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ON CONSTRAINED EXTREMA

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## ON CONSTRAINED EXTREMA

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**Assume that  $I$  and  $J$  are smooth functionals defined on a Hilbert space  $H$ . We derive sufficient conditions for  $I$  to have a local minimum at  $y$  subject to the constraint that  $J$  is constantly  $J(y)$ .**

The first order necessary condition for  $I$  to have a constrained minimum at  $y$  is that for some constant  $\lambda$ ,  $I'_y + \lambda J'_y$  is identically zero. Here  $I'_y$  and  $J'_y$  are the Fréchet derivatives of  $I$  and  $J$  at  $y$ . For the rest of the paper, we assume that  $y$  in  $H$  satisfies this necessary condition.

A common misapprehension (upon which much of the stability results for capillary surfaces has been based) is to assume that if the quadratic form  $I''_y + \lambda J''_y$  is positive definite on the kernel of  $J'_y$  then  $I$  has a local constrained minimum at  $y$ . This is not correct in a Hilbert space of infinite dimension; Finn [1] has supplied a counterexample in the unconstrained case, and the same difficulty will occur in the constrained case. In the unconstrained case, if (as often occurs in practice) the spectrum of  $I''_y$  is discrete and 0 is not a cluster point of the spectrum, then  $I''_y$  positive definite at a critical point  $y$  implies that  $I''_y$  is strongly positive, (i.e., there exists  $k > 0$  such that  $I''_y(x) \geq k\|x\|^2$  holds for all  $x$ ), and this in turn *does* imply that  $y$  is a local minimum (see [2]). However, in the constrained case, things are not so easy. Even if  $I''_y + \lambda J''_y$  has a nice spectrum (in some sense), it is not clear that  $I''_y + \lambda J''_y$  being positive definite on the kernel of  $J'_y$  implies that this quadratic form is strongly positive on the kernel, nor that strong positivity implies that  $y$  is a local minimum.

In [3], Maddocks obtained sufficient conditions for  $I''_y + \lambda J''_y$  to be positive definite on the kernel of  $J'_y$ . As Maddocks points out, this is not quite enough to say that  $I$  has a constrained minimum at  $y$ . Remarkably, essentially the same conditions as Maddocks obtained for positive definiteness do in fact imply that  $I$  has a strict local minimum at  $y$  subject to the constraint  $J = J(y)$ , as we shall see.

For any  $h \in H$  we may say  $J(y+h) - J(y) = J'_y(h) + \frac{1}{2}J''_y(h) + \epsilon_1(h)\|h\|^2$ , where  $\epsilon_1$  goes to zero as  $\|h\|$  goes to zero. If we consider an  $h$  for which  $J(y+h) = J(y)$ , then of course  $0 = J'_y(h) + \frac{1}{2}J''_y(h) + \epsilon_1(h)\|h\|^2$ . Now, for

that  $h$  we have

$$\begin{aligned}
 \Delta I &= I(y+h) - I(y) = I'_y(h) + \frac{1}{2}I''_y(h) + \epsilon_2\|h\|^2 \\
 (1) \qquad &= -\lambda J'_y(h) + \frac{1}{2}I''_y(h) + \epsilon_2\|h\|^2 \\
 &= \frac{1}{2}(I''_y + \lambda J''_y)(h) + (\lambda\epsilon_1 + \epsilon_2)\|h\|^2.
 \end{aligned}$$

Since  $I''_y + \lambda J''_y$  is a bilinear form, there is a linear operator  $A$  defined on  $H$  so that  $(I''_y + \lambda J''_y)(u, v) = \langle u, Av \rangle$ . Similarly there is some element of  $H$ , call it  $\nabla J$ , so that  $J'_y$  applied to a vector  $h$  is  $\langle h, \nabla J \rangle$ . Let  $\sigma(A)$  be the spectrum of  $A$ . There are three cases which often arise in practice:

**Theorem 1.** *If  $\sigma(A) \cap (-\infty, c] = \emptyset$  for some  $c > 0$ , then  $I$  has a constrained minimum at  $y$ .*

*Proof.* From (1) we may write  $\Delta I$  as  $\langle h, Ah \rangle + (\lambda\epsilon_1 + \epsilon_2)\|h\|^2$ . But  $\langle h, Ah \rangle \geq c\|h\|^2$  (this is easily verified using the spectral theorem, see [5]), so for  $h$  sufficiently small,  $\Delta I$  is positive.  $\square$

**Theorem 2.** *Suppose that  $\sigma(A) \cap (-\infty, \epsilon]$  consists of a single negative eigenvalue  $\lambda_0$  for some  $\epsilon > 0$ . Let  $\zeta$  solve  $A\zeta = \nabla J$ . ( $A$  will be invertible.)  $I$  has a constrained minimum at  $y$  if  $J'_y(\zeta) = \langle \zeta, A\zeta \rangle < 0$ , and  $I$  does not have a constrained minimum at  $y$  if  $J'_y(\zeta) = \langle \zeta, A\zeta \rangle > 0$ .*

The proof of Theorem 2 will proceed in a series of steps.

*Step 1.* Assume that  $\langle \zeta, A\zeta \rangle < 0$ . Then  $I''_y + \lambda J''_y$  is strongly positive on the kernel of  $J'_y$ .

*Proof.* Take  $x$  in the kernel of  $J'_y$ . As in [4],  $x$  may be written as  $v + \alpha\zeta$ , where  $v$  is perpendicular to  $\varphi_0$ , the eigenfunction corresponding to  $\lambda_0$ . (The key to this calculation is that  $\langle \zeta, \varphi_0 \rangle \neq 0$ . But if  $\zeta$  is orthogonal to  $\varphi_0$ , it can be shown that  $\langle \zeta, A\zeta \rangle > 0$ .) One can verify that  $\langle x, Ax \rangle = \langle v, Av \rangle - \alpha^2\langle \zeta, A\zeta \rangle$ , so that  $\langle x, Ax \rangle \geq \langle v, Av \rangle$ .

Let  $\{E_\lambda\}$  be the spectral family associated with  $A$ , so that  $A = \int_{-\infty}^{\infty} \lambda dE_\lambda$ . By our assumption on  $\sigma(A)$ ,  $A = \lambda_0 E_{\lambda_0} + \int_{\epsilon}^{\infty} \lambda dE_\lambda$ , where  $E_{\lambda_0}$  is orthogonal projection onto  $\varphi_0$ . Therefore,

$$\langle v, Av \rangle = \langle v, \lambda_0 E_{\lambda_0}(v) \rangle + \int_{\epsilon}^{\infty} \lambda d\|E_\lambda v\|^2.$$

The first term vanishes, so that

$$\langle v, Av \rangle \geq \epsilon \int_{\epsilon}^{\infty} d\|E_\lambda v\|^2 \geq \epsilon \int_{-\infty}^{\infty} d\|E_\lambda v\|^2 \geq \epsilon\|v\|^2.$$

Therefore,  $\langle x, Ax \rangle \geq \epsilon \|v\|^2$ .

To conclude the proof that  $I''_y + \lambda J''_y$  is strongly positive on the kernel of  $J'_y$ , we need to show that  $\|v\| \geq k\|x\|$  for some fixed positive constant  $k$ . Assume without loss of generality that  $\|x\| = 1$ . For any fixed  $x$ ,  $\|v\|$  is greater than or equal to the distance from  $x$  to the line  $\{c\zeta : c \in \mathbb{R}\}$ . Consider the projection of  $x$  onto  $\zeta$ . Its length is  $|\langle x, \zeta / \|\zeta\| \rangle|$ . We may write  $\zeta$  as  $\beta \nabla J + \hat{\zeta}$ , where  $\hat{\zeta}$  is perpendicular to  $\nabla J$ . We cannot have  $\beta$  equaling 0, since by assumption,  $\langle \zeta, A\zeta \rangle = \langle \zeta, \nabla J \rangle < 0$ .

Then the projection has length at most  $\|x\| \|\hat{\zeta}\| / \|\zeta\|$ . But  $\|\hat{\zeta}\| < \|\zeta\|$  (since  $\beta \neq 0$ ). Letting  $\gamma$  equal  $\|\hat{\zeta}\| / \|\zeta\|$ , we have  $\gamma < 1$  and the length of the vector component of  $x$  perpendicular to  $\zeta$  is greater than or equal to  $\sqrt{1 - \gamma^2}$ . But  $\|v\|$  is greater than or equal to the length of that component, so we get our  $k$  to be  $\sqrt{1 - \gamma^2}$ , concluding step 1.

*Step 2.* If  $\langle \zeta, A\zeta \rangle < 0$ , then  $I$  has a minimum at  $y$  subject to the constraint  $J = J(y)$ .

*Proof.* Take an  $h$  for which  $J(y + h) = J(y)$ . Now  $h$  need not be in the kernel of  $J'_y$ , but we may write  $h$  as  $h_1 + \alpha\zeta$ , where  $h_1$  is in the kernel of  $J'_y$ , by taking  $\alpha$  to be  $\langle h, \nabla J \rangle / \langle \zeta, \nabla J \rangle$ . (Note that  $\langle \zeta, \nabla J \rangle = \langle \zeta, A\zeta \rangle \neq 0$ .) Substituting into equation (1),

$$(2) \quad \Delta I = \frac{1}{2} \langle h_1, Ah_1 \rangle + \alpha \langle h_1, A\zeta \rangle + \frac{1}{2} \alpha^2 \langle \zeta, A\zeta \rangle + (\lambda \epsilon_1 + \epsilon_2) \|h\|^2.$$

However,  $\langle h_1, A\zeta \rangle = \langle h_1, \nabla J \rangle = 0$ , causing this term to vanish. We have  $0 = \Delta J = J'_y(h) + \epsilon_3 \|h\|$ , where  $\epsilon_3$  tends to 0 as  $\|h\|$  tends to 0. Thus  $\alpha^2 = \epsilon_3^2 \|h\|^2$ , and we conclude that

$$\Delta I = \frac{1}{2} \langle h_1, Ah_1 \rangle + \epsilon \|h\|^2$$

where  $\epsilon$  tends to zero as  $\|h\|$  tends to 0. From Step 1,  $A$  is strongly positive on the kernel of  $J'_y$ , so

$$\Delta I \geq \frac{k}{2} \|h_1\|^2 + \epsilon \|h\|^2.$$

Since  $h = h_1 + \alpha\zeta$ , with  $\alpha = -\epsilon_3 \|h\|$ , it is easy to see that for  $\|h\|$  sufficiently small there holds  $\|h_1\| \geq \frac{1}{2} \|h\|$ . Thus

$$\Delta I \geq \|h\| \left( \frac{k}{8} + \epsilon \right)$$

which must be greater than 0 for  $\|h\|$  sufficiently small. Therefore  $I$  has a minimum at  $y$  subject to the constraint  $J = J(y)$ , concluding the proof of step 2 and the first half of Theorem 2.

*Step 3.* Suppose that  $\langle \zeta, A\zeta \rangle > 0$ . Then  $I$  does not have a minimum at  $y$  subject to the constraint  $J = J(y)$ .

*Proof.* First,  $I''_y + \lambda J''_y$  is no longer positive definite on the kernel of  $J'$ . Indeed,  $\eta = \varphi_0 + c\zeta$  is in the kernel of  $J'_y$  if  $c = -\frac{\langle \varphi_0, \nabla J \rangle}{\langle \zeta, \nabla J \rangle} = -\frac{\langle \varphi_0, \nabla J \rangle}{\langle \zeta, A\zeta \rangle}$ , but one can verify that  $\langle \eta, A\eta \rangle < 0$ .

Now consider  $f(r, s) = J(y + r\eta + s\nabla J) - J(y)$ , a differentiable function of  $r$  and  $s$ . Then  $\nabla f(0, 0) = (0, \|\nabla J\|^2)$ , so the zero set of  $f$  is tangent to the  $r$  axis at the origin. From this we conclude that there is a function  $s(r)$  so that  $J(y + r\eta + s(r)\nabla J) - J(y) = 0$ , with  $\lim_{r \rightarrow 0} \frac{s(r)}{r} = 0$ . From equation (1), for  $h = r\eta + s(r)\nabla J$  we have

$$\Delta I = (I'' + \lambda J'')(r\eta + s(r)\nabla J) + (\lambda\epsilon_1 + \epsilon_2)\|r\eta + s(r)\nabla J\|^2$$

so that  $\Delta I = r^2\langle \eta, A\eta \rangle + o(r^2)$ . Thus, for all  $r$  sufficiently small  $\Delta I < 0$ , indicating that we do not have a constrained minimum, concluding the proof of Theorem 2. □

**Theorem 3.** *If  $\sigma(A) \cap (-\infty, 0)$  consists of more than one point,  $I$  does not have a constrained minimum at  $y$ .*

*Proof.* Suppose that  $\nu$  and  $\mu$  are in  $\sigma(A) \cap (-\infty, 0)$ , with  $\nu < \mu$ . Let  $E_\lambda$  be the spectral decomposition of  $A$ , so that  $E_\lambda$  is not constant in any neighborhood of  $\nu$  nor in any neighborhood containing  $\mu$ . Take an  $\epsilon > 0$  so that the two  $\epsilon$  neighborhoods around  $\nu$  and  $\mu$  are disjoint and contained in  $(-\infty, 0)$ . Then  $E_{\nu+\epsilon} - E_{\nu-\epsilon}$  is nonzero, i.e., is a nontrivial projection. Therefore there is some  $\varphi_0 \neq 0$  so that  $(E_{\nu+\epsilon} - E_{\nu-\epsilon})\varphi_0 = \varphi_0$ . I claim that  $\langle \varphi_0, A\varphi_0 \rangle < 0$ .

Indeed,  $\langle \varphi_0, A\varphi_0 \rangle = \langle \varphi_0, \int_{-\infty}^{\infty} \lambda dE_\lambda(\varphi_0) \rangle$ , which is  $\int_{-\infty}^{\infty} \lambda d\langle E_\lambda(\varphi_0), \varphi_0 \rangle$ , where the latter just a Stieljes integral. But beyond  $\nu + \epsilon$ ,  $E_\lambda(\varphi_0) = \varphi_0$ , so we only get a negative contribution. It is certainly strictly negative, since for  $\lambda < \nu - \epsilon$ ,  $E_\lambda(\varphi_0) = 0$ .

Now find a  $\varphi_1$  for  $\mu$  in the same fashion. We need to show that  $\langle \varphi_0, A\varphi_1 \rangle = 0$ . But  $\langle \varphi_0, A\varphi_1 \rangle = \int_{-\infty}^{\infty} \lambda d\langle \varphi_0, E_\lambda \varphi_1 \rangle$ , and it is routine to show that  $\langle \varphi_0, E_\lambda \varphi_1 \rangle = 0$  for all  $\lambda$ .

We may take  $c_0$  and  $c_1$ , not both zero, so that  $c_0\varphi_0 + c_1\varphi_1$  is perpendicular to  $\nabla J$ . Then  $\langle c_0\varphi_0 + c_1\varphi_1, A(c_0\varphi_0 + c_1\varphi_1) \rangle = c_0^2\langle \varphi_0, A\varphi_0 \rangle + c_1^2\langle \varphi_1, A\varphi_1 \rangle < 0$ . The proof now proceeds as in Step 3 of Theorem 2. □

**Note.** It often occurs in practice that the spectrum of  $A$  is discrete and may be written as  $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ , with 0 not a cluster point of  $\sigma(A)$ . In this special case, the parts of the hypotheses of the above theorems which relate to  $\sigma(A)$  are as follows. In Theorem 1 we require that  $0 < \lambda_0$ , in Theorem 2 we require that  $\lambda_0 < 0 < \lambda_1$  (in addition to the hypotheses on  $\zeta$ ), and in Theorem 3 we require that  $\lambda_0 < \lambda_1 < 0$ .

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