PROOF OF LONGUERRE’S THEOREM AND ITS EXTENSIONS BY THE METHOD OF POLAR COORDINATES

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There are several methods to prove the well-known Longuerre’s theorem and its extensions in plane geometry. We now prove them by the method of polar coordinates. Our proof is characterized by its directness, simplicity, regularity, originality, and no need for any auxiliary lines.

Longuerre’s Theorem. Let $A_1A_2A_3A_4$ be a quadrilateral inscribed in a circle, on which $p$ is an arbitrary point. Let $S_i$ denote the Simson line of point $p$ with respect to the triangle $A_jA_kA_l$ ($i,j,k,l$ distinct) and let $D_i$ denote the projection of $p$ on $S_i$.

The four points $D_1$, $D_2$, $D_3$, $D_4$ are collinear.

Proof. We establish a polar coordinates system (Fig. 1) with $p$ being the pole and the extension line of $po$ being the polar axis. Let $d$ be the diameter of the circle. Hence the equation of the circle is $\rho = d \cos \theta$. Let $(d \cos \theta_i, \theta_i)((i = 1, 2, 3, 4), \theta_i \in [0, 2\pi])$ be the coordinates of $A_1, A_2, A_3, A_4$. Hence the two-point form equation of $A_1A_2$ is

$$\frac{\sin(\theta_2 - \theta_1)}{\rho} = \frac{\sin(\theta_2 - \theta)}{d \cos \theta_1} + \frac{\sin(\theta - \theta_1)}{d \cos \theta_2},$$

$$\therefore \rho [\sin(\theta_2 - \theta) \cos \theta_2 + \sin(\theta - \theta_1) \cos \theta_1] = d \sin(\theta_2 - \theta_1) \cos \theta_1 \cos \theta_2.$$ 

$$\therefore \frac{1}{2} \rho [\sin(2\theta_2 - \theta) + \sin(\theta - 2\theta_1)] = d \sin(\theta_2 - \theta_1) \cos \theta_1 \cos \theta_2,$$

$$\therefore \rho \sin(\theta_2 - \theta_1) \cos(\theta - \theta_1 - \theta_2) = d \sin(\theta_2 - \theta_1) \cos \theta_1 \cos \theta_2,$$

$$\therefore \sin(\theta_2 - \theta_1) \neq 0,$$

$$\therefore \rho \cos(\theta - \theta_1 - \theta_2) = d \cos \theta_1 \cos \theta_2.$$

This is exactly the normal form equation of $A_1A_2$. Hence we have the coordinates of the foot $B_1$ at which $pB_1$ is normal to $A_1A_2 : B_1(d \cos \theta_1 \cos \theta_2, \theta_1 + \theta_2)$. By means of cyclic permutation of indices we get $B_2(d \cos \theta_2 \cos \theta_3$, 

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\[ \theta_2 + \theta_3 \), \( B_3(d \cos \theta_3 \cos \theta_1, \theta_3 + \theta_1). \] Obviously the coordinates of the three feet \( B_i \) satisfy the normal form equation
\[ \rho \cos(\theta - \theta_1 - \theta_2 - \theta_3) = d \cos \theta_1 \cos \theta_2 \cos \theta_3. \]

Hence we get the normal form equation of the Simson line \( S_1 \) of point \( p \) with respect to \( \Delta A_1A_2A_3 \):
\[ S_1: \rho \cos(\theta - \theta_1 - \theta_2 - \theta_3) = d \cos \theta_1 \cos \theta_2 \cos \theta_3. \]

\[ \text{Figure 1.} \]

Similarly, by the same means we can obtain the normal form equations of the other three Simson lines with respect to \( \Delta A_jA_kA_i \). They are
\[ S_2: \rho \cos(\theta - \theta_1 - \theta_2 - \theta_4) = d \cos \theta_1 \cos \theta_2 \cos \theta_4, \]
\[ S_3: \rho \cos(\theta - \theta_2 - \theta_3 - \theta_4) = d \cos \theta_2 \cos \theta_3 \cos \theta_4, \]
\[ S_4: \rho \cos(\theta - \theta_3 - \theta_4 - \theta_1) = d \cos \theta_3 \cos \theta_4 \cos \theta_1. \]

Hence the coordinates of the four projections \( D_i \) are:
\[ D_1(d \cos \theta_1 \cos \theta_2 \cos \theta_3, \theta_1 + \theta_2 + \theta_3), \]
It is obvious that the above-mentioned coordinates satisfy the normal form equation of the line

\[ S: \rho \cos(\theta - \theta_1 - \theta_2 - \theta_3 - \theta_4) = d \cos \theta_1 \cos \theta_2 \cos \theta_3 \cos \theta_4. \]

Thus the four points \( D_i \) are collinear.

The above equation of \( S \) represents a straight line containing the points \( D_i \) and this straight line is named the Simson line of a point \( p \) with respect to four concyclic points.

**Extension I.** Let \( A_1, A_2, A_3, A_4, A_5 \) be five points on a circle and let \( p \) be an arbitrary point on this circle. Let \( S_i \) denote the Simson line of \( p \) with respect to the 4-tuple \( A_jA_kA_lA_m \) (\( i, j, k, l, m \) distinct) and let \( D_i \) denote the projection of \( p \) on the line \( S_i \) (\( i = 1, 2, 3, 4, 5 \)). Then the five points \( D_i \) are collinear.

**Proof.** We establish a polar coordinates system (Fig. 2) with \( p \) being the pole and the extension line of \( po \) being the polar axis. Let \( d \) be the diameter of the circle. The equation of the circle is \( \rho = d \cos \theta \). Let \( (d \cos \theta_i, \theta_i)(\theta_i \in [0, 2\pi]) \) be the coordinates of \( A_i \). According to the above Longuerre’s theorem and its proof we can get the normal form equations of \( S_i \). They are:

\[
\begin{align*}
S_1 & : \rho \cos(\theta - \theta_1 - \theta_2 - \theta_3 - \theta_4) = d \cos \theta_1 \cos \theta_2 \cos \theta_3 \cos \theta_4, \\
S_2 & : \rho \cos(\theta - \theta_1 - \theta_2 - \theta_3 - \theta_5) = d \cos \theta_1 \cos \theta_2 \cos \theta_3 \cos \theta_5, \\
S_3 & : \rho \cos(\theta - \theta_2 - \theta_3 - \theta_4 - \theta_5) = d \cos \theta_2 \cos \theta_3 \cos \theta_4 \cos \theta_5, \\
S_4 & : \rho \cos(\theta - \theta_3 - \theta_4 - \theta_5 - \theta_1) = d \cos \theta_3 \cos \theta_4 \cos \theta_5 \cos \theta_1, \\
S_5 & : \rho \cos(\theta - \theta_4 - \theta_5 - \theta_1 - \theta_2) = d \cos \theta_4 \cos \theta_5 \cos \theta_1 \cos \theta_2.
\end{align*}
\]

Hence the coordinates of the five projections \( D_i \) are:

\[
\begin{align*}
D_1 & : d \cos \theta_1 \cos \theta_2 \cos \theta_3 \cos \theta_4, \quad \theta_1 + \theta_2 + \theta_3 + \theta_4, \\
D_2 & : d \cos \theta_1 \cos \theta_2 \cos \theta_3 \cos \theta_5, \quad \theta_1 + \theta_2 + \theta_3 + \theta_5, \\
D_3 & : d \cos \theta_2 \cos \theta_3 \cos \theta_4 \cos \theta_5, \quad \theta_2 + \theta_3 + \theta_4 + \theta_5, \\
D_4 & : d \cos \theta_3 \cos \theta_4 \cos \theta_5 \cos \theta_1, \quad \theta_3 + \theta_4 + \theta_5 + \theta_1, \\
D_5 & : d \cos \theta_4 \cos \theta_5 \cos \theta_1 \cos \theta_2, \quad \theta_4 + \theta_5 + \theta_1 + \theta_2.
\end{align*}
\]
Clearly the coordinates of $D_i$ satisfy the normal form equation of the line

$$S: \rho \cos \left( \theta - \sum_{i=1}^{5} \theta_i \right) = d \prod_{i=1}^{5} \cos \theta_i.$$ 

Thus the five points $D_i$ are collinear.

\[ \square \]

**Extension II.** Let $A_1, A_2, \ldots, A_n$ be $n$ points on a circle and let $p$ be an arbitrary point on this circle. Let $S_i$ denote the Simson line of $p$ with respect to the $(n-1)$-tuple $(n-1)$-gonal polygon $A_j A_k \cdots A_x$ ($i, j, k, \ldots, x$ distinct) and let $D_i$ denote the projection of $p$ on the line $S_i$ ($i = 1, 2, \ldots, n$).

Then the $n$ points $D_i$ are collinear.

**Proof.** We again establish a polar coordinates system with $p$ being the pole and the extension line of $po$ being the polar axis. One can immediately verify that

$$\rho \cos \left( \theta - \sum_{i=1}^{n} \theta_i \right) = d \prod_{i=1}^{n} \cos \theta_i.$$
represents a straight line containing the $n$ points $D_i$. Hence the $n$ points $D_i$ are collinear.

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