. By the argument as in [**S**, p. 581], Lemma 2.3 implies that there exists a Hermitian metric h_0 on E such that the following equality holds

$$\Phi(h_0,h^F) = \frac{1}{d}\|e\|_{L^2}^2.$$

Hence we have

$$\Phi(h^E,h^F) \le \Phi(h_0,h^F) = \frac{1}{d}\|e\|_{L^2}^2. \tag{2.3}$$

On the other hand, it follows from Lemma 2.2 that for any real number $t > 0$ and $s = t^{\frac{r+1}{r}}$,

$$\Phi(th,sh^F) = \Phi(h,h^F) - \frac{\log t}{r}\deg(F).$$

By (2.3), we have

$$\Phi(th^E,sh^F) \le \frac{\|e\|_{L^2}^2}{sd}.$$

Thus we obtain

$$\begin{aligned}\Phi(h^E,h^F) &= \frac{\log t}{r}\deg(F) + \Phi(th^E,sh^F)\\ &\le \frac{\log t}{r}\deg(F) + \frac{\|e\|_{L^2}^2}{t^{\frac{r+1}{r}}d}.\end{aligned}$$

Since we have $\deg(E) > 0$, the right-hand side of the above inequality, when considered as a function of t, takes its minimum at $t = \left(\frac{(r+1)\|e\|_{L^2}^2}{d\deg(F)}\right)^{\frac{r}{r+1}}$. Thus we obtain

$$\Phi(h^E,h^F) \le \frac{\deg(F)}{r+1}\left(1 + \log\frac{(r+1)\|e\|_{L^2}^2}{d\deg(F)}\right).$$

This completes the proof of the proposition since we have $m = d!\deg(F)$. □

3. Proof of Theorem 1.1.

Let X, $\overline{F}$ and $\overline{H}$ be as in Theorem 1.1. Let e be a non-torsion element of $H^1(X, F^\vee)$. Since there exists an isomorphism $H^1(X, F^\vee) \cong \mathrm{Ext}^1(F, \mathcal{O}_X)$, the element e corresponds to a non-trivial extension

$$\mathcal{E} : 0 \to \mathcal{O}_X \to E \to F \to 0.$$

Let $\tilde{\phi}(\mathcal{E}, h, h^F)$ denote the Bott-Chern secondary characteristic class associated to $\overline{\mathcal{E}} = (\mathcal{E}_\infty, \tau, h, h^F)$ where τ is the trivial metric on $\mathcal{O}_{X_\infty}$.

We claim that E_∞ admits an Einstein-Hermitian metric with respect to the Kähler form $c_1(H_\infty, k)$ under the assumption in the theorem. To see this, we need the following, which is a special case of [**K**, Lemma 3] (and its proof).

Lemma 3.1. *Let X be a nonsingular projective variety of dimension $d \geq 1$ defined over an algebraically closed field of characteristic zero and H an ample line bundle on X. Let F be a vector bundle on X and E is given by a non-split extension*

$$0 \to \mathcal{O}_X \to E \to F \to 0.$$

Assume that F is μ-stable with respect to H and one of the following conditions is satisfied:

(1) $c_1(F) \cdot c_1(H)^{d-1} = 1$;

(2) $\mathrm{Num}(X) \cong \mathbb{Z}[H]$ *and* $c_1(E) = c_1(H)$.

Then E is also μ-stable with respect to H.

It follows from Lemma 3.1 that E_∞ is μ-stable with respect to H_∞. Hence by a theorem of Donaldson ([**Do**]), E_∞ admits an Einstein-Hermitian metric with respect to μ_{H_∞}. Since the curvature of an Einstein-Hermitian metric is harmonic, there exists a unique Einstein-Hermitian metric h^E such that the canonical isomorphism

$$\det(E_\infty) \xrightarrow{\sim} \det(F_\infty)$$

induced by $\mathcal{E}$ becomes an isometry.

Setting $\overline{E} := (E, h^E)$ and $\overline{\mathcal{O}}_X := (\mathcal{O}_X, \tau)$, we have

$$\begin{aligned}\hat{c}_1(\overline{E}) &= \hat{c}_1(\overline{F}),\\ \hat{c}_2(\overline{E}) &= \hat{c}_2(\overline{\mathcal{O}}_X \oplus \overline{F}) - a(\tilde{c}_2(\mathcal{E}, h^E, h^F))\\ &= \hat{c}_2(\overline{F}) - a(\tilde{c}_2(\mathcal{E}, h^E, h^F))\end{aligned}$$

where we put $a(\tilde{c}_2(\mathcal{E}, h^E, h^F)) := [(0, \tilde{c}_2(\mathcal{E}, h^E, h^F))] \in \widehat{CH}^2(X)$. Thus we have

$$
\begin{aligned}
(\hat{c}_2(\overline{E}) - \hat{c}_2(\overline{F})) \cdot \hat{c}_1(\overline{H})^{d-1} &= -\frac{1}{2}\int_{X_\infty} \tilde{c}_2(\mathcal{E}, h^E, h^F) \cdot c_1(H_\infty, k)^{d-1} \\
&= \frac{d!}{2}\Phi(h^E, h^F).
\end{aligned}
$$

Furthermore, Proposition 1.2 yields

$$
\left(\hat{c}_2(\overline{E}) - \frac{r}{2(r+1)}\hat{c}_1(\overline{E})^2\right) \cdot \hat{c}_1(\overline{H})^{d-1} \geq 0.
$$

Therefore, by Proposition 2.4, we obtain

$$
\begin{aligned}
&\left(\frac{r}{2(r+1)}\hat{c}_1(\overline{F})^2 - \hat{c}_2(\overline{F})\right) \cdot \hat{c}_1(\overline{H})^{d-1} \\
&\leq (\hat{c}_2(\overline{E}) - \hat{c}_2(\overline{F})) \cdot \hat{c}_1(\overline{H})^{d-1} \\
&\leq \frac{m}{2(r+1)}\left(2\log\|e\|_{L^2} + 1 + \log\frac{(r+1)(d-1)!}{m}\right).
\end{aligned}
$$

Hence the theorem follows.

4. Applications.

In this section we shall give some consequences of Theorem 1.1. For a Hermitian vector bundle $\overline{F}$ on an arithmetic variety X, we denote by $h^0(H^1(X, F^\vee), \|\ \|_{L^2})$ the logarithm of the number of elements of L^2-norm less than or equal to one. Assume that $H^1(X, F^\vee)$ is a torsion-free $\mathbb{Z}$-module and let n be its rank. We define the number $C(n)$ as follows:

$$
C(n) = n\log 6 - \log V(B_n)
$$

where $V(B_n)$ denotes the volume of the standard unit ball in the n-dimensional Euclidean space. As a corollary of Theorem 1.1, we obtain the following "vanishing theorem".

Proposition 4.1. *Let X, $\overline{F}$ and $\overline{H}$ be as in Theorem 1.1. Assume that $H^1(X, F^\vee)$ is a torsion-free $\mathbb{Z}$-module of rank n and the following inequality holds*

$$
(r\hat{c}_1(\overline{F})^2 - 2(r+1)\hat{c}_2(\overline{F})) \cdot \hat{c}_1(\overline{H})^{d-1} \geq 0.
$$

Then we have

$$
h^0(H^1(X, F^\vee), \|\ \|_{L^2}) \leq \frac{n}{2}\left(1 + \frac{(r+1)(d-1)!}{m}\right) + 2C(n).
$$

Proof. By Theorem 1.1, the assumption implies that there is no element $e \in H^1(X, F^\vee)$ such that $\|e\|_{L^2} < \exp(-\frac{1}{2}(1 + \log \frac{(r+1)(d-1)!}{m}))$. Hence the claim is a consequence of [**S**, Lemma 3]. □

To state the next application, assume that $f : X \to \operatorname{Spec}\mathbb{Z}$ is smooth with the geometrically irreducible generic fiber. Let $\Omega^1_{X/\mathbb{Z}}$ be the sheaf of relative differentials and $T_{X/\mathbb{Z}} = (\Omega^1_{X/\mathbb{Z}})^\vee$ the relative tangent bundle. We assume that $\Omega^1_{X_\infty}$ is equipped with the dual metric $g^\vee$ and ω_{X_∞} with the determinant $\det g^\vee$. We denote these Hermitian bundles by $\overline{T}_{X/\mathbb{Z}} = (T_{X/\mathbb{Z}}, g)$, $\overline{\Omega}^1_{X/\mathbb{Z}} = (\Omega^1_{X/\mathbb{Z}}, g^\vee)$ and $\overline{\omega}_{X/\mathbb{Z}} = (\omega_{X/\mathbb{Z}}, \det g^\vee)$. We define the i-th arithmetic Chern class of (X, g) as follows

$$\hat{c}_i(X) = \hat{c}_i(\overline{T}_{X/\mathbb{Z}}).$$

Then we have $\hat{c}_1(X) = -\hat{c}_1(\overline{\omega}_{X/\mathbb{Z}})$ and $\hat{c}_2(X) = \hat{c}_2(\overline{\Omega}^1_{X/\mathbb{Z}})$. In fact, for any Hermitian vector bundle $\overline{E}$ on an arithmetic variety, the equality $\hat{c}_i(\overline{E}^\vee) = (-1)^i \hat{c}_i(\overline{E})$ holds (cf. [**GS2**]).

We recall that a projective algebraic manifold M is called a canonical variety(resp.a Fano variety) if its canonical bundle ω_M(resp.the anticanonical bundle $\omega_M^\vee$) is ample. It is well known that for such varieties the tangent bundle T_M and the cotangent bundle Ω^1_M are μ-semistable with respect to ω_M or $\omega_M^\vee$ (cf. [**Ti, Ts**]). We introduce arithmetic analogues of these varieties. Assume that g is a Kähler-Einstein metric which is normalized so that $c_1(\omega_{X_\infty}, \det g^\vee) = \lambda \omega_g$ where $\lambda = 1$ or -1. We define an arithmetic variety (X, g) to be *canonical* (resp. *Fano*) if $\overline{\omega}_{X/\mathbb{Z}}$ (resp. $\overline{\omega}^\vee_{X/\mathbb{Z}}$) is arithmetically ample. By Proposition 1.2, we obtain the inequality

$$(-1)^{d-1}((d-1)\hat{c}_1(X)^2 - 2d\hat{c}_2(X)) \cdot \hat{c}_1(X)^{d-1} \leq 0$$

if X is canonical and we have

$$((d-1)\hat{c}_1(X)^2 - 2d\hat{c}_2(X)) \cdot \hat{c}_1(X)^{d-1} \leq 0$$

if X is Fano.

It is natural to ask whether the following inequality of Miyaoka-Yau type

$$(-1)^{d-1}(d\hat{c}_1(X)^2 - 2(d+1)\hat{c}_2(X)) \cdot \hat{c}_1(X)^{d-1} \leq 0$$

holds for canonical arithmetic varieties. We have the following result in this direction.

Proposition 4.2. *Let (X, g) be a canonical arithmetic variety of dimension $d+1$ and assume that one of the following conditions is satisfied:*
(1) $m := c_1(\omega_{X_\infty})^d = 1$;

(2) $\mathrm{Num}(X_\infty) \cong \mathbb{Z}[\omega_{X_\infty}]$.
Then for every non-torsion element $e \in H^1(X, T_{X/\mathbb{Z}})$, the following inequality holds

$$(-1)^{d-1}(d\hat{c}_1(X)^2 - 2(d+1)\hat{c}_2(X)) \cdot \hat{c}_1(X)^{d-1} \leq m\left(2\log\|e\|_{L^2} + 1 + \log\frac{(d+1)!}{dm}\right).$$

Proof. The tangent bundle T_{X_∞} of X_∞ is μ-semistable with respect to ω_{X_∞}. It can be easily seen that under our assumptions T_{X_∞} is in fact μ-stable. Therefore applying Theorem 1.1 to $\overline{F} = \overline{\Omega}^1_{X/\mathbb{Z}}$ and $\overline{H} = \overline{\omega}_{X/\mathbb{Z}}$ we obtain the claim. □

It follows from the above proposition that under the assumptions (1) or (2), the existence of an element of small L^2-norm in $H^1(X, T_{X/\mathbb{Z}})$ would imply the Miyaoka-Yau inequality. It would be interesting to know whether we can remove such restrictive assumptions.

In [**KMM**], it has been proved that, for any Fano variety of dimension n, the self-intersection $c_1(X)^n$ is bounded by a universal constant depending only on n. One can ask whether an analogous bound exists for $\hat{c}_1(X)^{d+1}$ of arithmetic Fano varieties. The following result suggests that again we may reduce this problem to the existence of a cohomology class of small L^2-norm.

Proposition 4.3. *Let (X, g) be an arithmetic Fano variety of dimension $d+1$ and let $e \in H^1(X, \Omega^1_{X/\mathbb{Z}})$ be a non-torsion element. Assume that one of the following conditions is satisfied:*
(1) $m := c_1(X_\infty)^d = 1$;
(2) $\mathrm{Num}(X_\infty) \cong \mathbb{Z}[\omega^\vee_{X_\infty}]$;
(3) *e induces $e_\infty = c_1(X_\infty)$.*
Then the following inequality holds

$$(d\hat{c}_1(X)^2 - 2(d+1)\hat{c}_2(X)) \cdot \hat{c}_1(X)^{d-1} \leq m\left(2\log\|e\|_{L^2} + 1 + \log\frac{(d-1)!}{dm}\right).$$

Proof. In the case (1) or (2), the claim follows similarly as in Proposition 4.2. Under the assumption (3), [**Ti**, Theorem 0.1] implies that there is an Einstein-Hermitian metric on the vector bundle E_∞ corresponding to e_∞. Hence we are done using Proposition 2.4. □

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TOKYO METROPOLITAN UNIVERSITY
MINAMI-OHSAWA 1-1, HACHIOJI-SHI
TOKYO, 192-03, JAPAN
E-mail address: nakasima@math.metro-u.ac.jp
takeda@math.metro-u.ac.jp

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