PACIFIC JOURNAL OF MATHEMATICS
Vol. 178, No. 2, 1997

ON STABILITY OF CAPILLARY SURFACES IN A BALL

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We study stable capillary surfaces in a euclidean ball in the absence of gravity. We prove, in particular, that such a surface must be a flat disk or a spherical cap if it has genus zero. We also prove that its genus is at most one and it has at most three connected boundary components in case it is minimal. Some of our results also hold in $\mathbb{H}^3$ and $\mathbb{S}^3$.

Introduction.

Consider a smooth and compact convex body B in $\mathbb{R}^3$. Let ∂B and $\operatorname{int} B$ denote its boundary and its interior respectively. We are interested in embedded constant mean curvature surfaces M in $\mathbb{R}^3$ with non empty boundary such that $\operatorname{int} M \subset \operatorname{int} B$ and $\partial M \subset \partial B$ and which intersect ∂B at a constant angle $\gamma \in (0, \pi)$. Such surfaces, called capillary surfaces, are critical points of an energy functional under some constraints. The energy functional is defined as follows: the surface M separates B into two bodies, consider among these two bodies the one inside which the angle γ is measured and call Ω the part of its boundary that lies on ∂B. Denote by A the area of M and by T that of Ω. The energy function is then

$$E = A - \cos\gamma\, T.$$

The space of surfaces under consideration are compact orientable surfaces in $\mathbb{R}^3$ with boundary contained in ∂B and interior contained in $\operatorname{int} B$ and which divide B into two bodies of preassigned volumes. The Euler-Lagrange equation shows that a critical point of E under these constraints is a constant mean curvature surface that intersect ∂B at the constant angle γ, that is the angle between the exterior conormals to ∂M in M and Ω is everywhere equal to γ along ∂M. We say that such a surface is capillarily stable if it minimizes the energy up to second order. Capillary surfaces correspond to the physical problem of the behavior of an incompressible liquid in a container B in the absence of gravity. A great deal of work has been devoted to capillary phenomena from the point of view of existence and uniqueness of solutions mainly in the non-parametric case and in the more general situation of presence of gravity (see the book of R. Finn, [**F**], for an account of the subject). Stability has also drawn some attention, we quote the works of H.C. Wente ([**W1**]) and T.I. Vogel. This last author has, for example, studied stability in the absence of gravity for a capillary annulus between two parallel planes ([**V1**]) and of an infinite cylinder in a wedge ([**V2**]).

When $B \subset \mathbb{R}^3$ is a ball, examples of capillary surfaces for varying mean curvature and angle of contact are given by rotational ones, namely totally geodesic disks, spherical caps, pieces of catenoids and pieces of Delaunay surfaces. We must remark that we do not know any other examples. A consequence of one of our results is that pieces of catenoids and of Delaunay surfaces are not stable.

We here study stability of capillary surfaces in euclidean balls with no restrictions a priori about their topology. The orthogonal case, that is when the angle of contact is $\pi/2$, has been investigated by the first author and E. Vergasta ([**R-V**]) in the euclidean case and by the second author ([**S**]) in the spherical and hyperbolic cases. The techniques we use here are mainly borrowed from ([**R-V**]) but our results here are less precise than those obtained in the orthogonal case. This is due to the fact that in the non orthogonal case the geometry of the surface is involved in the boundary term in the second variation formula.

Our first result (Corollary 2.3) says that

"*A genus zero capillarily stable surface in a ball of* $\mathbb{R}^3$ *must be a totally geodesic disk or a spherical cap*".

Before proving this result we reprove a theorem of Nitsche [**N**] which states that a capillary disk (with no assumption on stability) in a euclidean ball must be a totally geodesic disk or a spherical cap. Nitsche proved his theorem in the orthogonal case and pointed out at the end of his paper that the result extends to general contact angles. More generally his argument actually extends to balls in $\mathbb{S}^3$ or $\mathbb{H}^3$ (Theorem 2.1). Moreover the proof of the above result gives some information for a more general class of metrics on $\mathbb{R}^3$, namely those that are invariant under the group of rotations $SO(3)$. We show that if B is a ball centered at the origin in $\mathbb{R}^3$ endowed with such a metric, then a capillary stable surface of genus zero in B must be a disk that is invariant under rotation around an axis that passes through the origin (Theorem 2.2).

Our second result (Theorem 3.3) concerns the case where the surface is assumed minimal. We show that

"*The only capillarily stable and minimal surfaces in a ball of* $\mathbb{R}^3$ *are the totally geodesic disks or surfaces of genus* 1 *with boundary having at most* 3 *connected components*".

We furthermore show that under these hypotheses the case of a surface of genus 1 and 1 boundary component cannot occur if the angle of contact is close to 0 or π. In the orthogonal case, A. Ros and E. Vergasta have shown that a capillarily stable and minimal surface in a euclidean ball is necessarily a totally geodesic disk (they have in fact proved this in every dimension, and the result is also true in the hyperbolic and spherical cases, see [**S**]). This result is probably true for any angle of contact but we have not been able to prove it. We have included in an appendix at the end of the paper a proof of the second variation formula. This formula is already known (see [**V1**] or [**W1**]) but its derivation doesn't seem to appear in the litterature.

1. Preliminaries.

Let W be an orientable riemannian manifold of dimension $n+1$ and B a smooth compact body in W that is diffeomorphic to a euclidean ball. Let M be an orientable n-dimensional compact manifold with non empty boundary ∂M and $\phi : M \longrightarrow W$ an embedding, smooth even at ∂M, that maps $\operatorname{int} M$ into $\operatorname{int} B$ and ∂M into ∂B. $\phi(\operatorname{int} M)$ then separates $\operatorname{int} B$ into two connected components. The boundary of each of these two components consists of the union of $\phi(M)$ and a domain on ∂B. Let us fix one of these two domains and call it Ω. By an admissible variation of ϕ we mean a differentiable map $\Phi : (-\epsilon, \epsilon) \times M \longrightarrow W$ such that $\Phi_t : M \longrightarrow W, t \in (-\epsilon, \epsilon)$, defined by $\Phi_t(p) = \Phi(t,p), p \in M$, is an embedding satisfying $\Phi_t(\operatorname{int} M) \subset \operatorname{int} B$ and $\Phi_t(\partial M) \subset \partial B$ for all t, and $\Phi_0 = \phi$. Fix an angle $\gamma \in (0, \pi)$. Given an admissible variation Φ, Ω then moves to $\Omega(t)$, for $t \in (-\epsilon, \epsilon)$. We define the energy function $E : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}$ by:

$$E(t) = A(t) - \cos\gamma\, T(t)$$

where $A(t)$ (resp. $T(t)$) denotes the volume of M (resp. of $\Omega(t)$) in the metric induced by Φ_t (resp. by the inclusion in W). The volume function $V : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}$ is defined by:

$$V(t) = \int_{[0,t]\times M} \Phi^* dV$$

where dV is the volume element of W. $V(t)$ represents the volume enclosed between the hypersurfaces ϕ and Φ_t. The variation is said to be volume - preserving if $V(t) = V(0)(= 0)$ for all t. The variation vector field of Φ is defined on M by:

$$Y(p) = \frac{\partial \Phi}{\partial t}(p)\Big|_{t=0}.$$

Let N be the unit normal vector field along ϕ that points into the domain bounded in W by $\phi(M)$ and Ω, and let $\bar{N}$ be the exterior unit normal to ∂B.

We let ν (resp. $\bar{\nu}$) denote the unit exterior normal to ∂M in M (resp. in Ω), ds the volume element of ∂M induced by ϕ and H the mean curvature of ϕ. The first variation formula for this energy function and for the volume are given by the following formulae:

$$E'(0) = -n \int_M Hf \; dA + \int_{\partial M} \langle Y, \nu - \cos\gamma\, \bar{\nu}\rangle \, ds \tag{1-1}$$

$$V'(0) = \int_M f \, dA \tag{1-2}$$

where $f = \langle Y, N\rangle$ and dA is the volume element on M induced by ϕ. (1-1) follows from the first variation formula for the area function (cf. [**Sp**, vol. 4]), for a proof of (1-2) see [**B-dC-E**]. The embedding is said to be capillary if $A'(0) = 0$ for any admissible volume - preserving variation of ϕ. It follows from (1-1) and (1-2) that ϕ is capillary if and only if ϕ has constant mean curvature and intersects ∂B at the constant angle γ; that is along ∂M the angle between the normals N and $\bar{N}$ or equivalently between ν and $\bar{\nu}$ is everywhere equal to γ.

Henceforth we shall assume that ϕ is capillary. Let σ denote the second fundamental form of ϕ with respect to the chosen unit normal field N, and let II denote the second fundamental form of ∂B in W with respect to the inwards pointing unit normal (i.e with respect to $-\bar{N}$). For an admissible volume-preserving variation we have (cf. the appendix):

$$E''(0) = -\int_M \left(f\Delta f + (|\sigma|^2 + n\,\mathrm{Ric}(N))f^2\right) \, dA + \int_{\partial M} f \left(\frac{\partial f}{\partial \nu} - q\, f\right) \, ds$$

where

$$q = \frac{1}{\sin\gamma}\; \mathrm{II}(\bar{\nu}, \bar{\nu}) + \cot\gamma\, \sigma(\nu, \nu).$$

Here Δ is the Laplacian in the metric induced by ϕ, Ric the Ricci curvature of W and $\frac{\partial f}{\partial \nu}$ the derivative of f in the direction of the exterior normal ν. Adapting the arguments in [**B-dC-E**], one can show that for each smooth function f on M with mean value zero, that is $\int_M f \, dA = 0$, there exists an admissible volume - preserving vartiation of ϕ with variation vector field having fN as normal part. We say that a capillary embedding $\phi : M \longrightarrow B$ is stable if $E''(0) \geq 0$ for all admissible volume - preserving variations of ϕ. Let $\mathcal{F} = \{f \in H^1(M), \int_M f dA = 0\}$ where $H^1(M)$ denotes the first Sobolev space of M, and let's define the index form I of ϕ as the symmetric bilinear form on $H^1(M)$ by

$$I(f, g) = \int_M \{\langle \nabla f, \nabla g\rangle - (|\sigma|^2 + n\,\mathrm{Ric}(N))fg\} \, dA - \int_{\partial M} q\, fg \, ds$$

where ∇f means the gradient of f in the metric induced by ϕ. It follows from what preceeds that the capillary embedding ϕ is stable if and only if $I(f,f) \geq 0$ for all $f \in \mathcal{F}$.

Given $f \in \mathcal{F}$, we say that the normal vector field fN is a Jacobi vector field of ϕ if $I(f,g) = 0$ for all $g \in \mathcal{F}$. It can be shown that fN, for an $f \in \mathcal{F}$ is a Jacobi field if and only if $f \in \mathcal{C}^\infty(M)$ and

$$\begin{cases} \Delta f \;+\; (|\sigma|^2 + n\,\mathrm{Ric}(N))\, f \;=\; \text{constant} & \text{on } \; M \\ \dfrac{\partial f}{\partial \nu} = q\, f & \text{on } \; \partial M. \end{cases}$$

We first prove the stability of totally geodesics disks and spherical caps in a ball $B \subset \mathbb{R}^3$.

Proposition 1.1. *Let $B \subset \mathbb{R}^3$ be a euclidean ball. Then totally geodesic disks and spherical caps in B with boundary in ∂B are capillarily stable.*

Proof. We may assume without loss of generality that B is a unit ball. Let Σ denote either a totally geodesic disk or a spherical cap contained in B with boundary in ∂B.

Assume Σ is a totally geodesic disk and let R denote its radius. The angle of contact then satisfies the relation $\sin\gamma = R$. We thus have to show that

$$\int_M |\nabla f|^2\, dA \geq \frac{1}{R} \int_{\partial M} f^2\, ds \quad \text{for each } \; f \in \mathcal{F}. \tag{1-4}$$

Consider now the ball B' of radius R having Σ as an equatorial totally geodesic disk. Thus Σ intersects $\partial B'$ orthogonally and we know (cf. [**R-V**]) that Σ is capillarily stable in B', but the stability condition of Σ in B' is exactly Condition (1-4). Hence Σ is stable in B.

Assume now that Σ is a spherical cap. Let R denote the radius of the sphere S containing Σ and let also γ denote the constant angle between Σ and ∂B measured inside the domain into which the spherical cap is curved. When $\gamma = \pi/2$, we know that Σ is stable and in fact minimizing (see [**R-V**]). One may be tempted to use this to prove stability for any angle of contact in the same way as we did it before for flat disks. Unfortunately it is not always possible to find a ball B' containing Σ and intersecting $\partial\Sigma$ orthogonally (this is in fact possible if and only if Σ is smaller than a hemisphere). Instead of this, consider the plane Π that contains $\partial\Sigma$. Σ then intersects Π at a constant angle which we denote by γ'. Σ can hence be seen as a capillary surface in a halfspace and it is known that spherical caps in a halfspace are capillarily stable and in fact minimize the energy function (see [**G-M-T**]). Thus

$$\int_\Sigma \left(|\nabla f|^2 - \frac{2}{R^2} f^2 \right) dA \geq \frac{\cot\gamma'}{R} \int_{\partial\Sigma} f^2\, ds \quad \text{for each } \; f \in \mathcal{F}. \tag{1-5}$$

Elementary geometry shows that

$$\frac{1}{\sin\gamma} + \frac{\cot\gamma}{R} = \frac{\cot\gamma'}{R}. \tag{1-6}$$

Stability of Σ in B follows from (1-5) and (1-6). $\square$

Before proceeding further, we make some remarks concerning capillary surfaces in euclidean balls regardless of their stability properties. As we mentioned in the introduction, Nitsche's theorem ([**N**]) is true for any angle of contact and says that a capillary disk in a euclidean ball is a totally geodesic disk or a spherical cap (see Theorem 2.1). Another more or less known fact is that Alexandrov's reflection technique is useful in this setting and gives the following:

Proposition 1.2. *Let $\phi : M \longrightarrow B$ be a capillary embedding in a euclidean ball. Assume that $\phi(\partial M)$ is contained in an open hemisphere of ∂B, then $\phi(M)$ is a totally geodesic disk or a spherical cap.*

Proof. We may assume that B is centered at the origin and that $\phi(\partial M)$ is contained in the open hemisphere defined by $x_1 < 0$. Suppose first that ϕ is not entirely contained in this hemisphere. In particular, by the maximum principle, ϕ is not minimal, so we call Λ the domain in $\mathbb{R}^3$ into which the mean curvature vector of $\phi(M)$ points. Alexandrov's reflection method, using the family of parallel planes $\{x_1 = t\}, t \geq 0$, shows that the reflected image of $\phi(M) \cap \{x_1 \geq 0\}$ with respect to the plane $\{x_1 = 0\}$ is contained in Λ and it cannot touch the boundary of $\phi(M)$ because of the geometry of the sphere. The fact that $\phi(M)$ intersects ∂B at a constant angle allows us to continue Alexandrov's reflection process this time by rotating the plane $\{x_1 = 0\}$ around lines containing the origin to conclude that $\phi(M)$ has to be a rotational disk (see [**W2**] for the details in a similar situation, one needs to use a maximum principle at a corner). In case $\phi(M)$ is entirely contained in a hemisphere we just begin our argument using the rotating planes. $\square$

2. The genus zero case.

We begin with an extension of Nitsche's theorem [**N**].

Theorem 2.1. *Let $\phi : M \longrightarrow B$ be a capillary embedding of a closed disk M into a ball B in $\mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$, then $\phi(M)$ is totally umbilical.*

Proof. We may parametrize M conformally by the closed unit disk $\bar{D}$. Consider the Hopf-function defined on $\bar{D}$ by:

$$h(z) = \sigma(\partial_z, \partial_z) = \sigma(\partial_x, \partial_x) - \sigma(\partial_y, \partial_y) - 2i\,\sigma(\partial_x, \partial_y); \quad z = x + iy \in \bar{D}.$$

It is well known (see [**Ch**]) that h is holomorphic when ϕ has constant mean curvature and vanishes precisely at umbilics. By hypothesis $\phi(M)$ intersects ∂B at a constant angle and since the latter is totally umbilical we conclude that each component of ∂M is a line of curvature in M (this follows from the classical terquem-Joachimsthal's theorem, cf. [**Sp**, vol. 3, p. 296]). We thus have

$$\sigma(x\partial_x + y\partial_y, -y\partial_x + x\partial_y) = 0 \qquad \text{on} \quad \partial\bar{D}.$$

This is equivalent to $\Im m(z^2h(z)) = 0$ on $\partial\bar{D}$, and since this is a harmonic function, it has to be identically zero in $\bar{D}$. It follows that its conjuguate function $\Re e(z^2h(z))$ is constant in $\bar{D}$ and hence identically zero since it vanishes at zero. We infer that h vanishes everywhere in $\bar{D}$ which means that $\phi(M)$ is totally umbilical. □

We now treat the problem of stability of capillary surfaces of genus zero in a ball in a three dimensional simply connected space. We first show a result that holds for a larger class of metrics on $\mathbb{R}^3$. The method of proof is similar to the one used in the orthogonal case in [**R-V**], we give the argument in detail for completeness.

Theorem 2.2. *Let μ be a metric on $\mathbb{R}^3$ that is invariant under the group of rotations $SO(3)$. Let $\phi : M \longrightarrow B$ be a stable capillary surface of genus zero in a ball of $(\mathbb{R}^3, \mu)$ centered at the origin, then M is a disk and $\phi(M)$ is a rotation surface around an axis passing through the origin.*

Proof. Let $p_0 \in M$ be a point in M such that $\phi(p_0)$ is at the minimum euclidean distance on $\phi(M)$ to the origin. Denote by N the unit normal field along ϕ with respect to the metric μ and call X the Killing field on $(\mathbb{R}^3, \mu)$ induced by rotations around the axis passing through the origin and directed by $N(p_0)$. More precisely for $x \in \mathbb{R}^3$, $X(x) = x \wedge N(p_0)$, where $\wedge$ denotes the usual cross product in $\mathbb{R}^3$. Consider then the function $\beta : M \longrightarrow \mathbb{R}$ defined by $\beta(p) = \mu\,(X(\phi(p)), N(p))$. As our problem is invariant under rotation around $N(p_0)$, it follows that βN is a Jacobi vector field of ϕ. Thus

$$\Delta\beta + (|\sigma|^2 + 2\,\mathrm{Ric}(N))\beta = 0 \tag{2-1}$$

and

$$\frac{\partial\beta}{\partial\nu} = q\,\beta \quad \text{on} \;\; \partial M. \tag{2-2}$$

Also, one can check that

$$\beta(p_0) = 0 \quad \text{and} \quad \nabla\beta(p_0) = 0. \tag{2-3}$$

We claim that $\beta = 0$ on M. Assuming the contrary, the nodal set $\beta^{-1}(0)$ of β is then a graph whose vertices are the critical points of β (cf [**C**]). The Gauss-Bonnet formula applied to each connected component M_i of $M \setminus \beta^{-1}(0)$ gives

$$\int_{M_i} K\, dA = 2\pi \mathcal{X}(M_i) - \int_{\partial M_i} k_g\, ds - \sum_{l_i} \theta_{l_i}$$

where θ_{l_i} denote the external angles of ∂M_i. Summing up the above equations for all i, we obtain

$$\int_{M} K\, dA = 2\pi \sum_i \mathcal{X}(M_i) - \int_{\partial M} k_g\, ds - \sum_{j} \theta_{j} \tag{2-4}$$

where the last term means the sum of all external angles for every connected component M_i. As ∂M is smooth, (2-4) and again the Gauss-Bonnet formula yield

$$2\pi(2 - 2g - r) = 2\pi \sum_i \mathcal{X}(M_i) - \sum_j \theta_j$$

where g and r denote respectively the genus of M and the number of its boundary components. By (2-3) p_0 is a vertex of the graph $\beta^{-1}(0)$. Moreover, let Γ be a connected component of ∂M, then any critical point on Γ of the height function in the direction of the axis $N(p_0)$ is a zero of β. Actually at such a point, $\phi \wedge N(p_0)$ is tangent to Γ. This shows that β has at least two zeroes on each component of ∂M. So

$$\sum_j \theta_j \geq 2\pi(1 + r).$$

Together with (2-4), this gives

$$\sum_i \mathcal{X}(M_i) \geq 3 - 2g.$$

So if the genus of M is zero, there at least three connected components in $M \setminus \beta^{-1}(0)$. Let us then define a function $\tilde{\beta} : M \to \mathbb{R}$ by

$$\tilde{\beta} = \begin{cases} \beta & \text{on } \ M_1 \\ a\beta & \text{on } \ M_2 \\ 0 & \text{on } \ M \setminus (M_1 \cup M_2) \end{cases}$$

where the constant a is chosen so that $\int_M \tilde{\beta}\, dA = 0$. Using (2-1) and (2-2), a direct computation gives $I(\tilde{\beta}, \tilde{\beta}) = 0$. Since ϕ is assumed stable, this implies easily that $\tilde{\beta}N$ is a Jacobi field. But $\tilde{\beta}$ vanishes outside $M_1 \cup M_2$, the classical unique continuation principle then shows that $\tilde{\beta} = 0$ identically on M. This contradiction shows that $\beta = 0$ on M, this means that M is a rotation surface around the axis $N(p_0)$ with fixed point p_0, and therefore M is a disk. □

Remark. Compact constant mean curvature hypersurfaces without boundary are, as it is well known, critical points of the area function for volume-preserving variations. J.L. Barbosa, M. do Carmo and J. Eschenburg have shown (see [**B-dC-E**] or [**E-I**] for another proof) that stable compact constant mean curvature hypersurfaces without boundary in a simply connected space form must be totally umbilical spheres. The proof of Theorem 2-2 shows that if μ is any $SO(3)$-invariant metric on $\mathbb{R}^3$ then a stable closed constant mean curvature surface in $(\mathbb{R}^3, \mu)$ of genus zero must be a rotation surface around an axis passing through the origin.

As a corollary, using Theorem 2.1, we obtain:

Corollary 2.3. *Let $\phi : M \longrightarrow B$ be a capillary stable surface of genus zero in a ball of $\mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$, then $\phi(M)$ is totally umbilical.*

3. Some restrictions in the general case.

We start with some preliminary results. Let k_g (resp.$\bar{k}_g$) denote the geodesic curvature of ∂M in M (resp. in Ω). We have:

Lemma 3.1.

(1) $k_g = \cos\gamma\ \bar{k}_g + \sin\gamma$.

(2) $\sigma(\nu,\nu) = 2H - \tan\gamma\ k_g + \frac{1}{\cos\gamma}$.

Proof. (1) follows from the relation $\nu = \cos\gamma\ \bar{\nu} + \sin\gamma\ \bar{N}$. Let now α be an arclength parametrization of a component of ∂M, then

$$\sigma(\nu,\nu) = 2H - \sigma(\alpha',\alpha') = 2H - \langle \alpha'', N\rangle.$$

(2) then follows from the relation $N = -\tan\gamma\,\nu + \frac{1}{\cos\gamma}\,\bar{N}$. □

Proposition 3.2. *Let M be a capillary surface in a unit ball of $\mathbb{R}^3$, then:*

$$H^2 A + \int_{\partial M} k_g ds \ \geq\ 2\pi\sin\gamma\ +\ 2\pi\cos\gamma\ (\mathcal{X}(\Omega) - 1).$$

Proof. For any surface, it is well-known (cf. [**L-Y**]) that the integral

$$\int_M (H^2 - K)\ dA$$

is invariant under conformal transformations of $\mathbb{R}^3 \cup \{\infty\}$. Assume B is centered at the origin and consider a point $x_0 \in \mathbb{S}^2$. Then there exists a one parameter family $\{\eta_\lambda / \lambda > 0\}$ of conformal transformations of $\mathbb{R}^3 \cup \{\infty\}$ which preserve the ball B and fix the points x_0 and $-x_0$, and such that when $\lambda \to \infty$, $\eta_\lambda(x)$ converges to $-x_0$ for any $x \in \mathbb{R}^3 \setminus \{x_0\}$. These properties are satisfied for instance if we take $\eta_\lambda = f^{-1} \circ h_\lambda \circ f$ where h_λ is the homothety in $\mathbb{R}^3$ with center at the origin and factor of dilation λ and f is the conformal transformation of $\mathbb{R}^3 \cup \{\infty\}$ which transforms the ball B into the upper halfspace, the point x_0 into the origin and the point $-x_0$ into ∞. Denoting by H_λ, K_λ and dA_λ respectively the mean curvature, the Gauss curvature and the area element induced on M by the embedding $\eta_\lambda \circ \phi$, we have

$$H^2 A \; - \int_M K \; dA \; = \int_M (H_\lambda^2 - K_\lambda) \; dA_\lambda. \tag{3-1}$$

The η_λ being conformal, $\eta_\lambda \circ \phi(M)$ still intersects $\mathbb{S}^2$ at the constant angle γ. Using Gauss-Bonnet formula and Lemma 3.1 (1), we get

$$\begin{aligned} 2\pi \mathcal{X}(M) \; &= \int_M K \; dA \; + \int_{\partial M} k_g \; ds \\ &= \int_M K_\lambda \; dA_\lambda \; + \int_{\partial M} k_g^\lambda \; ds_\lambda \\ &= \int_M K_\lambda \; dA_\lambda \; + \; \sin\gamma \; L_\lambda \; + \; \cos\gamma \int_{\partial M} \bar{k}_g^\lambda \; ds_\lambda \end{aligned} \tag{3-2}$$

where k_g^λ (resp. $\bar{k}_g^\lambda$) denotes the geodesic curvature of ∂M in M (resp. in Ω) with respect to the embedding $\eta_\lambda \circ \phi$ (resp. η_λ), ds_λ the induced metric on ∂M and L_λ its length. (3-1) and (3-2) give

$$H^2 A + \int_{\partial M} k_g ds = \int_M H_\lambda^2 \; dA_\lambda + \sin\gamma \; L_\lambda + \cos\gamma \int_{\partial M} \bar{k}_g^\lambda \; ds_\lambda.$$

If we take $x_0 = \phi(p_0)$ for some $p_0 \in \partial M$, then the integral $\int_{\partial M} \bar{k}_g^\lambda \, ds_\lambda$ which by Gauss-Bonnet formula equals $2\pi\mathcal{X}(\Omega) - \mathrm{area}(\eta_\lambda(\Omega))$ converges to $2\pi(\mathcal{X}(\Omega) - 1)$ as $\lambda \to \infty$. Moreover $\eta_\lambda \circ \phi(\partial M)$ converges to an equator of the unit sphere $\mathbb{S}^2$ and so $\lim_{\lambda \to \infty} L_\lambda = 2\pi$. This completes the proof of the proposition. □

We are now ready to prove the main result in this section. Henceforth we shall denote by g the genus of the surface M and by r the number of its boundary components.

Theorem 3.3. *Let $\phi : M \longrightarrow B$ be a stable capillary surface in a ball of $\mathbb{R}^3$. Assume ϕ is minimal, then the only possibilities are*

(1) *$\phi(M)$ is a totally geodisic disk*

(2) $g = 1$ *and* $r = 1$, 2 *or* 3.
Moreover, putting $\gamma_0 = \arcsin(\sqrt{13} - 3) = 0.65...$, *for* $\gamma \le \gamma_0$ *or* $\gamma \ge \pi - \gamma_0$, *the case* $g = 1$ *and* $r = 1$ *cannot occur.*

Proof. We begin the proof without assuming the minimality of ϕ. Let $\tilde{M}$ be a compact Riemann surface obtained from M by attaching a conformal disk at each connected component of ∂M. There exists a non constant holomorphic map $\tilde{\psi} : \tilde{M} \longrightarrow \mathbb{S}^2$ such that (cf. [**G-H**, p. 261])

$$\text{degree}\left(\tilde{\psi}\right) \le 1 + \left[\frac{g+1}{2}\right] \tag{3-3}$$

where $[.]$ denotes the integer part. Let $\psi : M \longrightarrow \mathbb{S}^2(1) \subset \mathbb{R}^3$ be the restriction of $\tilde{\psi}$ to M. Using an extended version of a result of Hersch (see [**H**] and [**L-Y**]) we may assume after composing ψ with a conformal diffeomorphism of $\mathbb{S}^2$ that its coordinate functions satisfy

$$\int_M \psi_i \, dA = 0, \qquad i = 1, 2, 3.$$

Stability of ϕ implies, using Lemma 3.1 (2), that

$$\begin{aligned} 0 \le \int_M \{|\nabla \psi_i|^2 \; - \; |\sigma|^2 \, \psi_i^2\} \; dA \; - \int_{\partial M} k_g \; \psi_i^2 \, ds \\ + \; \frac{2}{\sin\gamma} \, (1 + H\cos\gamma) \int_{\partial M} \psi_i^2 \, ds, \qquad i = 1, 2, 3. \end{aligned}$$

Summing up these inequalities and using (3-3), Gauss equation and Gauss-Bonnet formula, we obtain

$$8\pi\left(1 + \left[\frac{g+1}{2}\right]\right) + 4\pi(2 - 2g - r) > 4H^2 A + \frac{2L}{\sin\gamma}(1 + H\cos\gamma) + \int_{\partial M} k_g \, ds. \tag{3-4}$$

We now assume that ϕ is minimal. We may assume without loss of generality that B is centered at the origin. Consider then the support function $u = \langle \phi, N \rangle$ of M. It satisfies the following

$$\begin{cases} \Delta u \; + \; |\sigma|^2 u \; = \; 0 & \text{on} \;\; M \\ u \; = \; \cos\gamma & \text{on} \;\; \partial M. \end{cases}$$

Let's call M^+ (resp. M^-) the subset of M where u is positive (resp. negative) and define $u^+, u^- \in H^1(M)$ by

$$u^+(p) = \begin{cases} u(p) & \text{if} \;\; p \in M^+ \\ 0 & \text{if} \;\; p \in M \setminus M^+ \end{cases}, \qquad u^-(p) = \begin{cases} u(p) & \text{if} \;\; p \in M^- \\ 0 & \text{if} \;\; p \in M \setminus M^- \end{cases}.$$

We may suppose $\gamma \leq \pi/2$, the case $\gamma \geq \pi/2$ is similar. A direct computation gives

$$\begin{aligned} I(u^-, u^-) &= 0 \\ I(u^+, u^+) &= \int_{\partial M} k_g \, ds - \frac{1+\cos^2\gamma}{\sin\gamma} L. \end{aligned}$$

If u does not change sign in M, then as in [**R-V**] this implies that $\phi(M)$ is starshaped with respect to the center of the ball and so M has genus zero and hence (Corollary 2.3) $\phi(M)$ is totally geodesic. Now if u changes sign, we can consider the function $\tilde{u} = u^+ + au^-$, where a is the positive constant

$$a = -\frac{\int_M u^+ \, dA}{\int_M u^- \, dA}.$$

$\tilde{u}$ is then not identically zero, satisfies $\int_M \tilde{u} \, dA = 0$ and

$$\begin{aligned} I(\tilde{u}, \tilde{u}) &= I(u^+, u^+) + 2aI(u^+, u^-) + a^2 I(u^-, u^-) \\ &= \int_{\partial M} k_g \, ds - \frac{1+\cos^2\gamma}{\sin\gamma} L. \end{aligned}$$

Stability then implies that

$$\int_{\partial M} k_g \, ds \geq \frac{1+\cos^2\gamma}{\sin\gamma} L.$$

Putting this into Inequality (3-4), we get

$$8\pi\left(\left[\frac{g+1}{2}\right] - g\right) + 4\pi(4-r) > \frac{3+\cos^2\gamma}{\sin\gamma} L. \tag{3-5}$$

The only possibilities besides totally geodesics disks are thus

$$\begin{aligned} g &= 1 \quad &\text{and} \quad r &= 1, 2 \text{ or } 3 \\ g &= 2 \text{ or } 3 \quad &\text{and} \quad r &= 1. \end{aligned}$$

To finish the proof we analyze more closely the case $r = 1$, that is M having only one boundary component. In this case, if $L < 2\pi$, then $\phi(\partial M)$ is contained in some open hemisphere (for a proof, see [**Sp**, vol. 3, p. 427]). By Proposition (1.2), $\phi(M)$ must then be a totally geodesic disk. Hence if $\phi(M)$ is not a totally geodesic disk its boundary length satisfies $L \geq 2\pi$. Putting this into (3-5), we get the conclusion by direct calculation. □

If we do not assume minimality, denoting by A and L the area of M and the lenght of its boundary respectively, the same arguments give the following

Proposition 3.4. *Let $\phi : M \longrightarrow B$ be a stable capillary surface in a ball in $\mathbb{R}^3$. Assume that either $\cos^2\gamma\, L \leq 6\sin\gamma\, A$ or $1 + H\,\cos\gamma \geq 0$, then the only possibilities are*

(1) $\phi(M)$ *is a totally geodesic disk or a spherical cap*

(2) $g = 1$ *and* $r \leq 1 + \left[\frac{6-\sin\gamma}{2-|\cos\gamma|}\right] \leq 6$

(3) $g = 2$ *or* 3 *and* $r \leq 1 + \left[\frac{2-\sin\gamma}{2-|\cos\gamma|}\right] \leq 2.$

([.] *denotes the integer part.*)

Proof. Recall the Inequality (3-4)

$$8\pi\left(1 + \left[\frac{g+1}{2}\right]\right) + 4\pi(2-2g-r) > 4H^2A + \frac{2L}{\sin\gamma}(1 + H\cos\gamma) + \int_{\partial M} k_g\, ds.$$

The binomial in H, $3H^2A + 2L\cot\gamma\, H + \frac{2L}{\sin\gamma}$ is non negative provided that $\cos^2\gamma\, L \leq 6\sin\gamma\, A$. Assuming either this inequality satisfied or that $1 + H\cos\gamma \geq 0$ and using Lemma 3.2 together with the inequalities $2 - r \leq \mathcal{X}(\Omega) \leq r$, we obtain

$$4\left(2 - g + \left[\frac{g+1}{2}\right]\right) \;+\; r\,(|\cos\gamma| - 2) \;>\; \sin\gamma \;+\; |\cos\gamma|.$$

The conclusion follows. □

4. Appendix.

We give here a proof of the second variation Formula (1-3) for an admissible volume-preserving variation Φ of a capillary embedding ϕ. We keep the notations used in Section 1 and for each $t \in (-\epsilon, \epsilon)$, we use the subscript t for terms related to Φ_t, for example ds_t stands for the induced metric on ∂M by Φ_t etc..., and we shall omit this subscript for $t = 0$.

The first variation formula gives:

$$E'(t) = -n\int_M H(t)\langle Y_t, N_t\rangle\; dA_t + \int_{\partial M}\langle Y_t, \nu_t - \cos\gamma\,\bar{\nu}_t\rangle\; ds_t$$

where

$$Y_t = \frac{\partial\Phi}{\partial s}(p)\bigg|_{s=t}.$$

It follows that

$$E''(0) = -\,n\int_M H'(0)\,\langle Y, N\rangle\; dA - nH(0)\,\frac{d}{dt}\bigg|_{t=0}\left(\int_M \langle Y_t, N_t\rangle\; dA_t\right)$$

$$+\int_{\partial M}\left\langle \frac{D}{dt}\bigg|_{t=0} Y_t, \nu - \cos\gamma\,\bar\nu\right\rangle ds + \int_{\partial M}\left\langle Y, \frac{D}{dt}\bigg|_{t=0} \nu_t - \cos\gamma\,\bar\nu_t\right\rangle ds$$

$$+\int_{\partial M}\langle Y, \nu - \cos\gamma\,\bar\nu\rangle\,\frac{d}{dt}\bigg|_{t=0}(ds_t)$$

where D denotes the covariant derivative in W. Notice that

$$\frac{d}{dt}\bigg|_{t=0}\left(\int_M \langle Y_t, N_t\rangle\; dA_t\right) = V''(0) = 0$$

and $\langle Y, \nu - \cos\gamma\,\bar\nu\rangle = 0$ since Y is tangent to ∂B along ∂M and

$$\nu - \cos\gamma\,\bar\nu = \sin\gamma\,\bar N.$$

Moreover, we've the well known formula (cf. [**R**])

$$n\,H'(0) \;=\; \Delta f + \left(|\sigma|^2 + n\,\mathrm{Ric}(N)\right)\, f.$$

So to prove the formula for $E''(0)$ we need to compute

$$\left\langle \frac{D}{dt}\bigg|_{t=0} Y_t, \nu - \cos\gamma\,\bar\nu\right\rangle + \left\langle Y, \frac{D}{dt}\bigg|_{t=0} \nu_t - \cos\gamma\,\bar\nu_t\right\rangle.$$

Henceforth we shall denote simply by a "prime" the covariant derivative $\frac{D}{dt}\big|_{t=0}$.

Lemma 4.1. *Let $\widetilde\nabla$ denote the gradient on ∂M for the metric induced by ϕ and Y_0 (resp. Y_1) the tangent part of Y to M (resp. to ∂M). Let also S_0, S_1 and S_2 denote respectively the shape operator of M in W with respect to N, of ∂M in M with respect to ν and of ∂M in ∂B with respect to $\bar\nu$. Then*

(1) $N' = -\nabla f - S_0(Y_0)$

(2) $\nu' = \left(\frac{\partial f}{\partial \nu} + \sigma(Y_0,\nu)\right) N + f S_0(\nu) - f\sigma(\nu,\nu)\,\nu - S_1(Y_1) + \cot\gamma\,\widetilde\nabla f$

(3) $\bar\nu' = -II(Y,\bar\nu)\,\bar N - S_2(Y_1) + \frac{1}{\sin\gamma}\,\widetilde\nabla f.$

Proof. To prove (1), let $\{e_i\}, i = 1, \dots, n$ be an orthonormal frame of T_pM for some $p \in M$. Put $e_i(t) = d\Phi_t(e_i)$, then using the fact that N has norm one, we get

$$\langle N, D_{e_i} Y\rangle = df(e_i) + \langle N, D_{e_i} Y_0\rangle.$$

Now, since $\langle N_t, e_i(t)\rangle = 0$ and $[e_i(t), Y(t)] = 0$, we have

$$N' = \sum_{i=1}^n \langle N', e_i\rangle e_i$$

$$
\begin{aligned}
&= -\sum_{i=1}^{n}\langle N, {e_i}'\rangle e_i \\
&= -\sum_{i=1}^{n}\langle N, D_{e_i}Y\rangle e_i \\
&= -\sum_{i=1}^{n} df(e_i)e_i - \sum_{i=1}^{n}\langle N, D_{e_i}Y_0\rangle e_i \\
&= -\sum_{i=1}^{n} df(e_i)\, e_i + \sum_{i=1}^{n}\langle D_{Y_0}N, e_i\rangle e_i \\
&= -\nabla f - S_0(Y_0).
\end{aligned}
$$

As a consequence of (1) we have

$$
\langle \nu', N\rangle = -\langle \nu, N'\rangle = \frac{\partial f}{\partial \nu} + \sigma(Y_0, \nu). \tag{4-1}
$$

Let now $\{v_i\}, i = 1, \dots, n-1$ be an orthonormal frame of $T_p\partial M$ for some $p \in \partial M$. As before, put $v_i(t) = d\Phi_t(v_i)$, then one can check that along ∂M

$$
\begin{aligned}
Y &= fN + Y_1 - \cot\gamma\, f\,\nu \\
\langle \nu', v_i\rangle = -\langle \nu, {v_i}'\rangle &= -\langle \nu, D_{v_i}Y\rangle \\
&= -f\langle \nu, D_{v_i}N\rangle - \langle \nu, D_{v_i}Y_1\rangle + \cot\gamma\, df(v_i).
\end{aligned} \tag{4-2}
$$

The Formula (2) now follows from (4-1), (4-2) and the fact that $\langle \nu', \nu\rangle = 0$.

To prove (3), we notice that on ∂M, we have

$$
Y = Y_1 - \frac{1}{\sin\gamma}\, f\,\bar{\nu}. \tag{4-3}
$$

So, for $i = 1, \dots, n-1$

$$
\langle \bar{\nu}', v_i\rangle = -\langle \bar{\nu}, {v_i}'\rangle = -\langle \bar{\nu}, D_{v_i}Y\rangle = -\langle \bar{\nu}, D_{v_i}Y_1\rangle + \frac{1}{\sin\gamma}\, df(v_i).
$$

Moreover

$$
\langle \bar{\nu}', \bar{\nu}\rangle = 0 \quad \text{and} \quad \langle \bar{\nu}', \bar{N}\rangle = -\,\mathrm{II}(Y, \bar{\nu}).
$$

Thus

$$
\bar{\nu}' = -\,\mathrm{II}(Y, \bar{\nu})\,\bar{N} - \sum_{i=1}^{n-1}\langle \bar{\nu}, D_{v_i}Y_1\rangle v_i + \frac{1}{\sin\gamma}\sum_{i=1}^{n-1} df(v_i)\, v_i
$$

and this is exactly Formula (3). □

Combining Formulae (2) and (3) in Lemma 4.1, we obtain

$$\langle Y, \nu' - \cos\gamma\,\bar{\nu}'\rangle = f\,\frac{\partial f}{\partial \nu} + f^2\,\cot\gamma\,\sigma(\nu,\nu) + 2f\,\sigma(Y_0,\nu) + \sin\gamma\ \mathrm{II}(Y_1,Y_1). \tag{4-4}$$

Furthermore, the relation

$$Y_0 = Y_1 - \cot\gamma\, f\,\nu$$

implies that

$$\sigma(Y_0,\nu) = \sigma(Y_1,\nu) - f\cot\gamma\,\sigma(\nu,\nu).$$

Putting this into (4-4) we obtain

$$\langle Y, \nu' - \cos\gamma\,\bar{\nu}'\rangle = f\,\frac{\partial f}{\partial \nu} - f^2\,\cot\gamma\,\sigma(\nu,\nu) + 2f\,\sigma(Y_1,\nu) + \sin\gamma\ \mathrm{II}(Y_1,Y_1). \tag{4-5}$$

We now compute $\langle Y', \nu - \cos\gamma\,\bar{\nu}\rangle$. Since $\nu - \cos\gamma\,\bar{\nu} = \sin\gamma\bar{N}$, we have

$$\langle Y', \nu - \cos\gamma\,\bar{\nu}\rangle = -\sin\gamma\ \mathrm{II}(Y,Y). \tag{4-6}$$

Using the decomposition (4-3), we get

$$\mathrm{II}(Y,Y) = \mathrm{II}(Y_1,Y_1) - \frac{2}{\sin\gamma}\,f\ \mathrm{II}(Y_1,\bar{\nu}) + \frac{1}{\sin^2\gamma}\,f^2\ \mathrm{II}(\bar{\nu},\bar{\nu}). \tag{4-7}$$

It is straightforward to check that

$$\sigma(Y_1,\nu) + \mathrm{II}(Y_1,\bar{\nu}) = 0. \tag{4-8}$$

Finally, (4-5), (4-6), (4-7) and (4-8) give

$$\langle Y', \nu - \cos\gamma\,\bar{\nu}\rangle + \langle Y, \nu' - \cos\gamma\,\bar{\nu}'\rangle = f\frac{\partial f}{\partial \nu} - \left\{\cot\gamma\,\sigma(\nu,\nu) + \frac{1}{\sin\gamma}\ \mathrm{II}(\bar{\nu},\bar{\nu})\right\}\,f^2.$$

This completes the proof of the second variation formula.

References

[B-dC-E] J.L. Barbosa, M. do Carmo and J. Eschenburg, *Stability of hypersurfaces of constant mean curvature in Riemannian manifolds*, Math. Z., **197** (1988), 123-128.

[C] S.Y. Cheng, *Eigenfunctions and nodal sets*, Comment. Math. Helvitici, **51** (1976), 43-55.

[Ch] S.S. Chern, *On surfaces of constant mean curvature in a three dimensional space of constant curvature*, Selected papers IV, Springer Verlag, New York, 1989.

[E-I] A. El Soufi and S. Ilias, *Majoration de la seconde valeur propre d'un opérateur de Schrodinger sur une variété compacte et applications*, Jour. of Func. Anal., **103** (1992), 294-316.

[F] R. Finn, *Equilibrium capillary surfaces*, Springer-Verlag, New York, 1986.

[G-M-T] E. Gonzalez, U. Massari and I. Tamanini, *Existence and regularity problems of a pendant drop*, Pac. J. Math., **88** (1980), 399-420.

[G-H] P. Griffiths and M. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978.

[H] J. Hersch, *Quatre propriétés isopérimétriques des membranes sphériques homogènes*, C. R. Acad. Sci. Paris, **270** (1970), 1645-1648.

[L-Y] P. Li and S.T. Yau, *A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces*, Invent. Math., **69** (1982), 269-291.

[N] J.C.C. Nitsche, *Stationary partitioning of convex bodies*, Arch. Rat. Mech. An., **89** (1985), 1-19.

[R-V] A. Ros and E. Vergasta, *Stability for hypersurfaces of constant mean curvature with free boundary*, Geometriae Dedicata, **56** (1995), 19-33.

[R] H. Rosenberg, *Hypersurfaces of constant curvature in space forms*, Bull. Sc. Math. 2^{e} série, **117** (1993), 211-239.

[S] R. Souam, *On stability of stationary hypersurfaces for the partitioning problem for balls in space forms*, to appear in Math. Z.

[Sp] M. Spivak, *A comprehensive introduction to differential geometry*, Publish or Perish, Inc. Houstan, Texas, 1979.

[V 1] T.I. Vogel, *Stability of a liquid drop between two parallel planes*, SIAM J. Appl. Math., **47** (1987), 516-525.

[V 2] ———, *Stability and bifurcation of a surface of constant mean curvature in a wedge*, Indiana Univ. Math. J., **41**(**3**) (1992), 625-648.

[W 1] H.C. Wente, *The stability of the axially symmetric pendant drop*, Pac. J. Math., **88** (1980), 421-470.

[W 2] ———, *The symmetry of sessile and pendent drops*, Pac. J. Math., **88** (1980), 387-397.

Received July 25, 1995. The first author was partially supported by DGICYT grant PB94-0796 and the second author was partially supported by a GADGET grant. The second author wishes to thank the Departement of Geometry and Topology of Granada University, where part of this work was done, for its hospitality.

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