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BRANCHED SURFACES AND THE SIMPLEST FOLIATIONS OF 3-MANIFOLDS

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The set of foliations carried by the same branched surface as a foliation f includes, but is not restricted to, the set of foliations sufficiently close to f. We show that a branched surface may be constructed to ensure that no foliation carried by it has more dead-end components than f. In particular, if f is taut, all foliations carried by the branched surface will be taut. We use the result to classify the foliations of closed 3-manifolds having the fewest number of dead-end components. We then develop an algorithm to determine a lower bound on the depth of a foliation in a given class.

0. Introduction.

The topology of certain orientable Riemannian 3-manifolds has recently been studied by constructing transversely orientable foliations of these manifolds (see [4]-[7]). For this, it useful to work with the simplest foliations of the manifolds. These we consider to be the foliations with the least number of "dead-end components" (described in [13, p. 104] and [9]) that are at the lowest "depth" ([1]).

Branched surfaces were used in [16] to define an equivalence relation on codimension one orientable C^1 foliations of a compact 3-manifold M that are transverse to a flow ϕ. For the definition, we used the simplest branched surface that can be associated with each foliation. Such a branched surface gives a skeletal outline of the foliation and equivalent foliations correspond to the same branched surface. Under this relation, foliations in the same class need not be topologically equivalent but do share certain topological properties such as the existence of a compact leaf of a particular genus ([15]) or a covering by a trivial product of hyperplanes ([14]). In this paper we show that all foliations in the same class as a foliation f will have the same number of dead-end components as f. In particular, this relation classifies foliations of M that are transverse to ϕ and have the fewest number of dead-end components. The simplest branched surfaces associated with these foliations provide a combinatorial classification.

One of the consequences of the above mentioned result is that all foliations in the same equivalence class as a taut foliation are taut. (Tautness rules out dead-end components.) In a series of papers ([4], [6] and [7]), Gabai has discovered deep information about the topology of knot and link components by constructing taut foliations of these manifolds. The simplest of these foliations (i.e. those at the lowest depth) yield the most information, and much can be said about the topology of knot and link complements that have taut foliations at depth 1.

Most manifolds do not have a depth 0 foliation (i.e. do not fiber over a compact 1-manifold). Yet it often is not known whether a manifold admits a simple depth 1 foliation. In fact, Conlon and Cantwell ([1]) have shown that for any positive integer k, there exists a knot for which the simplest foliation of its complement has depth k. In this paper, we use weight systems as defined by Floyd and Oertel in [3] to define an algorithm that produces a lower bound on the depth of foliations associated with the same branched surface. In particular, we may use the algorithm to find a lower bound on depth for foliations in the same equivalence class. In this sense it determines the simplest structure that is possible for a foliation in this class. The algorithm may also be used to produce a lower bound on the depth of foliations sufficiently close to a given foliation.

Throughout this paper, M will be a closed orientable 3-manifold, although there appears to be no obstruction to carrying out the arguments in the case M is compact rather than closed and the foliation is transverse to the boundary. The branched surfaces we associate with transversely orientable codimension one foliations of M are constructed using the technique described by Christy and Goodman in [2]. The same branched surface W can be constructed from many different foliations of M, each of which is "carried" by W. It was shown in [15] that if a foliation f is carried by W, then all foliations of M that are sufficiently close to f are also carried by W. (We refer to the metric on the space of C^1 foliations, defined by Hirsch in [11].) In addition, if f is transverse to a continuous flow ϕ, then W may be constructed so that every foliation it carries is transverse to ϕ.

We may modify a branched surface W using moves that mimic Thurston's "zipping" and "unzipping" of train tracks ([18]). The foliations carried by the modification constitute a subset of foliations carried by W. We use this to show the following:

Theorem I. *Suppose f is a foliation carried by a compact branched surface W. There exists another branched surface W' carrying f such that every foliation carried by W' is carried by W and has no more dead-end components than does f.*

In particular, if f has the least number of dead-end components possible for a foliation of M, then every foliation carried by W' has this same number *of dead-end components.*

A foliation f of M is "taut" if each leaf is met by a loop transverse to f. A branched surface is "taut" if every foliation it carries is taut. By Theorem I, if f is taut, then it is carried by a taut branched surface. So in particular, taut foliations are stable under C^1 perturbations. This latter result is well-known and an earlier proof can be found in [**17**].

1. Branched surfaces constructed from foliations.

The idea of using branched surfaces to study foliations can be traced back to R.F. Williams ([**19**]). The branched surfaces constructed from foliations are the same as those constructed by Floyd and Oertel ([**3**]) from laminations of 3-manifolds. Therefore, they are more restricted than those used by Williams. We begin with a brief description of these branched surfaces and their relationship with the foliations they carry. An outline of Christy and Goodman's construction of a branched surface from a foliation can be found in [**14**] and [**16**], with more details given in [**2**]. Here we will simply review the fundamentals of this construction.

Christy and Goodman's construction of a branched surface from a foliation of a closed 3-manifold M requires a transverse flow ϕ, and the outcome W is dependent on which flow is used. To begin, we choose a "generating set" consisting of compact planar surfaces contained in leaves of the foliation that satisfy certain general position requirements with respect to ϕ. We then cut M open along the interior of each element in this generating set to obtain a foliated manifold with boundary which can be imbedded in M. Each orbit of ϕ gives rise to a union of interval orbits of a flow on this imbedding which is transverse to the induced foliation. We form a quotient space by identifying points on the same interval orbit of this flow; this is the branched surface W. It can be imbedded in M so its complement, $M - W$, is the finite union of disjoint lens-shaped 3-manifolds; throughout this paper, we assume W is the image of such an imbedding. The branched surface W is a connected 2-dimensional complex with a set of charts Ψ, each defining an orientation preserving local diffeomorphism onto one of the models in Figure 1.1, 1.2, 1.3 or 1.4; the transition maps are smooth and preserve transverse orientation. So W is a connected 2-manifold except on a dimension one subset called the "branch set" (indicated by the dotted lines). A "branch point" of W is a point in the branch set (e.g. x in Figure 1.2). The branch set is a 1-manifold except at a finite number of points called "crossings" where it intersects itself transversely.

In addition, there exists a flow ϕ'_w on M that is transverse to W in the positive direction. The branch set divides the boundary of each component of $M - W$ into two "hemispheres" which are both homeomorphic to the same compact planar surface, and points in one hemisphere (the "lower" hemisphere) flow along arcs in the orbits of ϕ'_w onto points in the other (the "upper" hemisphere).

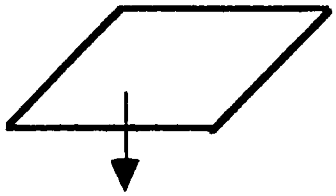


Figure 1.1.

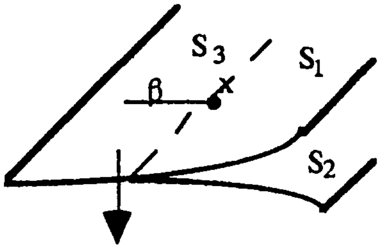


Figure 1.2.

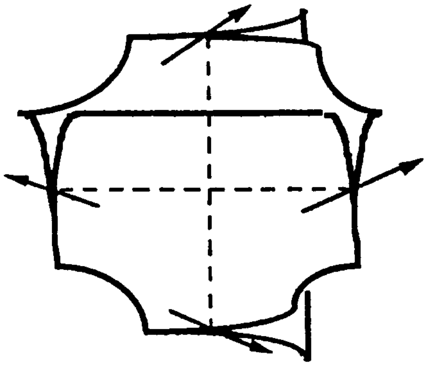


Figure 1.3.

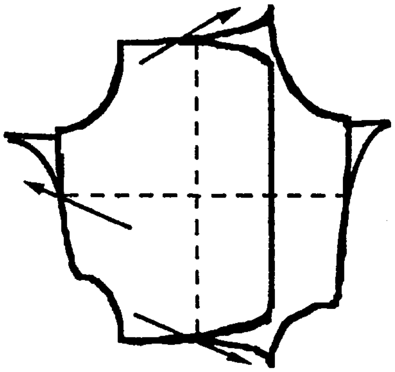
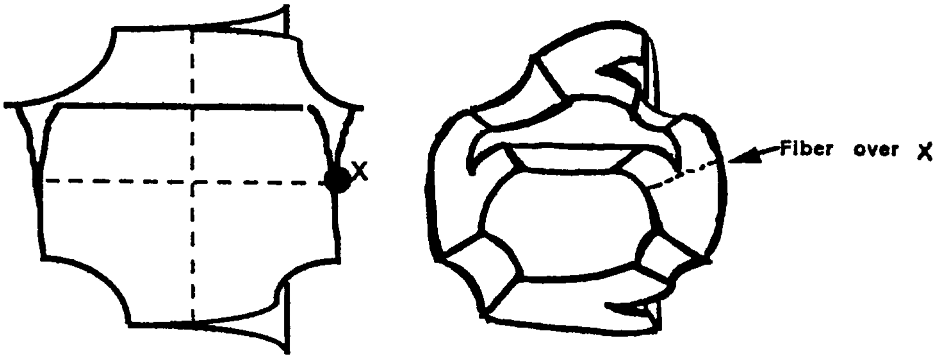


Figure 1.4.

The connected components of W-{branch points} are called the "sectors of W". At any branch point x that is not a crossing, there are locally 3 adjacent sectors, S_1, S_2 and S_3, such that $\mathrm{cl}(S_1) \cup \mathrm{cl}(S_3)$ and $\mathrm{cl}(S_2) \cup \mathrm{cl}(S_3)$ are smooth submanifolds of W (i.e. Ψ is a smooth imbedding in a planar subset of $\Re^3$ on $\mathrm{cl}(S_1) \cup \mathrm{cl}(S_3)$ and $\mathrm{cl}(S_2) \cup \mathrm{cl}(S_3)$). Suppose in a local neighborhood of x forward orbits (under ϕ'_w) of points in S_1 flow into S_2. Then for any curve β that begins or ends at x and contains a nontrivial curve in S_3 that is also bounded by x, we say S_2 is the "upper" sector branching from β at x and S_1 is "lower" sector branching from β at x. (See Figure 1.2 above.) If a curve α contains β and has the same orientation, then sectors branching from β at its beginning (i.e. from the negative boundary of β) are "incoming sectors branching from α" and sectors branching from β at its end (i.e. from the positive boundary of β) are "outgoing sectors branching from α".

We may uniformly thicken the closure of each sector along the transverse direction to obtain a trivial closed interval-fiber bundle. We then piece together these thickened sectors to obtain a submanifold $N(W)$ of M such that $M - N(W)$ is again the union of 3-manifolds. In particular, when we thicken W to $N(W)$, each component of the complement shrinks to a homeomorphic copy of itself. We refer to the interval in $N(W)$ obtained when we thicken a point x in W as the "fiber" over x (even though $N(W)$ is not quite a fibration). Accordingly, we say points in this fiber lie "over" x. (See Figure 1.5.) Throughout, $\pi_W \colon N(W) \to W$ is the map for which the inverse image of any point in W is the fiber lying over it. We say the image of a point under π_W is the "projection" of that point. It is worth noting that $N(W)$ is homeomorphic to the imbedded manifold obtained when we cut M open along surfaces in the generating set for W, and the homeomorphism maps fibers of $N(W)$ onto the interval orbits of the flow induced by ϕ.



Local W Thicken to local $N(W)$

Figure 1.5.

The boundary of $N(W)$ inherits a transverse orientation from W. We assume the fibers of $N(W)$ are oriented so that the positive boundary of each is contained in the positive (i.e. outwardly oriented) boundary of $N(W)$ and the negative boundary of each is contained in the negative (i.e. inwardly oriented) boundary of $N(W)$. A "furrow point" of $N(W)$ is a non-differentiable point of $\partial N(W)$. We consider only foliations of $N(W)$ by leaves (possibly branched) that are transverse to the fibers of $N(W)$. We require that each component of $\partial N(W)$ is contained in a leaf and that the set of branch points for the leaves is the same as the set of furrow points in $\partial N(W)$. (Figure 1.6 shows how a foliation appears locally.) The fibers of $N(W)$ induce a transverse orientation on each of these foliations. We may collapse each component of $M - N(W)$ by identifying the upper and lower hemispheres of its boundary in such a way that the fibers yield orbits of the flow ϕ used to construct W. In the process, each foliation of $N(W)$ gives rise to a foliation of M, which we say is "carried by W". Evidently, ϕ is transverse to all foliations carried by W, so there is a natural duality between the foliations of $N(W)$ and a subset of foliations of M that are transverse to ϕ (in particular, those that are carried by W). If W is constructed from f, then, since $N(W)$ is obtained (up to homeomorphism) by cutting M open along surfaces in leaves of f, there is a dual foliation f^* of $N(W)$ that yields f when we collapse the complement of $N(W)$; that is, f is carried by W. In addition, each foliation carried by W may be used to construct W using ϕ and the same generating set. Throughout we will use f^* to represent the foliation of $N(W)$ that is dual to a foliation f carried by W.

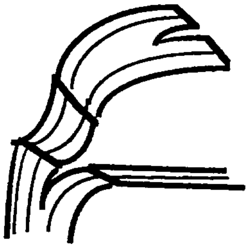


Figure 1.6.

It is important to note that there may be many foliations carried by W that are structurally different from f. However, it was shown in [15] that if a foliation is carried by a branched surface W, then each sufficiently close foliation is topologically equivalent to a foliation carried by W. (We refer to the metric on the space of C^1 foliations, defined by Hirsch in [11].)

An "integral" curve of a foliation will be a curve that is contained in a leaf. A curve $\kappa(t)_{0\leq t\leq 1}$ in W is a "connecting curve" of W if there are two incoming sectors branching from κ at $\kappa(0)$ and two outgoing sectors branching from κ at $\kappa(1)$. The image K of an immersion $\kappa^*\colon [0,1]X[-\varepsilon,\varepsilon] \to N(W)$ is a "connecting strip" of $N(W)$ if :

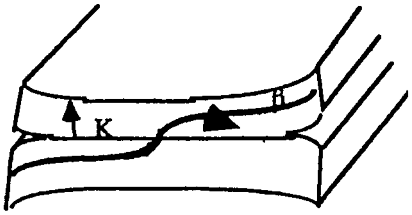
(1) only its ends, $\kappa^*(\{0\}X[-\varepsilon,\varepsilon])$ and $\kappa^*(\{1\}X[-\varepsilon,\varepsilon])$, are contained in the set of furrow points for $N(W)$,

(2) it is transverse to the fibers, and

(3) the projection of the curve $\kappa(t,0)|_{0\leq t\leq 1}$ is a connecting curve κ.

So for each connecting strip K there exists a corresponding connecting curve κ. We shall consider only those connnecting curves in W that correspond to some connecting strip.

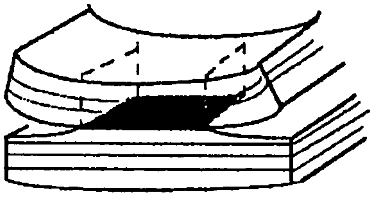
Suppose α is a curve in $N(W)$ whose projection to W is an immersion onto the connecting curve κ. Let β be any curve containing α with the same orientation. If α begins at a point below or contained in ∂K (where "above" and "below" are determined by the orientation of the fiber over $\kappa(0)$) and ends above ∂K, then β "crosses K with index 1". (See Figure 1.7.)



The curve β crosses the connection strip K with index 1.

Figure 1.7.

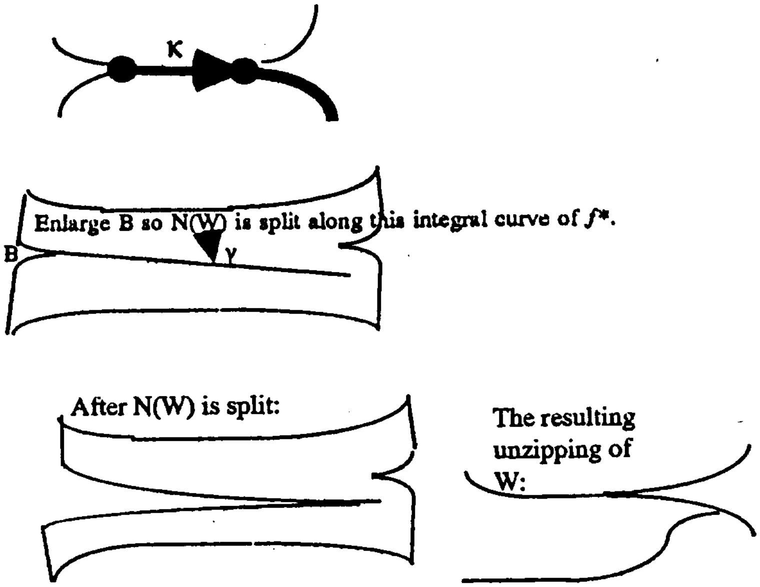
If α begins at a point above or contained in ∂K and ends at a point below ∂K, then β "crosses K with index -1". If α ends in ∂K, (e.g. if α is contained in K as in Figure 1.8), then β "crosses K with index 0". A surface S transverse to the fibers in $N(W)$ crosses K with a particular index if it contains a curve that crosses K with that index. If α begins in ∂K and is an integral curve of a foliation f^*, then we say "f^* crosses K" with the same index as α.



The shaded region is a connecting strip K. An integral curve of the foliation shown is contained in K and therefore crosses K with index 0.

Figure 1.8.

Suppose f^* crosses the connecting strip K. There is an integral curve $\gamma \supseteq \alpha$ begining in ∂K, that projects onto a slight extension $\kappa(t)|_{0\leq t\leq 1+\delta}$ of the corresponding connecting curve. Let B be the component of $M - N(W)$ whose boundary contains $\gamma(0)$ (so $\partial B \cap \kappa^*(\{0\}X[-\varepsilon,\varepsilon]) \neq \varnothing$). We may enlarge B by splitting the leaf of f^* containing γ along a bicollar neighborhood of γ. The projection W of $N(W)$ is consequently "unzipped" along κ to give a new branched surface W'. Figure 1.9 shows the splitting along γ and the branched surface W before and after it is unzipped along κ. We refer to this technique as modifying W by a "splitting in f". We will use it to prove the theorem. (Further details of this technique can be found in [16].)



- marks the ends of κ.

The bold line indicates the projection of the integral curve γ.

Figure 1.9.

Let W' represent the branched surface that results from modifying W by a splitting in f. By definition, each foliation of $N(W')$ has a leaf containing the boundary of the enlarged B. It follows that any foliation carried by W' is also carried by W, and the dual foliation of $N(W)$ crosses K with the same index as f^*.

In [3], Floyd and Oertel define a weight system on a branched surface W as an assignment of a nonnegative real number, called a "weight", to each sector. These weights are required to satisfy the obvious additive condition across the branch set. (See Figure 1.10.)

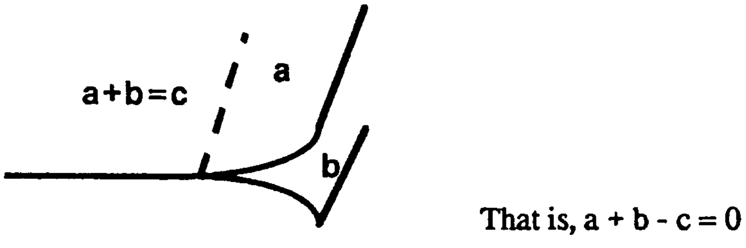


Figure 1.10.

For example, the Reeb foliation of S^3 is carried by the weighted branched surface shown below.

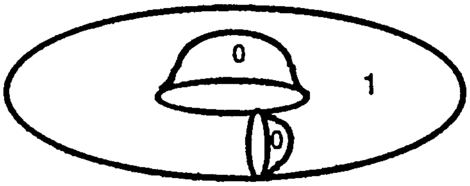


Figure 1.11.

Here we consider only those weight systems where each weight is a nonnegative integer. We may partition the set of all weight systems on a branched surface W into a finite number of equivalence classes such that two weight systems are in the same class if and only if they assign zeros to exactly the same sectors of W. An "elementary set of weight systems" is a set containing exactly one representative from each equivalence class.

For W compact, Floyd and Oertel show that a nontrivial weight system on W, (i.e. a weight system with at least one nonzero weight) may be used to produce a compact surface C imbedded transverse to the fibers in the interior of $N(W)$. This is done by taking copies of sectors (where the number of copies of a given sector equals the weight of that sector), imbedding these copies over the corresponding sectors so they are transverse to the fibers and parallel, then carefully joining them together over the branch set. Conversely, a surface C imbedded transverse to the fibers in the interior of $N(W)$ can be used to "induce" a weight system on W by letting the weight of a particular sector be the number of times C intersects a fiber over that sector.

2. Main results.

In what follows, M will be a closed orientable 3-manifold and f will be a codimension one transversely orientable C^1 foliation of M. We will let W represent a branched surface carrying f and f^* represent the dual foliation of $N(W)$.

Recall that two distinct leaves are in the same "Novikov component" if and only if there is a loop transverse to the foliation that meets both leaves. Novikov ([12]) showed there are only three possibilities:

(1) The (Novikov) component is the entire manifold,

(2) The (Novikov) component consists of a single leaf; in this case it is closed and there is no transverse loop passing through it,

(3) The closure of the (Novikov) component is a manifold with boundary, where the latter consists of finitely many components of type 2.

A Novikov component is "proper" if it is of type 3. A "dead-end component" is a proper Novikov component such that no transverse curve which enters it can escape. For example, a Reeb foliated solid torus is a dead-end component of any foliation that contains it. Other examples of dead-end components are described in [**13**, p. 104] and [**9**]. For sufficiently small $\varepsilon > 0$, the leaves belonging to a proper Novikov component occur in the ε-neighborhood of a point in its boundary on only one side of that point ([**12**, Lemma 1.3]). So a transverse curve cannot meet the boundary of a dead-end component more than once.

At each branching of a leaf L in f^*, there is a component of $\partial N(W)$ contained in L. We may restrict L at the branching so that it contains only one hemisphere of this component. After restricting L at its branchings, we have a surface in $N(W)$ that yields the same leaf in f (as L) when we collapse $M - N(W)$.

Lemma 2.1. *Let S be a surface in the interior of $N(W)$ that is transverse to the fibers and intersects each fiber at most once. If there is no set of leaves in f^* (possibly restricted at branchings) whose projection onto W equals $\pi_W(S)$, then there is a connecting strip K that is crossed by S and f^* with different indices. Moreover, K depends only on the projection $\pi_W(S)$ of S, not on S.*

It is evident that, under these conditions, a single splitting in f^* (which unzips W along the corresponding connecting curve) will give a branched surface W' that is also imbedded in M and, since K depends only on $\pi_W(S)$, any surface in the interior of $N(W)$ that transversely intersects each fiber at most once and has the same projection as S is not contained in $N(W')$.

Proof of Lemma 2.1. Suppose there is no set of leaves in f^* whose projection onto W equals $\pi_W(S)$, regardless of the restrictions at branchings. We show that there is a connecting strip K crossed by f^* and S, but with different indices.

By hypothesis, there exists a fiber I of $N(W)$ that is intersected by S such that no leaf of f^* meeting I projects onto a subset of $\pi_W(S)$. For each point x in I, there is an integral curve $\alpha_x(t)_{0\leq t\leq 1}$ of f^* that begins at x and projects onto a curve which branches away from $\pi_W(S)$ along an upper (or lower) sector D_x. That is, for some $0 \leq t_x < 1$, $\pi_W * \alpha_x(t)_{0\leq t\leq t_x}$ is contained in $\pi_W(S)$ and $\pi_W * \alpha_x(t_x)$ is a branch point such that an upper (respectively lower) outgoing sector D_x branching from $\pi_W * \alpha_x$ at $\pi_W * \alpha_x(t_x)$ is met by $\pi_W * \alpha_x(t_x + \varepsilon)$ (for $\varepsilon > 0$ small) but not by $\pi_W(S)$. We may assume $\alpha_x(t_x)$ is not a furrow point of $N(W)$. (Otherwise we may restrict the leaf through

x to the other hemisphere at this branching so that its projection no longer branches away from $\pi_W(S)$ at $\pi_W * \alpha_x(t_x)$.)

If the projection of some leaf of f^* through I branches away from $\pi_W(S)$ along an upper sector and also branches away from $\pi_W(S)$ along a lower sector, then clearly there exists a connecting strip K that is crossed by integral curves of f^* and by S, but with different indices. Take K to be minimal with respect to these properties. That is, given any other connecting strip with these properties, the corresponding connecting curve is not contained in κ (the connecting curve for K). It follows from this minimality assumption that f^* crosses K. So in this case, the result is immediate. Therefore, we assume that given any leaf through I, its projection branches away from $\pi_W(S)$ either entirely along upper sectors or entirely along lower sectors.

We order points in I according in the orientation of I. Let I_U be the set of points in I that are contained in leaves whose projection branches away from $\pi_W(S)$ along upper sectors, and let y be the greatest lower bound of I_U. (Reversing the transverse orientation of W if necessary, we may assume I_U is nonempty.) Choose some α_y as above and let I_y be the connected set consisting of points $x \in I$ such that for some choice of α_x, $\pi_W * \alpha_y(t)_{0 \leq t \leq t_y+\varepsilon} = \pi_W * \alpha_x(t)_{0 \leq t \leq t_y+\varepsilon}$, for $\varepsilon > 0$ small. That is, x is in I_y if α_x can be chosen to ensure $\pi_W * \alpha_x$ and $\pi_W * \alpha_y$ begin with the same arcs and branch away from $\pi_W(S)$ along the same sector.

Consider the case where $I_y = \{y\}$. Then since $\alpha_y(t)_{0 \leq t \leq t_y}$ is a compact arc, it must contain an arc $\alpha_y(t)_{t_0 \leq t \leq t_1}$ that extends from the positive (i.e. outwardly oriented) boundary of $N(W)$ to the negative (i.e. inwardly oriented) boundary of $N(W)$ or vice versa. But $\alpha_y(t)_{t_0 \leq t \leq t_1}$ is an integral curve, so a portion of it must be contained in a connecting strip K. That is, f^* crosses K with index 0. Now, $\alpha_y(t)_{t_0 \leq t \leq t_1}$ projects onto a curve in $\pi_W(S)$, so S also crosses K. However, S is contained in the interior of $N(W)$ so it crosses K with nonzero index.

Now consider the case where I_y contains points in addition to y. For some $0 \leq t_0 < t_y$, $\alpha_y(t_0) \in \partial N(W)$. (Otherwise there is a point in I_y above y and one below y, contradicting the assumption that y is the greatest lower bound of I_U.) Choose t_0 to be as close to 0 as possible (i.e. $\alpha_y(t_0)$ is the first place α_y meets $\partial N(W)$). Now $\alpha_y(t)_{t_0 \leq t \leq t_y}$ is not contained in $\partial N(W)$. For if $\alpha_y(t)_{t_0 \leq t \leq t_y}$ were contained in the positive boundary of $N(W)$, then points in I_y would lie below y. In addition, $\pi_W * \alpha_y$ would branch away from $\pi_W(S)$ along an upper sector, contradicting the assumption that y is a lower bound of I_U. If $\alpha_y(t)_{t_0 \leq t \leq t_y}$ were contained in the negative boundary of $N(W)$, then points in I_y would lie above y. In addition, $\pi_W * \alpha_y$ would branch away from $\pi_W(S)$ along an lower sector, contradicting the assumption that y is the *greatest* lower bound of I_U. So there exists $t_0 \leq t_1 < t_y$ such that $\alpha_y(t_1)$

is a furrow point of $N(W)$ and two incoming sectors branch from $\pi_W * \alpha_y$ at $\pi_W * \alpha_y(t_1)$. Now $\pi_W * \alpha_y(t)_{t_1 \le t \le t_y}$ has W branching from both its ends. That is $\pi_W * \alpha_y(t)_{t_1 \le t \le t_y} = \kappa(t)_{t_1 \le t \le t_y}$ is a connecting curve contained in $\pi_W(S)$. Let K be the corresponding connecting strip.

Consider the fiber I_1 over $\pi_W * \alpha_y(t_1)$. If S intersects I_1 above $\alpha_y(t_1)$, then points in I_y lie above y and $\pi_W * \alpha_y$ branches away from $\pi_W(S)$ along an upper sector (because y is the *greatest* lower bound of I_U). (See Figure 2.2.) If S intersects I_1 below $\alpha_y(t_1)$, then points in I_y lie below y and $\pi_W * \alpha_y$ branches away from $\pi_W(S)$ along a lower sector (because y is a lower bound of I_U). In either case, S crosses the connecting strip K.

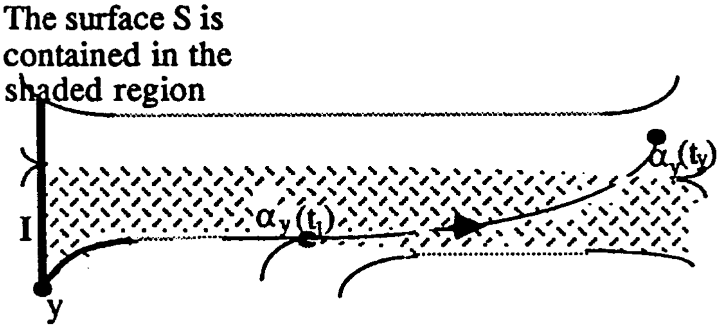


Figure 2.2.

Now, $\alpha_y(t_1) \in \partial K$ and for sufficiently small $\varepsilon > 0$, $\pi_W * \alpha_y(t_y + \varepsilon)$ is in an outgoing sector J branching from κ at $\kappa(t_y)$. So $\alpha_y(t)_{t_1 \le t \le 1}$ is an integral curve of f^* that begins in ∂K and crosses K. Since $\pi_W(S)$ does not meet J, f^* crosses the corresponding connecting strip with a different index from S. (See Figure 2.2.) ☐

Lemma 2.2. *Given a proper Novikov component N that is not a dead-end component, any transverse arc that enters N can be extended to a transverse curve that later escapes N.*

Proof. Let $\alpha(t)_{0 \le t \le 1}$ be a transverse arc that enters N. If $\alpha(1) \notin N$, then we are done. So suppose L_α is a leaf in N containing $\alpha(1)$. Since N is not dead-end, there exists a transverse arc $\beta(t)_{0 \le t \le 1+\varepsilon}$ entering N with the same orientation as α that later escapes N at $\beta(1)$. Choose a leaf L_β ($\neq L_\alpha$) in N that is met by β, and let b be a point in $L_\beta \cap \beta$. There is a transverse loop, with the same orientation as α, meeting both L_α and L_β. This loop contains an arc from L_α to L_β, and a standard modification of this arc produces a transversal γ from $\alpha(1)$ to b. Let β' be the arc in β that begins at b and ends at $\beta(1+\varepsilon)$. Now, $\beta' * \gamma * \alpha$ is a transverse curve that both enters and exits N. $\square$

Lemma 2.3. *Given a foliation f of M, let N be a saturated submanifold of M. If all the normals to f on ∂N are oriented toward the interior* (*exterior, respectively*) *of N, then N contains a dead-end component.*

Proof. Let N be a saturated submanifold of M. We may assume that there are a finite number of compact leaves in the closure of N. For if there are infinitely many, they accumulate in some fiber neighborhood LXI of a compact leaf L, with the foliation transverse to the I direction. We may assume the two copies of L in the boundary of this fiber neighborhood are leaves. Without affecting the number of dead-ends, we may remove the interior of LXI and identify the boundary leaves. After repeating this for all such neighborhoods of accumulation, the foliation of the resulting copy of N has only finitely many compact leaves. Since each Novikov component is bounded by compact leaves, there are finitely many Novikov components in N.

If N contains no dead-ends, then by Lemma 2.2 we could construct a transverse curve by starting in ∂N and every time we enter a new Novikov component continuing to extend the curve so that we leave it, which puts us in a new component, and continue ad infinitum. This transversal would have the same orientation as the normals to f on ∂N, so it would be contained in the closure of N. Consequently, this transversal would necessarily meet some leaf bounding a Novikov component more than once. A standard modification of the transversal would produce a transverse loop through this leaf, contradicting the fact that it is its own Novikov component. So N contains a dead-end component. $\square$

It is easily verified that any dead-end component N satisfies the hypothesis of Lemma 2.3.

Lemma 2.4. *Suppose χ is a finite union of compact surfaces imbedded in $N(W)$ that is transverse to the fibers and intersects each fiber at most once. If the projection $\pi_W(\chi)$ of χ is also the projection of a union of leaves in f^** (*possibly restricted at branching*), *then each connected component of χ is isotopic, in $N(W)$, to a leaf in f^*.*

Proof. For each component C of χ, we may choose a fiber I through C that does not contain any furrow points of $N(W)$ (e.g. take I over some sector met by $\pi_W(C)$). Given a curve γ in C, consider integral curves of f^* that shadow γ in $N(W)$; that is, integral curves α such that $\pi_W * \alpha = \pi_W * \gamma$. Let $\Gamma = \{x \in I \mid$ for every curve γ in C begining in I, there exists an integral curve from x that shadows $\gamma\}$. We claim that $\Gamma \neq \varnothing$. By hypothesis, there exists a leaf L' in f^* such that $L' \cap I \neq \varnothing$ and $\pi_W(L')$ is contained in $\pi_W(\chi)$. Suppose $L' \cap I$ is not contained in Γ. Then for some γ in C, there are outgoing sectors branching from $\pi_W * \gamma$ at a point $\pi_W * \gamma(t_0)$, one of which is met by $\pi_W * \gamma$ and the other by $\pi_W(L')$. Now χ meets the fiber over $\pi_W * \gamma(t_0)$ exactly once. So C is the only component of χ that meets an outgoing sector branching from $\pi_W * \gamma$ at $\pi_W * \gamma(t_0)$. This implies that the sector met by $\pi_W(L')$ is not met by $\pi_W(\chi)$, contradicting the fact that $\pi_W(L')$ is contained in $\pi_W(\chi)$. So $\Gamma \neq \varnothing$.

It was shown in [**15**] that $\Gamma \neq \varnothing$ guarantees the existence of a leaf L_y (possibly restricted at branchings) that is isotopic to C in $N(W)$. We give a brief outline of the argument.

To begin, note $\Gamma = \cap_{\gamma \subseteq C} \Gamma_\gamma$, where each γ begins in I and each $\Gamma\gamma = \{x \in I \mid \text{there exists an integral curve from } x \text{ that shadows } \gamma\}$. It is easily verified that $\Gamma\gamma$ is closed for every γ, so Γ is a closed subset of I. Therefore, there exists a point $y \in \partial\Gamma$ such that all points of Γ are above y (with respect to the orientation of I). Since $y \in \Gamma$, the leaf L_y through y may be restricted at branchings so that each curve in L_y is shadowed by a curve in C. Since the converse also holds, L_y (possibly restricted at branchings) projects along fibers of $N(W)$ onto C. This projection is a covering map that preserves transverse orientation. If γ is closed, then an integral curve from y that shadows a curve $\gamma' * \gamma \subseteq C$ contains an arc that begins over the end of γ and shadows γ'. It follows that the end of any integral curve from y that shadows a closed γ in C is also in Γ. So since no point in Γ lies below y, every integral curve from y shadowing a closed curve in C is closed. Therefore, L_y meets I exactly once and the covering map is injective on I, hence everywhere. So, L_y is isotopic to C in $N(W)$. □

Proof of Theorem 1. Consider a set of surfaces in $N(W)$ that meet each fiber at most once and induce weight systems corresponding to distinct equivalence classes; choose this set to be as large as possible (it is finite since the weight systems on W are partitioned into only finitely many distinct equivalence classes). Let Σ be the largest subset with the property that for every $S \in \Sigma$, $\pi_W(S)$ is not the projection of a union of leaves in f^* (regardless of restrictions at branchings). This condition guarantees that there is a connecting strip K crossed by S and f^*, but with different indices (Lemma 2.1). We use a splitting in f to unzip W along the corresponding connecting curve κ. Since K depends only on $\pi_W(S)$, any surface imbedded in $N(W)$ that meets every fiber at most once and has the same projection as S (i.e. corresponds to the same equivalence class as S) is not contained in the fiber neighborhood of the modified branched surface. Repeating this process for each element of Σ, we modify W with at most a finite number of splittings in f to obtain a branched surface W'.

Given a foliation f' carried by W', let f'^{*}_{W} be the dual foliation of $N(W)$. (Recall that every foliation carried by W' is carried by W.) When we cut f' open along the surfaces in the generating set for W to obtain $N(W)$, each dead-end component of f' yields a component which we refer to as a "dead-end component of f'^{*}_{W}". We think of $N(W)$ as imbedded in M, so each dead-end component of f'^{*}_{W} is the union of a saturated submanifold of $N(W)$ with certain components of $M - N(W)$.

We will define an injective mapping Φ from the set of dead-end components of f'^{*}_{W} into the set of dead-end components of f^*. Let χ be the boundary of a dead-end component $N\chi$ of f'^{*}_{W}. Restrict each branched component of χ to the upper hemisphere at each of its branchings. Now $N\chi$ is a dead-end component, so χ meets every fiber at most once. Further, f'_W* is dual to a foliation carried by W', so $\pi_W(\chi) \neq \pi_W(S)$ for any $S \in \Sigma$. So $\pi_W(\chi)$ is the projection of a union of leaves in f^* (after the appropriate restrictions at branchings).

By Lemma 2.4, each component C of χ is isotopic to some leaf in f^*, and the isotopy is obtained by flowing along fibers of $N(W)$. Since no two components of χ meet the same fiber, there exists a union $\Lambda = L_1 \cup \ldots \cup L_n$ of leaves in f^* that is isotopic to χ in $N(W)$. The set Λ bounds a submanifold N_Λ of M which has the same boundary orientation as $N\chi$. So all normals to f^* on ∂N_Λ are oriented toward the interior (exterior, respectively) of N_Λ. Since f^* is dual to f, N_Λ yields a saturated component of f when we collapse the complement of $N(W)$ in M. So by Lemma 2.3, the interior of N_Λ contains a dead-end component N of f^*. We let $\Phi(N\chi) = N$.

Now suppose $\Phi(N\chi) = \Phi(N\chi') = N$ for distinct dead-end components $N\chi$ and $N\chi'$ of f'^{*}_{W}. As above, $N\chi$ and $N\chi'$ have boundaries that are isotopic to a union of leaves, Λ and Λ' respectively, of f^*. Both isotopies are along fibers of $N(W)$, and the component N is contained in the union of $N\chi$ ($N\chi'$) with the region of isotopy between Λ and $\partial N\chi = \chi$ (Λ' and $\partial N\chi' = \chi'$ respectively). Now $N - (N\chi \cap N)$ is contained in $N(W)$ (specifically in the region of isotopy between Λ and χ). Similarly, $N - (N\chi' \cap N)$ is contained in $N(W)$. Since N cannot be entirely contained in $N(W)$ (the boundary of a dead-end component can meet a fiber at most once), this implies that $N\chi \cap N\chi' \cap N$ contains a component of $M - N(W)$; in particular, some furrow point x of $N(W)$ is contained in $\mathrm{cl}(N\chi) \cap \mathrm{cl}(N\chi') \cap \mathrm{cl}(N)$. Consider the fiber I through x. Since N is a dead-end component of f^*, $\mathrm{cl}(N) \cap I$ is connected and has nonempty interior. When we collapse the complement of $N(W)$ in M, $N\chi$ and $N\chi'$ yield disjoint components, so $N\chi \cap N\chi' \cap I = \varnothing$. It follows that $x \in \partial N\chi$ and $x \in \partial N\chi'$. Now $\partial N \cap I$ contains at most one point, so some $y \in \partial I$ is contained in N. Furthermore, either $N\chi$ or $N\chi'$

contains y, since both χ and χ' already meet I at x. If y is contained in $N\chi'$, then y is also contained in a portion of the isotopy region between χ and Λ that is not met by $N\chi$. This isotopy is along fibers of $N(W)$, so $y \in \Lambda = \partial N_\Lambda$, contradicting the fact that $y \in N \subseteq N_\Lambda$. Similarly, we get a contradiction if y is contained in $N\chi$. So Φ is injective. □

For a simple illustration of Theorem 1, we use a lower dimensional analog. Figure 2.3 shows a branched 1-manifold (left) imbedded in a planar model of the torus. A foliation f^* of $N(W)$ is shown in Figure 2.4. It has one dead-end component. The bold lines in Figure 2.5 indicate the projections of the two leaves in f^* that bound this dead-end component. There is only one surface S in Σ (shown in Figure 2.6). We may modify W by a splitting in f (as indicated in Figure 2.7), to obtain a different branched surface W' carrying f. No surface in $N(W)$ with the same projection as S is embedded in $N(W')$. Therefore, every foliation carried by W' has at most 1 dead-end component.

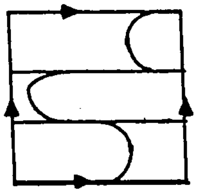


Figure 2.3.

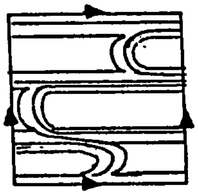


Figure 2.4.

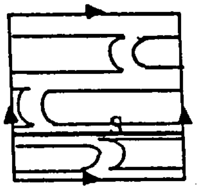


Figure 2.6.

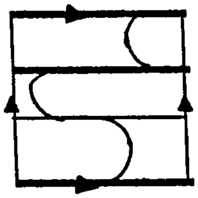


Figure 2.5.

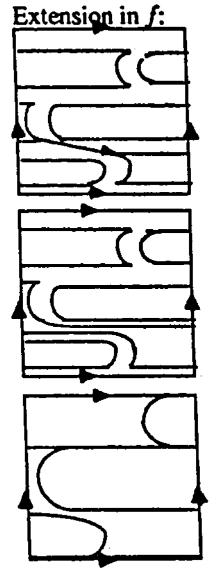


Figure 2.7.

Corollary 2.3. *If no foliation carried by W has fewer dead-end components than f, then every foliation carried by the modified branched surface W' has the same number of dead-end components.*

Now suppose f has N dead-end components. If some foliation carried by W' has fewer than N dead-end components, say $N - k$, we may use this foliation to modify W' (in the same way we modified W in Theorem I) so that all foliations carried by the new branched surface have at most $N - k$

dead-end components. Continuing in this manner, we obtain a branched surface V such that each foliation carried by V has J dead-end components, for some $J < N$. Each foliation f' carried by V is carried by W and all intermediate branched surfaces. (However, V does not necessarily carry f.) So we have:

Corollary 2.4. *For some foliation f' carried by the branched surface W, we may modify W using splittings in f' to obtain a branched surface V such that all foliations carried by V have the same number of dead-end components.*

We note that one choice for f' is the foliation with the least number of dead-end components. However, this condition on f' is not necessary.

In [**16**], branched surfaces were used to define an equivalence relation on foliations of a manifold M that are transverse to a nonsingular flow ϕ. We let $[f, \phi]$ represent the equivalence class of the foliation f under this relation. One of the properties of the relation is that we may associate a branched surface W with the eqivalence class of f such that any modification of W by splittings in $f' \in [f, \phi]$ carries all foliations in $[f, \phi]$. In particular, we may modify W by splittings in f' to produce a branched surface W' carrying only foliations which have no more dead-end components than does f' (Theorem I) and W' carries f. So there is no foliation in $[f, \phi]$ that has fewer dead-end components than does f. That is, all foliations in the same class as f will have the same number of dead-end components. Specifically, if we consider the subset $\Omega(\phi)$ of foliations of M (transverse to ϕ) having the fewest number of dead-end components, this relation can be used to classify the elements of $\Omega(\phi)$, since each equivalence class of a foliation in $\Omega(\phi)$ contains only foliations in $\Omega(\phi)$.

A splitting is one of the several "moves" in f (described in [**16**]) that can be used to modify a branched surface. For a branched surface W associated with the equivalence class of f, we call the set of all modifications of W by moves in f the "distinguished set" of (f, ϕ). This set consists of the simplest branched surfaces carrying f (i.e. those with least number of components in their complement). It was shown in [**16**] that the distinguished set of (f_1, ϕ) is the same as the distinguished set of (f_2, ϕ) if and only if $[f_1, \phi] = [f_2, \phi]$. So the distinguished set is a combinatorial representative of the equivalence class. For example, distinguished sets provide a combinatorial classification of foliations in $\Omega(\phi)$.

Corollary 2.5. *If f has N dead-end components, then all foliations sufficiently close to f have at most N dead-end components.*

This follows from Theorem I and the fact that for every foliation of M that is sufficiently close to f (in the C^1 metric defined by Hirsch [**11**]), there is a topologically equivalent foliation that is carried by W' ([**15**]).

The injective map constructed in the proof of Theorem I sends each Reeb component to a dead-end component bounded by a torus, which one can chose to be a Reeb component. So we have the following specialization of the theorem:

Corollary 2.6. *The number of Reeb components does not increase under sufficiently small perturbations of a foliation.*

A branched surface is "taut" if every foliation that it carries is taut. So we have:

Corollary 2.7. *If a foliation is taut, then it is carried by a taut branched surface. In particular, if f is a taut foliation, then every foliation sufficiently close to f is taut.*

We recall from [**1**] that a leaf L of a foliation is at depth 0 if it is compact. Inductively, L is at depth $k \geq 1$ if $Cl(L) - L$ consists of leaves at depth strictly less than k, at least one of which is depth $k-1$. Further, a foliation has finite depth if every leaf is proper and there is an upper bound on the depth of the leaves; in this case, the smallest upper bound k is called the "depth" of the foliation. Otherwise, the foliation has infinite depth.

Definition. A branched surface W has "depth j" if every foliation carried by W has depth at least j and j is the greatest integer for which this is true. If W does not carry a foliation at finite depth (i.e. if no such integer j exists), then W has infinite depth.

It was shown in [**14**] that if W as above admits a strictly positive weight system, then it has depth 0. That is, it carries a fibration over S^1. Here we develop an algorithm for determining a lower bound k on the depth of W. So in particular, the depth for all foliations sufficiently close to the foliation f will be bounded below by k. Given the equivalence class $[f, \phi]$, we may apply the algorithm to any branched surface W' in the corresponding distinguished set to determine the simplest structure possible for a foliation in this class. That is, if k is a lower bound on the depth of W', then any foliation in this class has the same number of dead-end components and is at depth at least k.

If the only weight system on W is that which assigns a 0 to every sector, then W has infinite depth. In particular, no foliation carried by W has a leaf at depth 0 (since after restricting all branchings to the same hemisphere, a compact leaf in a foliation of $N(W)$ induces a non-trivial weight system on W). So we assume W carries a foliation with a depth 0 leaf. Specifically, W admits a non-trivial weight system.

To begin, we consider possible foliations of $N(W)$. Each leaf at depth 0 lies over the (closure of) sectors that are nonzero in the weight system it induces. Hence, we use the non-trivial weight systems on W to locate the regions of $N(W)$ that may contain leaves that are at depth 0.

Choose a finite set Σ of surfaces (not necessarily connected) inducing an elementary set of weight systems. (Recall that if S is the union of compact leaves in a foliation f^*, then there is a surface $S_0 \in \Sigma$ such that $\pi_W(S_0) = \pi_W(S)$.) Next, select some $S_0 \in \Sigma$. If the weight system induced by S_0 assigns a nonzero weight to every sector, then as noted above, W is of depth 0. So assume $\pi_W(S_0) \neq W$.

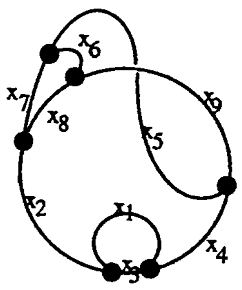
Our objective is to find a lower bound k on the depth of foliations of $N(W)$. So we assume there exists a foliation f^* of $N(W)$ such that if $\pi_W(S_0) \supseteq \pi_W(L)$ for some leaf L, then L is compact. That is, assume all noncompact leaves intersect a fiber that is not met by S_0. Next, take the subset $W - \pi_W(S_0)$ of W. Since $\pi_W(S_0) \neq W$, this is a union W_1 of finitely many branched submanifolds. Let $N(W_1)$ be the restriction of $N(W)$ to the fibers over W_1. Choose a finite set Σ_1 of compact surfaces in $N(W_1)$ inducing an elementary set of weight systems on W_1. (The surfaces in Σ_1 are transverse to the boundary fibers of $N(W_1)$.) Given $S_1 \in \Sigma_1$, if $\pi_W(S_1) \supseteq \pi_W(L|_{N(W_1)})$ for some leaf L at depth 0, then some weight system on W is induced by a surface containing both S_0 and L. This case will be considered when we change our initial choice for S_0. So here select $S_1 \in \Sigma_1$ such that each component meets $\partial N(W_1)$, and assume that if $\pi_W(S_1) \supseteq \pi_W(L|_{N(W_1)})$ for some leaf L, then $L|_{N(W_1)}$ is compact and L is at depth 1 in f^*. Next, take $W_2 = W_1 - \pi_{W_1}(S_1)$.

Inductively, given W_i choose a finite set of surfaces Σ_i inducing an elementary set of weight systems on W_i. Select $S_i \in \Sigma_i$ such that each component meets $\partial N(W_i)$, and assume that if $\pi_W(S_i) \supseteq \pi_W(L|_{N(W_i)})$ for some leaf L, then $L|_{N(W_i)}$ is compact and L has depth i in f^*. This is a harmless assumption since a leaf in a foliation of $N(W)$ at depth less than i will be considered when we choose a different elementary surface at an earlier stage. Then let $W_{i+1} = W_i - \pi_{W_i}(S_i)$.

We continue until for some $k \geq 0$, $W_{k+1} = W_k - \pi_{W_k}(S_k)$ is empty (i.e. W_k has a strictly positive weight system induced by S_k). We assign the number k to the associated string of surfaces $\{S_0, S_1, \ldots, S_k\}$ of length $k+1$. For every i, different choices for $S_i \in \Sigma_i$ will be followed by a different substring of surfaces. We generate all possible strings beginning with elements of Σ and take the smallest k over all strings. This is our lower bound on the depth of foliations carried by W.

For example, all non-trivial weight systems of the branched surface in Figure 1.11 are in the same equivalence class. So each choice for Σ has one element. Moreover, there exists a strictly positive weight system on W_1. Therefore, $k = 1$. It can be shown that all foliations carried by this branched surface are equivalent and have 2 dead-end components. So, the Reeb foliation (which is depth 1) is among the simplest foliations carried by this branched surface.

To illustrate with a more complex example, we consider a lower dimensional analog. Figure 2.8 shows a branched 1-manifold W that has nine sectors.



• indicates a branch point

Figure 2.8.

The additive condition across the branch set yields the following system of equations:

$$\begin{aligned} x_1 + x_2 &= x_3 \\ x_1 + x_4 &= x_3 \\ x_7 + x_8 &= x_2 \\ x_6 + x_9 &= x_8 \\ x_5 + x_6 &= x_7 \\ x_4 + x_5 &= x_9. \end{aligned}$$

Let $(Z^+)^9$ be the set of vectors in R^9 with nonnegative integral entries. There exists a 9×9 matrix A, such that $v \in (Z^+)^9$ is a solution to the homogeneous system $Ax = 0$ if and only if there exists a weight system on W that assigns the i-th coordinate of v to the i-th sector of W, for each $1 \leq i \leq 9$. We may use A to identify an elementary set of weight systems on

W as follows:

The homogeneous system is:

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Using the reduced row echelon form of the above matrix, we have:

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

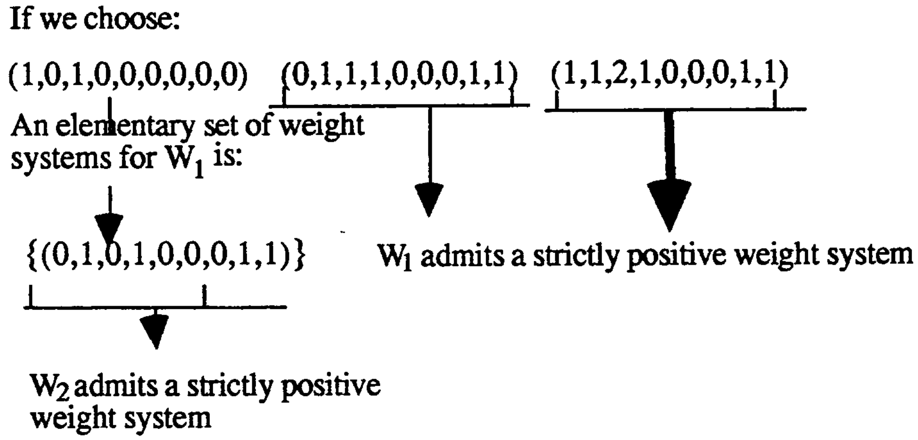
The null space is:

$$\{x_3(1,0,1,0,0,0,0,0,0) + x_8(-1,1,0,1,-1,1,0,1,0) \\ + x_9(0,0,0,0,1,-1,0,0,1)\}.$$

So another representation of the null space is

$$\{a(1,0,1,0,0,0,0,0,0) + b(-1,1,0,1,0,0,0,1,1) + c(0,1,1,1,0,0,0,1,1)\}.$$

Since we only consider weight systems where each weight is a nonnegative integer, choosing $a < b$ does not yield a weight system. So a set of elementary weight systems is given by $\{(1,0,1,0,0,0,0,0,0),\ (0,1,1,1,0,0,0,1,1),\ (1,1,2,1,0,0,0,1,1)\}$.



Paths following the three initial choices of elementary weight systems on W are indicated in the diagrams below. The sectors for W_1 are also sectors of W so we represent the weight systems on W_1 in the same manner as we represent weight systems on W. For example, if $(0,1,1,1,0,0,0,1,1)$ is chosen for W, then W_1 contains sectors x_1, x_5, x_6 and x_7. We represent its weight systems with 9-tuples where all entries except possibly the first, fifth, sixth and seventh, are 0. In this case, W_1 admits a strictly positive weight system, so the algorithm terminates at the second stage. (The bold arrow indicates this string.) Since no choice of elementary weight system gives a string of length 1, (i.e. there are no strictly positive weight systems on W), this is the shortest path possible. So $k = 1$.

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