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**A UNIQUENESS THEOREM FOR THE MINIMAL SURFACE  
EQUATION**

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## A UNIQUENESS THEOREM FOR THE MINIMAL SURFACE EQUATION

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In 1991, Collin and Krust proved that if  $u$  satisfies the minimal surface equation in a strip with linear Dirichlet data on two sides, then  $u$  must be a helicoid. In this paper, we give a simpler proof of this result and generalize it.

### 1. Introduction.

Let  $\Omega_\alpha \subset \mathbb{R}^2$  be a sector domain with angle  $0 < \alpha < \pi$ . Consider the minimal surface equation

$$(1) \quad \operatorname{div} Tu = 0$$

where  $Tu = \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}$  and  $\nabla u$  is the gradient of  $u$ . In 1965, Nitsche [7] announced the following results:

- (1) Given a continuous function  $f$  on  $\partial\Omega_\alpha$ , there always exists a solution  $u$  which satisfies the minimal surface equation in  $\Omega_\alpha$  with Dirichlet data  $f$  on  $\partial\Omega_\alpha$ ;
- (2) If  $u$  satisfies the minimal surface equation with vanishing boundary value in  $\Omega_\alpha$ , then  $u \equiv 0$ .

Nitsche thus raised the following question: Let  $\Omega \subset \Omega_\alpha$  and let  $f$  be an arbitrary continuous function on  $\partial\Omega$ . If the Dirichlet problem

$$\begin{cases} \operatorname{div} Tu = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega \end{cases}$$

has a solution, is it unique?

We notice that similar questions for higher dimensions are raised in [6]. Results in this direction were obtained by Miklyukov [5] and Hwang [4] independently, in which the following result was established:

**Theorem 1.** *Let  $\Omega \subset \mathbb{R}^2$  be an unbounded domain and let  $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ . For every  $R > 0$ , set  $B_R = \{x \in \mathbb{R}^2 \mid |x| < R\}$  and  $\Gamma_R = \partial(\Omega \cap B_R) \cap$*

$\partial B_R$ . Denote  $|\Gamma_R|$  as the length of  $\Gamma_R$ . And suppose that

$$\begin{cases} \text{(i)} & \operatorname{div} Tu = \operatorname{div} Tv & \text{in } \Omega, \\ \text{(ii)} & u = v & \text{on } \partial\Omega, \\ \text{(iii)} & \max_{\Omega \cap B_R} |u - v| = O\left(\sqrt{\int_{R_0}^R \frac{1}{|\Gamma_r|} dr}\right) & \text{as } R \rightarrow \infty, \text{ for some} \\ & & \text{positive constant } R_0. \end{cases}$$

Then  $u \equiv v$  in  $\Omega$ .

A stronger version of Theorem 1 was discovered by Collin and Krust [2] independently, which is the following:

**Theorem 1\*.** Let  $\Omega, u, v, B_R, \Gamma_r$  and  $|\Gamma_r|$  as in Theorem 1. And suppose that

$$\begin{cases} \text{(i)} & \operatorname{div} Tu = \operatorname{div} Tv & \text{in } \Omega, \\ \text{(ii)} & u = v & \text{on } \partial\Omega, \\ \text{(iii)} & \max_{\Omega \cap B_R} |u - v| = o\left(\int_{R_0}^R \frac{1}{|\Gamma_r|} dr\right) & \text{as } R \rightarrow \infty, \text{ for some} \\ & & \text{positive constant } R_0. \end{cases}$$

Then  $u \equiv v$  in  $\Omega$ .

In fact, for any unbounded domain  $\Omega$ , we have  $|\Gamma_R| = O(R)$ , and condition (iii) in Theorem 1\* becomes

$$\max_{\Omega \cap B_R} |u - v| = o(\log R) \quad \text{as } R \rightarrow \infty.$$

In the special case when  $\Omega$  is a strip, then  $|\Gamma_R| \leq \text{constant}$ , and condition (iii) becomes  $\max_{\Omega \cap B_R} |u - v| = o(R)$ .

On the other hand, in a strip domain  $\Omega$ , Collin [1] showed that there exist two different solutions for the minimal surface equation such that  $u = v$  on  $\partial\Omega$  and  $\max_{\Omega \cap B_R} |u - v| = O(R)$  as  $R \rightarrow \infty$ . So condition (iii) is necessary.

This counterexample also answers Nitsche’s question in the negative.

In contrast, the following result is also given in [2].

**Theorem 2.** Let  $\Omega = (0, 1) \times \mathbb{R}$  be a strip. Suppose that

$$\begin{cases} \operatorname{div} Tu = 0 & \text{in } \Omega, \\ u(0, y) = ay + b, \\ u(1, y) = cy + d \end{cases}$$

where  $a, b, c, d$  are constant. Then  $u$  must be a helicoid.

The following inequality was discovered by Miklyukov [5, p. 265], Hwang [4, p. 342] and Collin and Krust [2, p. 452]:

$$(Tu - Tv) \cdot (\nabla u - \nabla v) \geq \frac{\sqrt{1 + |\nabla u|^2} + \sqrt{1 + |\nabla v|^2}}{2} |Tu - Tv|^2$$

$$(2) \quad \geq |Tu - Tv|^2.$$

Using this inequality, Miklyukov [5] and Hwang [4] proved Theorem 1 independently, and Collin and Krust [2] proved Theorem 1\* also based on this inequality.

It seems that the method of proof of Theorem 1\* can not be used to prove Theorem 2, and so Collin and Krust [2] resorted to the theory of Gauss maps instead.

In this paper, we will point out that the method of proof of Theorem 1 and Theorem 1\* could be use to give a simpler proof of Theorem 2. Moreover, we shall generalize Theorem 1\* and Theorem 2 to get the more general results as stated in Theorem 3 and Theorem 4. And we will make a remark after Theorem 3 to point out why Collin and Krust [2] could get a better result then Miklyukov [5] and Hwang [4].

## 2. A new proof for Theorem 2 and its generalization.

Without loss of generality, we may rephrase Theorem 2 in the following form:

**Theorem 2\*.** *Let  $\Omega = (b, a) \times \mathbb{R}$  be a strip domain in  $\mathbb{R}^2$  where  $a, b$  are two constants with  $-\frac{\pi}{2} < b < a < \frac{\pi}{2}$ , and let  $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ . Suppose that*

$$\begin{cases} \operatorname{div} Tu = 0 & \text{in } \Omega, \\ u = y \tan x & \text{on } \partial\Omega. \end{cases}$$

*Then  $u \equiv y \tan x$  in  $\Omega$ ; in other words,  $u$  must be a helicoid.*

*Proof.* For any  $y > 0$ , let

$$\begin{aligned} \Omega_y &= (b, a) \times (-y, y), \\ \Gamma_y &= \{(b, a) \times \{y\}\} \cup \{(b, a) \times \{-y\}\} \end{aligned}$$

and, set

$$\begin{aligned} g(y) &= \int_{\Gamma} (u - v)(Tu - Tv) \cdot \nu \, ds \\ &= \oint_{\partial\Omega_y} (u - v)(Tu - Tv) \cdot \nu \, ds \\ &= \int \int_{\Omega_y} (\nabla u - \nabla v) \cdot (Tu - Tv) \end{aligned}$$

where  $v \equiv y \tan x$  and  $\nu$  is the unit outward normal of  $\Gamma_y$  and  $\partial\Omega_y$ . Since  $(\nabla u - \nabla v) \cdot (Tu - Tv) \geq 0$ , Fubini's Theorem yields that the derivative  $g'(y)$

exists for almost all  $y > 0$  and

$$g'(y) = \int_{\Gamma_y} (\nabla u - \nabla v) \cdot (Tu - Tv)$$

whenever  $g'(y)$  exists. Thus, in view of (2), for these  $y$ ,

$$\begin{aligned} g'(y) &\geq \int_{\Gamma_y} \frac{\sqrt{1 + |\nabla u|^2} + \sqrt{1 + |\nabla v|^2}}{2} |Tu - Tv|^2 \\ &\geq \left( \min_{\Gamma_y} \frac{\sqrt{1 + |\nabla v|^2}}{2} \right) \int_{\Gamma_y} |Tu - Tv|^2, \end{aligned}$$

in which, as  $v_x = y \sec^2 x$ , we have

$$\frac{\sqrt{1 + |\nabla v|^2}}{2} \geq \frac{y \sec^2 x}{2} \geq \frac{y}{2}.$$

Furthermore, by means of Schwarz's inequality,

$$|\Gamma_y| \int_{\Gamma_y} |Tu - Tv|^2 \geq \left( \int_{\Gamma_y} |Tu - Tv| \right)^2,$$

and  $|\Gamma_y| = 2(a - b)$  (in virtue of the special geometry of  $\Omega$ ), thus

$$\int_{\Gamma_y} |Tu - Tv|^2 \geq \frac{1}{2(a - b)} \left( \int_{\Gamma_y} |Tu - Tv| \right)^2.$$

Hence, for any  $y$  where  $g'(y)$  exists,

$$\begin{aligned} (3) \quad g'(y) &\geq \frac{y}{4(a - b)} \left( \int_{\Gamma_y} |Tu - Tv| \right)^2 \\ &\geq \frac{y}{4(a - b)} \left( \frac{1}{\pi} \int_{\Gamma_y} \tan^{-1}(u - v)(Tu - Tv) \cdot \nu \right)^2. \end{aligned}$$

Now, for all  $y > 0$ , set

$$\begin{aligned} h(y) &= \int_{\Gamma_y} \tan^{-1}(u - v)(Tu - Tv) \cdot \nu \\ &= \int \int_{\Omega_y} \frac{(\nabla u - \nabla v) \cdot (Tu - Tv)}{1 + (u - v)^2}. \end{aligned}$$

We note that  $h \geq 0$  and  $h(y)$  increases as  $y$  increases. Thus, if  $h \equiv 0$ , it is easy to see that Theorem 2\* holds. Hence we may assume that  $h \not\equiv 0$  and

that there exist two positive constants  $y_1$  and  $c_1$  such that  $h(y) \geq c_1$  for all  $y \geq y_1$ .

Substituting this into (3), we obtain  $g'(y) \geq \frac{c_1^2}{4(a-b)\pi^2}y$  for almost all  $y \geq y_1$ , which yields  $g(y) - g(y_1) \geq \frac{c_1^2}{4(a-b)\pi^2}(y - y_1)^2$ . Since  $|u| = O(|y|)$  on  $\partial\Omega$  as  $|y| \rightarrow \infty$ , by [7, p. 256], we have  $|u| = O(|y|)$  in  $\Omega$  as  $|y| \rightarrow \infty$ . Since for all  $y > 0$ ,  $g(y) = \int_{\Gamma_y} (u - v)(Tu - Tv) \cdot \nu$  and  $|Tu - Tv| \leq 2$ , we have  $g(y) = O(y)$  as  $y \rightarrow \infty$ , which gives a contradiction and completes our proof.  $\square$

By modifying the proof of Theorem 2\*, we can derive the following

**Theorem 3.** *Let  $\Omega \subseteq \mathbb{R}^2$  be an unbounded domain and let  $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ . Let  $B_R, \Gamma_R$  and  $|\Gamma_R|$  be as in Theorem 1. Suppose that*

$$\begin{cases} \text{(i)} & \operatorname{div} Tu = \operatorname{div} Tv & \text{in } \Omega, \\ \text{(ii)} & u = v & \text{on } \partial\Omega, \\ \text{(iii)} & \max_{\Omega \cap B_R} |u - v| = o\left(\int_{R_0}^R \frac{1}{|\Gamma_R|} \min_{\Gamma_R} \sqrt{1 + |\nabla v|^2} dR\right) & \text{as } R \rightarrow \infty, \end{cases}$$

where  $R_0$  is a positive constant. Then we have  $u \equiv v$  in  $\Omega$ .

**Remark.**

- (a) Notice that condition (iii) depends on  $|\nabla v|$  only, without assuming any condition on  $|\nabla u|$ .
- (b) In Theorem 2\*, since  $\operatorname{div} Tu = 0$  in  $\Omega$  and  $u = y \tan x$  on  $\partial\Omega$ , by [7, p. 256], we have  $u = O(|y|)$  in  $\Omega$  as  $|y| \rightarrow \infty$ . And so, condition (iii) of Theorem 3 holds.

*Proof of Theorem 3.* The proof is similar to that of Theorem 2\*. For every  $R > 0$ , let

$$\begin{aligned} M(R) &= \max_{\Omega \cap B_R} |u - v| = \max_{\Gamma_R} |u - v|, \\ Q(R) &= \min_{\Gamma_R} \frac{\sqrt{1 + |\nabla v|^2}}{2}, \\ g(R) &= \int_{\Gamma_R} (u - v)(Tu - Tv) \cdot \nu = \int \int_{\Omega_R} (\nabla u - \nabla v) \cdot (Tu - Tv) \end{aligned}$$

and

$$h(R) = \int_{\Gamma_R} \tan^{-1}(u - v)(Tu - Tv) \cdot \nu.$$

As in the proof of Theorem 2\*, we may assume that  $h \not\equiv 0$  and that there exist two positive constants  $R_1$  and  $C_1$  such that  $R_1 > R_0$  and

$$(4) \quad h(R) \geq C_1 \quad \text{for all } R \geq R_1.$$

For almost all  $R > 0$ , we have

$$\begin{aligned}
 (5) \quad g'(R) &= \int_{\Gamma_R} (\nabla u - \nabla v) \cdot (Tu - Tv) \\
 &\geq \int_{\Gamma_R} Q(R) |Tu - Tv|^2 \\
 &\geq Q(R) |\Gamma_R|^{-1} \left( \int_{\Gamma_R} |Tu - Tv| \right)^2.
 \end{aligned}$$

Thus  $g'(R) \geq (\frac{\pi}{2})^2 C_1^2 |\Gamma_R|^{-1} Q(R)$ , for almost all  $R > R_1$ . Hence, for every  $R$  and  $R_2$  such that  $R > R_2 \geq R_1$ , we have

$$(6) \quad g(R) - g(R_2) \geq \left( \frac{2C_1}{\pi} \right)^2 \int_{R_2}^R \frac{Q(r)}{|\Gamma_r|} dr.$$

By (4), we have  $M(R) > 0$  for all  $R \geq R_1$ , hence (5) yields, for almost all  $R \geq R_1$ ,

$$\begin{aligned}
 g'(R) &\geq Q(R) |\Gamma_R|^{-1} \int |Tu - Tv|^2 \\
 &\geq \frac{g^2(R) Q(R)}{M^2(R) |\Gamma_R|};
 \end{aligned}$$

and so, for every  $R$  and  $R_2$  such that  $R > R_2 \geq R_1$ ,

$$-\frac{1}{g} \Big|_{R_2}^R \geq \int_{R_2}^R \frac{g'}{g^2} \geq \int_{R_2}^R \frac{Q(r)}{M^2(r) |\Gamma_r|} dr \geq \frac{1}{M^2(R)} \int_{R_2}^R \frac{Q(r)}{|\Gamma_r|} dr,$$

and then

$$(7) \quad \frac{1}{g(R_2)} \geq \frac{1}{M^2(R)} \int_{R_2}^R \frac{Q(r)}{|\Gamma_r|} dr.$$

Now, since  $M(R) > 0$  for all  $R \geq R_1$ ,  $M(R)$  is an increasing function of  $R$  and, in view of condition (iii),

$$(M(R))^{-1} \int_{R_1}^R \frac{Q(r)}{|\Gamma_r|} dr \rightarrow \infty \quad \text{as } R \rightarrow \infty,$$

and also

$$\int_{R_1}^R \frac{Q(r)}{|\Gamma_r|} dr \rightarrow \infty \quad \text{as } R \rightarrow \infty;$$

hence we can choose a constant  $R_3 > R_1$  such that

$$(M(R))^{-1} \int_{R_1}^R \frac{Q(r)}{|\Gamma_r|} dr \geq \sqrt{2\pi} C_1^{-1}, \quad \text{for every } R \geq R_3,$$

and a constant  $R_4, R_4 > R_3$ , which depends on  $R_3$ , such that

$$\int_{R_1}^{R_4} \frac{Q(r)}{|\Gamma_r|} dr = 2 \int_{R_1}^{R_3} \frac{Q(r)}{|\Gamma_r|} dr.$$

With this choice of  $R_3$  and  $R_4$ , we have

$$\begin{aligned} 1 &\geq \frac{g(R_3) - g(R_1)}{g(R_3)} \\ &\geq \left[ \left( \frac{2C_1}{\pi} \right)^2 \int_{R_1}^{R_3} \frac{Q(r)}{|\Gamma_r|} \right] \left[ (M^2(R_4))^{-1} \int_{R_3}^{R_4} \frac{Q(r)}{|\Gamma_r|} \right] \quad (\text{by (6), (7)}) \\ &= \left[ \left( \frac{2C_1}{\pi} \right)^2 (M^2(R_4))^{-1} \right] \frac{1}{4} \left( \int_{R_1}^{R_4} \frac{Q(r)}{|\Gamma_r|} \right)^2 \quad (\text{by the choice of } R_3, R_4) \\ &\geq \frac{C_1^2}{\pi^2} (2\pi^2) C_1^{-2} \quad (\text{again by the choice of } R_3 \text{ and } R_4) \\ &\geq 2, \end{aligned}$$

which is desired contradiction.  $\square$

**Remark.** The above proof is to show (6), which is the lower bound of  $g(R)$ , and (7), which is the upper bound of  $g(R)$ . And from (6) and (7), we get contradiction and so prove the theorem. Miklyukov [5] and Hwang [4] only observed the upper bound of  $g(R)$ , and so could not derive the better result as in Collin and Krust [2].

Let  $\Omega$  be a domain in  $\mathbb{R}^2$ . Consider the following equation in divergence form

$$\operatorname{div} A(x, u, \nabla u) = f(x, u, \nabla u),$$

where

$$\begin{aligned} A &= (A_1, A_2), \quad A_i: \Omega \times \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}, \quad i = 1, 2, \\ f &: \Omega \times \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}, \end{aligned}$$

and

$$A_i \in C^0(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^2) \cap C^1(\Omega \times \mathbb{R} \times \mathbb{R}^2), \quad i = 1, 2, \quad f \in C^0(\Omega \times \mathbb{R} \times \mathbb{R}^2).$$

We rewrite  $A(x, u, \nabla u)$  briefly as  $Au$ .

Suppose that  $Au$  satisfies the following structural condition:

$$(8) \begin{cases} (Au - Av) \cdot (\nabla u - \nabla v) \geq |Au - Av|^2 Q(R), \\ \quad \text{where } R = \sqrt{x^2 + y^2} \text{ and } Q(R) \text{ is a positive function,} \\ (\nabla u - \nabla v) \cdot (Au - Av) = 0, \quad \text{iff } \nabla u = \nabla v. \end{cases}$$

Now we have the following result:

**Theorem 4.** *Let  $\partial\Omega = \Sigma^\alpha + \Sigma^\beta$  be a decomposition of  $\partial\Omega$  such that  $\Sigma^\beta \in C^1$ . Let  $u, v \in C^2(\Omega) \cap C^1(\Omega \cup \Sigma^\beta) \cap C^0(\bar{\Omega})$  and let  $M(R) = \max_{\Omega \cap B_R}(u - v, 0)$ . Suppose that*

$$\left\{ \begin{array}{ll} \text{(i)} & A \text{ satisfies the structural condition (8)} \\ \text{(ii)} & \operatorname{div} Au \geq \operatorname{div} Av \text{ in } \Omega \\ \text{(iii)} & u \leq v \text{ on } \Sigma^\alpha \\ \text{(iv)} & Au \cdot \nu \leq Av \cdot \nu \text{ on } \Sigma^\beta \\ \text{(v)} & M(R) = o\left(\int_{R_0}^R \frac{|Q_r|}{|\Gamma_r|} dr\right) \text{ as } R \rightarrow \infty, \text{ where } R_0 \text{ is} \\ & \text{a positive constant.} \end{array} \right.$$

*Then, if  $\partial\Omega = \Sigma^\beta$ , we have either  $u(x) \equiv v(x) + a$  positive constant or else  $u(x) \leq v(x)$ . Otherwise,  $u(x) \equiv v(x)$ .*

The proof of Theorem 4 is exactly the same as that of Theorem 3. The interested readers may consult [4].

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