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DIFFERENTIAL GALOIS GROUPS OF CONFLUENT GENERALIZED HYPERGEOMETRIC EQUATIONS: AN APPROACH USING STOKES MULTIPLIERS

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In memory of Ellis Kolchin

We explicitly compute the differential Galois groups of some families of generalized confluent hypergeometric equations by a method based on the asymptotic analysis of their irregular singularity at infinity. We obtain the Galois group directly from a particular set of topological generators. These are formal and analytic invariants of the equation, reflecting the asymptotic behaviour of the solutions. Our calculations yield classical groups as well as as the exceptional group G_2.

0. Introduction.

The differential Galois groups of all irreducible confluent hypergeometric differential equations have been determined by Katz and Gabber, after previous work of Beukers, Brownawell and Heckman (see [**BBH**] and [**K2**]). Their proofs use purely algebraic arguments, and rely on global characterizations of semisimple algebras. The aim of the present paper is to recover these differential Galois groups in a number of cases by explicitly giving some of their topological generators. It is indeed a classical result of Schlesinger that the local differential Galois group of a meromorphic differential operator at a regular singular point is topologically generated by the monodromy acting on a fundamental solution. The presence of exponential factors in formal solutions at an irregular singularity and the fact that the corresponding formal series may be divergent give rise to new Galois automorphisms, reflecting the asymptotic behaviour of the solutions. These particular automorphisms of a solution field are the elements of the exponential torus, the Stokes multipliers and the formal monodromy, which together generate the differential Galois group topologically by a theorem of Ramis (cf. [**R1**], [**MR2**]). We show how Ramis's theorem can be applied to determine explicitly the differential

Galois group over $\mathbf{C}(z)$ of generalized confluent hypergeometric equations $D_{qp}(y) = 0$ where

$$D_{qp} = (-1)^{q-p} z \prod_{j=1}^{p} (\partial + \mu_j) - \prod_{j=1}^{q} (\partial + \nu_j - 1),$$

with $\partial = z\frac{d}{dz}$, $1 \leq p < q$, and with complex parameters μ_j, ν_j such that all μ_i are distinct modulo $\mathbf{Z}$ for $1 \leq i \leq p$.

The results were announced in [**M1**], [**M2**]. Our method, relying on the computation of Stokes matrices, allowed us in a joint work [**DM**] with A. Duval to determine the Galois group of equations D_{31} and D_{32}, including in the reducible cases. The present paper deals with higher order irreducible equations.

Our aim is to illustrate on these examples how the analytic invariants introduced by Birkhoff ([**Bi**]) and generalized by Sibuya [**Si**], Balser, Jurkat, Lutz ([**BJL**]) and Malgrange ([**Ma**]), as well as the formal invariants obtained from the canonical form of Levelt-Turrittin ([**L1**], [**Tu**]), play an essential rôle in the algebraic structure of the differential Galois group.

Let us detail the content of each section. In Section 1 we recall the main results in differential Galois theory and complex analysis needed in the proofs. In Section 2 we define a formal fundamental solution for the general equation D_{qp}. In Section 3 we prove some general results for the Galois group $G = \mathrm{Gal}_{\mathbf{C}(z)}(D_{qp})$. We explicitely compute G for equations D_{42} in Section 4, for selfdual equations $D_{2q,2q-2}$ in Section 5, for equations D_{51} in Section 6, and for equations D_{71} in Section 7.

1. Generalities.

For basic facts on differential Galois theory we refer to [**B**], [**Ka**], [**Ko**], [**L2**], [**MR1**] and [**S**]. An overview of recent developments is presented in Bertrand's Bourbaki lecture [**Be**] on the subject.

We consider a linear differential system

$$\Delta Y = \partial Y - AY = 0$$

where A is an $n \times n$ matrix with entries in some differential field (K, ∂) whose field of constants is $\mathbf{C}$, and Y is an unknown matrix function with entries in some differential field extension of K. Up to *K-equivalence* (that is a change $Y = PZ$ with $P \in \mathrm{GL}(n, K)$ of Δ into a system in Z) one may suppose that Δ is the system naturally obtained from a linear differential operator

$$D = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_0$$

with coefficients in K.

Definition 1.1. The *differential Galois group* $\mathrm{Gal}_K(\Delta)$ of Δ over K is the group of differential K-automorphisms of a Picard-Vessiot extension of K relative to D. It is defined up to isomorphism and represented as an algebraic subgroup of $\mathrm{GL}(n, \mathbf{C})$ with respect to a fundamental solution of Δ.

Definition 1.2. Let Δ be a linear differential system with coefficients in $\mathbf{C}(x)$. The *global* (resp. *local* at a point $a \in \mathbf{P}^1(\mathbf{C})$) differential Galois group G (resp. G_a) of Δ over $\mathbf{P}^1(\mathbf{C})$ (resp. at $a \in \mathbf{P}^1(\mathbf{C})$) is the group $\mathrm{Gal}_K(\Delta)$ where $K = \mathbf{C}(x)$ (resp. K is the field $\mathbf{C}\{x_a\}[{x_a}^{-1}]$ of germs of meromorphic functions at $a \in \mathbf{P}^1(\mathbf{C})$, where x_a denotes a local uniformisant at a).

1.2.1. Let us fix a base-point x_0 of $\mathbf{P}^1(\mathbf{C}) \setminus S$, where S denotes the set of singular points of D and let Σ_{x_0} denote an analytic germ of fundamental solution of D at x_0. We may analytically continue Σ_{x_0} to a "sectorial" germ of fundamental solution at every $a \in S$, provided that we fix a number of choices. More precisely let U_a, for every $a \in S$, be an open disc with center a, together with a local parameter x_a at a, and such that $U_a \cap S = \{a\}$. Let d_a be a fixed ray from a in U_a, together with a point $b_a \in d_a$ in U_a and a path γ_a from x_0 to b_a. Analytic continuation of Σ_{x_0} along γ_a and d_a provides an analytic germ Σ_a of fundamental solution on a germ of open sector with vertex a, bisected by d_a. Let G_a be the local Galois group of D over $\mathbf{C}\{x_a\}[x_a]^{-1}$ with respect to Σ_a. If we "conjugate" elements of G_a by the analytic continuation described above, we get an injective morphism of algebraic groups $G_a \hookrightarrow G$ with respect to the representation of these groups in $\mathrm{GL}(n, \mathbf{C})$ given by Σ_a and Σ_{x_0} respectively. Thus all G_a, $a \in S$, can be simultaneously identified with closed subgroups of G and we have the following result.

Proposition 1.3. *The global Galois group G is topologically generated in $\mathrm{GL}(n, \mathbf{C})$ by the subgroups G_a, where a runs over S.*

Proof. Let u be an element of the Picard-Vessiot extension of $\mathbf{C}(x)$ generated by the entries of Σ_{x_0} in $\mathbf{C}\{x - x_0\}[(x - {x_0}^{-1})]$ (we choose x_0 in the finite plane). With notations as above, suppose that u is invariant by the subgroups G_a, $a \in S$, all these subgroups of $\mathrm{GL}(n, \mathbf{C})$ being defined with respect to Σ_{x_0} (via the prescribed analytic continuations). Then, by normality of Picard-Vessiot extensions, u extends to a meromorphic function on $\mathbf{P}^1(\mathbf{C})$, that is, a rational function $u \in \mathbf{C}(x)$. Equivalently, by differential Galois correspondence, G is the Zariski closure in $\mathrm{GL}(n, \mathbf{C})$ of the subgroup generated by the (finite) union of the closed subgroups G_a, that is, G is generated as a group by the G_a. □

When $K = \mathbf{C}\{x\}[x^{-1}]$, we know from the classical theory that the system $\Delta Y = 0$ admits a *formal* fundamental solution

$$\widehat{Y}(x) = \widehat{H}(x)x^L e^{Q(t)} \tag{1.4}$$

where $t^\sigma = x$ ($\sigma \in \mathbf{N}^*$), $L \in M_n(\mathbf{C})$, $\widehat{H} \in \mathrm{GL}(n; \mathbf{C}[[x]][x^{-1}])$ and $Q = \mathrm{diag}(q_1, \dots, q_n)$, where $q_i \in t^{-1}\mathbf{C}[t^{-1}]$, $i = 1, \dots, n$.
We may moreover suppose that $t \mapsto te^{2i\pi/\sigma}$ permutes the polynomials q_i and that $[e^{2i\pi\sigma L}, Q] = 0$ if $\sigma \neq 1$, while $[L, Q] = 0$ if $\sigma = 1$. Note that $\widehat{Y}$ is given here in terms of formal series in x, rather than t. This can easily be deduced from ([**BJL**], Th.I, p. 199).

Notations 1.5. A solution $\widehat{Y}$ being given as in (1.4), let $\widehat{K} = \mathbf{C}[[x]][x^{-1}]$, $K_t = \mathbf{C}\{t\}[t^{-1}]$, $\widehat{K_t} = \mathbf{C}[[t]][t^{-1}]$, $\zeta = e^{2i\pi/\sigma}$ and let $\mathcal{K}$ denote the differential extension of $\widehat{K_t}$ generated by the entries of x^L and e^Q (considered as symbols as long as no determination has been fixed for $\log(x)$). Let $\mathcal{L} = K\langle\widehat{Y}\rangle$ denote the Picard-Vessiot extension of K generated by the entries of $\widehat{Y}$ in $\mathcal{K}$ and let G denote the differential Galois group $\mathrm{Gal}_K(\Delta)$.

Definition 1.6. The *formal monodromy* $\widehat{M} \in G$ of Δ relative to $\widehat{Y}$ is defined by $\widehat{Y}(t\zeta) = \widehat{Y}(t)\widehat{M}$ and the *formal monodromy group* G_M is the closed subgroup of G topologically generated by $\widehat{M}$.

Definition 1.7. The *exponential torus* $\mathcal{T}(\Delta)$ of Δ relative to $\widehat{Y}$ is the group of differential $\widehat{K_t}$-automorphisms of the differential extension $\widehat{K_t}\langle e^Q\rangle = \widehat{K_t}\langle e^{q_1}, \dots, e^{q_n}\rangle$ of $\widehat{K_t}$. By extending these automorphisms to $\mathcal{K}$, we can identify $\mathcal{T}(\Delta)$ with a subgroup of G. It is a torus $\mathcal{T} \simeq (\mathbf{C}^*)^r$, where r denotes the rank of the $\mathbf{Z}$-module generated by the polynomials q_i. More precisely, we may order these q_i, or equivalently the columns of $\widehat{Y}$, in such a way that $\{q_1, \dots, q_r\}$ is a $\mathbf{Z}$-basis for $\sum_{i=0}^n \mathbf{Z}q_i$. Then there exist monomial functions $a_{r+1}, \dots, a_n$ such that $\mathcal{T}(\Delta)$ consists of all matrices $T(\lambda_1, \dots, \lambda_r) = \mathrm{diag}(l_1, \dots, l_n)$, where $\lambda_1, \dots, \lambda_r$ are arbitrary complex parameters and

$$\begin{aligned} l_i &= \lambda_i, & i &\le r \\ l_i &= a_i(\lambda_1, \dots, \lambda_r), & i &> r. \end{aligned}$$

Remark 1.8. The formal monodromy and the exponential torus are *formal* invariants, depending only on the $\widehat{K}$-equivalence class of Δ or, with other words, on the connection defined by Δ over $\widehat{K}$. They generate topologically the *formal* Galois group $\mathrm{Gal}_{\widehat{K}}(\Delta)$ corresponding to the Picard-Vessiot extension $\widehat{K}\langle\widehat{Y}\rangle$ of $\widehat{K}$ (cf. [**Be**]).

Lemma 1.9. *The formal monodromy* $\widehat{M}$ *acts by conjugation on* $\mathcal{T}$. *Hence* $\mathcal{T}$ *is a normal subgroup of* $\mathrm{Gal}_{\widehat{K}}(\Delta)$.

Proof. For a given choice of $\widehat{Y}$ as in (1.2.1) there exists a permutation $\alpha \in \mathfrak{S}_n$ such that $q_i(te^{\frac{2i\pi}{\sigma}}) = q_{\alpha(i)}$. The change of variable $t \mapsto te^{\frac{2i\pi}{\sigma}}$ operates on polynomials in t (considered as functions of $x = t^\sigma$) as the usual monodromy, hence preserving any possible algebraic relation between the functions q_i. If we order these as in 1.7 we get, for every $T(\lambda_1, \dots, \lambda_r) = \mathrm{diag}(l_1, \dots, l_n) \in \mathcal{T}$

$$\widehat{M}T(\lambda_1, \dots, \lambda_r)\widehat{M}^{-1} = \mathrm{diag}(l_{\alpha(1)}, \dots, l_{\alpha(n)}) = T(l_{\alpha(1)}, \dots, l_{\alpha(r)})$$

whence $\widehat{M}T\widehat{M}^{-1} \in \mathcal{T}$. □

From now on we shall assume that all non-zero $(q_i - q_j)$ for $1 \leq i, j \leq n$ have the same degree k. This is enough for our purpose.

Definition 1.10. All angular directions and sectors are to be considered on the universal covering of the unit circle or, with other words, on the Riemann surface of the logarithm. A *singular direction* for Δ with respect to $\widehat{Y}$ is a bisecting ray of any maximal angular sector where $\mathrm{Re}(q_i(x^{\frac{1}{\sigma}}) - q_j(x^{\frac{1}{\sigma}})) < 0$ for some $i, j = 1, \dots, n$.

Definition 1.11. For a given $k > 0$ and a given direction d the formal series $\hat{f} = \sum_{n \geq 0} a_n x^n \in \mathbf{C}[[x]]$ is *k-summable in the direction d* if there exists a germ V of open sectors bisected by d with opening greater than π/k and a holomorphic function f on V, satisfying inequalities:

$$|x|^{-n} \left| f(x) - \sum_{p=0}^{n-1} a_p x^p \right| < C_W (n!)^{1/k} A_W^n$$
$$\forall x \in W, \forall n \in \mathbf{N}^*,$$

on every proper subsector $W \subset V$, with constants A_W and C_W depending on W only. The function f is then unique, since the difference $f - g$ with another function g satisfying the same inequalities would be "exponentially flat" of degree k on a sector with opening greater than π/k, whence identically zero by Watson's lemma (cf. [**W**], p. 295). The function f is called the "*k-sum of* $\hat{f}$ *along d* ". These definitions easily extend to formal Laurent series.

In order to define the Stokes multipliers as elements of the Galois group we need the following fundamental result (cf. [**R4**], [**Tu**]):

Theorem 1.12 (Turrittin, Martinet-Ramis). *With notations as before, the matrix* $\widehat{H}$ *is k-summable in every non-singular direction d and if H denotes the k-sum of* $\widehat{H}$ *along d, then* $Y = Hx^L e^Q$ *is an actual solution of* Δ. *Moreover H can be obtained from* $\widehat{H}$ *by a Borel-Laplace transform, which* *yields an injective morphism of differential algebras from the algebra of k-summable series along d to the algebra of germs of holomorphic functions on germs of sectors bisected by d.*

Let α be a singular direction for Δ at 0 and let $\alpha^+ = \alpha + \varepsilon$ (resp. $\alpha^- = \alpha - \varepsilon$) where $\varepsilon > 0$ be non-singular neighbouring directions of α. Let Y_{α^+} (resp. Y_{α^-}) be actual solutions of Δ obtained from $\widehat{Y}$ by (1.12) on a germ of sector bisected by α^+ (resp.α_-).

Definition 1.13. With respect to a given formal fundamental solution $\widehat{Y}$ as in (1.4), the *Stokes matrix* (or *multiplier*) $S_\alpha \in \mathrm{GL}(n, \mathbf{C})$ corresponding to the singular line α of Δ at 0 is defined by $Y_{\alpha^-} = Y_{\alpha^+} S_\alpha$ on α.

Note that this definition is independent of the choice of α^+ and α^- as soon as these are sufficiently near α (cf. [**M2**], p. 158). Moreover, if we identify S_α with its conjugate by Borel-Laplace transform then the map $\widehat{Y} \mapsto \widehat{Y} S_\alpha$ appears as a *differential* automorphism of $\mathcal{L}$, that is an element of G (cf. [**M2**], 2.3.10, Th. 2.3.11). The formal monodromy $\widehat{M}$ clearly operates by conjugation on the set of Stokes matrices by $\widehat{M} S_\alpha \widehat{M}^{-1} = S_{\alpha+2\pi}$.

Note that the Stokes multipliers are *analytic* invariants, which only depend on the K-equivalence class of Δ.

The *actual monodromy* M can be recovered from the formal monodromy and the Stokes matrices by the formula

$$M = \widehat{M} S_{\alpha_1} \dots S_{\alpha_r} \tag{1.14}$$

where $0 \leq \alpha_1 \dots \leq \alpha_r < 2\pi$ are the singular rays lying in the x-plane.

All results in the forthcoming sections are based on the following description of G (cf. [**Be**], [**MR**], [**R1**], [**R2**], [**R3**] and also [**LR**] for a different proof of this theorem).

Theorem 1.15 (Ramis). *With respect to a formal solution $\widehat{Y}$ given as in (1.2.1) the analytic differential Galois group G of Δ at 0 is the Zariski closure in* $\mathrm{GL}(n, \mathbf{C})$ *of the subgroup generated by the formal monodromy $\widehat{M}$, the exponential torus $\mathcal{T}(\Delta)$ and the Stokes matrices S_α for all singular rays α.*

2. Confluent generalized hypergeometric equations.

We consider the family of differential equations:

$$D_{qp} = (-1)^{q-p} z \prod_{j=1}^{p} (\partial + \mu_j) - \prod_{j=1}^{q} (\partial + \nu_j - 1) \tag{2.1}$$

where $\partial = z\frac{d}{dz}$, $1 \le p < q$ and μ_j, ν_j are complex parameters. Throughout this paper we will assume that

$$\text{(2.1.1)} \qquad \text{all } \mu_i \text{ are distinct modulo } \mathbf{Z}, \quad \text{for } 1 \le i \le p.$$

These equations have exactly two singularities, a regular singular one at 0 and an irregular one at ∞. Let G denote the global differential Galois group $\mathrm{Gal}_{\mathbf{C}(z)}(D_{qp})$ and G_0 (respectively G_∞) the local Galois group at 0 (repectively ∞). The local differential Galois subgroup G_0 corresponding to the regular singularity at 0 is topologically generated by the monodromy operator at 0. This is a well-known result due to Schlesinger(cf. [**Sch**]). As outlined in 1.2.1, both G_0 and G_∞ appear as subgroups of G relatively to a given germ of fundamental solution Σ_{x_0} at a base-point $x_0 \in \mathbf{P}^1(\mathbf{C}) \setminus S$, and with respect to prescribed sectors and paths. We see that the action on Σ_{x_0} of the monodromy round 0 coincides here with the (inverse) monodromy round ∞, so that G_0 actually is a subgroup of G_∞, that is, $G = G_\infty$ in this case. To compute G, we shall therefore first determine a fundamental solution of D_{qp} at infinity.

Notations 2.2. For a given equation D_{qp} as in (2.1) let

$$\sigma = q - p, \qquad \zeta = e^{2i\pi/\sigma}, \qquad \lambda = \frac{1}{2}(\sigma + 1) + \sum_{j=1}^{p} \mu_j - \sum_{j=1}^{q} \nu_j.$$

For $\underline{x} = (x_1, \dots, x_p) \in \mathbf{C}^p$ and $a \in \mathbf{C}$, let

$$\begin{aligned} (\underline{x} + a) &= (x_1 + a, \dots, x_p + a) \\ \underline{x}_i^\wedge &= (x_1, \dots, \hat{x}_i, \dots, x_p) \in \mathbf{C}^{p-1} \\ \underline{x}_i^* &= (\underline{x} - x_i)_i^\wedge \\ \Gamma(\underline{x}) &= \prod_{i=1}^{n} \Gamma(x_i) \\ (\underline{x})_n &= \prod_{i=1}^{n} x_i(x_i + 1) \cdots (x_i + n - 1). \end{aligned}$$

For $\underline{a} \in \mathbf{C}^p$ and $\underline{b} \in (\mathbf{C} \backslash \mathbf{Z}^-)^q$ let

$${}_pF_q(\underline{a}, \underline{b}; z) = \sum_{n \ge 0} \frac{(\underline{a})_n}{(\underline{b})_n} \frac{z^n}{n!}$$

and

$$G_{pq}^{mn}\left(z \,\middle|\, \begin{matrix} \underline{a} \\ \underline{b} \end{matrix}\right) = \frac{1}{2i\pi} \int_\gamma \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} z^s ds$$

for a suitable path γ.

A classical computation yields a *formal fundamental solution*:

$$(2.3)\qquad \widehat{Y}(z)=\begin{cases}\Phi_k(z)=z^{-\mu_k}{}_qF_{p-1}(1+\mu_k-\underline{\nu},1-\mu_k^*;z^{-1})\\ \text{for}\ \ k=1,\dots,p,\\ \Theta_j=e^{-\sigma\zeta^j z^{1/\sigma}}\Theta(ze^{2i\pi j})\\ \text{for}\ \ j\in\mathbf{Z}\quad\text{and}\quad 1-[-\frac{\sigma}{2}]\leq j\leq[\frac{\sigma}{2}].\end{cases}$$

where Θ is a formal series in the ramified variable $z^{\frac{1}{\sigma}}$.

The *Mellin transform* of D_{qp} is a difference equation of order one. Special solutions of it can therefore be expressed by means of the Γ-function. The corresponding solutions of D_{qp} are G-functions in the sense of Meijer (cf. [**Me**]), namely

$$\begin{aligned}G_k(z)&=G_{pq}^{q1}\left(-z\left|\begin{matrix}1-\mu_k,(1-\underline{\mu})_k^{\wedge}\\ 1-\underline{\nu}\end{matrix}\right.\right)\\ &=\frac{1}{2i\pi}\int_{\gamma_k}\frac{\Gamma(1-\underline{\nu}-s)\Gamma(\mu_k+s)}{\Gamma((1-\underline{\mu}-s)_k^{\wedge})}\,e^{i\pi s}z^s ds\end{aligned}$$

on one hand ($k=1,\dots,p$), and

$$G_0(z)=G_{pq}^{q0}\left(z\left|\begin{matrix}1-\underline{\mu}\\ 1-\underline{\nu}\end{matrix}\right.\right)=\frac{1}{2i\pi}\int_{\gamma_0}\frac{\Gamma(1-\underline{\nu}-s)}{\Gamma(1-\underline{\mu}-s)}z^s ds$$

on the other hand, where the paths γ_k and γ_0 from $-i\infty$ to $+i\infty$ both leave the points $(-\nu_j+n)$, $j=1,\dots,q$, $n\in\mathbf{N}$, on the right whereas γ_k also has to leave the points $(-\mu_k-n)$, $k=1,\dots,p$, $n\in\mathbf{N}$, on the left.

As $|z|$ tends to infinity, the asymptotic expansion of G_k (respectively G_0) is

$$e^{-i\pi\mu_k}\frac{\Gamma(1+\mu_k-\underline{\nu})}{\Gamma(1+\mu_k-\underline{\mu})}\,z^{-\mu_k}{}_qF_{p-1}(1+\mu_k-\underline{\nu}\ ;\ 1-\mu_k^*\ ;\ z^{-1}),$$

(respectively $e^{-\sigma z^{1/\sigma}}z^{\lambda/\sigma}\Theta(z)$) on suitable sectors of the universal covering of the z-plane at the origin. Analytic continuation of the G-functions G_0 and G_k (or of linear combinations of these) on the t-plane ($t^\sigma=z$) yields 1-sums of $\widehat{Y}$ along all non-singular lines.

The singular lines of D_{qp} at ∞ are the bisecting lines of maximal sectors of decay for $e^{-\sigma\zeta^j t}$ or $e^{\sigma(\zeta^i-\zeta^k)t}$ for $j,k\in\mathbf{N}$ and $j\neq k$. From the natural fundamental solution of D_{qp} at 0 consisting of generalized hypergeometric series, Meijer (cf. [**Me**]) deduced useful linear formulas relating the different determinations of the multivalued functions G_0 and G_k. These formulas were used in [**DM**] to compute the Stokes matrices of D_{qp}.

Notation 2.4. As in 1.13, we define the Stokes matrices relatively to the formal fundamental solution $\widehat{Y}$ of D_{qp} given in 2.3. Let $S_d \in G$ denote the Stokes matrix corresponding to a singular ray d and let s_d denote the corresponding element of the Lie algebra $\mathfrak{G}$ of G. These *infinitesimal Stokes matrices* are well-defined by $\exp s_d = S_d$ since the Stokes matrices are easily seen to be unipotent from their definition.

The formal monodromy $\widehat{M}$ is represented by the matrix

$$\widehat{M} = \begin{pmatrix} D & 0 \\ 0 & R \end{pmatrix}$$

where $D = \operatorname{diag}(e^{-2i\pi\underline{\mu}})$ and $R = e^{2i\pi\lambda/\sigma}P_\sigma$, if we denote by P_σ the permutation matrix of order σ

$$P_\sigma = \begin{pmatrix} 0 & 0 & \cdots & & 1 \\ 1 & 0 & \cdots & & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & 0 \end{pmatrix}.$$

Let n_{ik}, for $k \in \mathbf{Z}$ and $0 \leq i \leq \varphi(\sigma) - 1$, ($\varphi$ denoting the Euler function) be integers defined by

$$\zeta^{k+[-\frac{\sigma}{2}]} = \sum_{i=0}^{\varphi(\sigma)-1} n_{ik}\zeta^i.$$

The exponential torus $\mathcal{T} = \mathcal{T}(D_{qp})$ has dimension $\varphi(\sigma)$ and consists of all diagonal matrices $\operatorname{diag}(l_1, \ldots, l_q)$ where

$$\begin{cases} l_k = 1 & \text{when} \quad 1 \leq k \leq p\,, \\ l_{p+k} = \prod_{i=0}^{\varphi(\sigma)-1} \lambda_i^{n_{ik}} & \text{when} \quad 1 \leq k \leq \sigma\,, \end{cases} \tag{2.5}$$

for arbitrary complex parameters $\lambda_0, \ldots, \lambda_{\varphi(\sigma)-1}$.

Remark 2.6. The torus $\mathcal{T}$ is a subgroup of $\mathrm{SL}(q, \mathbf{C})$ if and only if $\sum_{j=1}^{\sigma} \zeta^j = 0$, or equivalently if $\sigma \neq 1$.

3. General results.

Let us first give a useful irreducibility criterion due to Beukers, Brownawell and Heckmann (cf. [**BBH**]).

Theorem 3.1 (Beukers, Brownawell, Heckmann). *The equation D_{qp} or equivalently the corresponding representation of G in* $\mathrm{GL}(q, \mathbf{C})$ *is irreducible if and only if $\mu_i \not\equiv \nu_j \bmod \mathbf{Z}$ for all $1 \leq i \leq p$ and $1 \leq j \leq q$.*

For a different proof of this result, see [**DM**].

The following result is a consequence of ([**K2**], Cor. 3.6.1). We give a different proof, using the invariants introduced in Section 2.

Theorem 3.2 (Katz). *If the equation D_{qp} is irreducible and such that the complex numbers $\{\mu_1, \dots, \mu_p, \lambda, 1\}$, with λ as in* 2.2, *are* $\mathbf{Z}$*-linearly independent, then G is ismorphic to* $\mathrm{GL}(q, \mathbf{C})$.

Proof. The formal monodromy group G_M is isomorphic to $G_M^0 \times G_P$ where

$$G_M^0 = \{\mathrm{diag}(t_1, \dots, t_p, t, \dots, t),\ t_1, \dots, t_p, t \in \mathbf{C}^*\}$$

denotes the identity component of G_M and G_P is the subgroup generated by the permutation matrix

$$m_\sigma = \begin{pmatrix} I & 0 \\ 0 & P_\sigma \end{pmatrix}.$$

The Lie algebra of G_M is generated by

$$\{E_{ii}\}_{1 \leq i \leq p} \quad \text{and} \quad \sum_{i=p+1}^{q} E_{ii}$$

where E_{ij}, for $1 \leq i, j \leq q$ denote the elementary matrices of $\mathfrak{gl}(q, \mathbf{C})$.

i) If σ is *odd*, we know (cf. [**DM**], Th. 5.1, 5.3, p. 37) that the Stokes matrices relative to the singular rays $\arg t = 0$ and $\arg t = \frac{\pi}{\sigma}$ can be written as follows:

$$S_0 = I + \sum_{i=1}^{p} \alpha_i E_{p+\frac{\sigma+1}{2}, i}$$

$$S_{\frac{\pi}{\sigma}} = I + \sum_{i=1}^{p} \beta_i E_{i,q}$$

where all coefficients α_i and β_i are non-zero (cf. [**DM**], proof of Th.6.2, p. 42). We clearly have $s_0 = S_0 - I$ and $s_{\frac{\pi}{\sigma}} = S_{\frac{\pi}{\sigma}} - I$. An easy computation shows that for $1 \leq i \leq p$ the matrices $E_{p+\frac{\sigma+1}{2},i}$ and $E_{i,q}$ belong to $\mathfrak{G}$. If we conjugate S_0 and $S_{\frac{\pi}{\sigma}}$ repeatedly by m_σ we prove in the same way that all $E_{i,j}$ for $(i,j) \in \{1,\dots,p\} \times \{p+1,\dots,q\} \cup \{p+1,\dots,q\} \times \{1,\dots,p\}$, are elements of $\mathfrak{G}$ and generators of $\mathfrak{gl}(q,\mathbf{C})$ at the same time.

ii) If σ is *even*, then (cf. [**DM**], Th. 5.1.b, 5.2.a, 5.2.b, p. 37) we have

$$S_0 = \sum_{i=1}^{p} \alpha_i E_{p+\frac{\sigma}{2},i} + \sum_{i=1}^{p} \beta_i' E_{i,q} + N$$

where all coefficients β_i', and α_i as before, are non-zero whereas N is a linear combination of elements $\{E_{i,j}\}_{p+1<i,j\leq q}$ and satisfies $N^2 = 0$, so that we get

$$s_0 = S_0 - I - \sum_{i=1}^{p} \frac{\alpha_i \beta_i'}{2} E_{p+\frac{\sigma}{2},q}$$

and for $i = 1,\dots,p$

$$\left[s_0 + \sum_{j=1}^{p} [s_0, E_{jj}], E_{ii} \right] = 2\alpha_i E_{p+\frac{\sigma}{2},i}$$

$$\left[s_0 - \sum_{j=1}^{p} [s_0, E_{jj}], E_{ii} \right] = 2\beta_i' E_{i,q}\,.$$

All conjugates of S_0 by m_σ satisfy similar formulas and we conclude as in (i). □

We can slightly refine this generic result under some irreducibility conditions for the Lie algebra representation of G.

Notations 3.3. Let G_S denote the (connected) subgroup of G topologically generated

(i) by the Stokes matrices alone if $q - p = 1$,

(ii) by the Stokes matrices and the exponential torus otherwise,

and let $\mathfrak{G}_S$ denote the Lie algebra of G_S. Note that $\mathfrak{G}_S$ is a subalgebra of $\mathfrak{sl}(q,\mathbf{C})$.

Lemma 3.4. *The subgroup G_S is normal in the Galois group G and G is topologically generated by G_S, $\widehat{M}$ and $\mathcal{T} \simeq \mathbf{C}^*$ if $q - p = 1$, by G_S and $\widehat{M}$ otherwise.*

Proof. By 1.9 and 1.14 we know that $\widehat{M}$ acts by conjugation on G_S. If $q - p = 1$ then $\mathcal{T} = \{\operatorname{diag}(1,\dots,1,\lambda),\ \lambda \in \mathbf{C}^*\}$, which is easily seen in this case to act on G_S by conjugation (cf. [**DM**], Th.5.1, p. 37). The result then follows from Theorem 1.15. $\square$

Remark 3.5. If $\sigma \neq 1$ the *Stokes subgroup* G_{st} topologically generated by the Stokes matrices may not be normal in G. This is due to the fact that the adjoint action of $\mathcal{T}$ on the infinitesimal Stokes matrices, which may be interpreted as a "Fourier expansion" of these (cf. [**MR2**]) yield elements of $\mathfrak{G}$ which do not necessarily belong to $\mathfrak{G}_{st} = \mathrm{Lie}\ G_{st}$, as shown below.

Example 3.6. Consider an equation of order three

$$D_{31} = z(\partial + \mu) - \prod_{i=1}^{3}(\partial + \nu_j - 1).$$

The Stokes group G_{st} in this case is generated by

$$S_0 = I + \alpha E_{21} + \beta E_{13} + \gamma E_{23}$$

where $\alpha, \beta, \gamma \in \mathbf{C}$ (cf. [**DM**]), and all conjugates of S_0 by $\widehat{M} = mE_{11} + l(E_{13} + E_{32})$ where $m, l \in \mathbf{C}^*$. One may choose the parameters of D_{31} in such a way that $\beta = 0$, $\alpha\gamma \neq 0$ and $l = m$ (take for instance $D_{31} = z(\partial + \frac{1}{2}) - (\partial - 1)^2(\partial - \frac{1}{2})$). Let $S_1 = \widehat{M} S_0 \widehat{M}^{-1} = I + \alpha E_{31} + \gamma E_{32}$. The Lie algebra $\mathfrak{G}_{st}$ is generated by $s_0 = \alpha E_{21} + \gamma E_{23}$ and $s_1 = \alpha E_{31} + \gamma E_{32}$ and is isomorphic to $\mathfrak{sl}(2, \mathbf{C})$, with basis $\{s_0, s_1, [s_0, s_1]\}$.

Let $T_t = \mathrm{diag}(1, t, t^{-1})$ be an element of $\mathcal{T}$. We get

$$\mathrm{Ad}(T_t)(s_0) = t\alpha E_{21} + t^2\gamma E_{23} \quad \text{and} \quad \mathrm{Ad}(T_t)(s_1) = t\alpha E_{31} + t^2\gamma E_{32}$$

for all $t \in \mathbf{C}^*$ and conclude that $\mathfrak{G}_{st}$ is not invariant by the adjoint action of $\mathcal{T}$, since otherwise E_{21}, E_{23}, E_{31}, E_{32} would belong to $\mathfrak{G}_{st}$ which has dimension 3.

We denote by e_r (resp. e'_r), for $1 \leq r \leq q$, the elementary symmetric functions on $(e^{-2i\pi\nu_j})_{1\leq j\leq q}$ (resp. on $(e^{2i\pi\nu_j})_{1\leq j\leq q}$) and by h_r (resp. h'_r) the complete symmetric functions on $(e^{-2i\pi\mu_j})_{1\leq j\leq p}$ (resp. on $(e^{2i\pi\mu_j})_{1\leq j\leq p}$) defined by

$$h_r = \sum_{\substack{\alpha_1+\cdots+\alpha_p=r \\ \underline{\alpha}\in\mathbf{N}^p}} e^{-2i\pi\sum_{j=1}^{p}\alpha_j\mu_j}.$$

For $r = 1, \ldots, q$ let

$$A_r = \sum_{i=0}^{r}(-1)^{i+1}e_i h_{r-i} \quad \text{and} \quad B_r = \sum_{i=0}^{r}(-1)^i e'_i h'_{r-i}. \tag{3.7}$$

Theorem 3.8. *If the parameters $\underline{\mu}$, $\underline{\nu}$ of D_{qp} are such that $A_r \neq 0$ for $1 \leq r \leq [\frac{q-p}{2}]$ and $B_r \neq 0$ for $1 \leq r \leq [\frac{q-p-1}{2}]$, then the representation of G_S in $\mathbf{C}^q$ relative to D_{qp} is irreducible and so are consequently the representations of $\mathfrak{G} = \mathrm{Lie}\ G$ and G. Moreover, in this case, $\mathfrak{G}_S$ is semisimple.*

Proof. It is close to the proof of ([**DM**], Th. 6.2, p. 41). Let $\{e_i\}_{1\leq i\leq q}$ denote the canonical basis of $\mathbf{C}^q$, and V_p (resp. V_σ) the subspace generated by $e_1, \dots, e_p$ (resp. by $e_{p+1}, \dots, e_q$). Let F denote an invariant subspace of $\mathbf{C}^q$ by G_S. There are three cases:

(i) If $F \subset V_p$, then for any $v = \sum_{k=1}^{p} v_k e_k \in F$, and $n \in \mathbf{Z}$ we have

$$S_{\frac{2\pi n}{\sigma}}(v) = \widehat{M} S_0 \widehat{M}^{-1}(v) = v + \left(\sum_{k=1}^{p} v_k \gamma_k e^{-ni\pi(2\mu_k + \frac{\lambda}{\sigma})}\right) e_{l_n}$$

where $e_{l_n} \in V_\sigma$ and $\gamma_k \neq 0,\ k = 1, \dots, p$ (cf. [**DM**]), whence $S_{\frac{2\pi n}{\sigma}}(v)$ belongs to F for all $n \in \mathbf{Z}$ if and only if $v = 0$.

(ii) If $V_\sigma \subset F$, let S_1 denote the matrix S_0 (resp. $S_{\frac{\pi}{\sigma}}$) if σ is even (resp. odd). Then

$$S_1(e_q) = \sum_{k=1}^{p} \delta_k e_k + \omega$$

where $\omega \in V_\sigma$ and $\delta_k \neq 0, k = 1, \dots, p$. The image of e_q by the Stokes matrice $\widehat{M} S_1 \widehat{M}^{-1}$ lies in F for all $n \in \mathbf{Z}$ if and only if

$$\sum_{k=1}^{p} \delta_k e^{-ni\pi(\frac{\lambda}{\sigma} + 2\mu_k)} e_k$$

lies in F for all $n \in \mathbf{Z}$. Therefore $V_p \subset F$ and $F = \mathbf{C}^q$ by (ii).

(iii) Assume $F \cap V_\sigma \neq (0)$. If $\sigma = 1$ we are in case (ii), so let $\sigma \neq 1$. For a proper choice of the parameters $\{\lambda_i\}_{1\leq i\leq \phi(\sigma)}$ as in (2.5), the corresponding matrix $T_0 = \mathrm{diag}(l_1, \dots, l_q)$ of $\mathcal{T}$ has distinct eigenvalues $l_{p+1}, \dots, l_q$ in V_σ, so there exists i_0, with $p+1 \leq i_0 \leq q$, such that the eigenvector e_{i_0} of T_0 lies in $F \cap V_\sigma$.

a) Let $\sigma \not\equiv 2 \bmod 4$. Since A_r and B_r are non-zero, it is possible for every $i = 1, \dots, q$ to find a Stokes matrix S such that

$$S(e_{i_0}) = e_{i_0} + m_i e_i, \quad m_i \in \mathbf{C}^*$$

in the following way. One can find integers j, l, with $p+1 \leq j, l \leq q$ and $(l-j) \equiv (i - i_0) \bmod \sigma$ and a Stokes matrix S' equal to S_0, $S_{\frac{\pi}{2\sigma}}$, $S_{\frac{\pi}{\sigma}}$ or $S_{\frac{3\pi}{2\sigma}}$ according to the parity of σ such that

$$S'(e_l) = e_l + m_j e_j, \quad m_j \in \mathbf{C}^*$$

and one can take $S = \widehat{M}^{j_0 - j} S' \widehat{M}^{j - j_0}$.

b) If $\sigma \equiv 2 \bmod 4$, then for every i with $p+1 \leq i \leq q$, $i \neq i_0$ and $i \not\equiv (i_0 + \frac{\sigma}{2}) \bmod \sigma$, there is a Stokes matrix S such that $S(e_{i_0}) = e_{i_0} + m_i e_i$ where $m_i \in \mathbf{C}^*$. This implies that $e_i \in F \cap V_\sigma$. Let j_0 be such that $p+1 \leq j_0 \leq q$ and $j_0 \equiv (i_0 + \frac{\sigma}{2}) \bmod \sigma$ and let $S_r = \widehat{M}^r S_0 \widehat{M}^{-r}$ where $r = q - i_0$. We get

$$S_r(e_{i_0}) = e_{i_0} + \sum_{k=1}^{p} \delta_k e_k + \delta_{j_0} e_{j_0}$$

where $\delta_1, \ldots, \delta_r, \delta_{j_0} \in \mathbf{C}^*$. Let $t = l_{i_0}$ in $T_0 = \operatorname{diag}(l_1, \ldots, l_q)$ as above. In the present case, σ being even, we can write

$$T_0 = \operatorname{diag}(1, \ldots, 1, l_{p+1}, \ldots, l_{p+\frac{\sigma}{2}}, l_{p+1}^{-1}, \ldots, l_{p+\frac{\sigma}{2}}^{-1})$$

and we may choose T_0 such that $t \neq 1$. We get

$$T_0 S_0 T_0^{-1}(e_{i_0}) = e_{i_0} + t \sum_{k=1}^{p} \delta_k e_k + t^2 \delta_{j_0} e_{j_0}.$$

Since e_{i_0}, $S_r(e_{i_0})$ and $T_0 S_0 T_0^{-1}(e_{i_0})$ belong to $F \cap V_\sigma$ we get $e_{j_0} \in F \cap V_\sigma$ and finally $V_\sigma \subset F$, hence $F = \mathbf{C}^q$ by (ii).

The representation of the connected subgroup G_S being irreducible, so are the corresponding Lie algebra representations of $\mathfrak{G}_S = \operatorname{Lie} G_S$ and of $\mathfrak{G} = \operatorname{Lie} G$. Since $\mathfrak{G}_S \subset \mathfrak{sl}(q, \mathbf{C})$, the Lie algebra $\mathfrak{G}_S$ must be semisimple (cf. [**Hu**], p. 102). This ends the proof of 3.8. $\square$

The following result is to be compared with ([**K2**], Th. 3.6). Note that our hypothesis imply "Lie-irreducibility" in the sense of [**K2**], so that we, as expected, recover part (2) of Katz's result.

Theorem 3.9. *If $q - p$ is odd and if the representation of $\mathfrak{G} \cap \mathfrak{sl}(q, \mathbf{C})$ is irreducible, then*

(i) *$G \simeq \mathrm{GL}(q, \mathbf{C})$ if $q - p = 1$ or if $\sum_{j=1}^{q} \nu_j \notin \mathbf{Q}$*

(ii) *$G \simeq \mathrm{SL}(q, \mathbf{C}) \times \mathbf{Z}/\nu\mathbf{Z}$ if $q - p \neq 1$ and $\sum_{j=1}^{q} \nu_j = \frac{r}{\nu}$ where r, ν are coprime integers.*

Proof. Let G_1 be the connected subgroup of G corresponding to the subalgebra $\mathfrak{G} \cap \mathfrak{sl}(q, \mathbf{C})$ of $\mathfrak{G}$. It is clearly generated by G_S and the identity component of $G_M \cap \mathrm{SL}(q, \mathbf{C})$. Since $\sigma = q - p$ is odd, the subgroup G_1 contains the Stokes matrix

$$S_{\frac{\pi}{\sigma}} = I + \sum_{i=1}^{p} \beta_i E_{iq}$$

(cf. [**DM**], Th. 5.1 (a) and Th. 5.3 (ii), p. 37-38), where the coefficients β_i are all non-zero since both (2.1.1) and the condition of Th 3.1 are satisfied. A result by Beukers and Heckman (cf. [**BH**], Prop. 6.4) states that the only irreducible connected subgroups of $\mathrm{SL}(q, \mathbf{C})$ which can be normalized by an element $v \in \mathrm{SL}(q, \mathbf{C})$, with $\operatorname{rank}(v - I) = 1$, are $\mathrm{SL}(q, \mathbf{C})$ and also $\mathrm{Sp}(q, \mathbf{C})$ if q is even. But for $G_1 \subset \mathrm{SL}(q, \mathbf{C})$ and $v = S_{\frac{\pi}{\sigma}}$ we claim that this last case does not occur. To prove the claim, we suppose that there exists an antisymmetric invertible matrix $U = (u_{ij})$ such that $g^t U g = U$ for all $g \in G_1$, in particular for the conjugates $S_r = I + \sum_{i=1}^{p} \alpha_{ri} E_{ri}$ of S_0 by $\widehat{M}^r$, where $\alpha_{ri} \in \mathbf{C}^*$, $1 \le r \le \sigma$ and $1 \le i \le p$. This implies that $u_{ij} = 0$ for $1 \le i, j \le p$, whence $\sigma \ne 1$. The exponential torus $\mathcal{T}$ must then belong to G_1. Take $T = \operatorname{diag}(1, \dots, 1, l_{p+1}, \dots, l_q) \in \mathcal{T}$ such that $l_i \ne 1$ for $p + 1 \le i \le q$. Then $T^t U T = U$ implies that $u_{ij} = u_{ji} = 0$ for all i, j, $1 \le i \le p$ and $1 \le j \le q$, so that $\operatorname{rank}(u) \le q - 1$, a contradiction. It follows that $G_1 = \mathrm{SL}(q, \mathbf{C})$. If $q - p = 1$, then $\det(\mathcal{T}) = \mathbf{C}^*$, so that $G = \mathrm{GL}(q, \mathbf{C})$ in this case. Since $G_1 = \mathrm{SL}(q, \mathbf{C})$ and $\det \widehat{M} = e^{-2i\pi \sum_{j=1}^{q} \nu_j}$, we can write $\widehat{M} = e^{-\frac{2i\pi}{q} \sum_{j=1}^{q} \nu_j} \widehat{M}_1$, with $\widehat{M}_1 \in \mathrm{SL}(q, \mathbf{C}) \subset G$. If $q - p \ne 1$, this shows that G is topologically generated by $\mathrm{SL}(q, \mathbf{C})$ and the scalar matrix $e^{-\frac{2i\pi}{q} \sum_{j=1}^{q} \nu_j} I$. The latter generates a cyclic group of order ν if $\sum_{j=1}^{q} \nu_j$ is a reduced rational number $\frac{r}{\nu}$, and generates topologically $\mathbf{C}^*$ if $\sum_{j=1}^{q} \nu_j \notin \mathbf{Q}$. This proves (i) and (ii) in this case. $\square$

We recall that two linear differential equations of the same order with coefficients in $\mathbf{C}(z)$ are *rationally equivalent* if the corresponding differential systems are $\mathbf{C}(z)$-equivalent, which implies that their differential Galois groups are equal.

Definition 3.10. The differential equation

$$D = \partial^n + \sum_{i=1}^{n-1} a_i \partial^i$$

where $\partial = \frac{d}{dz}$ and $a_i \in \mathbf{C}(z)$ for $1 \le i \le n - 1$ is *selfdual* if it is rationally equivalent to the *dual* equation

$$D^* = (-\partial)^n + \sum_{i=1}^{n-1} (-\partial)^i a_i \,.$$

The following criterion (cf. [**K2**], Prop. 3.2, p. 93) will be useful.

Proposition 3.11 (Katz). *Two irreducible equations D_{qp} and D'_{qp} with parameters $(\underline{\mu}, \underline{\nu})$ and $(\underline{\mu}', \underline{\nu}')$ respectively are rationally equivalent if and* *only if*

$$(\underline{\mu},\underline{\nu}) \equiv (\underline{\mu}',\underline{\nu}') \mod \mathbf{Z}^{p+q}$$

for a suitable ordering of the parameters.

In what follows we shall only consider irreducible equations. We refer to Katy Boussel's thesis for an algebraic study of the Galois group of reducible hypergeometric equations (cf. [**Bo1**], [**Bo2**]).

4. Equations D_{42}.

We now consider irreducible equations of the form

$$D_{42} = z(\partial + \mu_1)(\partial + \mu_2) - \prod_{i=1}^{4}(\partial + \nu_i - 1)$$

where the complex parameters μ_1, μ_2, $\nu_1, \dots, \nu_4$ satisfy as before the conditions $\mu_1 \not\equiv \mu_2$ and $\mu_i \not\equiv \nu_j \mod \mathbf{Z}$ for all $i = 1, 2$ and $j = 1, \dots, 4$.

Notations 4.1. Let:

$$\begin{aligned}
\lambda &= \frac{3}{2} + \sum_{j=1}^{2}\mu_j - \sum_{j=1}^{4}\nu_j \qquad \text{(as in 2.2)}\\
\eta &= \sum_{j=1}^{4} e^{-2i\pi(\nu_j + \frac{\lambda}{2})} - \sum_{j=1}^{2} e^{-2i\pi(\mu_j + \frac{\lambda}{2})}\\
\theta_j &= e^{(\lambda + 2\mu_j)}, \;\; j = 1, 2\\
\alpha &= \frac{2i\pi\Gamma(1 + \mu_1 - \mu_2)}{\prod_{j=1}^{4}\Gamma(1 + \mu_1 - \nu_j)}\\
\beta &= \frac{2i\pi\Gamma(1 + \mu_2 - \mu_1)}{\prod_{j=1}^{4}\Gamma(1 + \mu_2 - \nu_j)}\\
\gamma &= \frac{4i\pi^2\Gamma(\mu_2 - \mu_1)}{\prod_{j=1}^{4}\Gamma(\nu_j - \mu_1)}\\
\delta &= \frac{4i\pi^2\Gamma(\mu_1 - \mu_2)}{\prod_{j=1}^{4}\Gamma(\nu_j - \mu_2)}\\
m &= \alpha\gamma + \beta\delta.
\end{aligned}$$

Lemma 4.2. *For any irreducible equation D_{42} the following holds:*

(i) $\alpha, \beta, \gamma, \delta$ *are defined and non-zero*

(ii) *and*

$$\alpha\gamma + \varepsilon\beta\delta = \frac{8}{\sin\pi(\mu_1-\mu_2)}\left(\prod_{j=1}^{4}\sin\pi(\nu_j-\mu_1) - \varepsilon\prod_{j=1}^{4}\sin\pi(\nu_j-\mu_2)\right).$$

Proof. Assertion (i) follows from 3.1 and assertion (ii) from the formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}$$

where $z \in \mathbf{C}\backslash\mathbf{Z}$. □

We begin with the selfdual case, where G can be completely determined.

4.3. Invariants. The Stokes matrices are the conjugates S_r of

$$S_0 = I + \alpha E_{31} + \beta E_{32} + \gamma E_{14} + \delta E_{24} + \eta E_{34}$$

by $\widehat{M}^r$, where $r \in \mathbf{Z}$. More precisely, if $S_r = \exp s_r$ we get

(i) for even r

$$\begin{aligned} S_r &= I + \theta_1^r\alpha E_{31} + \theta_2^r\beta E_{32} + \theta_1^{-r}\gamma E_{14} + \theta_2^{-r}\delta E_{24} + \eta E_{34} \\ s_r &= \theta_1^r\alpha E_{31} + \theta_2^r\beta E_{32} + \theta_1^{-r}\gamma E_{14} + \theta_2^{-r}\delta E_{24} + nE_{34}, \end{aligned}$$

(ii) for odd r

$$\begin{aligned} S_r &= I + \theta_1^r\alpha E_{41} + \theta_2^r\beta E_{42} + \theta_1^{-r}\gamma E_{13} + \theta_2^{-r}\delta E_{23} + \eta E_{43} \\ s_r &= \theta_1^r\alpha E_{41} + \theta_2^r\beta E_{42} + \theta_1^{-r}\gamma E_{13} + \theta_2^{-r}\delta E_{23} + nE_{43} \end{aligned}$$

where $n = \eta - \frac{m}{2}$.

The exponential torus in this case is

$$\mathcal{T} = \{\mathrm{diag}(1,1,t,t^{-1}),\ t \in \mathbf{C}^*\}$$

and its Lie algebra is generated by $\tau = E_{33} - E_{44}$. The Lie algebra $\mathfrak{G}_S$ is generated by τ and the infinitesimal Stokes matrices s_r where $r \in \mathbf{Z}$. The formal monodromy is

$$\widehat{M} = e^{-2i\pi\mu_1}E_{11} + e^{-2i\pi\mu_2}E_{22} + e^{i\pi\lambda}(E_{34} + E_{43}).$$

In the following we recover, for the identity component G^0 of G, a result of Katz (cf. [**K2**], Cor. 3.6.1 and Th. 3.4). We will say that a semidirect product $A \rtimes B$ is *trivial* if the action of B on A is trivial, that is, $A \rtimes B$ is actually a direct product.

Theorem 4.4. *Any irreducible selfdual equation D_{42} is rationally equivalent to one of the following equations:*

$$\text{(a)} \qquad D_{42} = z(\partial+\mu)(\partial-\mu) - \prod_{j=1}^{2}(\partial+\nu_j-1)(\partial-\nu_j-1)$$

with $\mu, \nu_1, \nu_2 \in \mathbf{C}$, $2\mu \notin \mathbf{Z}$ and $\mu \pm \nu_i \notin \mathbf{Z}$, $i = 1,2$. In this case the differential Galois group is isomorphic to

(i) *a non-split extension of* $\mathbf{Z}/2\mathbf{Z}$ *by a non-trivial semidirect product* $\mathrm{SL}(2,\mathbf{C}) \rtimes \mathbf{C}^*$ *if* $2\mu \equiv \frac{1}{2}$ *and* ($\nu_1+\nu_2 \equiv \frac{1}{2}$ *or* $\nu_1-\nu_2 \equiv \frac{1}{2}$) $\bmod \mathbf{Z}$,

(ii) $\mathrm{Sp}(4,\mathbf{C})$ *otherwise.*

$$\text{(b)} \qquad D_{42} = z\partial\left(\partial+\frac{1}{2}\right) - \prod_{j=1}^{2}(\partial+\nu_j-1)(\partial-\nu_j-1)\,.$$

Here G is isomorphic to

(i) $\mathrm{SO}(4,\mathbf{C})$ *if* $\nu_1+\nu_2 \not\equiv \frac{1}{2}$ *and* $\nu_1-\nu_2 \not\equiv \frac{1}{2}$ $\bmod \mathbf{Z}$,

(ii) *a non-trivial semidirect product* $(\mathrm{SL}(2,\mathbf{C}) \rtimes \mathbf{C}^*) \rtimes \mathbf{Z}/2\mathbf{Z}$ *otherwise.*

$$\text{(c)} \qquad D_{42} = z(\partial+\mu)(\partial-\mu) - \partial\left(\partial-\frac{1}{2}\right)(\partial+\nu-1)(\partial-\nu-1)$$

with $2\mu \notin \mathbf{Z}$, and G is isomorphic to $\mathrm{O}(4,\mathbf{C})$.

Proof. By 3.1, 3.10 and 3.11 the given equation is selfdual if and only if, in addition to the irreducibility conditions 3.1, the parameters $\underline{\mu}$ on one hand, $\underline{\nu}$ on the other hand can be rearranged in pairs either of opposite numbers modulo $\mathbf{Z}$ or of half integers. It must therefore be of type (a), (b) or (c).

Case (a): We have $\lambda = \frac{3}{2}$ and $m = \alpha\gamma + \beta\delta = 0$ by Lemma 4.2 and also $\theta_1 = \theta$, $\theta_2 = -\theta^{-1}$ where $\theta = e^{-i\pi(\frac{3}{2}+2\mu)}$.

To prove (i), let us show that $\mathfrak{G}_S$ is isomorphic to a semidirect product $\mathfrak{sl}(2,\mathbf{C}) \rtimes \mathbf{C}$. Since in this case $\theta = 0$ and $\theta^2 = 1$, the set of infinitesimal Stokes matrices reduces to $\{s_0, s_1\}$ where

$$s_0 = \alpha E_{31} + \delta E_{24} + \beta E_{32} + \gamma E_{14}$$

and

$$s_1 = \theta(\alpha E_{41} - \delta E_{23} - \beta E_{42} + \gamma E_{13})\,.$$

Denoting $h = [s_0, s_1]$, we get $h = 2\theta(\alpha\delta E21 - \beta\gamma E_{12} + \alpha\gamma(E_{33} - E_{44}))$, whence

$$[h, s_0] = 4\alpha\gamma\theta s_0 \quad \text{and} \quad [h, s_1] = -4\alpha\gamma\theta s_1.$$

Since $\alpha\gamma \neq 0$, the matrices s_0 and s_1 generate a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbf{C})$ on which Lie $\mathcal{T}$ acts by

$$[\tau, s_0] = s_0 \quad \text{and} \quad [\tau, s_1] = -s_1,$$

the elements s_0, s_1, h and τ being linearly independent. We clearly have $G_{st} \cap \mathcal{T} = (I)$. Here the Stokes group G_{st} is generated by S_0 and S_1, so that we get $G_S = G_{st} \rtimes \mathcal{T}$. In the new basis $\{v_1, v_2, v_3, v_4\}$ of $\mathbf{C}^4$ where

$$v_1 = \begin{pmatrix} \beta \\ -\alpha \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} \beta \\ -\alpha \\ 0 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} -\beta\theta \\ -\alpha\theta \\ 0 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} -\beta\theta \\ -\alpha\theta \\ 1 \\ 0 \end{pmatrix},$$

S_0 and S_1 are represented by

$$\bar{S}_0 = \begin{pmatrix} 1 & \frac{\gamma}{\beta} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1+u & u \\ 0 & 0 & -u & 1-u \end{pmatrix} \quad \bar{S}_1 = \begin{pmatrix} 1-u & -u & 0 & 0 \\ u & 1+u & 0 & 0 \\ 0 & 0 & 1 & -\frac{\gamma}{\beta} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $u = 2\alpha\beta\theta$. It is easy to check that G_{st} acts as $\mathrm{SL}(2, \mathbf{C})$ on each subspace $\langle v_1, v_2 \rangle$ and $\langle v_3, v_4 \rangle$. The formal monodromy generates a finite subgroup $G_M \simeq \mathbf{Z}/4\mathbf{Z}$ acting non-trivially on G_S and $\widehat{M}^2 = -I \in G_S$ whereas $\widehat{M} \notin G_S$. This proves (i).

To prove part (ii) of (a), let us show that $\mathfrak{G}_S$ admits the following basis:

$$\mathcal{B} = \{\alpha E_{31} + \delta E_{24},\ \alpha E_{42} + \delta E_{13},\ \alpha E_{32} - \delta E_{14},\ \alpha E_{41} - \delta E_{23},$$
$$E_{12},\ E_{21},\ E_{34},\ E_{43},\ E_{11} - E_{22},\ E_{33} - E_{44}\}.$$

The conditions of (ii) imply that $2\mu \not\equiv \frac{1}{2} \bmod \mathbf{Z}$ or $\eta \neq 0$. To see this, let us first assume that $2\mu \equiv \frac{1}{2} \bmod \mathbf{Z}$. Then

$$\eta = e^{-i\pi\lambda} \sum_{j=1}^{2} \left(e^{-2i\pi\nu_j} + e^{2i\pi\nu_j}\right) = 4e^{-i\pi\lambda} \cos\pi(\nu_1 + \nu_2) \cos\pi(\nu_1 - \nu_2),$$

whence $\eta \neq 0$. Note also that $m = 0$, so that $n = \eta$ and $s_r = S_r - I$ for $r \in \mathbf{Z}$. The adjoint action of $\mathcal{T}$ on $\mathfrak{G}_S$ gives

$$[\tau, s_r] = s_r + \eta E_{34} \text{ if } r \text{ is even} \quad \text{and} \quad [s_r, \tau] = s_r + \eta E_{43} \text{ if } r \text{ is odd},$$

so we see that ηE_{34}, ηE_{43}, $(s_r - 2\eta E_{34})$ belong to $\mathfrak{G}_S$ for all even r while $(s_r - 2\eta E_{43})$ belong to $\mathfrak{G}_S$ for all odd r. We have

$$s_r - \eta E_{34} = \theta^r(\alpha E_{31} + \delta E_{24}) + \theta^{-r}(\beta E_{32} + \gamma E_{14}) \quad \text{if } r \text{ is even},$$

and

$$s_r - \eta E_{43} = \theta^r(\alpha E_{41} - \delta E_{23}) - \theta^{-r}(\beta E_{42} - \gamma E_{13}) \quad \text{if } r \text{ is odd}.$$

Now assume that $2\mu \not\equiv \frac{1}{2} \bmod \mathbf{Z}$. Since $\lambda = \frac{3}{2}$ and $\mu \notin \frac{1}{2}\mathbf{Z}$ by (2.1.1) we get $\theta^2 \neq \theta^{-2}$. Comparing $(s_r - \eta E_{34})$ (resp. $(s_r - \eta E_{43})$) and $(s_{r+2} - \eta E_{34})$ (resp. $(s_{r+2} - \eta E_{43})$) for all even (resp. odd) r, it is easy to show that the elements $\alpha E_{31} + \delta E_{24}$, $\alpha E_{42} + \delta E_{13}$ which is proportional to $\beta E_{42} - \gamma E_{13}$, $\alpha E_{41} - \delta E_{23}$, $\alpha E_{32} - \delta E_{14}$ which is proportional to $\beta E_{32} + \gamma E_{14}$, all belong to $\mathfrak{G}_S$. Taking Lie brackets of these elements we get E_{12}, E_{21}, $E_{11} - E_{22}$, E_{34}, E_{43} as elements of $\mathfrak{G}_S$ in view of $\alpha\delta \neq 0$.

If $2\mu \equiv \frac{1}{2} \bmod \mathbf{Z}$, then $\eta \neq 0$ and $\theta^2 = 1$. The set of matrices s_r reduces to $\{s_0, s_1\}$ and since ηE_{34} and ηE_{43} are elements of $\mathfrak{G}_S$ so are E_{34}, E_{43} and

$$\begin{aligned} s_0 - \eta E_{34} &= \alpha E_{31} + \delta E_{24} + \beta E_{32} + \gamma E_{14}, \\ \theta(s_1 - \eta E_{34}) &= \alpha E_{41} - \delta E_{23} - \beta E_{42} + \gamma E_{13}, \\ [E_{43}, s_0 - \eta E_{34}] &= \alpha E_{41} - \delta E_{23} + \beta E_{42} - \gamma E_{13}, \\ \theta[E_{34}, s_1 - \eta E_{43}] &= \alpha E_{31} + \delta E_{24} - \beta E_{32} - \gamma E_{14}. \end{aligned}$$

We conclude as before that $\mathcal{B}$ is a basis of $\mathfrak{G}_S$. Therefore the latter consists of all matrices u such that $u^t L + Lu = 0$ where L is the antisymmetric matrix

$$L = \alpha(E_{12} - E_{21}) + \delta(E_{34} - E_{43}).$$

We get $G_S \simeq \mathrm{Sp}(4, \mathbf{C})$. Since the formal monodromy also satisfies $\widehat{M}^t L \widehat{M} = L$, it follows that $G = G_S \simeq \mathrm{Sp}(4, \mathbf{C})$.

Case (b): We have $\lambda = 2$ and

$$\eta = 4\cos\pi(\nu_1 + \nu_2)\cos\pi(\nu_1 - \nu_2) = \frac{\alpha\gamma + \beta\delta}{2} = \frac{m}{2}$$

and by Lemma 4.2

$$\alpha\gamma \pm \beta\delta = -8((\sin\pi\nu_1 \sin\pi\nu_2)^2 \pm (\cos\pi\nu_1 \cos\pi\nu_2)^2)$$

whence $n = \eta - \frac{m}{2} = 0$, $\theta_1 = 1$ and $\theta_2 = -1$. The set of Stokes matrices reduces to

$$S_0 = I + \alpha E_{31} + \beta E_{32} + \gamma E_{14} + \delta E_{24} + \eta E_{34}$$

and

$$S_1 = I + \alpha E_{41} - \beta E_{42} + \gamma E_{13} - \delta E_{23} + \eta E_{43},$$

and the formal monodromy is

$$\widehat{M} = E_{11} - E_{22} + E_{34} + E_{43}.$$

To prove part (i) of (b), let $\nu_1 + \nu_2 \not\equiv \frac{1}{2}$ and $\nu_1 - \nu_2 \not\equiv \frac{1}{2} \bmod \mathbf{Z}$. Since $n = 0$, the Lie algebra $\mathfrak{G}_S$ is generated by

$$s_0 = \alpha E_{31} + \beta E_{32} + \gamma E_{14} + \delta E_{24}\,,$$
$$s_1 = \alpha E_{41} - \beta E_{42} + \gamma E_{13} - \delta E_{23}$$

and $\tau = E_{33} - E_{44}$. From

$$[s_0, s_1] = 2\alpha\delta E_{21} - 2\beta\gamma E_{12} + (\alpha\gamma - \beta\delta)(E_{33} - E_{44})$$

we deduce that $k = \alpha\delta E_{21} - \beta\gamma E_{12}$ belongs to $\mathfrak{G}_S$. Moreover

$$[k, s_0] = \alpha\gamma(\beta E_{32} + \delta E_{24}) - \beta\delta(\gamma E_{14} + \alpha E_{31})$$

and

$$[k, s_1] = \alpha\gamma(\beta E_{42} + \delta E_{23}) + \beta\delta(\gamma E_{13} + \alpha E_{41})\,.$$

Since $\alpha\gamma + \beta\delta \neq 0$, by comparing s_0 and $[k, s_0]$ (resp. s_1 and $[k, s_1]$), we get the following elements of $\mathfrak{G}_S$

$$\{\alpha E_{31} + \gamma E_{14},\ \beta E_{32} + \delta E_{24},\ \alpha E_{41} + \gamma E_{13},\ \beta E_{42} + \delta E_{23}\}\,.$$

Together with k and τ, these elements form a basis of $\mathfrak{G}_S$, which therefore consists of all matrices u such that $u^t R + Ru = 0$, where R is the symmetric matrix

$$R = \alpha\delta E_{11} + \beta\gamma E_{22} - \gamma\delta(E_{34} + E_{43}).$$

It is easy to show that $U^t R U = R$ for $U = S_0,\ S_1,\ \widehat{M}$ and for $U \in \mathcal{T}$, so that $G \simeq \mathrm{SO}(4, \mathbf{C})$.

To prove part (ii) of (b) we assume that $\nu_1 + \nu_2 \equiv \frac{1}{2}$ and $\nu_1 - \nu_2 \equiv \frac{1}{2} \bmod \mathbf{Z}$. Then $m = \alpha\gamma + \beta\delta = 0$, whence $\alpha\gamma - \beta\delta \neq 0$. We have

$$[[s_0, s_1], s_0] = 2(\beta\delta - \alpha\gamma)s_0 \quad \text{and} \quad [[s_0, s_1], s_1] = 2(\alpha\gamma - \beta\delta)s_1\,.$$

This implies that the subalgebra generated by s_0 and s_1 is isomorphic to $\mathfrak{sl}(2, \mathbf{C})$ with Lie $\mathcal{T}$ acting by $[\tau, s_0] = s_0$, $[\tau, s_1] = -s_1$. As in (a), we prove that G_S is isomorphic to a semidirect product $\mathrm{SL}(2, \mathbf{C}) \rtimes \mathbf{C}^*$. We have $\widehat{M}^2 = I$ whereas clearly $\widehat{M} \notin G_S$, and $\widehat{M}$ acts non-trivially on G_S. This proves (ii).

Case (c): We have $\lambda = 1,\ \alpha\gamma - \beta\delta = 0,\ m = \alpha\gamma + \beta\delta \neq 0$ and

$$\eta = \frac{m}{2} = 4 \sin \pi(\nu + \mu) \sin \pi(\nu - \mu)$$

whence $n = 0$. Denoting $\theta = -e^{-2i\pi\mu}$ we have $\theta_1 = \theta,\ \theta_2 = -\theta^{-1}$ and

$$s_r = \theta^r \alpha E_{31} + \theta^{-r} \beta E_{32} + \theta^{-r} \gamma E_{14} + \theta^r \delta E_{24} \quad \text{if } r \text{ is even}$$

and

$$s_r = \theta^r \alpha E_{41} + \theta^{-r} \beta E_{42} + \theta^{-r} \gamma E_{13} + \theta^r \delta E_{23} \quad \text{if } r \text{ is odd}.$$

If $2\mu \not\equiv \frac{1}{2} \bmod \mathbf{Z}$ then $\theta^2 \neq \theta^{-2}$ since $2\mu \notin \mathbf{Z}$. If $2\mu \equiv \frac{1}{2} \bmod \mathbf{Z}$, then all s_r are collinear either to s_0 or to s_1 since $\theta^2 = -1$. We have

$$s_0 = \alpha E_{31} + \beta E_{32} + \gamma E_{14} + \delta E_{24} \quad \text{and} \quad s_1 = \theta(\alpha E_{41} - \beta E_{42} - \gamma E_{13} + \delta E_{23}).$$

Since $\alpha\gamma = \beta\delta$ it is easy to show in both cases that

$$\begin{aligned} \mathcal{B} = \{\alpha E_{31} + \delta E_{24},\ \alpha E_{32} + \delta E_{14},\ \alpha E_{42} + \delta E_{13},\ \alpha E_{41} + \delta E_{23}, \\ E_{11} - E_{22},\ E_{33} - E_{44}\} \end{aligned}$$

is a basis of $\mathfrak{G}_S$, which is therefore isomorphic to the Lie algebra $\mathfrak{so}(4, \mathbf{C})$ consisting of all matrices $u \in \mathfrak{sl}(4, \mathbf{C})$ such that $u^t Q + Qu = 0$. Here Q is the symmetric matrix

$$Q = \alpha(E_{12} + E_{21}) - \delta(E_{34} + E_{43}).$$

It follows from $\widehat{M}^t Q \widehat{M} = Q$ and $\det \widehat{M} = -1$ that $G_S \simeq \mathrm{SO}(4, \mathbf{C})$ and $G \simeq \mathrm{O}(4, \mathbf{C})$, which completes the proof of Theorem 4.4. $\square$

In the general case of an irreducible equation D_{42} we would compute G following the same scheme: we first determine the Lie algebra $\mathfrak{G}_S$, then the corresponding connected subgroup G_S of G and then we obtain G by letting $\widehat{M}$ (or $\widehat{M}$ and $\mathcal{T} \simeq \mathbf{C}^*$ if $\sigma = 1$) act on G_S or we determine $\mathfrak{G}$ by the action of $\mathfrak{G}_M$ (or $\mathfrak{G}_M$ and $\mathbf{C}$ if $\sigma = 1$) on $\mathfrak{G}_S$.

4.5. Table for $\mathfrak{G}_S$. In the table below we list all possible Lie algebras $\mathfrak{G}_S$ obtained for different values of the parameters $\underline{\mu}$ and $\underline{\nu}$.

Let $l = 2(\mu_1+\mu_2) - \sum_{j=1}^4 \nu_j$ and $m = \alpha\gamma + \beta\delta$, $n = \eta - \frac{m}{2}$ as before.

<table>
<tr><td></td><td colspan="8">$\mu_1 - \mu_2 \in \frac{1}{2} + \mathbf{Z}$</td></tr>
<tr><td></td><td colspan="6">$l \in \frac{1}{2}\mathbf{Z}$</td><td colspan="2">$l \notin \frac{1}{2}\mathbf{Z}$</td></tr>
<tr><td rowspan="4"></td><td colspan="5">$n = 0$</td><td rowspan="4">$n \neq 0$</td><td>$n \neq 0$ or $m \neq 0$</td><td>$n = 0$ and $m = 0$</td></tr>
<tr><td colspan="2">$l \in \frac{1}{2} + \mathbf{Z}$</td><td colspan="3">$l \in \mathbf{Z}$</td><td rowspan="3"></td><td rowspan="3"></td></tr>
<tr><td>$m = 0$</td><td>$m \neq 0$</td><td>$m = 0$</td><td colspan="2">$m \neq 0$</td></tr>
<tr><td></td><td colspan="2"></td><td>$\alpha\gamma = \beta\delta$</td><td>$\alpha\gamma \neq \beta\delta$</td></tr>
<tr><td>$\mathfrak{G}_S$</td><td>$\mathfrak{sl}(2,\mathbf{C}) \rtimes \mathbf{C}$</td><td colspan="3">$\mathfrak{so}(4,\mathbf{C})$</td><td colspan="3">$\mathfrak{sl}(4,\mathbf{C})$</td><td>$\mathfrak{so}(4,\mathbf{C})$</td></tr>
</table>

<table>
<tr><td></td><td colspan="6">$\mu_1 - \mu_2 \notin \frac{1}{2} + \mathbf{Z}$</td></tr>
<tr><td rowspan="3"></td><td colspan="3">$l \in \frac{1}{2} + \mathbf{Z}$</td><td colspan="2">$l \in \mathbf{Z}$</td><td>$l \notin \frac{1}{2}\mathbf{Z}$</td></tr>
<tr><td colspan="2">$n = 0$</td><td>$n \neq 0$</td><td>$m = 0$</td><td>$m \neq 0$</td><td rowspan="2"></td></tr>
<tr><td>$\alpha\gamma = \beta\delta$</td><td>$\alpha\gamma \neq \beta\delta$</td><td></td><td></td><td></td></tr>
<tr><td>$\mathfrak{G}_S$</td><td>$\mathfrak{so}(4,\mathbf{C})$</td><td colspan="2">$\mathfrak{sl}(4,\mathbf{C})$</td><td>$\mathfrak{sp}(4,\mathbf{C})$</td><td colspan="2">$\mathfrak{sl}(4,\mathbf{C})$</td></tr>
</table>

Sketch of the proof of the results in 4.5: We first prove that $\mathfrak{G}_S$ is generated by $\tau = E_{33} - E_{44}$, nE_{34}, nE_{43} and

$$\sigma_r = \theta_1^r \alpha E_{31} + \theta_2^r \beta E_{32} + \theta_1^{-r} \gamma E_{14} + \theta_2^{-r} \delta E_{24}, \quad r \in 2\mathbf{Z}$$

and

$$\sigma_r = \theta_1^r \alpha E_{41} + \theta_2^r \beta E_{42} + \theta_1^{-r} \gamma E_{13} + \theta_2^{-r} \delta E_{23}, \quad r \in 2\mathbf{Z} + 1\,,$$

where $\theta_j = e^{-i\pi(\lambda+2\mu_j)}$ for $j = 1,2$ and where nE_{34}, nE_{43} are obtained by the adjoint action of $\mathcal{T}$ on $\mathfrak{G}_{st}$. The rank over $\mathbf{C}$ of each family $(\sigma_r)_{r\in 2\mathbf{Z}}$ and $(\sigma_r)_{r\in 2\mathbf{Z}+1}$ is < 4 if and only if one of the following holds:

$$\theta_1^2\theta_2^2 = 1, \quad \theta_1^2\theta_2^{-2} = 1, \quad \theta_1^4 = 1 \quad \text{or} \quad \theta_2^4 = 1\,.$$

This is equivalent (still under assumptions of 2.1) to $\theta_1^2 = \theta_2^2$ or $\theta_1^2 = \theta_2^{-2}$ where

$$\theta_1^2 = \theta_2^2 \;\Leftrightarrow\; \theta_1 = -\theta_2 \;\Leftrightarrow\; \mu_1 - \mu_2 \in \frac{1}{2} + \mathbf{Z}$$

and

$$\theta_1^2 = \theta_2^{-2} \;\Leftrightarrow\; (\lambda + \mu_1 + \mu_2) \in \frac{1}{2}\mathbf{Z} \;\Leftrightarrow\; 2(\mu_1 + \mu_2) - \sum_{j=1}^{4} \nu_j \in \frac{1}{2}\mathbf{Z}\,.$$

Remark.

(i) The only case where the representation of $\mathfrak{G}_S$ is reducible is $\mathfrak{G}_S \simeq \mathfrak{sl}(2,\mathbf{C}) \rtimes \mathbf{C}$ where $\eta = 0$ (the irreducibility condition of 3.8 does not hold).

(ii) The different degeneracy cases reflect the inclusions

$$\mathfrak{sl}(2,\mathbf{C}) \rtimes \mathbf{C} \subset \mathfrak{so}(4,\mathbf{C}) \simeq \mathfrak{sl}(2,\mathbf{C}) \times \mathfrak{sl}(2,\mathbf{C}) \subset \mathfrak{sp}(4,\mathbf{C}) \subset \mathfrak{sl}(4,\mathbf{C}).$$

As an example we will compute G for an irreducible *non selfdual* equation D_{42}.

Proposition 4.6. *The differential Galois group of*

$$D_{42} = z(\partial + \mu)\left(\partial + \mu + \frac{1}{2}\right) - \left(\partial + \mu - \frac{1}{4}\right)^3 \left(\partial + \mu + \frac{1}{4}\right)$$

where $\mu \in \mathbf{C}$, *is isomorphic to*

(i) $(\mathrm{SO}(4,\mathbf{C}) \rtimes \mathbf{Z}/2\mathbf{Z}) \times \mathbf{C}^*$ *if* $\mu \notin \mathbf{Q}$ (*the semidirect product beeing non-trivial*)

(ii) *an extension of* $\mathbf{Z}/\nu\mathbf{Z}$ *by* $\mathrm{SO}(4,\mathbf{C})$ *if* $\mu = \frac{r}{s}$ *where* r, s *are coprime integers and*

$$\nu = \begin{cases} 2r & \text{if } r \in 1 + 2\mathbf{Z}\,, \\ r & \text{if } r \in 2 + 4\mathbf{Z}\,, \\ \frac{r}{2} & \text{if } r \in 8\mathbf{Z}\,, \\ \frac{r}{4} & \text{if } r \in 4 + 8\mathbf{Z}\,. \end{cases}$$

Proof. We have

$$\lambda = -2\mu - \frac{3}{2},\; \theta_2 = -\theta_1 = i,\; \eta = 2,\; \alpha\gamma = \beta\delta = 2,\; m = 4,\; n = 0$$

and the formal monodromy is

$$\widehat{M} = e^{-2i\pi\mu}(E_{11} - E_{22} + i(E_{34} + E_{43})).$$

The Lie algebra $\mathfrak{G}_S$ is isomorphic to $\mathfrak{so}(4, \mathbf{C})$ since it admits the following basis

$$\mathcal{B} = \{E_{33} - E_{44},\ E_{11} - E_{22},\ \alpha E_{31} + \delta E_{24},\ \alpha E_{32} + \delta E_{14},\\ \alpha E_{41} + \delta E_{23},\ \alpha E_{42} + \delta E_{13}\}$$

which leaves invariant the symmetric bilinear form given by the matrix

$$L = \alpha(E_{12} + E_{21}) - \delta(E_{34} + E_{43}).$$

If $\mu \notin \mathbf{Q}$, the formal monodromy group G_M is isomorphic to $G_M^0 \times (M_1)$ where the identity component G_M^0 is the group of scalar matrices and the subgroup $(M_1) \simeq \mathbf{Z}/4\mathbf{Z}$ is generated by $M_1 = \operatorname{diag}(1, -1, i, i)$ with $M_1^2 \in \mathcal{T} \subset G_S$ but $M_1 \notin G_S$. Replacing M_1 by TM_1 where $T = \operatorname{diag}(1, 1, i, -i) \in \mathcal{T}$ we see that the quotient $\mathbf{Z}/2\mathbf{Z}$ splits off, so that we finally get

$$G \simeq (\mathrm{SO}(4, \mathbf{C}) \rtimes \mathbf{Z}/2\mathbf{Z}) \times \mathbf{C}^*.$$

If $\mu \in \mathbf{Q}$, let r denote the order of $e^{2i\pi\mu}$. We get (ii) by computing $G_M \cap G_S$ for different values of r. □

5. Equations $D_{2q,2q-2}$.

The calculations of Section 4 can be extended to the more general case of an equation $D_{2q,2q-2}$. We shall treat the case of an irreducible selfdual equation, in a "generic" case where the parameters $\underline{\mu}$ and $\underline{\nu}$ respectively can be arranged in pairs of opposite numbers modulo $\mathbf{Z}$. Note that our assumptions actually imply that the equation is not "Kummer induced" in the sense of ([**K2**], Lemma 3.5.6) and that Th. 3.6 of Katz ([**K2**]) could be applied here to determine the identity component G^0 of G. Computing G directly from a set of generators, we get:

Theorem 5.1. *The differential Galois group of*

$$D_{2q,2q-2} = z\prod_{k=1}^{q-1}(\partial + \mu_{2k-1})(\partial - \mu_{2k-1}) - \prod_{k=1}^{q}(\partial + \nu_{2k-1} - 1)(\partial - \nu_{2k-1} - 1)$$

where for all $1 \leq k, l \leq q-1$

$$\mu_{2k-1} + \mu_{2l-1} \notin \frac{1}{2}\mathbf{Z}, \quad \textit{and} \quad \mu_{2k-1} - \mu_{2l-1} \notin \frac{1}{2}\mathbf{Z} \quad \textit{if } k \neq l,$$

is isomorphic to $\mathrm{Sp}(2q, \mathbf{C})$.

Proof. Let $\mu_{2k} = -\mu_{2k-1}$ for $1 \le k \le q-1$ and $\nu_{2k} = -\nu_{2k-1}$ for $1 \le k \le q$. We have in this case $\lambda = \frac{1}{2}(\sigma+1) + \sum_{j=1}^{p} \mu_j - \sum_{j=1}^{q} \nu_j = \frac{3}{2}$. The exponential torus is $\mathcal{T} = \{\mathrm{diag}(1, \ldots, 1, t, t^{-1}),\ t \in \mathbf{C}^*\}$ and Lie $\mathcal{T}$ is generated by $\tau = E_{2q-1,2q-1} - E_{2q,2q}$. The Stokes matrices are the conjugates by $\widehat{M}$ of

$$S_0 = I + \sum_{k=1}^{2q-2} \alpha_k E_{2q-1,k} + \sum_{k=1}^{2q-2} \beta_k E_{k,2q} + \eta E_{2q-1,2q}$$

where

$$\alpha_k = 2i\pi \frac{\Gamma(1+\mu_k - \underline{\mu})}{\Gamma(1+\mu_k - \underline{\nu})}, \quad \beta_k = 4i\pi^2 \frac{\Gamma(\underline{\mu}_k^*)}{\Gamma(\underline{\nu} - \mu_k)},$$

$$\eta = e^{-i\pi\lambda} \left(\sum_{j=1}^{2q-2} e^{-2i\pi\mu_j} - \sum_{j=1}^{2q} e^{-2i\pi\nu_j} \right).$$

Let $\theta_j = e^{-i\pi(\lambda + 2\mu_j)}$ for $1 \le j \le 2q-2$, $\quad m = \sum_{k=1}^{2q-2} \alpha_k \beta_k$ and $n = \eta - \frac{m}{2}$. As in Section 4, we show that $\mathfrak{G}_S$ is generated by τ, $nE_{2q-1,2q}$, $nE_{2q,2q-1}$ and

$$\sigma_r = \sum_{k=1}^{2q-2} \theta_k^r \alpha_k E_{2q-1,k} + \sum_{k=1}^{2q-2} \theta_k^{-r} \beta_k E_{k,2q} \quad \text{for all even } r$$

$$\sigma_r = \sum_{k=1}^{2q-2} \theta_k^r \alpha_k E_{2q,k} + \sum_{k=1}^{2q-2} \theta_k^{-r} \beta_k E_{k,2q-1} \quad \text{for all odd } r.$$

Under the assumptions of the theorem we have $\theta_k^2 \ne \theta_l^2$ and $\theta_k^2 \ne \theta_l^{-2}$ for $1 \le k, l \le 2q-2$ and $k \ne l$. The elements $\{\sigma_r\}_{r \in \mathbf{Z}}$ generate the same Lie algebra as the elements

$$\begin{aligned}
&\alpha_i E_{2q-1,i} + \beta_{i+1} E_{i+1,2q} \\
&\alpha_{i+1} E_{2q-1,i+1} + \beta_i E_{i,2q} \\
&\alpha_i E_{2q,i} - \beta_{i+1} E_{i+1,2q-1} \\
&\alpha_{i+1} E_{2q,i+1} - \beta_i E_{i,2q-1}
\end{aligned}$$

for all odd i such that $1 \le i \le 2q-3$. We get the following set of generators of $\mathfrak{G}_S$:

$$\begin{aligned}
&\alpha_i E_{2q-1,i} + \beta_{i+1} E_{i+1,2q} \\
&\alpha_i E_{2q,i+1} + \beta_{i+1} E_{i,2q-1} \\
&\alpha_i E_{2q-1,i+1} - \beta_{i+1} E_{i,2q} \\
&\beta_{i+1} E_{i+1,2q-1} - \alpha_i E_{i+1,2q-1}
\end{aligned}$$

for all odd i such that $1 \leq i \leq 2q-3$,

$$\begin{gathered} \alpha_i\beta_{j+1}E_{j+1,i} + \alpha_j\beta_{i+1}E_{i+1,j} \\ \alpha_i\beta_{j+1}E_{j,i+1} + \alpha_j\beta_{i+1}E_{i,j+1} \end{gathered}$$

for all odd i, j such that $1 \leq i \leq j \leq 2q-3$,

$$\alpha_i\beta_{j+1}E_{j,i} - \alpha_j\beta_{i+1}E_{i+1,j+1}$$

for all odd i, j such that $1 \leq i, j \leq 2q-3$ and

$$E_{2q-1,2q}\,, \quad E_{2q,2q-1}\,, \quad E_{2q-1,2q-1} - E_{2q,2q}\,.$$

One can check that these q^2+q generators are $\mathbf{C}$-linearly independent and form a basis of $\mathfrak{G}_S$. Up to scalar multiplication, the following permutation of the canonical basis of $\mathbf{C}^q$

$$\begin{pmatrix} 1\ 2, \dots & q & q+1 \dots, 2q \\ 1\ 3 & 2q-1 & 2 \quad \dots, 2q \end{pmatrix}$$

changes the previous set of matrices into the classical basis of $\mathfrak{sp}(2q, \mathbf{C})$:

$$\{E_{q+i,j} + E_{q+j,i}\,,\ E_{i,q+j} + E_{j,q+i}\}_{1 \leq i,j \leq q} \quad \cup \quad \{E_{ij} - E_{q+i,q+j}\}_{1 \leq i,j \leq q}\,.$$

More precisely, with respect to the given fundamental solution, $\mathfrak{G}_S$ consists of all matrices u such that $u^tL + Lu = 0$, where L is the block-diagonal antisymmetric matrix $L = \operatorname{diag}(m_1, m_2, \dots, m_{q-1}, m)$, defined by

$$m_i = \begin{pmatrix} 0 & -\alpha_{2i-1}\widehat{\beta}_i \\ \alpha_{2i-1}\widehat{\beta}_i & 0 \end{pmatrix} \quad \text{and} \quad m = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

with

$$\beta = \prod_{j=1}^{q-1} \beta_{2j} \quad \text{and} \quad \widehat{\beta}_i = \beta(\beta_{2i})^{-1} \quad \text{for} \quad 1 \leq i \leq q-1\,.$$

Since $e^{i\pi\lambda} = -i$, the formal monodromy also leaves the form L invariant and we get $G \simeq G_S \simeq \mathrm{Sp}(2q, \mathbf{C})$. □

6. Equations D_{51}.

In this section we consider particular cases of equations

$$D_{51} = z(\partial + \mu) - \prod_{j=1}^{5}(\partial + \nu_j - 1).$$

Notations 6.1. With the notations of 2.2 we get here

$$\sigma = 4\,, \quad \lambda = \frac{5}{2} + \mu - \sum_{j=1}^{5} \nu_j\,, \quad \zeta = e^{\frac{i\pi}{2}}$$

$$\alpha = \frac{2i\pi}{\prod_{j=1}^{5} \Gamma(1+\mu-\nu_j)}\,, \quad \beta = \frac{(2\pi)^4 i}{\prod_{j=1}^{5} \Gamma(\nu_j - \mu)}\,.$$

If A_r and B_r are defined as in 3.7, let

$$c = \zeta^{-\lambda} A_1\,,\ d = e^{i\pi\lambda}\zeta^{-\lambda} B_1\,,\ \eta = \zeta^{-2\lambda} A_2\,,\ n = \eta - \frac{\alpha\beta}{2}\,,\ \text{and } \theta = e^{-i\pi(\frac{\lambda}{2}+2\mu)}\,.$$

Part (b) of the following result is to be compared with ([**K2**], Cor. 3.6.1).

Theorem 6.2. *Consider an irreducible equation D_{51} with parameters $\mu, \nu_1, \dots, \nu_5$ such that $cd \neq 0$. The following holds.*

(a) *The Lie algebra $\mathfrak{G}_S$ is isomorphic to*
   (i) $\mathfrak{so}(5, \mathbf{C})$ *if the following conditions $\alpha\beta = 2\eta$, $5\mu - \sum_{j=1}^{5} \nu_j \in \frac{1}{2} + \mathbf{Z}$ and $c\theta^2 + d = 0$ hold,*
   (ii) $\mathfrak{sl}(5, \mathbf{C})$ *otherwise.*

(b) *Furthermore, if the equation is selfdual, the differential Galois group is isomorphic to*
   (i) $\mathrm{O}(5, \mathbf{C})$ *if $\mu \in \mathbf{Z}$*
   (ii) $\mathrm{SO}(5, \mathbf{C})$ *otherwise.*

Proof. The exponential torus relative to D_{51} is

$$\mathcal{T} = \{\mathrm{diag}(1, t_1, t_2, t_1^{-1}, t_2^{-1}),\ t_1, t_2 \in \mathbf{C}^*\}$$

and its Lie algebra is generated by $\tau_1 = E_{22} - E_{44}$, $\tau_2 = E_{33} - E_{55}$. The formal monodromy is

$$\widehat{M} = e^{-2i\pi\mu} E_{11} + e^{\frac{i\pi\lambda}{2}} \left(E_{25} + \sum_{i=0}^{2} E_{3+i,2+i} \right)\,.$$

With notations as in 6.1, the Stokes matrices are the all successive conjugates by $\widehat{M}$ of

$$S_0 = I + \alpha E_{31} + \beta E_{15} + \eta E_{35} \quad \text{and} \quad S_{\frac{\pi}{4}} = I + cE_{34} + dE_{25}\,.$$

We have $S_0 = \exp s_0$ and $S_{\frac{\pi}{4}} = \exp s_{\frac{\pi}{4}}$ where

$$s_0 = \alpha E_{31} + \beta E_{15} + nE_{35} \quad \text{and} \quad s_{\frac{\pi}{4}} = cE_{34} + dE_{25}\,.$$

The adjoint action of $\mathcal{T}$ on the infinitesimal Stokes matrices yields the following elements of $\mathfrak{G}_S$

$$(\Sigma) \qquad \begin{cases} \sigma_0 = \theta^{4r}\alpha E_{31} & +\theta^{-4r}\beta E_{15} \\ \sigma_1 = \theta^{4r+1}\alpha E_{21} & +\theta^{-4r-1}\beta E_{14} \\ \sigma_2 = \theta^{4r+2}\alpha E_{51} & +\theta^{-4r-2}\beta E_{13} \\ \sigma_3 = \theta^{4r+3}\alpha E_{41} & +\theta^{-4r-3}\beta E_{12} \end{cases}$$

for $r \in \mathbf{Z}$ and

$$\{cE_{34} + dE_{25},\ \ cE_{23} + dE_{54},\ \ cE_{52} + dE_{43},\ \ cE_{45} + dE_{32},\\ nE_{35},\ nE_{53},\ nE_{24},\ nE_{42}\}$$

which together with τ_1 and τ_2 generate $\mathfrak{G}_S$.

Case 1: Suppose that $\theta^4 \neq \theta^{-4}$ or equivalently that $\frac{1}{2} + 5\mu - \sum_{j=1}^{5} \nu_j \notin 2\mathbf{Z}$. Then it is easy to show that $\mathfrak{G}_S \simeq \mathfrak{sl}(5, \mathbf{C})$.

Case 2: Suppose that $\theta^4 = \theta^{-4}$ (and therefore $= \pm 1$). Then, up to scalar multiplication, the first family (Σ) reduces to

$$(\Sigma') \qquad \begin{cases} \alpha E_{31} & +\beta E_{15} \\ \theta\alpha E_{21} & +\theta^{-1}\beta E_{14} \\ \theta^2\alpha E_{51} & +\theta^{-2}\beta E_{13} \\ \theta^3\alpha E_{41} & +\theta^{-3}\beta E_{12}\,. \end{cases}$$

The family (Σ') generates the following elements of the Lie algebra

$$\{\theta^{-1}E_{34} - \theta E_{25}\,,\quad \theta^{-3}E_{32} - \theta^3 E_{45}\,,\quad \theta^{-1}E_{23} - \theta E_{54}\,,\quad \theta^{-1}E_{52} - \theta E_{43}\,,\\ \theta^{-2}(E_{33} - E_{11}) + \theta^2(E_{11} - E_{55})\,,\quad \theta^{-2}(E_{22} - E_{11}) + \theta^2(E_{11} - E_{44})\}\,.$$

If $\theta^2 \neq \theta^{-2}$, then $\theta^4 = -1$ whence $\theta = \varepsilon i\theta^{-1}$, with $\varepsilon = \pm 1$. We get the following set of generators of $\mathfrak{G}_S$:

$$\alpha E_{31} + \beta E_{15}\,,\ \theta\alpha E_{21} + \theta^{-1}\beta E_{14}\,,\ \alpha E_{51} - \beta E_{13}\,,\ \theta\alpha E_{41} - \theta^{-1}\beta E_{12}\,,\\ E_{33} - 2E_{11} + E_{55}\,,\ E_{22} - 2E_{11} + E_{44}\,,\\ E_{22} - E_{44}\,,\ E_{33} - E_{55}\,.$$

An easy calculation in the Lie algebra shows that we get all E_{1j} and E_{j1} for $2 \leq j \leq 5$ whence $\mathfrak{G}_S \simeq \mathfrak{sl}(5, \mathbf{C})$. If $\theta^2 = \theta^{-2}$, then $\theta^2 = \pm 1 = \varepsilon$. If $n \neq 0$, we show as before that $\mathfrak{G}_S \simeq \mathfrak{sl}(5, \mathbf{C})$. Now suppose that $n = 0$. The following elements are $\mathbf{C}$-linear generators of $\mathfrak{G}_S$:

$$\begin{gathered}\alpha E_{31}+\beta E_{15}\,,\ \alpha E_{21}+\beta E_{14}\,,\ \alpha E_{51}+\beta E_{13}\,,\ \alpha E_{41}+\beta E_{12}\,,\\ cE_{34}+dE_{25}\,,\ cE_{23}+dE_{54}\,,\ cE_{52}+dE_{43}\,,\ cE_{45}+dE_{32}\,,\\ E_{34}-\varepsilon E_{25}\,,\ E_{32}-\varepsilon E_{45}\,,\ E_{23}-\varepsilon E_{54}\,,\ E_{52}-\varepsilon E_{43}\,,\\ E_{22}-E_{44}\,,\ E_{33}-E_{55}\,.\end{gathered}$$

We get $\mathfrak{G}_S \simeq \mathfrak{sl}(5,\mathbf{C})$ if $c\varepsilon + d \neq 0$, and $\mathfrak{G}_S \simeq \mathfrak{so}(5,\mathbf{C})$ otherwise. In the latter case $\mathfrak{G}_S \simeq \mathfrak{so}(5,\mathbf{C})$ leaves the symmetric bilinear form $L = -\alpha E_{11} + \alpha\beta(E_{24}+E_{42}) + \beta(E_{35}+E_{53})$ invariant.

To prove (b), consider a selfdual equation D_{51}. Still under conditions 2.1.1 and 3.1, we can rearrange the parameters in such a way that

(1) either $\mu = 0$ and $\underline{\nu} = (\nu_1, -\nu_1, \nu_2, -\nu_1, \frac{1}{2})$

(2) or $\mu = \frac{1}{2}$ and $\underline{\nu} = (\nu_1, -\nu_1, \nu_2, -\nu_1, 0)$.

In case (1) we have

$$\lambda = 2\,, \quad \varepsilon = 1\,, \quad \zeta^{-\lambda} = -1\,, \quad c + d = 0$$

and in case (2)

$$\lambda = 3\,, \quad \varepsilon = -1\,, \quad \zeta^{-\lambda} = -1\,, \quad c = d\,.$$

Let us show that $n = 0$ in both cases, which implies that $\mathfrak{G}_S \simeq \mathfrak{so}(5,\mathbf{C})$ by (a). We have

$$\alpha\beta = -2^5 \prod_{j=1}^{5} \sin \pi(\nu_j - \mu)$$

and

$$A_2 = -\left(b^2 - b\sum_{j=1}^{5} a_j + \sum_{1\le i<k\le 5} a_j a_k\right)$$

where $b = e^{-2i\pi\mu}$ and $a_j = e^{-2i\pi\nu_j}$ for $1 \le j \le 5$. In case (1) we get

$$\eta = \zeta^{-2\lambda} A_2 = -4(1 - \cos 2i\pi\nu_1 - \cos 2i\pi\nu_2 + \cos 2i\pi\nu_1 \cos 2i\pi\nu_2)$$

which is equal to

$$\frac{\alpha\beta}{2} = -2^4 (\sin \pi\nu_1)^2 (\sin \pi\nu_2)^2$$

whence $n = \eta - \frac{\alpha\beta}{2} = 0$. The proof is similar in case (2). Now to compute G we note that the formal monodromy is $\widehat{M} = \operatorname{diag}(1, P)$ in case (1) and

$\widehat{M} = \operatorname{diag}(-1, -iP)$ in case (2), where P denotes the permutation matrix P_4 defined in 3.2. In both cases we have $\widehat{M}^t L \widehat{M} = L$, where L is the invariant symmetric bilinear form of $\mathfrak{G}_S \simeq \mathfrak{so}(5, \mathbf{C})$. Hence $\widehat{M}$ belongs to G_S if and only if $\det \widehat{M} = 1$, that is $G \simeq \mathrm{O}(5, \mathbf{C})$ in case (1) and $G \simeq \mathrm{SO}(5, \mathbf{C})$ in case (2). This ends the proof. □

7. Equations D_{71}.

In this section we consider equations

$$D_{71} = z(\partial + \mu) - \prod_{j=1}^{7} (\partial + \nu_j - 1)$$

with the irreducibility condition $\mu \not\equiv \nu_i \bmod \mathbf{Z}$, for all $i = 1, \dots, 7$.

Notations 7.1. With notations as in 2.2, we get

$$\sigma = 6, \quad \lambda = \frac{7}{2} + \mu - \sum_{j=1}^{7} \nu_j, \quad \zeta = e^{\frac{i\pi}{3}},$$

$$\alpha = \frac{2i\pi}{\prod_{j=1}^{7} \Gamma(1 + \mu - \nu_j)}, \quad \beta = \frac{(2\pi)^6 i}{\prod_{j=1}^{7} \Gamma(\nu_j - \mu)}.$$

Let e_r (resp. e'_r) for $1 \le r \le 7$ denote the elementary symmetric functions on $(e^{-2i\pi\nu_j})_{1 \le j \le 7}$ (resp. on $(e^{2i\pi\nu_j})_{1 \le j \le 7}$) and let

$$b = e^{-2i\pi\mu}, \quad c = -e^{-2i\pi\frac{\lambda}{3}}(e_2 - be_1 + b^2), \quad d = e^{\frac{2i\pi\lambda}{3}}(e'_2 - b^{-1}e'_1 + b^{-2})$$
$$\gamma = e^{\frac{i\pi\lambda}{3}}(b^{-1} - e'_1), \quad \delta = -e^{-\frac{i\pi\lambda}{3}}(b - e_1), \quad \eta = -\zeta^{-3\lambda}(b^3 - b^2 e_1 + be_2 - e_3)$$
$$\theta = e^{-i\pi(\frac{\lambda}{3} + 2\mu)}, \quad m = \alpha\gamma + \beta\delta \quad \text{and} \quad n = \eta - \frac{\alpha\beta}{2}.$$

For all $k \in \mathbf{Z}$, we define $\underline{k}$, $2 \le \underline{k} \le 7$, by $\underline{k} \equiv k$ modulo 6.

The exponential torus in this case is

$$\mathcal{T} = \{\operatorname{diag}(1, t_1, t_2, t_1^{-1}t_2, t_1^{-1}, t_2^{-1}, t_1 t_2^{-1}),\ t_1, t_2 \in \mathbf{C}^*\}$$

and Lie $\mathcal{T}$ is generated by

$$\tau_1 = E_{22} - E_{44} - E_{55} + E_{77} \quad \text{and} \quad \tau_2 = E_{33} + E_{44} - E_{66} - E_{77}.$$

The formal monodromy is $\widehat{M} = \operatorname{diag}(e^{-2i\pi\mu}, e^{\frac{i\pi\lambda}{3}} P)$ where P denotes the permutation matrix P_6 defined in 3.2. The Lie algebra $\mathfrak{G}_S$ is generated by

τ_1, τ_2 and the infinitesimal Stokes matrices consisting of all conjugates by $\widehat{M}^r$, $r \in \mathbf{Z}$, of

$$s_0 = S_0 - I - \frac{\alpha\beta}{2} E_{47} = \alpha E_{41} + \beta E_{17} + \gamma E_{32} + \delta E_{56} + n E_{47}$$

and

$$s_{\frac{\pi}{6}} = S_{\frac{\pi}{6}} - I = c E_{46} + d E_{37} \,.$$

The adjoint action of $\mathcal{T}$ on these matrices yields the following family of generators for $\mathfrak{G}_S$

$$\begin{aligned} \sigma_{i,r} = \theta^{6r+i}\alpha E_{\underline{4-i},1} + \theta^{-6r-i}\beta E_{1,\underline{7-i}} + \gamma E_{\underline{3-i},\underline{2-i}} + \delta E_{\underline{5-i},\underline{6-i}} & \quad \text{for } 0 \leq i \leq 5\,, \\ \psi_i = c E_{i,\underline{i+2}} + d E_{\underline{i-1},\underline{i+3}} & \quad \text{for } 2 \leq i \leq 7\,, \\ E_{i,\underline{i+3}} & \quad \text{for } 2 \leq i \leq 7\,, \end{aligned}$$

and τ_1 τ_2.

Theorem 7.2. *Consider an irreducible equation D_{71} with $cd\gamma\delta \neq 0$.*

(i) *If the parameters moreover satisfy the following conditions:*

$$\left(7\mu - \sum_{j=1}^{7} \nu_j\right) \in \frac{1}{2} + \mathbf{Z},\ \alpha\beta = 2\eta,\ c\delta = d\gamma,\ c^3 + d^3 = 0 \text{ and } c\theta^2 + d = 0\,,$$

then the Lie algebra $\mathfrak{G}_S$ is isomorphic to

(1) $\mathfrak{g}_2(\mathbf{C})$ *if* $\alpha\beta\theta + 2\gamma^2 = 0$

(2) $\mathfrak{so}(7, \mathbf{C})$ *else.*

(ii) *$\mathfrak{G}_S$ is isomorphic to $\mathfrak{sl}(7, \mathbf{C})$ otherwise.*

Proof. To prove (ii), suppose that the conditions of (i) do not hold.

Case 1: If $\left(7\mu - \sum_{j=1}^{7} \nu_j\right) \notin \frac{1}{2} + \mathbf{Z}$ or, equivalently, if $\lambda + 6\mu \notin \mathbf{Z}$, then:

a) if $\lambda + 6\mu \in \frac{1}{2} + \mathbf{Z}$, the following elements

$$\theta^r \alpha E_{\underline{i-r},1} + \theta^{-r}\beta E_{1,\underline{i+3-r}}\,, \quad \text{for } 0 \leq i \leq 5$$

belong to $\mathfrak{G}_S$ and generate $\mathfrak{sl}(7, \mathbf{C})$,

b) if $\lambda + 6\mu \notin \frac{1}{2} + \mathbf{Z}$ then $\theta^6 \neq \theta^{-6}$. From $\{\sigma_{i,1},\ \sigma_{i,2},\ \sigma_{i,3}\}_{0 \leq i \leq 5}$, we easily get all elementary matrices $E_{1,j}$ and $E_{j,1}$, for $2 \leq j \leq 7$, as elements of $\mathfrak{G}_S$ generating $\mathfrak{sl}(7, \mathbf{C})$.

Case 2: If $\lambda + 6\mu \in \mathbf{Z}$ then $\theta^3 = \theta^{-3}$. To show that $\mathfrak{G}_S \simeq \mathfrak{sl}(7,\mathbf{C})$ it is sufficient to show, in all following cases, that the elements $\left\{E_{i,\underline{i+3}}\right\}_{2\le i\le 7}$ belong to $\mathfrak{G}_S$ since we then get all $E_{1,j}$ and $E_{j,1}$ for $2 \le j \le 7$ by computing $\left[\sigma_{i,0}, E_{\underline{4-i},\underline{7-i}}\right]$ for all $i = 0, \dots, 5$.

a) If $n \neq 0$ the result follows easily.

b) If $n = 0$, we consider three cases.

If $c^3 + d^3 \neq 0$ we get the result by taking Lie brackets of elements

$$\left\{E_{i,\underline{i+2}} + E_{\underline{i-1},\underline{i+3}}\right\}_{2\le i\le 7}.$$

If $c\delta \neq d\gamma$ we note that

$$\left[\sigma_{i,0}, \psi_{\underline{6-i}}\right] = (c\delta - d\gamma)E_{\underline{5-i},\underline{2-i}} \quad \text{for } 0 \le i \le 5\,.$$

If $c^3 + d^3 = 0$ and $c\delta = d\gamma$, suppose that the last condition $c\theta^2 + d = 0$ of (i) does not hold, or equivalently that $\theta \neq \xi\theta^{-1}$, with $\xi = -\frac{d}{c}$. Consider all elements $[\sigma_{5,0}, \sigma_{0,0}]$ and $[\sigma_{i,0}, \sigma_{i+1,0}]$ for $0 \le i \le 4$. For instance, for $i = 0$ we get

$$[\sigma_{0,0}, \sigma_{1,0}] = (\theta^{-1}E_{46} - \theta E_{37})\alpha\beta - \xi^2(E_{46} - \xi E_{37})$$

and since $(E_{46} - \xi E_{37})$ belongs to $\mathfrak{G}_S$, so do E_{46} and E_{37}. This proves (ii).

Now suppose that all the conditions of (i) are satisfied. With ξ as above, we have $\theta = \xi\theta^{-1}$ whence $\theta = \varepsilon\xi^{-1}$ with $\varepsilon = \pm 1$. Let $\alpha' = \frac{\alpha}{\gamma}$ and $\beta' = \frac{\beta}{\gamma}$. The following elements then belong to $\mathfrak{G}_S$

$$\begin{aligned}
\sigma_0' &= \alpha' E_{41} + \beta' E_{17} + E_{32} - \xi E_{56}\\
\sigma_1' &= (\varepsilon\xi\alpha' E_{31} + \varepsilon\beta' E_{16})\xi + E_{27} - \xi E_{45}\\
\sigma_2' &= (\xi^2\alpha' E_{21} + \beta' E_{15})\xi^2 + E_{76} - \xi E_{34}\\
\sigma_3' &= \varepsilon\alpha' E_{71} + \varepsilon\beta' E_{14} + E_{65} - \xi E_{23}\\
\sigma_4' &= (\xi\alpha' E_{61} + \beta' E_{13})\xi + E_{54} - \xi E_{72}\\
\sigma_5' &= (\varepsilon\xi^2\alpha' E_{51} + \varepsilon\beta' E_{12})\xi^2 + E_{43} - \xi E_{67}\,.
\end{aligned}$$

Case 1: If $\alpha'\beta' \neq -2\varepsilon\xi$, consider all elements $\{[\sigma_i', \sigma'_{r(i+2)}]\}_{0\le i\le 5}$ and $\{\sigma'_{r(i+1)}\}_{0\le i\le 5}$ where $r(j)$, for $0 \le r(j) < 6$, denotes the remainder of an integer j modulo 6. These elements generate the following ones

$$\left\{\xi^{i-1}\alpha E_{i1} + \beta E_{1,\underline{i+3}}, E_{i,\underline{i+5}} - \xi E_{\underline{i+2},\underline{i+3}}\right\}_{2\le i\le 7}$$

which together with

$$\{\psi_i\}_{2\le i\le 7}\,, \quad E_{22} - E_{55}, \quad E_{33} - E_{66} \quad \text{and} \quad E_{44} - E_{77}$$

form a $\mathbf{C}$-basis of $\mathfrak{G}_S$. We show that $\mathfrak{G}_S$ is isomorphic to $\mathfrak{so}(7,\mathbf{C})$ since it leaves invariant the bilinear symmetric form

$$L = -\alpha\xi^2 E_{11} + \beta(E_{25}+E_{52}) + \beta\xi(E_{36}+E_{63}) + \beta\xi^2(E_{47}+E_{74})\,.$$

Case 2: If $\alpha'\beta' = -2\varepsilon\xi$ then the elements $\{\sigma_i',\ \psi_i\}_{2\le i\le 7}$ and $\tau_1,\ \tau_2$ form a $\mathbf{C}$-basis of $\mathfrak{G}_S$. We know by theorem 3.8 that $\mathfrak{G}_S$ is semisimple in this case. It is easy to see that the subalgebra generated by τ_1 and τ_2 is a Cartan subalgebra of $\mathfrak{G}_S$ and we deduce from the classification of semisimple Lie algebras (cf. [**T**]) that $\mathfrak{G}_S$, which is of rank 2 and of dimension 14, is of type $\mathfrak{g}_2$. □

For the following statements 7.3, 7.4 and 7.5, we refer to corresponding results of Katz in ([**K2**], Th. 3.6, Cor. 3.6.1 and Section 4.1).

Theorem 7.3. *Consider an irreducible equation D_{71} such that $cd\gamma\delta \neq 0$.*

(1) *If the following conditions hold:*

$$\left(7\mu - \sum_{j=1}^{7}\nu_j\right) \in \frac{1}{2}+\mathbf{Z},\ \ \alpha\beta = 2\eta,\ \ c\delta = d\gamma,\ \ c^3+d^3=0 \ \ \textit{and}\ \ c\theta^2+d=0\,,$$

then

(i) *if $\alpha\beta\theta + 2\gamma^2 = 0$, the differential Galois group is*

$$G \simeq \begin{cases} \mathrm{G}_2 \times \mathbf{Z}/m\mathbf{Z} & \textit{if } \mu+\frac{1}{2} = \frac{r}{m},\ \textit{where } r,m \textit{ are coprime integers,}\\ \mathrm{G}_2 \times \mathbf{C}^* & \textit{if } \mu \notin \mathbf{Q}.\end{cases}$$

(ii) *Otherwise*

$$G \simeq \begin{cases} \mathrm{O}(7,\mathbf{C}) & \textit{if } \mu \in \mathbf{Z}\\ \mathrm{SO}(7,\mathbf{C}) \times \mathbf{Z}/m\mathbf{Z} & \textit{if } \mu+\frac{1}{2} = \frac{r}{m},\ \textit{where } r,m \textit{ are coprime integers,}\\ \mathrm{SO}(7,\mathbf{C}) \times \mathbf{C}^* & \textit{if } \mu \notin \mathbf{Q}\,.\end{cases}$$

(2) *Otherwise*

$$G \simeq \begin{cases} \mathrm{SL}(7,\mathbf{C}) \times \mathbf{Z}/m\mathbf{Z} & \textit{if } \sum_{i=1}^{7}\nu_i = \frac{r}{m},\ \textit{where } r,m \textit{ are coprime integers}\\ \mathrm{GL}(7,\mathbf{C}) & \textit{else.}\end{cases}$$

To prove this theorem, we need the following fact (also used by Katz in [**K2**], Lemma 4.1.1).

Lemma 7.4. *Let H be a subgroup of* $\mathrm{SO}(7, \mathbf{C})$ *which is isomorphic to* G_2. *Then H is equal to its normalizer in* $\mathrm{SO}(7, \mathbf{C})$.

Proof. The group of outer automorphisms of G_2 is trivial (cf. [**He**], IX.5.4, X.3.29), as can be deduced from the Dynkin diagram of G_2. If $m \in \mathrm{SO}(7, \mathbf{C})$ acts by conjugation on H, there exists $h \in H$ such that $m^{-1}gm = h^{-1}gh$ for all $g \in H$. Since mh^{-1} acts trivially on H and the representation of G_2 is irreducible, we get by Schur's lemma that mh^{-1} is a scalar matrix, whence $mh^{-1} = I$. □

Proof of Theorem 7.3. To prove part (i) of (1), we may write $\widehat{M} = -e^{-2i\pi\mu}\widehat{M}_1$, with $\widehat{M}_1 \in \mathrm{SL}(7, \mathbf{C})$. Since $\alpha\beta\theta + 2\gamma^2 = 0$ we know that G_S is isomorphic to G_2 and that G_S leaves the form L (defined in the proof of 7.2) invariant. It is easy in this case to check that $\widehat{M}_1$ also leaves L invariant, that is, $\widehat{M}_1 \in \mathrm{SO}(7, \mathbf{C})$. But $\widehat{M}_1$, as well as $\widehat{M}$, acts by conjugation on G_S, hence $\widehat{M}_1 \in G_S$ by Lemma 7.4. The Galois group G, topologically generated by G_S and $\widehat{M}$, is therefore isomorphic to $\mathrm{G}_2 \times K$, where K denotes the Zariski closure in $\mathbf{C}^*$ of the subgroup generated by $-e^{-2i\pi\mu}$.

To prove part (ii) of (1), suppose that $\alpha\beta\theta + 2\gamma^2 \neq 0$. We get $G \simeq \mathrm{G_S} \times K$ as before, where we know by theorem 7.2 that $G_S \simeq \mathrm{SO}(7, \mathbf{C})$.

The proof of (2) is similar to the proof of Th. 3.9. We know by Th. 7.2 that $G_S = \mathrm{SL}(7, \mathbf{C})$ in this case and we can write $\widehat{M} = e^{-\frac{i\pi}{3}\sum_{j=1}^{7}\nu_j}\widehat{M}_1$, where $\widehat{M}_1 \in G_S$. This implies that G is topologically generated by G_S and the scalar matrix $e^{-\frac{i\pi}{3}\sum_{j=1}^{7}\nu_j} I$, which proves (2). □

These results take an easier form in the selfdual case.

Corollary 7.5. *Consider a selfdual equation*

$$D_{71} = z(\partial + \mu) - (\partial + \nu - 1)\prod_{j=1}^{3}(\partial + \nu_j - 1)(\partial - \nu_j - 1)$$

where $\partial = z\dfrac{d}{dz}$ *and* $\{\mu, \nu\} = \left\{0, \dfrac{1}{2}\right\}$. *Let* $(s_i)_{1\leq i\leq 3}$ *denote the elementary symmetric functions in* $(\cos 2\pi\nu_j)_{1\leq j\leq 3}$ *and let* $\varepsilon = e^{2i\pi\mu}$. *Suppose that* $1 - \varepsilon s_1 \neq 0$ *and* $5 + 4(s_2 - \varepsilon s_1) \neq 0$.

(i) *If* $s_1^2 - 2s_2 + 2\varepsilon s_3 = 1$ *or, equivalently, if* $\nu + \nu_1 + \nu_2 + \nu_3 \in \mathbf{Z}$ *for a proper ordering of the parameters* ν_j, *then*

$$G \simeq \begin{cases} \mathrm{G}_2 & \text{if } \mu = \frac{1}{2} \\ \mathrm{G}_2 \times \mathbf{Z}/2\mathbf{Z} & \text{if } \mu = 0. \end{cases}$$

(ii) *Otherwise*

$$G \simeq \begin{cases} \mathrm{SO}(7,\mathbf{C}) & \text{if } \mu = \frac{1}{2} \\ \mathrm{O}(7,\mathbf{C}) & \text{if } \mu = 0 \end{cases}$$

Note that the condition $s_1^2 - 2s_2 + 2\varepsilon s_3 = 1$ of (i), reflecting a condition on entries of the Stokes matrices S_0 and S_π, is actually equivalent to the condition $\nu + \nu_1 + \nu_2 + \nu_3 \in \mathbf{Z}$ given by Katz in ([**K2**], 4.1).

Proof of the corollary: Under the conditions of 7.5, we may order the parameters in such a way that $\underline{\nu} = (\nu_1, -\nu_1, \nu_2, -\nu_2, \nu_3, -\nu_3, \nu)$. There are two cases:
Case (a): $\mu = 0$, $\nu = \frac{1}{2}$
Case (b): $\mu = \frac{1}{2}$, $\nu = 0$.

Let $\varepsilon = e^{2i\pi\mu}$. It is easy to check that the condition $cd\gamma\delta \neq 0$ is here equivalent to

$$1 - \varepsilon s_1 \neq 0 \quad \text{and} \quad 5 + 4(s_2 - 2\varepsilon s_1) \neq 0\,.$$

In case (a) we get

$$\lambda = 3, \quad \theta = -1, \quad \frac{d}{c} = \frac{\delta}{\gamma} = -1, \quad \xi = 1, \quad \gamma = 2\left(\sum_{j=1}^{3} \cos 2\pi\nu_j - 1\right),$$

$$\alpha\beta = \frac{-(2\pi)^7}{(\underline{\nu} - \mu)(1 + \mu - \underline{\nu})} = 2^7 \prod_{j=1}^{3} \sin^2 \pi\nu_j\,.$$

We get $\eta = \frac{\alpha\beta}{2} = 8(1 - s_1 + s_2 - s_3)$, so that the conditions of Th. 7.2 are satisfied. We show that

$$\alpha\beta\theta + 2\gamma^2 = -(\alpha\beta - 2\gamma^2) = -8\left(1 - \sum_{j=1}^{3} \cos 2\pi\nu_j - 1\right).$$

In case (b) we have

$$\lambda = 4\,, \quad \theta = e^{-\frac{i\pi}{3}} = -j \quad \frac{d}{c} = \frac{\delta}{\gamma} = -j^2\,, \quad \xi = j^2\,,$$

$$\alpha\beta = 2^7 \prod_{j=1}^{3} \cos^2 \pi\nu_j\,, \quad \gamma = 2e^{\frac{i\pi}{3}}(1 + s_1)\,,$$

$$\eta = \frac{\alpha\beta}{2} = -8(1 + s_1 + s_2 + s_3)\,,$$

$$\alpha\beta - 2\gamma^2 j^2 = 8\left(1 - \sum_{j=1}^{3} \cos^2 2\pi\nu_j + 2\prod_{j=1}^{3} \cos 2\pi\nu_j\right).$$

In both cases we get $\alpha\beta + 2\theta^{-1}\gamma^2 = 1 - s_1^2 + 2s_2 - 2\varepsilon s_3$. The condition $\alpha\beta\theta + 2\gamma^2 = 0$ equivalently says that $\nu + \nu_1 + \nu_2 + \nu_3$ is an integer (for such an ordering of the parameters). To see it we show that

$$1-s_1^2+2s_2-2\varepsilon s_3 = -\big(\varepsilon\cos 2\pi\nu_3+\cos 2\pi(\nu_1+\nu_2)\big)\big(\varepsilon\cos 2\pi\nu_3+\cos 2\pi(\nu_1-\nu_2)\big).$$

If $\varepsilon = -1$, that is if $\nu = 0$, the former expression is equal to zero if and only if

$$\nu_3 \equiv \pm(\nu_1+\nu_2) \quad \text{or} \quad \nu_3 \equiv \pm(\nu_1-\nu_2) \quad \text{mod } \mathbf{Z}.$$

If $\varepsilon = 1$, that is if $\nu = \frac{1}{2}$, the same holds if

$$\pm\nu_3 \equiv \frac{1}{2} - (\nu_1+\nu_2) \quad \text{or} \quad \pm\,\nu_3 \equiv \frac{1}{2} - (\nu_1-\nu_2) \quad \text{mod } \mathbf{Z}.$$

This ends the proof. □

We illustrate the criterions of Th. 7.3 and Cor. 7.5 with the following examples.

Proposition 7.6. *The differential Galois group of*

$$(1) \qquad D_{71} = z\left(\partial + \frac{1}{2}\right) - \partial^3\left(\partial^2 - \frac{1}{16}\right)^2 \quad \textit{is} \quad G \simeq \mathrm{G}_2$$

$$(2) \qquad D_{71} = z\partial - \prod_{j=1}^{7}\left(\partial - \frac{2j-1}{14}\right) \quad \textit{is} \quad G \simeq \mathrm{G}_2 \times \mathbf{Z}/2\mathbf{Z}$$

$$(3) \qquad D_{71} = z\left(\partial \pm \frac{1}{14}\right) - \prod_{r=0}^{6}\left(\partial - \frac{r}{7}\right) \quad \textit{is} \quad G \simeq \mathrm{G}_2 \times \mathbf{Z}/7\mathbf{Z}.$$

Proof. Both equations (1) and (2) are selfdual and it is easy to check that their parameters satisfy the conditions of Cor. 7.5. (i). To prove (3), we first replace D_{71} by an equivalent equation where $\mu = \pm\frac{1}{14}$ and $\underline{\nu} = (-\frac{3}{7}, -\frac{2}{7}, -\frac{1}{7}, 0, \frac{1}{7}, \frac{2}{7}, \frac{3}{7})$. If we express all conditions of Th. 7.3 in terms of $b = e^{-2i\pi\mu} = e^{\pm\frac{i\pi}{7}}$, we get $\eta = \frac{\alpha\beta}{2} = -1$, $d^3 = -c^3 = 1$, $c\delta = d\gamma = 1$, $\theta^2 = -\frac{d}{c} = e^{-\frac{i\pi}{3}}$ and $\frac{\gamma^2}{\theta} = -\frac{\alpha\beta}{2} = 1$. This is case (1) (i) of Th. 7.3, with $\mu + \frac{1}{2} = \frac{3}{7}$ or $\frac{4}{7}$. □

7.7. Application: An example of Katz. In ([**K2**], Th. 2.10.5), Katz̄ computes the Galois group of

$$L = \frac{d^7}{dt^7} - t\frac{d}{dt} - \frac{1}{2}$$

(and more generally of $\frac{d^7}{dt^7} - f\frac{d}{dt} - \frac{1}{2}f'$, where f is a polynomial in t) and shows that it is G_2, using in particular the fact that L is selfdual. This equation has a single irregular singularity at ∞ and in Section 6.1 of [**K2**], Katz shows how L is a particular case of equations arising as "Kummer pullbacks" of hypergeometric equations, in the same way as the Airy equation is related to the classical (hypergeometric of order 2) Kummer equation.

Our method enables us to recover the Galois group of L directly from Prop. 7.6 as follows.

Corollary 7.8. *The differential Galois group of*

$$L = \frac{d^7}{dt^7} \pm t\frac{d}{dt} \pm \frac{1}{2}$$

is isomorphic to G_2.

Proof. We get L from the hypergeometric equation $D_{71} = z(\partial \pm \frac{1}{14}) - \prod_{r=0}^{6}(\partial - \frac{r}{7})$ by a change of variable $z = \pm 7(\frac{t}{7})^7$. (More precisely, the equation D_{71} with $\mu = \frac{\varepsilon'}{14}$, $\varepsilon' = \pm 1$, is changed into $-\frac{d^7}{dt^7} + \varepsilon t\frac{d}{dt} + \frac{\varepsilon\varepsilon'}{2}$ by $z = 7\varepsilon(\frac{t}{7})^7$, $\varepsilon = \pm 1$). By Th. 7.2 we can check that this D_{71} has $\mathfrak{G}_S \simeq \mathfrak{g}_2$, where $\mathfrak{G}_S$ as before denotes the Lie algebra of the subgroup G_S topologically generated by the Stokes matrices and the exponential torus of D_{71}. It is easy to see that D_{71} and L have the same set of Stokes matrices and the same exponential torus, but different monodromy. The (actual) monodromy of D_{71} at ∞ (or equivalently at 0) is an element of order 7, since the monodromy exponents at 0 are $\{e^{\frac{2i\pi r}{7}}\}_{0 \leq r \leq 6}$. The monodromy of L at ∞ is therefore trivial. By 1.14.2, we see that the Galois group of L is generated by the group G_S of D_{71} and the monodromy of L at ∞, that is, $\mathrm{Gal}(L) = G_S = \mathrm{G}_2$. □

As a byproduct of this example, we get the following simple topological generators of G_2.

Corollary 7.9. *The group* G_2 *can be topologically generated by*

$\{S_\pi, -P, -e^{\frac{i\pi}{7}}M\}$ *or by* $\{S_\pi, S_0, -P\}$, *where*

$$S_\pi = \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&1\\ 0&0&0&1&0&-1&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix} \quad S_0 = \begin{pmatrix} 1&0&0&0&0&0&-2\alpha^{-1}\\ 0&1&0&0&0&0&0\\ 0&1&1&0&0&0&0\\ \alpha&0&0&1&0&0&-1\\ 0&0&0&0&1&-1&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1&0&0&0&0&0&-2\alpha^{-1}\\ 0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0\\ 0&1&1&0&0&0&0\\ \alpha&0&0&1&0&-1&-1\\ 0&0&0&0&1&-1&0\\ 0&0&0&0&0&1&0 \end{pmatrix} \quad P = \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0 \end{pmatrix}.$$

Here S_0 and S_π denote the Stokes matrices of $D_{71} = z(\partial + \frac{1}{14}) - \partial\prod_{r=1}^{3}(\partial^2 - \frac{r^2}{49})$ corresponding to the Stokes rays $\arg z = 0$ and $\arg z = \pi$ respectively, and M denotes the monodromy at ∞, all with respect to the same fundamental solution 2.2, with $\alpha = \frac{15}{14^3}\pi^{\frac{7}{2}}\prod_{j=0}^{2}\left(\sin\left(\frac{(2j+1)\pi}{14}\right)\right)^{-1}$. The proof of this result was given in [**M2**].

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