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# A GEOMETRIC CONSTRUCTION OF THE IWAHORI-HECKE ALGEBRA FOR UNRAMIFIED GROUPS

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**This paper concerns constructing the Iwahori-Hecke algebra for certain $p$-adic groups. It can be regarded as a natural extension of the ideals laid down in [KL2]; the objective (and conclusion) of that paper was the proof of Deligne-Langlands conjecture for Hecke algebras arising from split $p$-adic groups (with connected centers). The key observation of that paper was the geometric construction of the Iwahori-Hecke algebra for a split $p$-adic group.**

## 1. Introduction.

An unramified $p$-adic group is a linear algebraic group over a $p$-adic field $k$, which has a Borel subgroup defined over $k$, and which splits over an unramified extension of $k$. The collection of all such groups contains the split $p$-adic groups as a proper subset. Let $G$ be the $k$-points of a connected reductive unramified group over $k$, and let $I$ be an Iwahori subgroup. Then the Iwahori-Hecke algebra of $G$ is the convolution algebra of smooth compactly supported functions on the set of $I$-double cosets of $G$.

The purpose of this paper is to give a geometric construction of the Iwahori-Hecke algebra of an unramified $p$-adic group in terms of the equivariant $K$-theory of a certain variety in terms of the $L$-group of $G$. The basic ideas in this paper are extensions of and motivated by the path-breaking work of Kazhdan and Lusztig in the mid 1980's relating to the Iwahori-Hecke algebra of a split $p$-adic group. The importance of this work derives from the fact that in the case of split groups, Kazhdan and Lusztig the analogous construction was the first and key step in classifying the representations of the Iwahori-Hecke algebra, which in turn provides a classification of all representations of the corresponding $p$-adic group with non-trivial $I$-fixed vectors.

Let $k$ be a $p$-adic field such that its residue field $\mathbf{k}$ has $q$ elements. Let $\mathcal{G}$ be a $k$-group and let $G$ be its group of $k$-rational points. We shall assume that $\mathcal{G}$ is unramified, i.e., satisfies the following conditions:

(1) $\mathcal{G}$ splits over an unramified extension $F$ of $k$,

(2) $\mathcal{G}$ has a Borel subgroup defined over $k$.

Let $\Gamma$ be the Galois group of $F$ over $k$. Then by the above, $\Gamma$ is a cyclic group with a canonical generator $\sigma$ (the Frobenius element).

The case in which $\mathcal{G}$ is split, or equivalently when $\Gamma$ is trivial has was worked out in full in [**KL2**]. It is in this sense that this paper may be regarded as a generalization of some of the results in [**KL2**].

The approach taken here is quite similar to that in [**KL2**]. We first construct a certain canonical quotient of equivariant $K$-homology which depends on $\sigma$. It is denoted by ${}^{\sigma}K$-homology. We do this by decomposing the representation ring of ${}^{L}G \times \mathbb{C}^*$ into a direct sum of rings, one for each component of ${}^{L}G$. Then for any ${}^{L}G \times \mathbb{C}^*$-variety $X$ we have $\mathbf{K}_i^{{}^{L}G \times \mathbb{C}^*}$ ($i = 0$ or $1$) is an $\mathbf{R}_{{}^{L}G \times \mathbb{C}^*}$-module. We decompose $\mathbf{K}_i^{{}^{L}G \times \mathbb{C}^*}(X)$ according to the decomposition of $\mathbf{R}_{{}^{L}G \times \mathbb{C}^*}$ and write ${}^{\sigma}\mathbf{K}_i^{{}^{L}G \times \mathbb{C}^*}(X)$ for the component corresponding to $G\sigma$. Then ${}^{\sigma}\mathbf{K}_i^{{}^{L}G \times \mathbb{C}^*}(X)$ has the following properties:

(1) For any ${}^{L}G \times \mathbb{C}^*$-variety $X$, ${}^{\sigma}\mathbf{K}_i^{{}^{L}G \times \mathbb{C}^*}(X)$ ($i = 0$ or $1$) has all the functorial properties of $\mathbf{K}_i^{{}^{L}G \times \mathbb{C}^*}(X)$ which do not assume ${}^{L}G \times \mathbb{C}^*$ is connected.

(2) For $X$ a point ${}^{\sigma}\mathbf{K}_0^{{}^{L}G \times \mathbb{C}^*}(X)$ is isomorphic to the center of $\mathbf{H}$.

(3) If $Z$ is the variety of triples, then ${}^{\sigma}\mathbf{K}_0^{{}^{L}G \times \mathbb{C}^*}(Z)$ is isomorphic to $\mathbf{H}$ and ${}^{\sigma}\mathbf{K}_1^{{}^{L}G \times \mathbb{C}^*}(Z) = 0$.

(4) If $\sigma$ is the identity automorphism, then ${}^{\sigma}\mathbf{K}_0^{M}(X) = \mathbf{K}_0^{M}(X)$ for any $M$-variety $X$.

With this new functor we are able to generalize the Kazhdan-Lusztig construction of the Iwahori-Hecke algebra. One would like to be able to carry out the entire proof of the Deligne-Langlands conjecture, but in fact certain technical problems arise regarding ${}^{\sigma}K$-theory: in particular it is an open question whether a general Künneth formula spectral sequence exists for this functor (see [**KL2**]). Even without the Künneth spectral sequence one can prove the Deligne-Langlands conjecture using equivariant algebraic ${}^{\sigma}K$-theory and the methods outlines in [**CG**]. This will be handled in another paper.

We now give an in depth description of the contents of this paper. In Chapter 2 we give a definition of a root system in terms of the character group of an arbitrary complex algebraic torus. This formulation will be useful in later chapters. We also state the Pittie-Steinberg theorem (see [**ST2**]) in terms of this formulation. We actually only need a special case of this theorem which states that for a semisimple, simply connected complex algebraic group the representation ring of a maximal torus is a free module over the representation ring of the group.

In Chapter 3 we fix notation and collect some basic properties of $L$-groups following [**Bo**]. The $L$-group defined in this chapter would appear as the $L$-

group of an unramified $p$-adic group, but specific reference to the $p$-adic group is not necessary and so is omitted. The basic idea is to fix a complex algebraic group $G$. Let $\mathcal{S} = \{(B,T),\{X_\alpha\}\}$ be a splitting of $G$ (here $B$ is a Borel subgroup, $T$ a maximal torus of $B$ and the $X_\alpha$ are simple root vectors in the Lie algebra of $G$). Let $\sigma$ be an automorphism of $G$ preserving $\mathcal{S}$ and $\Gamma$ the group it generates. Then ${}^L G = G \rtimes \Gamma$ is the $L$-group.

In Chapter 4 we define the ${}^\sigma\mathbf{K}$-functor and fix notation. In Chapter 5 we describe the properties of equivariant ${}^\sigma\mathbf{K}$-homology which we shall need. Most of the properties are direct consequences of the analogous properties for equivariant $\mathbf{K}$-homology proved in [**KL2**]. One notable exception is the formula for the pushforward pullback of a line-bundle along an equivariant $\mathbb{P}^1$ bundle (see [**KL2**, 1.3 (o2)]). This formula is in some sense the link between $K$-homology and representations of the Hecke algebra. This is proved using the Borel-Weil-Bott theorem and the Weyl character formula. In this paper we need an analogous formula not for $\mathbb{P}^1$ bundles but for higher dimensional flag variety bundles. In order to prove this we need to give a 'twisted' version of the Weyl character formula whose proof depends on the Lefschetz fixed point theorem for elliptic complexes, or alternatively using the techniques in [**CG**]. This formula is also used to verify the Künneth formula isomorphism for the special case of the flag variety.

In Chapter 6 we give a definition of the Hecke algebra in terms of generators and relations, following Bernstein, Zelevinskii and Lusztig (see [**L4**]). These are a generalization of the relations given in [**KL2**].

In Chapter 10 we analyse the equivariant ${}^\sigma\mathbf{K}$-homology of the variety of triples and show that it gives a model for the regular representation of $\mathbf{H}$. This chapter bears a strong resemblance to [**KL2**, Sec. 3], and has been published with the permission of Kazhdan and Lusztig. The reason for this is that the set up in [**KL2**] is in some sense not the most general possible. In fact if we take $\sigma$ to be trivial in the above then our construction specializes to the construction in [**KL2**].

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## 2. Root Systems.

**2.1. Algebraic Tori.** Let $T$ be an algebraic torus over $\mathbb{C}$ and $X^*(T) = \mathrm{Hom}(T,\mathbb{C}^*)$, the group of *algebraic* homomorphisms from $T$ to $\mathbb{C}^*$. Set $V = X^*(T)\otimes\mathbb{Q}$ and identify $X^*(T)$ with $X^*(T)\otimes 1$. Given $\Sigma\subset X^*(T)$, we say that $\Sigma$ is a root system with respect to $T$ if $\Sigma$ is an abstract root system in the vector space $V$.

Suppose that $W$ is a finite group which acts on $T$. Then $W$ acts on $X^*(T)$ by $w \cdot \alpha(t) = \alpha(w^{-1}t)$ $(\alpha \in X^*(T), w \in W)$ and hence on $V$. If the $W$-action on $V$ coincides with the action of the Weyl group of the abstract root system $\Sigma$ on $V$, then we say that $W$ is the Weyl group of $\Sigma$ with respect to $T$.

Let $(\ ,\ )$ be a $W$-invariant positive definite symmetric bilinear form on $V$.

Let

$$L^*(\Sigma) = \left\{ \lambda \in V | \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \quad \text{for all} \ \ \alpha \in \Sigma \right\}.$$

Say that $\Sigma$ is *simply connected* if $L^*(\Sigma) = X^*(T)$. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a basis for the root system $\Sigma$.

**2.2. The Pittie-Steinberg Theorem.** Write $\lambda_i \in V$ for the *i-th* fundamental dominant weight of $\Sigma$ with respect to $\Pi$: $\lambda_i$ is defined to be such that

$$\frac{2(\alpha_j, \lambda_i)}{(\alpha_j, \alpha_j)} = \delta_{ij}.$$

We know that the set $\{\lambda_1, \dots, \lambda_n\}$ is a basis for $V$ and generates the lattice $L^*(\Sigma)$.

**Proposition 2.1.** *Let $\Sigma, T, V, \{\lambda_i\}_{1 \le i \le n}$ be as above. The following are equivalent.*

(1) *$\Sigma$ is simply connected,*

(2) *$\lambda_i \in X^*(T)$ for all $i$.*

*Proof.* Follows immediately from the definitions. □

The following theorem is proved in [**ST2**], and is a finer result of a theorem by Pittie.

**Theorem 2.2.** *Let $\Sigma, \Pi, T, X^*(T), W$ be as above and assume that $\Sigma$ is simply connected with respect to $T$. Let $\mathbf{X} = X^*(T)$ and write $\mathbb{C}[\mathbf{X}]$ for the group ring of $\mathbf{X}$ over $\mathbb{C}$, and $\mathbb{C}[\mathbf{X}]^W$ for the $W$ invariant elements therein. Then*

(1) *The natural inclusion of $\mathbb{C}[\mathbf{X}]^W$ into $\mathbb{C}[\mathbf{X}]$ makes $\mathbb{C}[\mathbf{X}]$ into a free $\mathbb{C}[\mathbf{X}]^W$-module of rank $|W|$.*

(2) *There are elements $e_v$ one for each $v \in W$, such that the $e_v$ $(v \in W)$ form a basis of $\mathbb{C}[\mathbf{X}]$ as a $\mathbb{C}[\mathbf{X}]^W$ module, and $det(u(e_v))_{(u,v) \in W \times W} = \Delta^{|W|/2}$, where $\Delta = \Pi(\alpha^{1/2} - \alpha^{-1/2}) \in \mathbb{C}[\mathbf{X}]$ and the product is taken over all positive roots.*

## 3. Preliminaries on the group ${}^LG$.

**3.1. The Definition of ${}^LG$.** Fix $G$ a connected, reductive linear algebraic group over $\mathbb{C}$ with simply connected derived group. Let $\mathcal{B}$ denote the variety of all Borel subgroups of $G$. We equip $G$ with the following data:

(1) a *pair* $(B, T)$, where $B$ is a Borel subgroup and $T$ is a maximal torus in $B$,

(2) a *splitting* $(B, T, \{X_\alpha\})$, where $(B, T)$ is the above pair and $\{X_\alpha\}$ is a collection of root vectors, one for each simple root $\alpha$ of $T$ with respect to $B$,

(3) an automorphism $\sigma$ of $G$ which preserves the splitting $(B, T, \{X_\alpha\})$.

Write $\Gamma$ for the (necessarily finite) group generated by $\sigma$, and let ${}^LG = G \rtimes \Gamma$. Also write ${}^LT$ for $T \rtimes \Gamma$ and ${}^LB$ for $B \rtimes \Gamma$. The subrgoups $B$ and $T$ will remain fixed for the rest of this paper.

**3.2. Orbits of $G$ on $\mathcal{B}\times\mathcal{B}$.** The orbits of $G$ on $\mathcal{B}\times\mathcal{B}$ are naturally in $1-1$ correspondence with the elements of $W$. A map

$$W \to \{G \text{ orbits in } \mathcal{B}\times\mathcal{B}\}$$

is obtained as follows. Let $w \in W$, and let $n \in N_G(T)$ be a representative of $w$. Map $w$ to the orbit of $(B, nBn^{-1})$. It is clear that this map is independent of the representative of $w$ chosen. As in [**KL2**] we write $B' \xrightarrow{w} B''$ whenever the orbit of $(B', B'')$ corresponds to $w$ ander the above bijection.

Let $\leq$ be the Bruhat order on $W$. It is well known that $w \leq w'$ if and only if the closure of the $G$-orbit on $\mathcal{B}\times\mathcal{B}$ corresponding to $w$ is contained in the closure of the $G$-orbit corresponding to $w'$ (this may also be taken as the defintion of the Bruhat order).

We write $B' \xrightarrow{\geq w} B''$ to mean: $B' \xrightarrow{w'} B''$ for some $w' \geq w$, and similarly we write $B' \xrightarrow{\leq w} B''$ to mean $B \xrightarrow{w'} B'$ for some $w' \leq w$ (see 6.4 for an alternative description of $B' \xrightarrow{\leq w} B''$).

**3.3. The $\sigma$-Action on $\mathcal{B}\times\mathcal{B}$.** It is clear that $\sigma$ acts on the $G$ orbits in $\mathcal{B}\times\mathcal{B}$. In this way we get a natural action of $\sigma$ on $W$ and we write $W^\Gamma$ for the group of fixed points of $W$ under this action.

If we let $(B, T)$ be as in 3.1 and let $\tilde{W} = N_G(T)/T$ then $\tilde{W} \cong W$ and $\sigma$ acts on $\tilde{W}$. This action coincides with the action of $\sigma$ on $W$ given in 3.3. The verification of this fact is trivial and we leave it to the reader.

**3.4. A Connectedness Theorem.** We know that $\sigma$ is a semisimple automorphism of $G$ and therefore by [**ST1**, Sec. 8] that the group $G^\sigma$ (resp. $T^\sigma$) the group of fixed points of $\sigma$ on $G$ (resp. $T$) is connected.

**3.5. The Root Systems.** Let $X^*(T) = \mathrm{Hom}(T, \mathbb{C}^*)$ where the homomorphisms are taken to be algebraic. Similarly let $X_*(T) = \mathrm{Hom}(\mathbb{C}^*, T)$. Write $\Sigma$ for the root system of $G$ with respect to $T$: we have $\Sigma \subset X^*(T)$. Write $\Sigma^\vee$ for the coroots of $G$ with respect to $T$: $\Sigma^\vee \subset X_*(T)$.

Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ (resp. $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$) be the basis of $\Sigma$ (resp. $\Sigma^\vee$) defined by $B$, and let $S$ be the corresponding set of simple reflections in $W$. Write $\Sigma^+$ for the positive system of roots determined by $\Pi$.

We see that $\Sigma$ is a root system with respect to $T$ (see 2.1) with Weyl groups $W$.

**3.6. A Form.** Set $V = X^*(T) \otimes \mathbb{Q}$ and let $(\ ,\ )$ be a positive definite, symmetric bilinear form on $V$ invariant under both $W$ and $\Gamma$ (such a form exists because the subgroup of $Aut(V)$ generated by $W$ and $\Gamma$ is finite).

**3.7. The $\sigma$-Action on $\Sigma$.** The natural action of $\sigma$ on $T$ (resp. $B$) induces an action of $\sigma$ on $\Sigma$ (resp. $\Sigma^+$ and $\Pi$). We denote this action by $\sigma : \alpha \mapsto \sigma(\alpha)$ for $\alpha \in \Sigma$. We identify $\sigma$ with an element of $S_n$ the symmetric group on $n$ letters in such a way that $\sigma(\alpha_i) = \alpha_{\sigma(i)}$.

**3.8. Type I Roots.** Call a simple root $\alpha \in \Pi$ type I if the roots in the $\sigma$-orbit of $\alpha$ are mutually perpendicular with respect to $(\ ,\ )$, see 3.6.

**3.9. Type II Roots.** Call a simple root $\alpha \in \Pi$ type II if each component $D$ in the Dynkin diagram of the $\sigma$-orbit of $\alpha$ is of type $A_2$ and for some positive integer $a$, $\sigma^a$ preserves and acts nontrivially on $D$. In this case the cardinality of the $\sigma$-orbit of $\alpha$ is $2a$ and $\alpha + \sigma^a\alpha, \sigma\alpha + \sigma^{a+1}\alpha, \ldots, \sigma^{a-1}\alpha + \sigma^{2a-1}\alpha$ are mutually perpendicular roots with respect to $(\ ,\ )$.

**3.10. Type III Roots.** Call a root $\alpha \in \Sigma$ type III if $\alpha = \beta + \sigma^a\beta$ with $\beta$ of type II and $a$ as in 3.9.

**3.11. The Norm of a Root.** Let $l_\alpha$ be the cardinality of the $\sigma$-orbit of $\alpha \in \Sigma$ and set

$$N\alpha = \sum_{i=0}^{l_\alpha - 1} \sigma^i \alpha,$$

and call this the norm of $\alpha$. For $\alpha \in \Pi$ set

$$N'\alpha = \begin{cases} N\alpha & \text{if } \alpha \text{ is of type I,} \\ 2N\alpha & \text{if } \alpha \text{ is of type II.} \end{cases} \tag{3.11a}$$

**3.12. The Root Systems $\Sigma_1$ and $\Sigma_2$.** We consider the following two sets:

$$\Pi_1 = \{N'\alpha | \alpha \in \Pi\}, \quad \text{and}$$
$$\Pi_2 = \{N\alpha | \alpha \in \Pi\}.$$

Let $\pi_1$ (resp. $\pi_2$) be the map $\Pi \to \Pi_1$ (resp. $\Pi \to \Pi_2$) given by $\alpha \to N'\alpha$ (resp. $\alpha \to N\alpha$). Call a root $\beta \in \Pi_1$ (resp. $\beta' \in \Pi_2$) type I or II according to whether it is the image of a root of type I or II under the map $\pi_1$ (resp. $\pi_2$). Set

$$\Sigma_1 = W^\Gamma(\Pi_1) = \{w\alpha \,| w \in W^\Gamma, \alpha \in \Pi_1\}, \tag{3.12b}$$

$$\Sigma_2 = W^\Gamma(\Pi_2) = \{w\alpha \,| w \in W^\Gamma, \alpha \in \Pi_2\}. \tag{3.12c}$$

Then $\Sigma_1$ and $\Sigma_2$ are root systems which are subsets of $X^*(T)^\Gamma$, the fixed points of $\sigma$ in $X^*(T)$; they have bases $\Pi_1$ and $\Pi_2$ respectively.

**3.13. Type I and II Roots in $\Sigma_1$.** Call a root $\beta \in \Sigma_1$ (resp. $\beta' \in \Sigma_2$) type I or II according to whether it is the image under $W^\Gamma$ of a root $\alpha \in \Pi_1$ (resp. $\alpha' \in \Pi_2$) of type I or II respectively.

**3.14. Positive Roots in $\Sigma_1$ and $\Sigma_2$.** Write $\Sigma_1^+$ (resp. $\Sigma_2^+$) for the positive system of roots in $\Sigma_1$ (resp. $\Sigma_2$) with respect to $\Pi_1$ (resp. $\Pi_2$).

Clearly we have that $\Pi_1$ (resp. $\Pi_2$) is a basis for $\Sigma_1$ (resp. $\Sigma_2$).

**3.15. $\Sigma_1$ and $\Sigma_2$ are Reduced.** It is obvious from the above observations that $\Sigma_1$ and $\Sigma_2$ are reduced root systems.

**3.16. The Root System $\Phi$.** The union $\Phi$ of $\Sigma_1$ and $\Sigma_2$ is a (possibly non-reduced) root system. If it is non-reduced, then $\Sigma_1$ is the root system obtained by taking the non-multipliable roots of $\Phi$ while $\Sigma_2$ is the root system obtained by taking the indivisible roots of $\Phi$.

The above facts (3.12, 3.11, 3.14, 3.15, and 3.16) can be deduced with very simple modifications from the results in [**ST1**].

**3.17. The Torus $T_\Gamma$.** The map $A : T \to T, x \to x\sigma(x^{-1})$ is an endomorphism of $T$. Let

$$U = \text{im } A,$$

and write $T_\Gamma$ for $T/U$. Note that im $A$ is connected and $T_\Gamma$ is a complex algebraic torus, called the *coinvariants* of $T$ with respect to $\Gamma$.

Since elements of $X^*(T)^\Gamma$ are trivial on $U$ there is a natural map $X^*(T)^\Gamma \to X^*(T_\Gamma)$ and this map is an isomorphism. By the definition of $\Sigma_1$ and $\Sigma_2$ we know that any $\alpha \in \Sigma_i$ ($i = 1$ or $2$) is $\sigma$-invariant and hence can be identified with an element of $X^*(T_\Gamma)$. From now on we will identify $\Sigma_1$ and $\Sigma_2$ with their images in $X^*(T_\Gamma)$.

**3.18. The $W^\Gamma$-Action on $T_\Gamma$.** It is easy to see that the action of $W^\Gamma$ on $T$ stabilizes $U$ (see 3.17) and therefore $W^\Gamma$ acts on $T_\Gamma$. From the induced action of $W^\Gamma$ on $T_\Gamma$ we obtain an action of $W^\Gamma$ on $X^*(T_\Gamma)$, and an easy calculation shows that $W^\Gamma$ is the Weyl group of the abstract root systems $\Sigma_1$ and $\Sigma_2$ (see 2.1).

**3.19. Type I and II Fundamental Dominant Weights.** Let $\Pi^* = \{\lambda_1, \dots, \lambda_n\}$ be the fundamental dominant weights (see 2.2) of $\Sigma$ with respect to $\Pi$. We will call $\lambda_i$ type I or type II according to whether the corresponding root $\alpha_i$ is of type I or type II respectively. Since $\sigma$ permutes $\Pi$ and $(\ ,\ )$ is invariant under $\sigma$ we have that the fundamental dominant weights are also permuted by $\sigma$. In fact, it is easy to see that $\sigma(\lambda_i) = \lambda_{\sigma(i)}$ (see 3.7).

**3.20. The Simple Reflections $S_1$.** Write $S_1$ for the set of simple reflections in $W^\Gamma$ with respect to the basis $\Pi_1$. Note that as a subset of $W^\Gamma$ this is the same set as if we took the simple reflections corresponding to the set $\Pi_2$. Call $r \in S_1$ type I or type II according to whether $r$ corresponds to a simple root of type I or II (see 3.13). The set $S_1$ plays an important role in what follows.

**3.21. The Vector Space $D$.** Write $D = X^*(T_\Gamma) \otimes \mathbb{Q}$. The surjection $T \to T_\Gamma$ induces an injection $X^*(T_\Gamma) \to X^*(T)$ which identifies $D$ as a subspace of $V$ (see 3.6). Restricting $(\ ,\ )$ from $V$ to $D$ we obtain a $W^\Gamma$ invariant form on $D$ compatible with the form on $V$.

For convenience order the elements of $\Pi$ (and correspondingly the elements of $\Pi^*$, see 3.20) so that $\{\alpha_1, \dots, \alpha_m\}$ $(m \le n)$ are representatives of the different $\sigma$-orbits in $\Pi$. We continue to identify $\sigma$ with an element of $S_n$ in such a way that $\sigma(\alpha_i) = \alpha_{\sigma(i)}$. Set $l_{\lambda_i}$ equal to the cardinality of the $\sigma$-orbit of $\lambda_i$. Write

$$N'\lambda_i = \sum_{k=0}^{l_{\lambda_i}-1} \sigma^k \lambda_i, \tag{3.21d}$$

viewing each $N'\lambda_i \in X^*(T)^\Gamma$ as an element of $X^*(T_\Gamma) \otimes \mathbb{Q}$ under the isomorphism in 3.17. Let

$$\Pi_1^* = \{N'\lambda_1, \dots, N'\lambda_m\}.$$

**Proposition 3.1.** *With respect to the basis $\Pi_1$ the set $\Pi_1^*$ is the set of fundamental dominant weights for the root system $\Sigma_1$.*

*Proof.* We must show (see 2.2) that for all $1 \le i, j \le m$

$$\frac{2(N'\lambda_i, N'\alpha_j)}{(N'\alpha_j, N'\alpha_j)} = \delta_{ij}.$$

If $1 \leq i \neq j \leq m$ then for any positive integers $a, b$, $(\sigma^a \lambda_i, \sigma^b \alpha_j) = 0$. It is therefore sufficient to only consider the case $i = j$.

Write $\alpha = \alpha_i, \lambda = \lambda_i$ and $l = l_\alpha$. Suppose $\alpha$ is of type I. We have

$$\begin{aligned}\frac{2(N'\alpha, N'\lambda)}{(N'\alpha, N'\alpha)} &= \frac{2(N\alpha, N'\lambda)}{(N\alpha, N\alpha)} \\ &= \frac{2\left(\sum_{a=0}^{l-1} \sigma^a \alpha, \sum_{b=0}^{l-1} \sigma^b \lambda\right)}{\left(\sum_{c=0}^{l-1} \sigma^c \alpha, \sum_{d=0}^{l-1} \sigma^d \alpha\right)} \\ &= \frac{2l(\alpha, \lambda)}{l(\alpha, \alpha)} \\ &= 1.\end{aligned}$$

The third equality follows from 3.8 and the $\sigma$-invariance of the form.

Now suppose that $\alpha$ is of type II. In this case $l = 2a$ for some positive integer $a$ (see 3.9). We have

$$\begin{aligned}\frac{2(N'\alpha, N'\lambda)}{(N'\alpha, N'\alpha)} &= \frac{2(2N\alpha, N'\lambda)}{(2N\alpha, 2N\alpha)} \\ &= \frac{4\left(\sum_{b=0}^{l-1} \sigma^b \alpha, \sum_{c=0}^{l-1} \sigma^c \lambda\right)}{4\left(\sum_{d=0}^{l-1} \sigma^d \alpha, \sum_{e=0}^{l-1} \sigma^e \alpha\right)} \\ &= \frac{l(\alpha, \lambda)}{l(\alpha, \alpha) + l(\alpha, \sigma^a \alpha)} \\ &= 1.\end{aligned}$$

The second equality follows from 3.9 and the $\sigma$-invariance of the form. The last equality follows from 3.9 and explicitly evaluating the second to last expression. $\square$

Now the following follows from 2.1:

**Proposition 3.2.** *The root system $\Sigma_1$ is simply connected with respect to $T_\Gamma$ (see Section 2).*

We have the following corollary.

**Corollary 3.3.** *$\Sigma_1 \subset X^*(T_\Gamma)$ satisfies the hypotheses of Steinberg's theorem 2.2. Setting $\mathbf{X} = X^*(T_\Gamma)$ we have:*

(1) *$\mathbb{C}[\mathbf{X}]$ is a free $\mathbb{C}[\mathbf{X}]^{W^\Gamma}$-module of rank $|W^\Gamma|$.*

(2) *There exist elements $e_v \in \mathbb{C}[\mathbf{X}]$ one for each $v \in W^\Gamma$, such that the $e_v$ form a basis for $\mathbb{C}[\mathbf{X}]$ as a $\mathbb{C}[\mathbf{X}]^{W^\Gamma}$-module, and*

$$\det(u(e_v))_{(u,v) \in W^\Gamma \times W^\Gamma} = \Delta^{|W^\Gamma|/2},$$

$$\textit{where } \Delta = \Pi(\alpha^{1/2} - \alpha^{-1/2}) \in \mathbb{C}[\mathbf{X}] \ (\textit{product over all } \alpha \in \Sigma_1^+).$$

## 4. The ${}^\sigma$K-functor.

This section assumes familiarity with equivariant **K**-homology as in [**KL2**]. Unless otherwise stated $M$ will denote an arbitrary complex algebraic group.

**4.1. Equivariant K-homology.** Let $X$ be an algebraic variety over $\mathbb{C}$ and $M$ a linear algebraic group acting algebraically on $X$. Then we have the groups $\mathbf{K}_i^M(X)$ $(i = 0, 1)$ as in [**KL2**]; these are homology groups with *complex* coefficients.

Let $R_M$ be the Grothendieck group of finite dimensional rational representations of the algebraic group $M$, and set $\mathbf{R}_M = R_M \otimes \mathbb{C}$. This is a commutative $\mathbb{C}$-algebra of finite type (see [**KL2**, Sec. 1.3]).

Recall the following basic facts from [**KL2**]:

**4.2. Characters.** By taking characters of representations we define an isomorphism of $\mathbf{R}_M$ with the $\mathbb{C}$-algebra of all regular functions $M \to \mathbb{C}$ which are constant on each coset of the unipotent radical of $M$ and are invariant under inner automorphism by $M$.

**4.3. The Maximal Ideals of $\mathbf{R}_M$.** The maximal ideals of $\mathbf{R}_M$ are in $1-1$ correspondence with conjugacy classes of semisimple elements in $M$; if $s$ is a semisimple element then the corresponding maximal ideal is denoted $\mathbf{I}_s$ and is identified with ideal of functions in $\mathbf{R}_M$ vanishing on $s$.

**4.4. The Finiteness of $\mathbf{K}_i^M(X)$.** For any $M$-variety $X$, $\mathbf{K}_i^M(X)$ is an $\mathbf{R}_M$-module of finite type ($i = 0$ or $1$).

**4.5. The Frobenius Component of ${}^LG$.** Let ${}^LG$ be as in 3.1. As an algebraic variety ${}^LG$ is the disjoint union of the connected varieties $G, G\sigma, \dots, G\sigma^{|\Gamma|-1}$. The action of $\sigma$ on ${}^LG$ by inner automorphism preserves each component $G\sigma^n$. We will be particularly interested in $G\sigma$, which for the purposes of this paper will be called the *Frobenius component* of ${}^LG$.

**4.6. Definition of ${}^\sigma\mathbf{R}_{{}^LG}$.** Consider the ideal ${}^\sigma\mathbf{R}_{{}^LG} \subset \mathbf{R}_{{}^LG}$

$${}^\sigma\mathbf{R}_{{}^LG} = \{f \in \mathbf{R}_{{}^LG} |\, f \text{ is supported on } G\sigma\}.$$

Let $e \in \mathbf{R}_{{}^LG}$ be the characteristic function of the Frobenius component of ${}^LG$; $e$ is an idempotent element of $\mathbf{R}_{{}^LG}$. Multiplication by $e$ is naturally a projection $\mathbf{R}_{{}^LG} \to {}^\sigma\mathbf{R}_{{}^LG}$, and ${}^\sigma\mathbf{R}_{{}^LG}$ is a ring with identity element $e$.

**4.7. Modules over ${}^\sigma\mathbf{R}_{{}^LG}$.** Given any $\mathbf{R}_{{}^LG}$-module $A$, let ${}^\sigma A$ denote the ${}^\sigma\mathbf{R}_{{}^LG}$-module $eA$ obtained by projection. In this way we have an exact functor $A \to {}^\sigma A$ taking $\mathbf{R}_{{}^LG}$-modules and $\mathbf{R}_{{}^LG}$-morphisms to ${}^\sigma\mathbf{R}_{{}^LG}$-modules and ${}^\sigma\mathbf{R}_{{}^LG}$-morphisms. For $x \in A$ we will write ${}^\sigma x$ for the image of $x$ in ${}^\sigma A$.

In particular for any ${}^LG$-variety $X$ we obtain the finite ${}^\sigma\mathbf{R}_{{}^LG}$-modules ${}^\sigma\mathbf{K}_i^{{}^LG}(X)$ $(i = 0$ or $1)$.

**4.8. Admissible Subgroups of ${}^L G$.** Let $M$ be a subgroup of ${}^L G$. We say that $M$ is *admissible* if $M$ is closed and its intersection with the Frobenius component of ${}^L G$ (see 4.5) is non-trivial. In this case we will call the intersection of $M$ with the Frobenius component of ${}^L G$ the Frobenius component of $M$.

By restriction of functions the group $\mathbf{R}_M$ is naturally an $\mathbf{R}_{{}^L G}$-module so that we may form the ${}^\sigma\mathbf{R}_{{}^L G}$-module ${}^\sigma\mathbf{R}_M$. On the other hand, let $e_M \in \mathbf{R}_M$ be the characteristic function of the Frobenius component of $M$. Form the ideal $e_M\mathbf{R}_M$ of ${}^\sigma\mathbf{R}_M$ as in 4.6. It is easy to see that the map $e_M\mathbf{R}_M \to {}^\sigma\mathbf{R}_M$ given by $e_M f \to ef$ ($f \in \mathbf{R}_M$) is well defined and an isomorphism.

In particular we have the rings ${}^\sigma\mathbf{R}_{{}^L B}$ and ${}^\sigma\mathbf{R}_{{}^L T}$ (see 3.1). Call an element of ${}^\sigma\mathbf{R}_{{}^L T}$ a $\sigma$-*character* if it is the image under the map $\mathbf{R}_{{}^L T} \to {}^\sigma\mathbf{R}_{{}^L T}$ of a character.

Call an element $E \in {}^\sigma\mathbf{R}_M$ a $\sigma$-*representation* if it is the image of a representation under the natural map $\mathbf{R}_M \to {}^\sigma\mathbf{R}_M$. Observe that calling an element of ${}^\sigma\mathbf{R}_M$ a $\sigma$-character or $\sigma$-representation can naturally lead to muddy thinking.

We note that if $M$ is not admissible then clearly ${}^\sigma\mathbf{R}_M = 0$ and ${}^\sigma\mathbf{K}_i^M = 0$ ($i = 0$ or 1).

**4.9. Equivariant $K$-Theory and $\sigma$-Bundles.** Let $M$ be an admissible subgroup of ${}^L G$. For an $M$-variety $X$ let $\mathbf{K}_M^i(X)$ ($i = 0$ or 1) be the equivariant $K$-cohomology of $X$ with *complex coefficients* as defined in [**Se**]. We know that these are $\mathbf{R}_M$-modules and so we may construct the ${}^\sigma\mathbf{R}_M$-modules ${}^\sigma\mathbf{K}_M^i(X)$ ($i = 0$ or 1) as in 4.7.

Let $E$ be an element of ${}^\sigma\mathbf{K}_0^M(X)$ for an $M$-variety $X$. Then we say that $E$ is a $\sigma$-vector bundle if $E$ is the image of a vector bundle under the canonical map $\mathbf{K}_M^0(X) \to {}^\sigma\mathbf{K}_M^0(X)$. In particular we say that $E$ is a $\sigma$-*line bundle* if it is the image of a line bundle. Note that a $\sigma$-line bundle may also be the image of a vector bundle of higher rank.

## 5. Basic Properties of ${}^\sigma$K-theory.

Let $G$ and ${}^L G$ be as in 3.1. Then let $\sigma$ act on $G \times \mathbb{C}^*$ with $\sigma$ acting trivially on $\mathbb{C}^*$. It is obvious that ${}^L G \times \mathbb{C}^* = {}^L(G \times \mathbb{C}^*)$ so that all the constructions in chapter 4 carry over replacing ${}^L G$ by ${}^L G \times \mathbb{C}^*$.

In this chapter unless otherwise specified $M$ will be an admissible subgroup of ${}^L G \times \mathbb{C}^*$ (see 4.8). We present a list of properties of ${}^\sigma\mathbf{R}_M$ and ${}^\sigma\mathbf{K}_i^M(X)$ ($X$ an $M$-variety) which we will need in this paper. Unless otherwise specified properties 5.1 through 5.19 follow immediately from the analogous properties for $\mathbf{K}_i^M$ ($i = 0$ or 1) given in [**KL2**] by the exactness of the functor $\mathbf{K}_i^M(X) \to {}^\sigma\mathbf{K}_i^M(X)$ ($i = 0$ or 1). The properties beyond 5.19 are analogous to properties discussed in [**KL2**] but require different proofs.

**5.1. Maximal Ideals of ${}^{\sigma}\mathbf{R}_M$.** The maximal ideals of ${}^{\sigma}\mathbf{R}_M$ are in 1-1 correspondence with the conjugacy classes of semisimple elements in the Frobenius component of $M$ (see 4.3). Given a semisimple element $s \in$ (Frobenius component of $M$) we write $\mathbf{I}_s$ for the maximal ideal in ${}^{\sigma}\mathbf{R}_M$ corresponding to $s$. For $s \in$ (Frobenius component of M), $\mathbf{I}_s$ is the set of all $\phi \in {}^{\sigma}\mathbf{R}_M$ such that $\phi(s) = 0$ (note that this makes sense because $\phi \in {}^{\sigma}\mathbf{R}_M$ is a function supported on the Frobenius component of $M$).

**5.2. Pullback.** If $f : X \to X'$ is an $M$-equivariant smooth morphism of $M$-varieties then there exists a natural ${}^{\sigma}\mathbf{R}_M$-morphism $f^* : {}^{\sigma}\mathbf{K}_i^M(X') \to {}^{\sigma}\mathbf{K}_i^M(X)$.

If we also have $g : X' \to X''$ is an $M$-equivariant smooth morphism of $M$-varieties, then $f^* \circ g^* = (g \circ f)^*$.

**5.3. Pushforward.** If $f : X \to X'$ is an $M$-equivariant proper morphism of $M$-varieties, then there is a natural ${}^{\sigma}\mathbf{R}_M$-homomorphism $f_* : {}^{\sigma}\mathbf{K}_i^M(X) \to {}^{\sigma}\mathbf{K}_i^M(X')$.

If we also have g : $X' \to X''$ is an $M$-equivariant proper morphism of $M$-varieties, then $g_* \circ f_* = (g \circ f)_*$.

**5.4. External Tensor Product.** Let $X, X'$ be two $M$-varieties, and let $X \times X'$ be equipped with the diagonal action of $M$. There is a natural ${}^{\sigma}\mathbf{R}_M$-homomorphism

$$\boxtimes : {}^{\sigma}\mathbf{K}_0^M(X) \underset{{}^{\sigma}\mathbf{R}_M}{\otimes} {}^{\sigma}\mathbf{K}_i^M(X') \to {}^{\sigma}\mathbf{K}_i^M(X \times X').$$

**5.5. Unipotent Bundles.** Let $B'$ be a Borel subgroup of an algebraic group and $T'$ a maximal torus in $B'$. Thomason (see [**T1**]) has proved that if $f : X \to Y$ is an $M$-equivariant morphism of algebraic varieties which is a locally trivial $B'/T'$-bundle (called a *unipotent bundle*) then

$$f^* : {}^{\sigma}\mathbf{K}_i^M(Y) \to {}^{\sigma}\mathbf{K}_i^M(X) \quad \text{is an isomorphism.}$$

**5.6. Tensor Product with a Complex.** Let $X$ be an $M$-variety and let $\mathcal{E} = (\ldots 0 \to E_n \to \ldots \to E_0 \to 0 \ldots)$ be an $M$-equivariant complex of algebraic vector bundles on $X$ (see 4.9) : each $E_i$ is an $M$-equivariant algebraic vector bundle and each map is an $M$-equivariant morphism of vector bundles. Let $X_0$ be a closed $M$-stable subvariety of $X$ such that $\mathcal{E}$ is exact off of $X_0$. Then there is a natural ${}^{\sigma}\mathbf{R}_M$-homomorphism

$$\mathcal{E}\otimes : {}^{\sigma}\mathbf{K}_0^M(X) \to {}^{\sigma}\mathbf{K}_0^M(X_0).$$

This is compatible with the operations 5.2, 5.3, and 5.4 as follows. Let $f : X' \to X$ be an $M$-equivariant morphism of $M$-varieties, let $\mathcal{E}'$ be the pullback of $\mathcal{E}$ under $f$ and let $X'_0 = f^{-1}(X_0)$.

We have $\mathcal{E}'\otimes : {}^{\sigma}\mathbf{K}_0^M(X') \to {}^{\sigma}\mathbf{K}_0^M(X'_0)$. Let $f_0 : X'_0 \to X_0$ be the restriction of $f$. Then

**5.7.** If $f$ is proper then $(\mathcal{E}\otimes)f_* = (f_0)_*(\mathcal{E}'\otimes) : {}^\sigma\mathbf{K}_0^M(X') \to {}^\sigma\mathbf{K}_0^M(X_0)$.
**5.8.** If $f$ is smooth then $f_0^*(\mathcal{E}\otimes) = (\mathcal{E}'\otimes)f^* : {}^\sigma\mathbf{K}_0^M(X) \to {}^\sigma\mathbf{K}_0^M(X_0')$.
**5.9.** If $\mathcal{E} = (\dots 0 \to E \to 0 \dots)$, then we may take $X = X_0$ and we write $E \otimes \xi$ instead of $\mathcal{E} \otimes \xi$ for all $\xi \in {}^\sigma\mathbf{K}_0^M(X)$. Thus $E \otimes \xi \in {}^\sigma\mathbf{K}_0^M(X)$.
**5.10.** If $V$ is a rational finite dimensional $M$-module and $\mathbf{V}$ is the corresponding $M$-equivariant vector bundle over $X$ (see [**Se**]) then identify $\mathbf{V}$ with its image in ${}^\sigma\mathbf{K}_M^i(X)$. We have $\mathbf{V} \otimes \xi = V\xi$ $(\xi \in {}^\sigma\mathbf{K}_0^M(X))$ where $V\xi$ is given by the ${}^\sigma\mathbf{R}_M$-module structure of ${}^\sigma\mathbf{K}_0^M(X)$, see 4.8.
**5.11.** If $\mathcal{E} = (\dots 0 \to E_n \to \dots \to E_0 \to 0 \dots)$ is as above and $X = X_0$, then $\mathcal{E} \otimes \xi = \sum_i (-1)^{i\sigma} E_i \otimes \xi$, $(\xi \in {}^\sigma\mathbf{K}_0^M(X))$. Here ${}^\sigma E_i$ is as in 4.7.
**5.12.** Let $\tilde{\mathcal{E}}$ be the pullback of $\mathcal{E}$ under $pr_1 : X \times X' \to X$. This gives rise to

$$\tilde{\mathcal{E}}\otimes : {}^\sigma\mathbf{K}_0^M(X \times X') \to {}^\sigma\mathbf{K}_0^M(X_0 \times X').$$

If $\xi \in {}^\sigma\mathbf{K}_0^M(X), \xi' \in {}^\sigma\mathbf{K}_0^M(X')$, then

$$\tilde{\mathcal{E}} \otimes (\xi \boxtimes \xi') = (\mathcal{E} \otimes \xi) \boxtimes \xi' \in {}^\sigma\mathbf{K}_0^M(X_0 \times X'),$$

and analogous result holds with respect to the other variable.
**5.13. Thom Isomorphism.** Let $\pi : E \to X$ be an $M$-equivariant algebraic vector bundle over $X$ and let

$$\mathcal{E} = (\dots 0 \to \ \wedge^n \tilde{\pi}E^* \to \dots \to \wedge^0 \tilde{\pi}E^* \to 0 \dots)$$

be the usual Koszul complex of vector bundles over $E$ (see [**Se**, Sec. 3]). Here $\tilde{\pi}E^*$ denotes the pull-back of the dual of $E$ under $\pi$. This is acyclic outside the zero section $j : X \hookrightarrow E$ of $\pi$ (which is identified with $X$). Hence we have $\mathcal{E}\otimes : {}^\sigma\mathbf{K}_0^M(E) \to {}^\sigma\mathbf{K}_0^M(X)$.

Then

$$\mathcal{E} \otimes \text{ is the inverse of } \pi^* : {}^\sigma\mathbf{K}_0^M(X) \xrightarrow{\sim} {}^\sigma\mathbf{K}_0^M(E). \tag{5.13e}$$

We have

$$(\pi^*)^{-1} j_*\xi = \sum_i (-1)^i \wedge^i E^* \otimes \xi, \qquad (\xi \in {}^\sigma\mathbf{K}_0^M(X)). \tag{5.13f}$$

**5.14. Base Change.** Consider the following fiber square:

$$\begin{tikzcd} X_4 \arrow[r, "p_2"] \arrow[d, "p_1"] & X_2 \arrow[d, "f_2"] \\ X_1 \arrow[r, "f_1"] & X_3 \end{tikzcd} \tag{5.14g}$$

where $f_1$ and $p_2$ are proper and $p_1$ and $f_2$ are smooth, and $X_4 \cong X_1 \times_{X_3} X_4$ by definition. Then as ${}^\sigma\mathbf{R}_M$-module homomorphisms ${}^\sigma\mathbf{K}^M(X_1) \to {}^\sigma\mathbf{K}^M(X_2)$ we have

$$(f_2)^*(f_1)_* = (p_2)_*(p_1)^*.$$

**5.15. Localization.** We need the following version of the localization theorem which is more general than what is stated in [**KL2**]. We will give a short proof of the result.

Let $M_0$ be any maximal compact subgroup of $M$ and let $s \in M$ be a semisimple element in the Frobenius component of $M$ (see 4.8) and $\mathbf{I}_s$ as in 5.1. For an $M$-variety $X$ write ${}^\sigma\mathbf{K}_i^M(X)_{\mathbf{I}_s}$ for the localization of the ${}^\sigma\mathbf{R}_M$-module ${}^\sigma\mathbf{K}_i^M(X)$ at $\mathbf{I}_s$.

Write $X^{(s)}$ for the $M_0$-saturation of the set of fixed points of $s$ on $X$. In other words

$$X^{(s)} = M_0 \cdot X^s, \tag{5.15h}$$

where $X^s$ is the fixed points of $s$ on $X$. Then $X^{(s)}$ is a closed subset of $X$ (closed in the complex topology) and we write $j : X^{(s)} \hookrightarrow X$ for the natural inclusion. Let $Y$ be any closed $M$-invariant subvariety of $X$ which contains $X^{(s)}$; note that $Y$ is automatically $M_0$-invariant. Let $i : Y \hookrightarrow X$ be the natural embedding. Then we claim that the ${}^\sigma\mathbf{R}_M$-homomorphism $i_* : {}^\sigma\mathbf{K}_i^M(Y) \to {}^\sigma\mathbf{K}_i^M(X)$ induces an isomorphism on the localizations of ${}^\sigma\mathbf{K}_i^M(X)$ and ${}^\sigma\mathbf{K}_i^M(Y)$ at the maximal ideal $\mathbf{I}_s$ (see 5.1), i.e.,

$$i_* : {}^\sigma\mathbf{K}_i^M(Y)_{\mathbf{I}_s} \xrightarrow{\sim} {}^\sigma\mathbf{K}_i^M(X)_{\mathbf{I}_s}.$$

We sketch a proof of this fact. First note that by definition $Y^{(s)} = X^{(s)}$.

Recall that by definition [**KL2**, 1.4] ${}^\sigma\mathbf{K}_i^M(X) = {}^\sigma\mathbf{K}_i^{M_0}(X)$. It follows from [**Se**, Proposition 4.1] that

$$i_* : {}^\sigma\mathbf{K}_i^{M_0}(X^{(s)})_{\mathbf{I}_s} \xrightarrow{\sim} {}^\sigma\mathbf{K}_i^{M_0}(X)_{\mathbf{I}_s}; \tag{5.15i}$$

$$j_* : {}^\sigma\mathbf{K}_i^{M_0}(X^{(s)})_{\mathbf{I}_s} = {}^\sigma\mathbf{K}_i^{M_0}(Y^{(s)})_{\mathbf{I}_s} \xrightarrow{\sim} {}^\sigma\mathbf{K}_i^{M_0}(Y)_{\mathbf{I}_s}. \tag{5.15j}$$

We have the following commutative diagram of ${}^\sigma\mathbf{R}_M$-modules.

$$\begin{tikzcd}
{}^\sigma\mathbf{K}_i^{M_0}\left(Y^{(s)}\right)_{\mathbf{I}_s} \arrow[rr, equal] \arrow[drrr, "\sim"'] & & {}^\sigma\mathbf{K}_i^{M_0}\left(X^{(s)}\right)_{\mathbf{I}_s} \arrow[rr, dash, "\sim"] \arrow[dr] & & {}^\sigma\mathbf{K}_i^{M_0}(X)_{\mathbf{I}_s} \\
 & & & {}^\sigma\mathbf{K}_i^{M_0}(Y)_{\mathbf{I}_s} \arrow[ur] &
\end{tikzcd}$$

where each map is the map induced by push-forward from the obvious inclusion and the isomorphisms follow from 5.15i and 5.15j. The desired result is now immediate.

**5.16. Exact Sequences.** Let $X$ be an $M$-variety, and $F$ be a closed $M$-stable subvariety of $X$. Let $j : F \hookrightarrow X$, $j' : X - F \hookrightarrow X$ be the inclusions. Then there is a natural exact hexagon of ${}^\sigma\mathbf{R}_M$-modules:

$$\begin{tikzcd}
 & {}^{\sigma}\mathbf{K}_0^M(X) \arrow[dr, "j'^*"] & \\
{}^{\sigma}\mathbf{K}_0^M(F) \arrow[ur, "j_*"] & & {}^{\sigma}\mathbf{K}_0^M(X-F) \arrow[d] \\
{}^{\sigma}\mathbf{K}_1^M(X-F) \arrow[u] & & {}^{\sigma}\mathbf{K}_1^M(F) \arrow[dl, "j_*"] \\
 & {}^{\sigma}\mathbf{K}_1^M(X) \arrow[ul, "j'^*"] &
\end{tikzcd}$$

Note that this is a 'periodic' version of the standard homology long exact sequence.

**5.17. Poincaré Duality.** Suppose $X$ is a compact, smooth $M$-variety. Let ${}^{\sigma}\mathbf{K}_M^i(X)$ ($i = 0$ or $1$) be as in 4.9. There is a natural isomorphism

$$ {}^{\sigma}\mathbf{K}_i^M(X) \cong {}^{\sigma}\mathbf{K}_M^i(X). \tag{5.17k}$$

Let $X'$ be a second compact, smooth $M$-variety. Let $f : X \to X'$ be an $M$-equivariant morphism. Then $f_* : {}^{\sigma}\mathbf{K}_i^M(X) \to {}^{\sigma}\mathbf{K}_i^M(X')$ (see 5.3) corresponds under 5.17k to the pushforward ${}^{\sigma}\mathbf{K}_M^i(X) \to {}^{\sigma}\mathbf{K}_M^i(X')$ obtained from the standard pushforward in equivariant $\mathbf{K}$-theory (see, e.g. [**KL1**]). If in addition $f$ is smooth, then $f^* : {}^{\sigma}\mathbf{K}_i^M(X') \to {}^{\sigma}\mathbf{K}_i^M(X)$ corresponds under 5.17k to the pullback map ${}^{\sigma}\mathbf{K}_M^i(X') \to {}^{\sigma}\mathbf{K}_M^i(X)$ obtained from the standard pullback map in $\mathbf{K}$-theory.

**5.18. The Image of a Vector Bundle in ${}^{\sigma}K$-Theory.** Let $E$ be an $M$-equivariant vector bundle over an $M$-variety $X$. We write ${}^{\sigma}E$ for the image of $E$ in ${}^{\sigma}\mathbf{K}_M^0(X)$ (see also 4.7).

If $X$ is a compact, smooth $M$-variety then under the isomorphism 5.17k we will identify ${}^{\sigma}E$ with its image in ${}^{\sigma}\mathbf{K}_0^M(X)$.

**5.19. Homogeneous Spaces.** Given $M$ an admissible subgroup of ${}^{L}G$ (see 4.8) and $H$ a closed subgroup of $M$, there are natural isomorphisms

(5.19l)
$$ {}^{\sigma}\mathbf{K}_0^M(M/H) \cong {}^{\sigma}\mathbf{R}_H \text{ and } {}^{\sigma}\mathbf{K}_1^M(M/H) \cong 0, \text{ where } {}^{\sigma}\mathbf{R}_H \text{ is as in 4.8.}$$

If $H$ does not intersect the Frobenius component of $M$ (see 4.5) then it is clear that

$$ {}^{\sigma}\mathbf{K}_0^M(M/H) = {}^{\sigma}\mathbf{R}_H = 0. \tag{5.19m}$$

**5.20. Trivial $\mathbb{C}^*$-Actions.** Let $M = {}^L G \times \mathbb{C}^*$ and let $X$ be a compact, smooth $M$-variety on which $\mathbb{C}^*$ acts trivially. $\mathbf{K}^0_M(X) \cong \mathbb{C}[q,q^{-1}] \otimes \mathbf{K}^0_{{}^L G}(X)$ where $q$ is an indeterminant (see [**Se**, Prop. 2.2]). Now under the isomorphism 5.17k we have

$$\mathbf{K}_0^M(X) \cong \mathbb{C}[q,q^{-1}] \otimes \mathbf{K}_0^{{}^L G}(X), \tag{5.20n}$$

and therefore it is immediate that

$${}^\sigma\mathbf{K}_0^M(X) \cong \mathbb{C}[q,q^{-1}] \otimes {}^\sigma\mathbf{K}_0^{{}^L G}(X). \tag{5.20o}$$

**5.21. Twisted Euler Characteristic.** Let $X$ be a projective $M$-variety where $M$ is an admissible subgroup of ${}^L G \times \mathbb{C}^*$ and $\mathcal{O}_X$ its sheaf of regular functions. Let $\mathfrak{F}$ be an $M$-equivariant coherent sheaf of $\mathcal{O}_X$-modules on $X$ (see [**BBM**]). Write $H^i(X,\mathfrak{F})$ for its $i$-th cohomology group. Note that $H^i(X,\mathfrak{F})$ is a finite dimensional $M$-module (since $X$ is projective), and hence an element of $\mathbf{R}_M$. Set

$$\mathcal{X}_\sigma(\mathfrak{F}) = \sum (-1)^i \text{ trace}\left\{\sigma : H^i(X,\mathfrak{F}) \to H^i(X,\mathfrak{F})\right\}.$$

We call $\mathcal{X}_\sigma(\mathfrak{F})$ the twisted Euler characteristic of $\mathfrak{F}$ with respect to $\sigma$, or, when no confusion is likely to arise, the twisted Euler characteristic of $\mathfrak{F}$.

Let $L$ be a $\sigma$-line bundle (see 4.9) on $X$. By the twisted Euler characteristic of $L$ we will mean the twisted Euler characteristic of any line bundle $\tilde{L}$, regarded as a coherent sheaf, whose image under the canonical map $\mathbf{K}^0_M(X) \to {}^\sigma\mathbf{K}^0_M(X)$ (see 4.7) is $L$.

We claim that the twisted Euler characteristic is independent of the bundle $\tilde{L}$ chosen. Let $\pi : X \to pt$ be the natural $M$-equivariant projection from $X$ to a point. Suppose that $\tilde{L}_1$ and $\tilde{L}_2$ have the same images in ${}^\sigma\mathbf{K}^0_M(X)$. Then it suffices to show that images of $\sum(-1)^i H^i(X,\tilde{L}_1)$ and $\sum(-1)^i H^i(X,\tilde{L}_2)$ are equal in ${}^\sigma\mathbf{R}_M$. Since $\pi_*$ is an $\mathbf{R}_M$-module homomorphism it induces a ${}^\sigma\mathbf{R}_M$-module homomorphism $\pi_* : {}^\sigma\mathbf{K}^0_M(X) \to {}^\sigma\mathbf{R}_M$. Therefore $\pi_*(\tilde{L}_1) = \pi_*(\tilde{L}_2)$ in ${}^\sigma\mathbf{R}_M$, so that

$$\sum(-1)^i\, {}^\sigma H^i(X,\tilde{L}_1) = \sum(-1)^i\, {}^\sigma H^i(X,\tilde{L}_2).$$

The result now follows.

**5.22. Twisted Weyl Character Formula.** In this section we will use *exponential* notation to represent elements of $X^*(T)$, e.g., if $\alpha,\beta \in X^*(T)$ then we write $e^\alpha$, $e^\beta$ for $\alpha,\beta$ and $e^\alpha e^\beta = e^{\alpha+\beta}$. Let $\Sigma$, $\Sigma_1$, $\Pi$, $\Pi_1$, $\Pi^*$, $\Pi_1^*$, $T$ be as in Section 3.

Let $\omega$ be a $\sigma$-invariant, dominant weight in $X^*(T)$ (i.e., a non-negative integer linear combination of elements of $\Pi^*$ fixed by $\sigma$). Let $\rho : {}^L G \to GL(\mathcal{V})$ be the finite dimensional, irreducible representation of highest weight $\omega$. We assume that $\rho$ is normalized so that $\sigma$ is trivial on a highest weight vector. Let $f_\rho(t) : T \to \mathbb{C}$ be defined by

$$f_\rho(t) = \mathrm{trace}(\rho(t\sigma)).$$

We have $f_\rho = \sum c_\lambda e^\lambda$ is a finite linear combination of characters $\lambda \in X^*(T)$. A simple calculation shows that if $c_\lambda \neq 0$, then $\lambda$ is trivial on the kernel of the map $T \to T_\Gamma$ (see 6.10 for a proof) and hence can be regarded as an element of $X^*(T_\Gamma)$.

We have the following:

**Proposition 5.1.** *If we regard $f_\rho$ as an element of $\mathbb{C}[X^*(T_\Gamma)]$ then*

$$f_\rho = \sum_{w \in W^\Gamma} \frac{\epsilon_w e^{w(\rho+\delta)}}{\prod\limits_{\alpha \in \Sigma_1^+} (e^{\alpha/2} - e^{-\alpha/2})},$$

*$\epsilon_w = det(w)$ where $w$ is identified with an automorphism of the vector space $D = X^*(T_\Gamma) \otimes \mathbb{Q}$ (this is just the sign of $w \in W^\Gamma$); $\delta = \frac{1}{2}\sum_{\alpha \in \Sigma_1^+} \alpha$ (recall that $\Sigma_1^+$ is the set of positive roots in $\Sigma_1$ with respect to $\Pi_1$).*

*Proof.* This is a generalization of the arguments in [**AB**]. □

## 6. The Hecke Algebra.

In this section $M = {}^L G \times \mathbb{C}^*$, $\mathbf{X} = X^*(T_\Gamma)$ and $(B, T)$ are as in 3.1. Recall that $G = {}^L G^\circ$.

**6.1. The Flag Variety.** Let $\mathcal{B}$ be the flag variety of ${}^L G$ (see 3.1). As a set this is the collection of all Borel subgroups of ${}^L G$, which can be identified with the set of all Borel subgroups of ${}^L G^\circ = G$. It is well known that $\mathcal{B}$ is a complete variety and is isomorphic to $G/B$ (see 3.1). The action ${}^L G$ on $\mathcal{B}$ by conjugation makes $\mathcal{B}$ into a transitive ${}^L G$ variety. We regard $\mathcal{B}$ as an $M$-variety by letting $\mathbb{C}^*$ act trivially.

**6.2. Standard Parabolic Subgroups.** Let $\Pi$ and $\Pi_1$ be our bases of $\Sigma$ and $\Sigma_1$ respectively (see 3.5 and 3.12). Each element $\alpha \in \Pi_1$ or $\alpha \in \Pi_2$ corresponds to a $\sigma$-orbit $\omega_\alpha$ in $\Pi$. It is well known that the parabolic subgroups of $G$ containing $B$ are naturally in bijection with the subsets of $\Pi$. Let $P_\alpha$ be the unique parabolic subgroup of $G$ containing $B$ which corresponds to $\omega_\alpha$.

If $r = r_\alpha \in S_1$ (see 3.20) is the simple reflection in $W^\Gamma$ corresponding to the simple root $\alpha \in \Pi_1$, then we will write $P_r$ for $P_\alpha$ when it is convenient. In this paper $P_r$ is called the standard parabolic of type $r$. An arbitrary parabolic subgroup of ${}^L\!G$ is said to be of type $r$ if it is an ${}^L\!G$ conjugate of $P_r$.

It is clear that for $r \in S_1$ we have $P_r$ is stable under $\sigma$. Let $\mathcal{P}_r$ be the variety of all parabolic subgroups of type $r$. This is a complete ${}^L\!G$-variety isomorphic to $G/P_r$.

**6.3. The Map $\pi_r$.** If $\alpha \in \Pi_1$ is a simple root and $r = r_\alpha$ then there is a natural map

$$\pi_r : \mathcal{B} \to \mathcal{P}_r,$$

which is a locally trivial (in the Zariski topology) $P_r/B$ fibration ([**BoT**]). The map is obtained by sending a Borel subgroup to the unique parabolic subgroup of type $r$ containing it.

**6.4. The Bruhat Ordering.** Let $r \in S_1$ and regard it as an element of $W$. We may now give an alternative description of $B' \overset{\leq r}{\to} B''$ $(B', B'' \in \mathcal{B})$:

$$B' \overset{\leq r}{\to} B'' \quad \text{if and only if} \quad \pi_r(B') = \pi_r(B''). \tag{6.4a}$$

**6.5. The Bundles $T_r$ and $T_r{}'$.** Let $T_r$ be the tangent bundle along the fibers of $\pi_r$, and let $T_r'$ be the corresponding cotangent bundle. They are naturally ${}^L\!G$-equivariant vector bundles over $\mathcal{B}$, and we regard them as $M$-equivariant vector bundles with the trivial $\mathbb{C}^*$ action.

**6.6. The Characters $\Psi_B^L$.** Let $L$ be an ${}^L\!G$-equivariant line bundle over $\mathcal{B}$, and let $L_B$ be the fiber of $L$ over $B$. Since $B \rtimes \Gamma$ normalizes $B$ it maps $L_B$ to itself yielding a character of $B \rtimes \Gamma$ and hence of $T \rtimes \Gamma$. Write $\Psi_B^L$ for the character of $T \rtimes \Gamma$ obtained in this way.

**6.7. $\sigma$-Trivial Line Bundles.** Let $L$ be an ${}^L\!G$-equivariant line bundle over $\mathcal{B}$ (see 4.9). Then we will call $L$ *$\sigma$-trivial* if the following condition is satisfied: $\Psi_B^L(\sigma) = 1$ (see 6.6). Equivalently $L$ is $\sigma$-trivial if the action of $\sigma$ on the fiber of $L$ over $B$ is trivial.

We call the image of a $\sigma$-trivial line bundle into ${}^\sigma\mathbf{K}$-theory a *$\sigma$-trivial*, *$\sigma$-line* bundle.

**6.8. The Associated $\sigma$-Trivial Line Bundle.** We will need the following trivial but important fact. Given a line bundle over $\mathcal{B}$ there is a unique character $\chi$ of $\Gamma$ such that $L \otimes \chi$ is $\sigma$-trivial. Here $\chi$ is regarded as the trivial line bundle $\mathbb{C}$ equipped with the ${}^L\!G$ action $g\sigma : \mathbb{C} \to \mathbb{C}$, $z \to \chi(\sigma)z$.

In this case, we will call $L \otimes \chi$ the $\sigma$-trivial line bundle *associated with* $L$.

**6.9. The Character q.** Let $\mathbf{q} : M \to \mathbb{C}^*$ be the second projection. We will identify $\mathbf{q}$ with an $M$-equivariant $\sigma$-trivial, $\sigma$-line bundle over $\mathcal{B}$ as follows. Identify $\mathbf{q}$ with the trivial line bundle $\mathbb{C}$ over $\mathcal{B}$ equipped with the action of $M$ via multiplication by the character $\mathbf{q}$. Identify this with its image in ${}^\sigma\mathbf{K}_0^M(\mathcal{B}) \cong {}^\sigma\mathbf{K}_M^0(\mathcal{B})$ under the isomorphism 5.17k.

**6.10. An Isomorphism.** The group $\mathbb{C}[\mathbf{X}]$ may be identified with ${}^\sigma\mathbf{R}_{{}^LT}$ as follows. An element $\lambda \in X^*(T_\Gamma)$ is an algebraic function $f : T \to \mathbb{C}$ such that $f(yx\sigma(x^{-1})) = f(y)$ for all $x, y \in T$ (see 3.17). Let $f' : T\sigma \to \mathbb{C}$ be defined by $f'(x\sigma) = f(x)$ for $x \in T$. Then

$$\begin{aligned} f'(yx\sigma y^{-1}) &= f'(xy\sigma(y^{-1})\sigma) \\ &= f(xy\sigma(y^{-1})) \\ &= f(x) \\ &= f'(x\sigma), \end{aligned} \tag{6.10b}$$

and $f' \in {}^\sigma\mathbf{R}_{{}^LT}$. Conversely let $f \in {}^\sigma\mathbf{R}_{{}^LT}$. Then $f : T\sigma \to \mathbb{C}$, and is invariant under inner-automorphism. Define $f' : T \to \mathbb{C}$ by $f'(t) = f(t\sigma)$, $(t \in T)$. It is a finite linear combination $f' = \sum c_\lambda \lambda$ of characters $\lambda \in X^*(T)$.

It is easy to see that $f'(s^{-1}\,\sigma(s)\,t) = f'(t)$ for all $s, t \in T$. Now

$$f'(t) = \sum c_\lambda \lambda(t),$$

but

$$f'(t) = f'(s^{-1}\sigma(s)t) = \sum c_\lambda \lambda(s^{-1}\sigma(s)t),$$

so

$$\sum c_\lambda(\lambda(t) - \lambda(s^{-1}\sigma(s)t)) = 0,$$

and finally

$$\sum c_\lambda \lambda(t)(1 - \lambda(s^{-1}\sigma(s))) = 0.$$

By linear independence of characters it follows that if $c_\lambda \neq 0$, then $\lambda(s^{-1}\sigma(s)) = 1$. Therefore $\lambda \in \mathbf{X} = X^*(T_\Gamma)$ and $f \in {}^\sigma\mathbf{R}_{{}^LT}$.

We leave it to the reader to check that the above correspondance is a bijection.

**6.11. Another Isomorphism.** An immediate consequence of this bijection is that $\mathbb{C}[\mathbf{X}]$ may be identified with ${}^\sigma\mathbf{K}_0^{{}^LG}(\mathcal{B})$, and $\mathbb{C}[q, q^{-1}][\mathbf{X}]$ ($q$ an indeterminant) may be identified with ${}^\sigma\mathbf{K}_0^M(\mathcal{B})$ (see 5.20).

**6.12. The $\sigma$-Line Bundle $L_r$.** Under the above correspondence an element $p \in X^*(T_\Gamma)$ is associated with a unique $\sigma$-trivial $\sigma$-line bundle $L_p$: indeed $p$ corresponds to an element $\xi_p \in {}^\sigma\mathbf{R}_{{}^LT}$ such that $\xi_p(\sigma) = 1$. Therefore under the isomorphism 5.19l $\xi_p$ corresponds to an ${}^LG$-equivariant $\sigma$-trivial, $\sigma$-line bundle. Under this correspondence we will identify $\mathbf{X}$ with the set of all ${}^LG$-equivariant $\sigma$-trivial $\sigma$-line bundles over $\mathcal{B}$.

Let $r \in S_1$ be the simple reflection about the simple root $\alpha_r \in \Pi_2$, and write $L_r$ for the unique $\sigma$-trivial, $\sigma$-line bundle corresponding to $\alpha_r$. Recall that $T_r$ is the tangent bundle along the fibers of $\pi_r$. We may explicitly describe $L_r$ as follows.

If $r$ is of type I then $L_r$ is the unique $\sigma$-trivial $\sigma$-line bundle associated to the highest exterior power of $T_r$ (see 6.8).

If $r$ is of type II then $L_r$ is the unique $\sigma$-trivial $\sigma$-line bundle such that $L_r^2$ is the unique $\sigma$-trivial $\sigma$-line bundle associated to the highest exterior power of $T_r$.

**6.13. The $W^\Gamma$-Action on X.** By transfering the action of $W^\Gamma$ on $X^*(T_\Gamma)$ to the set of ${}^LG$-equivariant $\sigma$-trivial $\sigma$-line bundles on $\mathcal{B}$ (denoted from now on by $\mathbf{X}$) we obtain a $W^\Gamma$- action on $\mathbf{X}$ written $w : L \mapsto {}^wL$ $(w \in W^\Gamma, L \in \mathbf{X})$. Explicitly if $r \in S_1$ and $L \in \mathbf{X}$ is an ${}^LG$-equivariant $\sigma$-trivial $\sigma$-line bundle then

$$ {}^rL = \begin{cases} LL_r^{-d+1} & \text{if } r \text{ is of type I,} \\ LL_r^{-2d+2} & \text{if } r \text{ is of type II,} \end{cases} \tag{6.13c} $$

where $L_r$ is as in 6.12 and $d$ is the twisted Euler characteristic of $L$ restricted to any fiber of $\pi_r$ (see 5.21). We note that this is well defined because the set of type-$r$ parabolics fixed by $\sigma$ is a single orbit under $G^\sigma = \{g \in G | \sigma(g) = g\}$. This will be proved in chapter 8 (Corollary 8.2) and the proof is independent of the intervening chapters.

We leave the verification of (6.13c) to the reader.

**6.14. A Formula for $(\pi^*)(\pi_r)_*L$.** Let $L$ be an $M$-equivariant $\sigma$-line bundle over $\mathcal{B}$. Recall the map $\pi_r : \mathcal{B} \to \mathcal{P}_r$ $(r \in S_1)$ (see 6.3). Then with the identifications 6.10 and 6.12 we have

$$ \pi_r^*(\pi_r)_*(L) = \begin{cases} L\dfrac{(L_r)^{-d} - 1}{(L_r)^{-1} - 1} & \text{if } r \text{ is of type I,} \\ L\dfrac{(L_r)^{-2d} - 1}{(L_r)^{-2} - 1} & \text{if } r \text{ is of type II,} \end{cases} \tag{6.14d} $$

where $d$ is the twisted Euler characteristic of $L$ restricted to any fiber of $\pi_r$ and $L_r$ is as in 6.12. To prove (6.14d) we will reduce to the case $\pi : P_r/B \to$

*pt* where $P_r$ is as in 6.2: to obtain the reduction we apply the following theorem.

**Theorem 6.1.** (**Grauert's theorem** [**Ha**]). *Let $f : X \to Y$ be a proper morphism of complex varieties. Let $\mathcal{F}$ be a coherent algebraic sheaf on $X$, flat over $Y$ (i.e., for each $x \in X$, $\mathcal{F}_x$ is a $\mathcal{O}_{y,f(x)}$-flat module). For $y \in Y$ let $\mathcal{F}_y = \mathcal{F} \otimes_{\mathcal{O}_Y} \mathbb{C}_y$ (i.e., $\mathcal{F}_y$ is the restriction of $\mathcal{F}$ to $X_y$), and $X_y = f^{-1}(y)$. Then for all integers $p$, the following are equivalent.*

(1) *The map*

$$y \mapsto \dim_{\mathbb{C}} H^p(X_y, \mathcal{F}_y)$$

*is constant.*

(2) *The higher derived functor $R^pF_*(\mathcal{F})$ is a locally free sheaf $\mathcal{E}$ on $Y$, and for all $y \in Y$ the natural map*

$$\mathcal{E} \otimes_{\mathcal{O}_Y} \mathbb{C}_y \to H^p(X_y, \mathcal{F}_y)$$

*is an isomorphism. (Note that here $\mathcal{E} \otimes_{\mathcal{O}_y} \mathbb{C}_y$ is just the restriction of $\mathcal{E}$ to the point $y$.)*

First of all, the map $\pi_r : \mathcal{B} \to \mathcal{P}_r$ is flat, i.e., the structure sheaf $\mathcal{O}_{\mathcal{B}}$ is flat over $\mathcal{P}_r$. It is flat on a dense open set by the theorem of generic flatness, and because everything is ${}^LG$-equivariant and homogenous, the map is flat on all of $\mathcal{B}$. Therefore the vector bundle $L$, being *locally* free, is flat over $\mathcal{P}_r$ as well. Hence we may apply Grauert's theorem (6.1). It is clear that condition (1) (of (6.1)) is satisfied, and therefore for all integers $p$ and all $y \in \mathcal{P}_r$ we have the natural isomorphism

$$R^p(\pi_r)_*(L)|_y \cong H^p(X_y, L_y).$$

Now we may apply the Borel-Weil-Bott theorem (see [**Bt**]) and 5.17k along with the twisted Weyl character formula (to the root system $\Sigma_1$) 5.22 and the Atiyah-Hirzebruch form of the Riemann-Roch theorem (see [**KL1**, Sec. 1.7]) to obtain our result.

**6.15. Exterior Powers of $\mathbf{q}T_r{}'$.** Fix $r \in S_1$. If $r$ is of type I let $\nu$ be the rank of $T_r'$ (see 3.9). If $r$ is of type II then the rank of $T_r'$ is divisible by 3 (see 3.9), and we let $3\nu$ be the rank of $T_r'$. For any integer $i$ we recall that ${}^{\sigma}\!\wedge^i \mathbf{q}T_r'$ denotes the image of $\wedge^i \mathbf{q}T_r'$ in ${}^{\sigma}\mathbf{K}_0^M(\mathcal{B})$ (see 5.18).

Let $\mathbb{C}$ denote the trivial $\sigma$-line bundle over $\mathcal{B}$ and let $\mathbf{q}$ be as in 6.9. Then the following facts follow from simple calculations which we will leave to the reader.

If $r$ is of type I,

$$\text{(6.15e)} \qquad {}^{\sigma}\!\wedge^i(\mathbf{q}T'_r) = \begin{cases} \mathbb{C} & \text{if } i = 0, \\ 0 & \text{if } 1 \le i < \nu, \\ (-1)^{\nu+1}\mathbf{q}^{\nu}L_r^{-1} & \text{if } i = \nu. \end{cases}$$

If $r$ is of type II,

$$\text{(6.15f)} \qquad {}^{\sigma}\!\wedge^i(\mathbf{q}T'_r) = \begin{cases} \mathbb{C} & \text{if } i = 0, \\ (-1)^{\nu}\mathbf{q}^{\nu}L_r^{-1} & \text{if } i = \nu, \\ (-1)^{2\nu+1}\mathbf{q}^{2\nu}L_r^{-1} & \text{if } i = 2\nu, \\ (-1)^{3\nu}\mathbf{q}^{3\nu+1}L_r^{-2} & \text{if } i = 3\nu, \\ 0 & \text{otherwise.} \end{cases}$$

This proves

(6.15g)

$$\sum(-1)^i \, {}^{\sigma}\!\wedge^i(\mathbf{q}T'_r) = \begin{cases} \mathbb{C} - \mathbf{q}^{\nu}L_r^{-1} & \text{if } r \text{ is of type I,} \\ \mathbb{C} - (\mathbf{q}^{2\nu} - \mathbf{q}^{\nu})L_r^{-1} - \mathbf{q}^{3\nu}L_r^{-2} & \text{if } r \text{ is of type II.} \end{cases}$$

**6.16. A Parameter System on $\Sigma_2$.** Let $\mathbf{X} = X^*(T_\Gamma)$ (= set of ${}^LG$-equivariant $\sigma$-trivial $\sigma$-line bundles over $\mathcal{B}$). Let $\Sigma_1$, $\Sigma_2$, $\Pi_1$, $\Pi_2$, $S_1$, $W^\Gamma$ be as in Section 3. Let $\tilde{W} = W^\Gamma \ltimes \mathbf{X}$ be the semidirect product, $W^\Gamma$ acting on $\mathbf{X}$ as in 6.13. Then $\tilde{W}$ contains the affine Weyl group as a subgroup of finite index.

Let

$$\lambda : \Sigma_2^+ \to \mathbb{N}$$

be defined as follows. For $\alpha \in \Pi_2$ (see 3.12) write $\eta(\alpha)$ for the number of connected components in the Dynkin diagram of $\omega_\alpha$ (see 6.2). Let

$$\text{(6.16h)} \qquad \lambda(\alpha) = \begin{cases} \eta(\alpha) & \text{if } \alpha \text{ is of type I,} \\ 3\eta(\alpha) & \text{if } \alpha \text{ is of type II.} \end{cases}$$

In addition set $\lambda^*(\alpha) = \eta(\alpha)$ if $\alpha$ is type II. (see 3.11). This is called a parameter system on $\Sigma_2$ in the language of [**L4**].

We note that if we form the Dynkin diagram of $\Sigma_2$ from the basis $\Pi_2$ and label each vertex with the corresponding $\lambda(\alpha)$ then we obtain the diagrams found in [**Ti**] which correspond to unramified groups. Note also that if $\Gamma$ is trivial then $\lambda$ is identically 1, the case studied in [**KL2**].

We will also regard $\lambda$ as a map from $S_1 \to \mathbb{N}$ as follows. If $r \in S_1$ is the reflection about the simple root $\alpha \in \Pi_1$ then define $\lambda(r) = \lambda(\alpha)$; also define $\lambda^*(r) = \lambda^*(\alpha)$ when applicable.

We see easily that if $\nu$ is as in 6.15 then

$$\eta(r) = \nu. \tag{6.16i}$$

In addition if $r$ is of type II then

$$\frac{\lambda(r) - \lambda^*(r)}{2} = \nu, \tag{6.16j}$$

$$\frac{\lambda(r) + \lambda^*(r)}{2} = 2\nu, \tag{6.16k}$$

and

$$\lambda(r) = 3\nu. \tag{6.16l}$$

Now we have

(6.16m)

$$\begin{aligned} &\sum (-1)^i \ {}^{\sigma}\!\wedge^i (\mathbf{q} T'_r) \\ &= \begin{cases} \mathbb{C} - \mathbf{q}^{\lambda(r)} L_r^{-1} & \text{if } r \text{ is of type I,} \\ \mathbb{C} - \left( \mathbf{q}^{\frac{\lambda(r)+\lambda^*(r)}{2}} - \mathbf{q}^{\frac{\lambda(r)-\lambda^*(r)}{2}} \right) L_r^{-1} - \mathbf{q}^{\lambda(r)} L_r^{-2} & \text{if } r \text{ is of type II.} \end{cases} \end{aligned}$$

**6.17. The Bernstein-Lusztig Relations for H.** Let $\mathcal{A} = \mathbb{C}[q, q^{-1}]$, $q$ an indeterminate. According to Bernstein and Lusztig (see [**L4**, Sec. 3]) one can describe the Hecke algebra $\mathbf{H}$ corresponding to $\tilde{W}$ as follows. It is an algebra over $\mathcal{A}$ with generators

$$T_r \, (r \in S_1), \quad \text{and} \quad \theta_L \, (L \in \mathbf{X}),$$

subject to the following relations:

$$T_r^2 = q^{\lambda(r)} + (q^{\lambda(r)} - 1) T_r, \tag{6.17n}$$

(6.17o)

$$T_r T_{r'} T_r \ldots = T_{r'} T_r T_{r'} \ldots (\mu \text{ factors in both products}), \forall r \neq r' \in S_1,$$

where in 6.17o $\mu$ is the order of $rr'$ in $W^\Gamma$.

(6.17p)

$$\begin{aligned} &\theta_L T_r - T_r \theta_{{}^r L} \\ &= \begin{cases} \left(q^{\lambda(r)} - 1\right) \frac{\theta_L - \theta_{{}^r L}}{1 - \theta_{L_r^{-1}}} & \text{if } r \text{ is of type I,} \\ \left( (q^{\lambda(r)} - 1) + \theta_{L_r^{-1}} \left( q^{\frac{\lambda(r)+\lambda^*(r)}{2}} - q^{\frac{\lambda(r)-\lambda^*(r)}{2}} \right) \right) \frac{\theta_L - \theta_{{}^r L}}{1 - \theta_{L_r}^{-2}} & \text{if } r \text{ is of type II.} \end{cases} \end{aligned}$$

We now present a list of properties of the Hecke algebra which we will need.

**6.18. Basis I.** The elements $T_w\theta_L$ ($w \in W^\Gamma$ , $L \in \mathbf{X}$) form an $\mathcal{A}$-basis for $\mathbf{H}$.

**6.19. Basis II.** The elements $\theta_L T_w$ ($w \in W^\Gamma$ , $L \in \mathbf{X}$) form an $\mathcal{A}$-basis for $\mathbf{H}$.

**6.20. The Center of H.** The center of $\mathbf{H}$ is $\mathcal{A}[\mathbf{X}]^{W^\Gamma}$.

**6.21. The Anti-Involution** There is a unique $\mathcal{A}$-linear involutive anti-automorphism $h \to \tilde{h}$ of the algebra $\mathbf{H}$ such that $\tilde{T_r} = T_r$, ($r \in S_1$) and $\tilde{\theta}_L = \theta_L$ ($L \in \mathbf{X}$).

**6.22. The Involution.** There is a unique $\mathcal{A}$-linear involutive automorphism $h \to h^*$ of $\mathbf{H}$ such that

$$T_r^* = -q^{\lambda(r)}T_r^{-1},$$

and

$$\theta_L^* = \theta_{L^{-1}}.$$

The statements and proofs of these facts can be found in [**L4**, Sec. 3].

**6.23. ${}^\sigma\mathbf{R}_M$ is Isomorphic to the Center of H.** We cite the following fact from [**Bo**, Prop. 6.7]. The proof there is in a different context, but works equally well here:

**Proposition 6.2.** *There is a natural isomorphism*

$$ {}^\sigma\mathbf{R}_M \xrightarrow{\sim} \mathcal{A}[\mathbf{X}]^{W^\Gamma}.$$

*sending* $\mathbf{q} \in {}^\sigma\mathbf{R}_M$ *to the indeterminant* $q$.

Combining this with 6.20 we get an identification

$$\text{(center of } \mathbf{H}) = {}^\sigma\mathbf{R}_M = \mathcal{A}[\mathbf{X}]^{W^\Gamma}. \tag{6.23q}$$

In this way $\mathbf{H}$ is identified as an algebra over ${}^\sigma\mathbf{R}_M$.

We also have

**Corollary 6.3.** ${}^\sigma\mathbf{K}_0^M(\mathcal{B})$ *is a free* ${}^\sigma\mathbf{R}_M$*-module of rank* $|W^\Gamma|$.

*Proof.* Immediate from 3.3 and 6.11. ∎

## 7. Some $M$-varieties.

This chapter is completely analogous to [**KL2**, Sec. 3]. We use similar notation whenever possible to emphasize the analogy between [**KL2**] and this paper. Throughout this section $M = {}^L G \times \mathbb{C}^*$ and $\mathbf{X} = X^*(T_\Gamma)$ (= the set of ${}^L G$-equivariant $\sigma$-trivial, $\sigma$-line bundles on $\mathcal{B}$).

**7.1. The Variety $\Lambda$.** Let

$$\Lambda = \{(n, B') | B' \in \mathcal{B} \quad \text{and} \quad n \in \mathrm{Lie}(B'),\ n \text{ nilpotent}\}.$$

The group $M$ acts on $\Lambda$ by

$$(g, q) : (n, B') \to (\mathrm{Ad} g(q^{-1} n), g B' g^{-1}). \tag{7.1a}$$

**7.2. The Varieties $\hat{\Lambda}^r$ and $\Lambda^r$.** Let $r \in S_1$ (see 3.20), and let $\hat{\Lambda}^r$ be the variety of all pairs $(n, P)$ where $P \in \mathcal{P}_r$ and $n$ is a nilpotent element in $\mathrm{Lie}(P)$. Let $\Lambda^r$ be the variety of all pairs $(n, B')$ such that $B' \in \mathcal{B}$ and if $P$ is the unique parabolic subgroup of type $r$ containing $B'$ then $n \in P$; $M$ acts on $\Lambda^r$ in a manner analogous to 7.1a.

**7.3. The Complex $\hat{\mathcal{E}}^r$.** Let

$$\hat{\pi}_r : \Lambda^r \to \hat{\Lambda}^r$$

be defined by $(n, B') \to (n, \pi_r(B'))$ $(n, B') \in \Lambda^r$ (see 6.3).

Let $C^r$ be the tangent bundle along the fibers of $\hat{\pi}_r$ and let $\mu$ denote the rank of $C^r$. Also let $C'^{\,r}$ be the corresponding cotangent bundle along the fibers of $\hat{\pi}_r$.

As in [**KL2**, Sec. 3.1], the nilpotent portion of $\Lambda^r$ gives a section $\mathcal{N}$ of $C^r$ and we may form the complex of $M$-equivariant vector bundles

$$\text{(7.3a)}\hat{\mathcal{E}}^r = \left( \ldots 0 \to\ \wedge^{\mu} (\mathbf{q} C'^{\,r}) \xrightarrow{{}^t\mathcal{N}} \wedge^{\mu - 1} (\mathbf{q} C'^{\,r}) \ldots \xrightarrow{{}^t\mathcal{N}} \mathbb{C} \to 0 \ldots \right).$$

We have that the support of $\hat{\mathcal{E}}^r$ is precisely $\Lambda$. (The support of a complex is the set of points where it is not acyclic, see [**Se**].)

**7.4. The Variety of Triples.** We now define the famous Steinberg variety of triples. Let

$$Z = \{(n, B', B'') | B', B'' \in \mathcal{B} \text{ and } n \text{ is nilpotent with } n \in \mathrm{Lie}(B') \cap \mathrm{Lie}(B'')\}.$$

The group $M$ acts on $Z$ by

$$(g, q) : (n, B', B'') \to (\mathrm{Ad} g(q^{-1} n), g B' g^{-1}, g B'' g^{-1}). \tag{7.4a}$$

**7.5. The Varieties ${}^{r}Z$ and $Z^{r}$.** Let $r \in S_1$ be a simple reflection (see 3.20).

Let ${}^{r}Z$ (resp. $Z^{r}$) be the variety of all triples $(n, B', B'')$ where $(B', B'') \in \mathcal{B} \times \mathcal{B}$ and $n \in \mathrm{Lie}({}^{L}G)$ is a nilpotent element such that if $P'$ (resp. $P''$) is the unique parabolic subgroup of type $r$ containing $B'$ (resp. $B''$) then $n \in \mathrm{Lie}(P')$ (resp. $n \in \mathrm{Lie}(P'')$). The group $M$ acts on ${}^{r}Z$ and $Z^{r}$ in an obvious way analogous to 7.4a.

**7.6. The Varieties ${}^{r}\hat{Z}$ and $\hat{Z}^{r}$.** Let ${}^{r}\hat{Z}$ (resp. $\hat{Z}^{r}$) be the variety of all triples $(n, P', B')$ (resp. $(n, B'', P'')$) such that $B' \in \mathcal{B}$ and $P' \in \mathcal{P}_r$ (resp. $P'' \in \mathcal{P}_r$ and $B'' \in \mathcal{B}$) and $n$ is a nilpotent element in $\mathrm{Lie}(B' \cap P')$ (resp. $(\mathrm{Lie}(P'' \cap B'')))$. The group $M$ acts on ${}^{r}\hat{Z}$ and $\hat{Z}^{r}$ in a manner analagous to 7.4a.

**7.7. The Complexes ${}^{r}\mathcal{E}$ and $\mathcal{E}^{r}$.** Let ${}^{r}\pi : {}^{r}Z \to {}^{r}\hat{Z}$ be defined by $(n, B', B'') \to (n, \pi_r(B'), B'')$ (see 6.3) and $\pi^{r} : Z^{r} \to \hat{Z}^{r}$ be defined by $(n, B', B'') \to (n, B', \pi_r(B''))$.

Let ${}^{r}\mathcal{E}$ (resp. $\mathcal{E}^{r}$) be the complex of $M$-equivariant vector bundles on ${}^{r}Z$ (resp. $Z^{r}$) defined by pulling back $\hat{\mathcal{E}}^{r}$ by the map ${}^{r}Z \to \Lambda^{r}$, $(n, B', B'') \to (n, B')$ (resp. $Z^{r} \to \Lambda^{r}$, $(n, B', B'') \to (n, B''))$. We have the support of ${}^{r}\mathcal{E}$ (resp. $\mathcal{E}^{r}$) is precisely $Z$.

**7.8. The Maps ${}^{r}\tau$ and $\tau^{r}$.** Following [**KL2**] we call a locally closed subvariety $\mathfrak{V}$ of $Z$ *left-r-saturated* ($r \in S_1$) if

$$\mathfrak{V} = (({}^{r}\pi)^{-1}({}^{r}\pi)\mathfrak{V}) \cap Z.$$

Similarly call $\mathfrak{V}$ *right-r-saturated* if

$$\mathfrak{V} = ((\pi^{r})^{-1}(\pi^{r})\mathfrak{V}) \cap Z.$$

Set $\hat{\mathfrak{V}} = {}^{r}\pi\mathfrak{V}$. Let $\mathfrak{V}$ be a left-$r$-saturated subvariety of $Z$ stable under the $M$-action 7.4a. Define a ${}^{\sigma}\mathbf{R}_M$-homomorphism

$$ {}^{r}\tau : {}^{\sigma}\mathbf{K}_0^M(\mathfrak{V}) \to {}^{\sigma}\mathbf{K}_0^M(\mathfrak{V}) \tag{7.8a}$$

by the composition

(7.8b)
$$ {}^{\sigma}\mathbf{K}_0^M(\mathfrak{V}) \xrightarrow{j_*} {}^{\sigma}\mathbf{K}_0^M\left(\tilde{\mathfrak{V}}\right) \xrightarrow{({}^{r}\pi)_*} {}^{\sigma}\mathbf{K}_0^M\left(\hat{\mathfrak{V}}\right) \xrightarrow{({}^{r}\pi)^*} {}^{\sigma}\mathbf{K}_0^M\left(\tilde{\mathfrak{V}}\right) \xrightarrow{{}^{r}\mathcal{E}\otimes} {}^{\sigma}\mathbf{K}_0^M(\mathfrak{V}).$$

Here $\tilde{\mathfrak{V}} = ({}^{r}\pi^{-1})\hat{\mathfrak{V}}$. The restriction of ${}^{r}\pi$ to $\mathfrak{V}$ is denoted again by ${}^{r}\pi$; $j : \mathfrak{V} \hookrightarrow \tilde{\mathfrak{V}}$ is the inclusion, ${}^{r}\mathcal{E}\otimes$ is as in 5.6 with respect to ${}^{r}\mathcal{E}\otimes$ restricted to $\tilde{\mathfrak{V}}$.

We defined in a similar manner for an $M$-stable, right-$r$-saturated subvariety of $Z$

$$\tau^r : {}^\sigma\mathbf{K}_0^M(\mathfrak{V}) \to {}^\sigma\mathbf{K}_0^M(\mathfrak{V})$$

by replacing in the previous definition ${}^r\pi$ and ${}^r\mathcal{E}\otimes$ by $\pi^r$ and $\mathcal{E}^r\otimes$ respectively.

In the forthcoming sections we will state properties of the maps ${}^r\tau$ and $\tau^r$. These properties are proved for analogous statements in [**KL2**, Sec. 3]. The proofs of these properties in all cases follow from the formal axioms of $\mathbf{K}$-homology and the various maps involved. Since the analogous axioms hold in ${}^\sigma\mathbf{K}$-homology and our maps are defined in the same manner as in [**KL2**] we omit the proofs and refer to the proofs in [**KL2**].

**7.9. An Alternative Definition of ${}^r\tau$.** We give an alternative definition of ${}^r\tau$. Let $\mathfrak{V}$ be as in 7.8.

Let $\mathfrak{V}' = \tilde{\mathfrak{V}} \underset{\hat{\mathfrak{V}}}{\times} \mathfrak{V}$, $\mathfrak{V}'' = \mathfrak{V} \underset{\hat{\mathfrak{V}}}{\times} \mathfrak{V}$. Let $pr_1' : \mathfrak{V}' \to \tilde{\mathfrak{V}}$, $pr_2' : \mathfrak{V}' \to \mathfrak{V}$, and $pr_1'' : \mathfrak{V}'' \to \mathfrak{V}$ be the projections.

Let ${}^r\tilde{\mathcal{E}}$ be the complex of $M$-equivariant vector bundles on $\mathfrak{V}'$ obtained by pulling back ${}^r\mathcal{E}|_{\tilde{\mathfrak{V}}}$ by the map $pr_1'$. The support of ${}^r\tilde{\mathcal{E}}$ is $\mathfrak{V}''$. Then ${}^r\tau$ is equal to the composition

$$ {}^\sigma\mathbf{K}_0^M(\mathfrak{V}) \xrightarrow{pr_2'^*} {}^\sigma\mathbf{K}_0^M(\mathfrak{V}') \xrightarrow{{}^r\tilde{\mathcal{E}}\otimes} {}^\sigma\mathbf{K}_0^M(\mathfrak{V}'') \xrightarrow{(pr_1'')_*} {}^\sigma\mathbf{K}_0^M(\mathfrak{V}). \tag{7.9a}$$

**7.10. Compatibilty Properties of ${}^r\tau$.** Now let $\mathfrak{V} \subset \mathfrak{V}'$ be two locally closed, $M$-stable left-$r$-saturated subvarieties of $Z$ and let $i : \mathfrak{V} \hookrightarrow \mathfrak{V}'$ be the inclusion. Then we have the following commutative diagrams.

If $\mathfrak{V}$ is closed in $\mathfrak{V}'$ then

$$\begin{array}{ccc} {}^\sigma\mathbf{K}_0^M(\mathfrak{V}) & \xrightarrow{{}^r\tau} & {}^\sigma\mathbf{K}_0^M(\mathfrak{V}) \\ {\scriptstyle i_*}\big\downarrow & & {\scriptstyle i_*}\big\downarrow \\ {}^\sigma\mathbf{K}_0^M(\mathfrak{V}') & \xrightarrow{{}^r\tau} & {}^\sigma\mathbf{K}_0^M(\mathfrak{V}') \end{array} \tag{7.10a}$$

if $\mathfrak{V}$ is open in $\mathfrak{V}'$ then

$$\begin{array}{ccc} {}^\sigma\mathbf{K}_0^M(\mathfrak{V}) & \xrightarrow{{}^r\tau} & {}^\sigma\mathbf{K}_0^M(\mathfrak{V}) \\ {\scriptstyle i^*}\big\uparrow & & {\scriptstyle i^*}\big\uparrow \\ {}^\sigma\mathbf{K}_0^M(\mathfrak{V}') & \xrightarrow{{}^r\tau} & {}^\sigma\mathbf{K}_0^M(\mathfrak{V}') \end{array} . \tag{7.10b}$$

Similar results apply to $\tau^r$.

**7.11. Examples.** We will now work out the maps ${}^r\tau$ and $\tau^r$ in the particular cases where $\mathfrak{V} = Z$ and $\mathfrak{V} = \mathcal{B}\times\mathcal{B}$. Here $\mathcal{B}\times\mathcal{B}$ is a subvariety of $Z$ by the closed immersion $(B', B'') \to (0, B', B'')$ $((B', B'') \in \mathcal{B}\times\mathcal{B})$. Both $\mathcal{B}\times\mathcal{B}$ and $Z$ are obviously left-$r$-saturated.

In the case $\mathfrak{V} = Z$ we have

$$\begin{aligned} {}^{\mathrm{r}}\tau &: {}^{\sigma}\mathbf{K}_0^M(Z) \to {}^{\sigma}\mathbf{K}_0^M(Z), \quad \text{and} \\ \tau^{\mathrm{r}} &: {}^{\sigma}\mathbf{K}_0^M(Z) \to {}^{\sigma}\mathbf{K}_0^M(Z) \end{aligned}$$

are the compositions

(7.11a)
$$ {}^{\sigma}\mathbf{K}_0^M(Z) \xrightarrow{j_*} {}^{\sigma}\mathbf{K}_0^M({}^rZ) \xrightarrow{{}^r\pi_*} {}^{\sigma}\mathbf{K}_0^M({}^r\hat{Z}) \xrightarrow{({}^r\pi)^*} {}^{\sigma}\mathbf{K}_0^M({}^rZ) \xrightarrow{{}^r\mathcal{E}\otimes} {}^{\sigma}\mathbf{K}_0^M(Z), $$

and

(7.11b)
$$ {}^{\sigma}\mathbf{K}_0^M(Z) \xrightarrow{j_*} {}^{\sigma}\mathbf{K}_0^M(Z^r) \xrightarrow{\pi_*^r} {}^{\sigma}\mathbf{K}_0^M(\hat{Z}^r) \xrightarrow{\pi^{r*}} {}^{\sigma}\mathbf{K}_0^M(Z^r) \xrightarrow{\mathcal{E}^r\otimes} {}^{\sigma}\mathbf{K}_0^M(Z) $$

respectively.

In the case where $\mathfrak{V} = \mathcal{B}\times\mathcal{B}$ we have

(7.11c)
$$ {}^r\tau : {}^{\sigma}\mathbf{K}_0^M(\mathcal{B}\times\mathcal{B}) \to {}^{\sigma}\mathbf{K}_0^M(\mathcal{B}\times\mathcal{B}), $$

and

(7.11d)
$$ \tau^r : {}^{\sigma}\mathbf{K}_0^M(\mathcal{B}\times\mathcal{B}) \to {}^{\sigma}\mathbf{K}_0^M(\mathcal{B}\times\mathcal{B}) $$

are given by

(7.11e)
$$ {}^{\sigma}\mathbf{K}_0^M(\mathcal{B}\times\mathcal{B}) \xrightarrow{{}_r\phi_*} {}^{\sigma}\mathbf{K}_0^M(\mathcal{P}_r\times\mathcal{B}) \xrightarrow{{}_r\phi^*} {}^{\sigma}\mathbf{K}_0^M(\mathcal{B}\times\mathcal{B}) \xrightarrow{{}^r\mathcal{E}\otimes} {}^{\sigma}\mathbf{K}_0^M(\mathcal{B}\times\mathcal{B}), $$

and

(7.11f)
$$ {}^{\sigma}\mathbf{K}_0^M(\mathcal{B}\times\mathcal{B}) \xrightarrow{(\phi_r)_*} {}^{\sigma}\mathbf{K}_0^M(\mathcal{B}\times\mathcal{P}_r) \xrightarrow{\phi_r^*} {}^{\sigma}\mathbf{K}_0^M(\mathcal{B}\times\mathcal{B}) \xrightarrow{\mathcal{E}^r\otimes} {}^{\sigma}\mathbf{K}_0^M(\mathcal{B}\times\mathcal{B}) $$

respectively. Here ${}_r\phi : \mathcal{B}\times\mathcal{B} \to \mathcal{P}_r\times\mathcal{B}$ is $\pi_r\times 1$, $\phi_{\mathrm{r}} : \mathcal{B}\times\mathcal{B} \to \mathcal{B}\times\mathcal{P}_r$ is $1\times\pi_r$, and ${}^r\mathcal{E}\otimes$ (resp. $\mathcal{E}^r\otimes$) denote ${}^r\mathcal{E}\otimes|_{\mathcal{B}\times\mathcal{B}}$ (resp. $\mathcal{E}^r\otimes|_{\mathcal{B}\times\mathcal{B}}$).

**7.12. The Element 1.** Define $\beta : \Lambda \to Z$ by $\beta(n, B') = (n, B', B')$ and $\beta' : \Lambda \to \mathcal{B}$ by $\beta'(n, B') = B'$.

Let $\mathbf{1} \in {}^{\sigma}\mathbf{K}_0^M(Z)$ be the image of the trivial $\sigma$-bundle $\mathbb{C}$ on $\mathcal{B}$ under the composition

$$ {}^{\sigma}\mathbf{K}_0^M(\mathcal{B}) \xrightarrow{\beta'^*} {}^{\sigma}\mathbf{K}_0^M(\Lambda) \xrightarrow{\beta_*} {}^{\sigma}\mathbf{K}_0^M(Z). \tag{7.12a} $$

We regard $\mathbb{C}$ as an element of ${}^{\sigma}\mathbf{K}^0_M(\mathcal{B}) \cong {}^{\sigma}\mathbf{K}_0^M(\mathcal{B})$ under the isomorphism 5.17k.

**7.13. The Bundles $L^{\cdot}$ and ${}^{\cdot}L$.** Identify each $L \in X^*(T_\Gamma)$ with the corresponding ${}^LG$-equivariant $\sigma$-trivial $\sigma$-line bundle (see 6.12). Denote by ${}^{\cdot}L$ (resp. $L^{\cdot}$) the pullback of $L$ along the $M$-equivariant map $Z \to \mathcal{B}, (n, B', B'') \to B'$ (resp. $(n, B', B'') \to B''$). We regard $L$ as an $M$-equivariant $\sigma$-trivial $\sigma$-line bundle on $\mathcal{B}$ with trivial action of $\mathbb{C}^*$. Hence ${}^{\cdot}L, L^{\cdot}$ are naturally $M$-equivariant $\sigma$-line bundles on $Z$.

## 8. The structure of $\mathcal{B}^{s\sigma}$.

We will now describe the structure of the variety $\mathcal{B}^{s\sigma}$. Let $B$ and $T$ be part of the splitting of $G$ and write $B = TU$ where $U$ is the unipotent radical of $B$. For each $w \in W$ let

$$ U^w = \prod_{\alpha>0, w^{-1}\alpha<0} U_\alpha. $$

For convenience identify $W$ with the set of fixed points of $T$ on $\mathcal{B}$. Then we have $\mathcal{B} = \sqcup_{w \in W} U^w w$ so that

$$ \mathcal{B}^\sigma = \bigsqcup_{w \in W^\Gamma} (U^w)^\sigma w. $$

Since every $w \in W^\Gamma$ is represented in $N(T)^\sigma$ (see [**Bo**]) we have for each $w \in W^\Gamma$ that $(U^w)^\sigma$ is non-empty (see [**ST1**]). Therefore we have that $G^\sigma$ acts transitively on $\mathcal{B}^\sigma$ and we have immediately

**Corollary 8.1.** *The variety $\mathcal{B}^\sigma$ is connected.*

**Corollary 8.2.** *The group $G^\sigma$ acts transitively on $\mathcal{P}_s^\sigma$ for any $s \in S_1$.*

*Proof.* Let $P_1, P_2 \in \mathcal{P}_s^\sigma$ be distinct. We must show that there exists an element $g \in G^\sigma$ such that $gP_1g^{-1} = P_2$. Let $B_1, B_2 \in \mathcal{B}^\sigma$ be such that $B_i \subset P_i$, and $B_1 \neq B_2$. Such $B_i$ exists since $\sigma$ fixes each $P_i$, see [**ST1**]. Then there exists $g \in G^\sigma$ such that $gB_1g^{-1} = B_2$. Now $gP_1g^{-1}$ is a parabolic subgroup of type $s$, and moreover it contains $gB_1g^{-1} = B_2$. Thus $gP_1g^{-1} = P_2$. □

Now let $s \in T$ and $W^\Gamma_s$ be the subgroup of $W^\Gamma$ generated by the set of $\alpha \in S_1$ which vanish on $s$.

**Corollary 8.3.** *Let $\mathcal{B}_{s\sigma}$ be the variety of Borel subgroups of the group $G^{s\sigma}$. Then we have*

$$\mathcal{B}^{s\sigma} \cong \mathcal{B}_{s\sigma} \underset{W^\Gamma_s}{\times} W^\Gamma.$$

*Proof.* This is essentially proved in [**ST3**]. □

Now let $s \in {}^L\!G$ be an arbitrary semisimple element, and let $n \in \operatorname{Lie} G$ be such that $\operatorname{Ad}(s)n = qn$. It is desirable to know that the variety $\mathcal{B}^s_n$ is non-empty. This was proved for the case ${}^L\!G = G$ in [**L4**]. We give here a generalization.

**Proposition 8.4.** *The variety $\mathcal{B}^s_n = \{B' \in \mathcal{B} | sB's^{-1} = B'$, and $n \in \operatorname{Lie}(B')\}$ is nonempty.*

*Proof.* It is well known that $\mathcal{B}^s$ is non-empty. Choose $B' \in \mathcal{B}^s$ and assume that $n \notin \operatorname{Lie}(B')$. Let $u = \exp(n)$. Then $sus^{-1} = u^q$. Consider the map

$$\phi : \mathbb{C} \to \mathcal{B}, \qquad \lambda \to u^\lambda B' u^{-\lambda}.$$

This is an algebraic embedding of $\mathbb{C} \hookrightarrow \mathcal{B}$. Let $\mathcal{S}$ denote the image of $\phi$. Write $\bar{\mathcal{S}}$ for the Zariski closure of $\mathcal{S}$. Because $\mathcal{B}$ is complete we have $\bar{\mathcal{S}} = \mathcal{S} \cup \{B_1\}$ where $B_1 \in \mathcal{B}$. Because $sus^{-1} = u^q$ the action of $s$ on $\mathcal{B}$ restricts to an action of $s$ on $\mathcal{S}$. This action extends to $\bar{\mathcal{S}}$ and therefore $s$ fixes $B_1$. The same argument applied to the $u$-action proves that $u$ fixes $B_1$. Therefore $n \in \operatorname{Lie}(B_1)$. This proves the proposition. □

Even when $s \in {}^L\!G^\circ$ this proof is different than the one in [**L4**].

## 9. A Splitting Theorem and Miscellany.

We can now state and prove an important theorem analogous to [**KL2**, prop. 1.6] in the $K$-theory case. The proof follows from a straightforward imitation of [**KL2**]. In this section $M = {}^L\!G \times \mathbb{C}^*$ , and $\mathbf{X} = X^*(T_\Gamma)$ (= the set of ${}^L\!G$-equivariant $\sigma$-trivial $\sigma$-line bundles on $\mathcal{B}$) are still in force. We let $G, \sigma, \Gamma, (B,T), {}^L\!G, {}^LB, {}^LT, \mathcal{B}, W, W^\Gamma$ be as in 3.1.

**Proposition 9.1.** *Let $\mathcal{B}_w$ be the ${}^L\!G$-orbit in $\mathcal{B} \times \mathcal{B}$ corresponding to the element $w \in W^\Gamma \subset W$ (see 7.4a); $\mathcal{B}_w$ is a locally closed, $M$-stable subvariety of $\mathcal{B} \times \mathcal{B}$. The projection*

$$\tilde{p}_w : \mathcal{B}_w \to \mathcal{B}, \quad (B', B'') \to B'$$

*is equivariant with respect to the $M$-action on $\mathcal{B}_w$ and on $\mathcal{B}$ and induces an isomorphism*

$$\tilde{p}_w^* : {}^{\sigma}\mathbf{K}_0^M(\mathcal{B}) \xrightarrow{\sim} {}^{\sigma}\mathbf{K}_0^M(\mathcal{B}_w). \tag{9.0a}$$

*Proof.* Immediate from 5.5 and 6.3. □

For $w \in W^{\Gamma} \subset W$ let

$$Z_w = \{(n, B', B'') \in Z | B' \xrightarrow{w} B'' \text{ (see 3.2)}\}. \tag{9.0b}$$

With respect to the $M$-action on $Z$ (see 7.4a) $Z_w$ is an $M$-stable subvariety of $Z$.

**Proposition 9.2.** *Let $w \in W^{\Gamma} \subset W$. Then the projection*

$$p_w : Z_w \to \mathcal{B}, \quad (n, B', B'') \to B'$$

*is equivariant with respect to the $M$-action on $Z_w$ and $\mathcal{B}$ respectively and induces an isomorphism*

$$(p_w)^* : {}^{\sigma}\mathbf{K}_0^M(\mathcal{B}) \xrightarrow{\sim} {}^{\sigma}\mathbf{K}_0^M(Z_w). \tag{9.0c}$$

*Proof.* Use 5.5. □

**9.1. Preparation.** For $w \in W$ let $\mathfrak{O}(w)$ be the $\sigma$-orbit of $w$ in $W$ and let $l_w$ be the cardinality of $\mathfrak{O}(w)$. Let

$$\mathcal{B}_{\mathfrak{O}(w)} = \bigsqcup_{w' \in \mathfrak{O}(w)} \mathcal{B}^{w'}, \tag{9.1d}$$

where $\mathcal{B}^{w'}$ is a copy of $\mathcal{B}$. Equip $\mathcal{B}_{\mathfrak{O}(w)}$ with a $\sigma$-action by letting $\sigma : \mathcal{B}^w \to \mathcal{B}^{\sigma(w)}$ be the identity; $\mathcal{B}_{\mathfrak{O}(w)}$ is an $M$-variety.

Let

$$Z_{\mathfrak{O}(w)} = \bigsqcup_{w' \in \mathfrak{O}(w)} Z_{w'}. \tag{9.1e}$$

It is clear that the $Z_{\mathfrak{O}(w)}$ form a partition of $Z$ into locally closed $M$-stable subvarieties.

**Proposition 9.3.** *The projection*

$$p_{\mathfrak{O}(w)} : Z_{\mathfrak{O}(w)} \to \mathcal{B}_{\mathfrak{O}_w}, \quad (n, B', B'') \to B'$$

*is $M$-equivariant morphism of varieties and induces the isomorphism*

$$p^*_{\mathfrak{O}(w)} : {}^\sigma\mathbf{K}_0^M(\mathcal{B}_{\mathfrak{O}_w}) \xrightarrow{\sim} {}^\sigma\mathbf{K}_0^M(Z_{\mathfrak{O}_w}). \tag{9.1f}$$

*Proof.* Use 5.5. □

We may now state

**Proposition 9.4.** *${}^\sigma\mathbf{K}_0^M(Z)$ is a projective ${}^\sigma\mathbf{R}_M$-module of rank $|W^\Gamma|^2$, and ${}^\sigma\mathbf{K}_1^M(Z) = 0$.*

*Proof.* We claim that for $w \in W$

$$ {}^\sigma\mathbf{K}_0^M(Z_{\mathfrak{O}(w)}) = \begin{cases} 0 & \text{if } l_w > 1 \text{ (see 9.1)}, \\ {}^\sigma\mathbf{K}_0^M(\mathcal{B}) & \text{if } l_w = 1. \end{cases} \tag{9.1g}$$

Consider the case where $l_w > 1$. We know that ${}^\sigma\mathbf{K}_0^M(\mathcal{B}_{\mathfrak{O}(w)}) \cong {}^\sigma\mathbf{K}_0^M(Z_{\mathfrak{O}(w)})$ (see 9.1f). The group $M$ acts transitively on $\mathcal{B}_{\mathfrak{O}(w)}$. If $B' \in \mathcal{B}_{\mathfrak{O}(w)}$ it is easily seen that the stabilizer of $B'$ in $M$ does not meet the Frobenius component of ${}^LG$. Therefore we have ${}^\sigma\mathbf{K}_0^M(Z_{\mathfrak{O}(w)}) = 0$ by 5.19m.

If $l_w = 1$ then $w \in W^\Gamma$ and $\mathfrak{O}(w) = w$ and the claim follows from 9.0c.

By 9.1g, 5.19m and 6.3 we easily see that for each $w \in W^\Gamma$:

$${}^\sigma\mathbf{K}_1^M(Z_w) = 0 \text{ and } {}^\sigma\mathbf{K}_0^M(Z_w) \text{ is a projective } {}^\sigma\mathbf{R}_M\text{-module of rank } |W^\Gamma|.$$

Now the proposition follows from the exact sequences of 5.16 applied to the partition of $Z$ by the $Z_{\mathfrak{O}(w)}$ $(w \in W)$. □

The same method of proof yields

**Proposition 9.5.** *The group ${}^\sigma\mathbf{K}_0^M(\mathcal{B}\times\mathcal{B})$ is a projective ${}^\sigma\mathbf{R}_M$-module of rank $|W^\Gamma|^2$.*

Recall (see 5.4) that we have a natural ${}^\sigma\mathbf{R}_M$-homomorphism

$$\boxtimes : {}^\sigma\mathbf{K}_0^M(\mathcal{B}) \underset{{}^\sigma\mathbf{R}_M}{\otimes} {}^\sigma\mathbf{K}_0^M(\mathcal{B}) \to {}^\sigma\mathbf{K}_0^M(\mathcal{B}\times\mathcal{B}).$$

In [**KL2**] it was proved that the analogous map in $K$-theory is an isomorphism. It turns out that the essense of this proof works equally well in the case of ${}^\sigma K$-theory. The proof is an almost word-for-word reproduction of the proof in [**KL2**].

**9.2. The Splitting Theorem.** Let $G$ and ${}^L\!G$ be as in 3.1.

**Proposition 9.6.** *The natural map*

$$\boxtimes : {}^{\sigma}\mathbf{K}^0_{{}^L G}(\mathcal{B}) \underset{{}^{\sigma}\mathbf{R}_{{}^L G}}{\otimes} {}^{\sigma}\mathbf{K}^0_{{}^L G}(\mathcal{B}) \to {}^{\sigma}\mathbf{K}^0_{{}^L G}(\mathcal{B} \times \mathcal{B})$$

*is an isomorphism.*

*Proof.* Let $(B, T), \sigma, \Gamma, {}^L B, {}^L T$ and $W$ be as in 3. Recall that $G$ is the connected component of the identity of ${}^L\!G$.

Recall that $\Sigma$ is the root system of $G$ with respect to $T$. The action of $B$ on the tangent space at $B$ of $\mathcal{B}$ determines a positive system of roots $\Sigma^+$ in $\Sigma$ opposite the positive system determined by $B$. Let $\Pi$ be the basis of $\Sigma$ determined by $\Sigma^+$. The action of $\sigma$ on the tangent bundle of $\mathcal{B}$ induces a $\sigma$-action on the fiber over $B$ and hence an action on $\Pi$. From this we may form the root system $\Sigma_1$ exactly as in (3.12). This system has a canonical basis $\Pi_1$ and a positive root in $\Sigma_1$ is understood to be positive with respect to $\Pi_1$. We identify $\Sigma_1$ with its image in ${}^{\sigma}\mathbf{R}_{{}^L T}$.

The natural homomorphism ${}^{\sigma}\mathbf{R}_{{}^L G} \to {}^{\sigma}\mathbf{R}_{{}^L T}$ is injective with image equal to the $W^{\Gamma}$-invariants ${}^{\sigma}\mathbf{R}^{W^{\Gamma}}_{{}^L T}$.

By 3.3 we have elements $e_v \in {}^{\sigma}\mathbf{R}_{{}^L T}$ one for each $v \in W^{\Gamma}$, such that

(1) The $e_v$ $(v \in W^{\Gamma})$ form a basis of ${}^{\sigma}\mathbf{R}_{{}^L T}$ as an ${}^{\sigma}\mathbf{R}^{W^{\Gamma}}_{{}^L T}$- module and

(2) $\det(u(e_v))_{(u,v)\in W^{\Gamma}\times W^{\Gamma}} = \Delta^{|W^{\Gamma}|/2}$ where $\Delta = \Pi(\alpha^{1/2} - \alpha^{-1/2}) \in {}^{\sigma}\mathbf{R}_{{}^L T}$, (product over all positive roots $\alpha$). (When $|W^{\Gamma}| = 1$, both sides of ( 2) are 1.)

Let $(\ ,\ ) : {}^{\sigma}\mathbf{K}^0_{{}^L G}(\mathcal{B}) \times {}^{\sigma}\mathbf{K}^0_{{}^L G}(\mathcal{B}) \to {}^{\sigma}\mathbf{R}_{{}^L G}$ be the pairing defined by $(E, E') = \pi_*(E \otimes E')$ where $\pi : \mathcal{B} \to$ point is the natural map and $\pi_*$ is the direct image in ${}^{\sigma}\mathbf{K}^0_{{}^L G}$-theory (this is derived from the direct image in $\mathbf{K}^0_{{}^L G}$-theory in the usual way).

Using the natural identification ${}^{\sigma}\mathbf{K}^0_M(\mathcal{B}) = {}^{\sigma}\mathbf{R}_{{}^L B} = {}^{\sigma}\mathbf{R}_{{}^L T}$ and 5.22 along with the Atiyah-Hirzebruch version [**KL1**, Sec. 1.7] of the Riemann-Roch theorem, the Borel-Weil-Bott theorem (see [**Bt**]) and the twisted Weyl character formula 5.21 the pairing becomes $(\ ,\ ) : {}^{\sigma}\mathbf{R}_{{}^L T} \times {}^{\sigma}\mathbf{R}_{{}^L T} \to {}^{\sigma}\mathbf{R}^{W^{\Gamma}}_{{}^L T}$ given by

$$(\alpha, \beta) = \Delta^{-1} \sum_{w \in W^{\Gamma}} \epsilon_w w(\alpha\beta\rho), \tag{9.2h}$$

where $\epsilon_w$ is the sign of $w \in W^{\Gamma}$ and $\rho^2$ is the product of all positive roots.

The proof of [**KL2**, Prop 1.6] shows that

$$\det((e_v, e_{v'}))_{(v,v')\in W^{\Gamma}\times W^{\Gamma}} = 1; \tag{9.2i}$$

where the determinant is a matrix with entries in ${}^\sigma\mathbf{R}_{{}^LT}^{W^\Gamma}$. It now follows that there is a unique basis $\hat{e}_v$ $(v \in W^\Gamma)$ of ${}^\sigma\mathbf{R}_{{}^LT}$ as an ${}^\sigma\mathbf{R}_{{}^LT}^{W^\Gamma}$-module such that

$$(e_v, \hat{e}_{v'}) = \delta_{v,v'}. \tag{9.2j}$$

We now define $F : {}^\sigma\mathbf{K}^0_{{}^LG}(\mathcal{B}\times\mathcal{B}) \to {}^\sigma\mathbf{K}^0_{{}^LG}(\mathcal{B}) \underset{{}^\sigma\mathbf{R}_{{}^LG}}{\otimes} {}^\sigma\mathbf{K}^0_{{}^LG}(\mathcal{B})$ by

$$F(\xi) = \sum_{v\in W^\Gamma} \hat{e}_v \boxtimes (\pi_2)_*(\pi_1^*(e_v)\otimes\xi), \tag{9.2k}$$

where $\pi_1, \pi_2 : \mathcal{B}\times\mathcal{B} \to \mathcal{B}$ are the two projections and $\pi_1^*$ (resp. $(\pi_2)_*$) is the inverse (resp. direct) image in ${}^\sigma\mathbf{K}^0_{{}^LG}$-theory. We regard $e_v, \hat{e}_v$ as elements in ${}^\sigma\mathbf{K}^0_{{}^LG}(\mathcal{B}) = {}^\sigma\mathbf{R}_{{}^LT}$ (see 5.19l). We now show that

$$F(\eta_1\boxtimes\eta_2) = \eta_1\boxtimes\eta_2, \quad \text{for any } \ \eta_1,\eta_2 \in {}^\sigma\mathbf{K}^0_{{}^LG}(\mathcal{B}). \tag{9.2l}$$

By using the projection formula in equivariant ${}^\sigma K$-theory and applying base change 5.14 to the diagram

$$\begin{array}{ccc} \mathcal{B}\times\mathcal{B} & \xrightarrow{\pi_2} & \mathcal{B} \\ {\scriptstyle\pi_1}\downarrow & & {\scriptstyle\pi}\downarrow \\ \mathcal{B} & \xrightarrow{\pi} & \mathrm{pt} \end{array} \tag{9.2m}$$

the left hand side of 9.2l equals

$$\begin{aligned} &\sum_{v\in W^\Gamma} \hat{e}_v \boxtimes (\pi_2)_*((\pi_1)^*(e_v\otimes\eta_1)\otimes(\pi_2)^*\eta_2) \\ &= \sum_{v\in W^\Gamma} \hat{e}_v \boxtimes (\pi_2)_*(\pi_1)^*(e_v\otimes\eta_1)\otimes\eta_2 \\ &= \sum_{v\in W^\Gamma} \hat{e}_v \boxtimes (\pi^*(\pi_*(e_v\otimes\eta_1))\otimes\eta_2) \\ &= \sum_{v\in W^\Gamma} \hat{e}_v \boxtimes (e_v,\eta_1)\eta_2 \\ &= \left(\sum_{v\in W^\Gamma} (e_v,\eta_1)\hat{e}_v\right)\boxtimes\eta_2 \\ &= \eta_1\boxtimes\eta_2, \end{aligned} \tag{9.2n}$$

and 9.2l is proved. From this we see that the map $\boxtimes$ in the proposition is injective and its image is a direct summand as a ${}^\sigma\mathbf{R}_{{}^LG}$-module. It is therefore enough to show that $\boxtimes$ is a map between two projective ${}^\sigma\mathbf{R}_{{}^LG}$-modules of the same rank $(= |W^\Gamma|^2)$. The ${}^\sigma\mathbf{R}_{{}^LG}$-module ${}^\sigma\mathbf{K}^0_{{}^LG}(\mathcal{B}) \underset{{}^\sigma\mathbf{R}_{{}^LG}}{\otimes} {}^\sigma\mathbf{K}^0_{{}^LG}(\mathcal{B})$ is free of rank $|W^\Gamma|^2$ by 3.3, and the ${}^\sigma\mathbf{R}_{{}^LG}$-module ${}^\sigma\mathbf{K}_0^{{}^LG}(\mathcal{B}\times\mathcal{B})$ is projective of rank $|W^\Gamma|^2$ by 9.5. This completes the proof. $\square$

**9.3. An Element of ${}^\sigma\mathbf{K}^0_{{}^LG}(\mathcal{B}\times\mathcal{B})$.** With the notations as in the previous section, let $i:\mathcal{B}\to\mathcal{B}\times\mathcal{B}$ be the diagonal embedding. Identify $i_*(\mathbb{C})\in {}^\sigma\mathbf{K}^0_{{}^LG}(\mathcal{B}\times\mathcal{B})$ with an element ${}^\sigma\mathbf{R}_{{}^LT}\underset{{}^\sigma\mathbf{R}_{{}^LG}}{\otimes}{}^\sigma\mathbf{R}_{{}^LT}$, using proposition 9.6. Let $\phi\in\mathrm{End}_{{}^\sigma\mathbf{R}_{{}^LG}}({}^\sigma\mathbf{R}_{{}^LT})$ and let ${}^t\phi$ be its transpose with respect to the inner product $(\ ,\ ):{}^\sigma\mathbf{R}_{{}^LT}\times{}^\sigma\mathbf{R}_{{}^LT}\to{}^\sigma\mathbf{R}_{{}^LG}$ in 9.6. We claim that

$$i_*(\mathbb{C})=\sum_{v\in W^\Gamma}\hat{e}_v\otimes e_v \tag{9.3o}$$

under the above identification. To see this note that

$$F(i_*(\mathbb{C}))=\sum_{v\in W^\Gamma}\hat{e}_v\boxtimes(\pi_2)_*(\pi_1^*(e_v)\otimes i_*\mathbb{C}).$$

But we claim that $(\pi_2)_*(\pi_1^*(e_v)\otimes i_*(\mathbb{C}))=e_v$. To see this let $D\subseteq\mathcal{B}\times\mathcal{B}$ be the diagonal subvariety and let $p_1,p_2:D\to\mathcal{B}$ be the projections. Then we have (via the Riemann-Roch theorem to switch to the algebraic theory)

$$(\pi_2)_*(\pi_1^*(e_v)\otimes i_*(\mathbb{C}))=(p_2)_*(p_1)^*(e_v)=e_v.$$

This proves the claim.

We now see immediately that

$$(\phi\otimes 1)(i_*\mathbb{C})=(1\otimes{}^t\phi)(i_*(\mathbb{C})). \tag{9.3p}$$

## 10. The Regular Represention of H.

In this section we let $M={}^LG\times\mathbb{C}^*$, and $\mathbf{X}=X^*(T_\Gamma)$ throughout.

### 10.1. The Main Theorem.

**Theorem 10.1.**

(a) *Regard ${}^\sigma\mathbf{K}_0^M(Z)$ as a ${}^\sigma\mathbf{R}_M=\mathcal{A}[\mathbf{X}]^{W^\Gamma}$–module (6.2). There is a unique left (resp. right) $\mathbf{H}$-module structure on ${}^\sigma\mathbf{K}_0^M(Z),(h,\xi)\to h\xi$ (resp. $(\xi,h)\to\xi h$), $h\in\mathbf{H},\xi\in{}^\sigma\mathbf{K}_0^M(Z)$ extending the $\mathcal{A}[\mathbf{X}]^{W^\Gamma}$–module structure such that for all $\xi\in{}^\sigma\mathbf{K}_0^M(Z),r\in S_1,L\in\mathbf{X}$ we have*

$$T_r\xi=(\mathbf{q}^{\lambda(r)}-{}^r\tau)(\xi)\ (\textit{resp.}\quad \xi T_r=(\mathbf{q}^{\lambda(r)}-\tau^r)(\xi)), \tag{10.1a}$$

*see 7.11a and 7.11b,*

$$\theta_L\xi={}^{\cdot}L\otimes\xi\ (\textit{resp.}\quad \xi\theta_L=L^{\cdot}\otimes\xi),\ \textit{see 7.13 and 5.9}. \tag{10.1b}$$

(b) *The two maps $\mathbf{H}\to{}^\sigma\mathbf{K}_0^M(Z)$, $h\to h\mathbf{1}$ and $h\to\mathbf{1}h$ (see 7.12) coincide.*

(c) *The two maps in (b) are isomorphisms.*

**10.2. An Auxillary Theorem.** As in [**KL2**, Sec. 3.6] we admit the following result for the moment.

**Proposition 10.2.**

(a) *There is a unique left (resp. right)* $\mathbf{H}$*-module structure on* ${}^{\sigma}\mathbf{K}_0^M(\mathcal{B}\times\mathcal{B})$, $(h,\xi)\to h\xi$ *(resp.* $(\xi,h)\to\xi h$*),* $h\in\mathbf{H}$*, extending the* $\mathcal{A}[\mathbf{X}]^{W^{\Gamma}}$*−module structure (see 6.2) such that*

$$T_r\xi = (\mathbf{q}^{\lambda(r)} - {}^{r}\tau)(\xi), \quad (\textit{resp. } \xi T_r = (\mathbf{q}^{\lambda(r)} - \tau^{r})\xi),$$

(10.2c)

*see 7.11e and 7.11f*

(10.2d)

$$\theta_L\xi = (L\boxtimes 1)\otimes\xi, \quad (\textit{resp. } \xi\theta_L = (1\boxtimes L)\otimes\xi).$$

(b) *Let* $\alpha:\mathcal{B}\to\mathcal{B}\times\mathcal{B}$ *be the diagonal embedding and let* $A\in{}^{\sigma}\mathbf{K}_0^M(\mathcal{B}\times\mathcal{B})$ *be defined as* $\alpha_*(\mathbb{C})$.

*For a positive root* $L\in\Sigma_2$ *set*

$$\gamma(L) = \begin{cases} q^{\lambda(L)} & \textit{if } L \textit{ is of type } \mathrm{I}, \\ q^{\frac{\lambda(L)+\lambda^*(L)}{2}} - q^{\frac{\lambda(L)-\lambda^*(L)}{2}} & \textit{if } L \textit{ is of type } \mathrm{II}. \end{cases} \tag{10.2e}$$

*Let*

$$\overline{q} = \left(q^{\frac{\lambda(r)+\lambda^*(r)}{2}} - q^{\frac{\lambda(r)-\lambda^*(r)}{2}}\right),$$

*and*

$$\phi^+ = \prod\left(1-q^{\lambda(r)}L_r\right)\prod\left(1-\overline{q}L_s - q^{\lambda(s)}L_s^2\right),$$

*where the first product is over all positive roots of type* I *while the second product is over all positive roots of type* II*. Note that if* $S_1$ *has no type* II *roots (which is the case, for insance, if* $\Sigma$ *is of type* $D_n$*) then the second product is vacuous. Let* $\theta_{\phi^+}$ *be the corresponding element of* $\mathbf{H}$*. Then, for any* $h\in\mathbf{H}$*, we have* $\theta_{\phi^+}hA = Ah\theta_{\phi^+}$*.*

**10.3. The Auxillary Theorem Implies the Main Theorem.** We now show how 10.2 implies 10.1(a) and (b). Fix a complex number $q$, $q^l\neq 1$ where $l$ is the order of $\sigma$ and consider the diagram

$$\begin{array}{ccc} {}^{\sigma}\mathbf{K}_0^M(\mathcal{B}\times\mathcal{B}) & \xrightarrow{j_*} & {}^{\sigma}\mathbf{K}_0^M(Z) \\ \big\downarrow & & \big\downarrow f_2 \\ {}^{\sigma}\mathbf{K}_0^M(\mathcal{B}\times\mathcal{B})_{\mathrm{loc}} & \xrightarrow[f_1]{\approx} & {}^{\sigma}\mathbf{K}_0^M(Z)_{\mathrm{loc}} \end{array}$$

where $j : \mathcal{B} \times \mathcal{B} \to Z, (B', B'') \to (0, B', B'')$. The symbol "loc" denotes localization with respect to the maximal ideal of ${}^{\sigma}\mathbf{R}_M$ corresponding to $(\sigma, q) \in M$ (see 5.1) and the vertical maps are the obvious maps. We have $f_1$ is the map induced by $j_*$ on the localizations. We know $f_1$ is an isomorphism because the set of fixed points of $(\sigma, q)$ on $Z$ is the set $j(\mathcal{B}^\sigma \times \mathcal{B}^\sigma)$, and therefore $Z^{(\sigma)} = j\left(\bigsqcup_{w \in W^\Gamma} \mathcal{B}_w\right)$ (see 9.1). Thus $j_* : {}^{\sigma}\mathbf{K}_0^M(\mathcal{B} \times \mathcal{B}) \to {}^{\sigma}\mathbf{K}_0^M(Z)$ induces an isomorphism on the localizations by 5.15. We claim that the map $f_2$ is injective: this follows because ${}^{\sigma}\mathbf{K}_0^M(Z)$ is a projective ${}^{\sigma}\mathbf{R}_M$-module (see Prop. 9.4).

Now $f_1$ and $f_2$ above have the stated properties. To see Theorem 10.1(a) we must see that certain identities hold for for the actions of the generators of $\mathbf{H}$ on ${}^{\sigma}\mathbf{K}_0^M(Z)$. These identities hold for the analogous actions on ${}^{\sigma}\mathbf{K}_0^M(\mathcal{B} \times \mathcal{B})$ (by 10.2) hence they hold on ${}^{\sigma}\mathbf{K}_0^M(\mathcal{B} \times \mathcal{B})_{\mathrm{loc}}$, hence they hold on ${}^{\sigma}\mathbf{K}_0^M(Z)_{\mathrm{loc}}$ (by the compatibility of these actions and since $f_1$ is an isomorphism), hence they hold on ${}^{\sigma}\mathbf{K}_0^M(Z)$, since $f_2$ is injective. The same argument shows that in the resulting $\mathbf{H}$-module structures the centre of $\mathbf{H}$ acts in the same way $\mathcal{A}[\mathbf{X}]^{W^\Gamma}$ acts by the ${}^{\sigma}\mathbf{R}_M$-module structure. This proves 10.1(a). We now prove 10.1(b). Let

$$\delta : \mathcal{B} \to \Lambda$$

be defined by $\delta(B') = (0, B')$. The calculations in 6.15 along with 5.13f combine to show that

$$\delta_*(\mathbb{C}) = \beta'^*(\phi^+),$$

where $\phi^+$ is regarded as an element of ${}^{\sigma}\mathbf{K}_0^M(\mathcal{B}) = \mathcal{A}[\mathbf{X}]$, (see 6.11).

With the notations of 10.2 and 7.12 we have:

(10.3f)
$$j_*A = j_*\alpha_*\mathbb{C} = \beta_*\delta_*(\mathbb{C}) = \beta_*\beta'^*(\phi^+) = \theta_{\phi^+}\beta_*\beta'^*(\mathbb{C}) = \theta_{\phi^+}\mathbf{1} = \mathbf{1}\theta_{\phi^+}.$$

From the definitions it follows immediately that $\theta_L\mathbf{1} = \mathbf{1}\theta_L$ for any $\sigma$-trivial $\sigma$-line bundle $L \in \mathbf{X}$. Let $h_1 \in \mathbf{H}$. By 10.2(b) we have $\theta_{\phi^+}h_1A = Ah_1\theta_{\phi^+}$. Apply $j_*$ to the last equality and use the compatibility of the $\mathbf{H}$-actions with $j_*$. We obtain $\theta_{\phi^+}h_1j_*(A) = j_*(A)h_1\theta_{\phi^+}$. Substituting $j_*A = \mathbf{1}\theta_{\phi^+} = \theta_{\phi^+}\mathbf{1}$ (by 10.3f), we get

$$\theta_{\phi^+}h_1\theta_{\phi^+}\mathbf{1} = \mathbf{1}\theta_{\phi^+}h_1\theta_{\phi^+}. \tag{10.3g}$$

We write 10.3g for $h_1$ in the form $\theta_{\phi^-}h_2\theta_{\phi^-}$ where $h_2 \in \mathbf{H}$ and $\phi^-$ and $\theta_{\phi^-}$ are defined just like $\phi^+$ and $\theta_{\phi^+}$ (see 10.2(b)) but using negative roots instead of postitive roots. Note that $\phi^+\phi^- \in \mathcal{A}[\mathbf{X}]^{W^\Gamma}$ hence $z = \theta_{\phi^+}\theta_{\phi^-}$ is in the center of $\mathbf{H}$ (see 6.20). It acts as multiplication by the corresponding element in ${}^\sigma\mathbf{R}_M$. Thus in our case 10.3g becomes $zh_2z\mathbf{1} = \mathbf{1}zh_2z$ and $z$ commutes with everything hence $z^2(h_2\mathbf{1}) = z^2(\mathbf{1}h_2)$. By 9.4 we know ${}^\sigma\mathbf{K}_0^M(Z)$ is a projective ${}^\sigma\mathbf{R}_M$-module, hence we can cancel $z^2$ from the last equality. Hence $h_2\mathbf{1} = \mathbf{1}h_2$ as desired.

We note that the above arguments were taken almost word for word from [**KL2**] (with the necessary modifications for ${}^\sigma\mathbf{K}$ theory). This will be a general pattern in this section. Once the basic notations are set, the arguments needed for most proofs are formal consequences of the axioms of $K$-homology.

**10.4. Two Lemmas.** We need some lemmas. Recall that to each simple reflection $r \in S_1$ there corresponds a $\sigma$-trivial $\sigma$-line bundle $L_r$ (see 6.12).

**Lemma 10.3.** *There is a unique left* $\mathbf{H}$*-module structure on* $\mathcal{A}[\mathbf{X}]$ (*denoted* $h \bullet \xi$) *extending the obvious* $\mathcal{A}$*-module structure and such that for* $r \in S_1$, *and* $L, L_1 \in \mathbf{X}$ *we have*

(10.4h)
$$T_r \bullet L = \begin{cases} \frac{L - {}^rL}{L_r - 1} - q^{\lambda(r)} \frac{L - {}^rLL_r}{L_r - 1} & \textit{if } r \textit{ is type I}, \\ \frac{L - {}^rL}{L_r^2 - 1} - q^{\lambda(r)} \frac{L - {}^rLL_r^2}{L_r^2 - 1} & \\ \quad - L_r \left( q^{\frac{\lambda(r) + \lambda^*(r)}{2}} - q^{\frac{\lambda(r) - \lambda^*(r)}{2}} \right) \frac{L - {}^rL}{L_r^2 - 1} & \textit{if } r \textit{ is type II}. \end{cases}$$

(10.4i)
$$\theta_{L_1} \bullet L = L_1^{-1} L.$$

*Proof.* The proof follows exactly as in [**KL2**, Prop. 3.9] using the Hecke algebra relation 6.17p. ☐

**Lemma 10.4.** *There is a unique left* $\mathbf{H}$*-module structure* (*denoted* $h \circ \xi$) *on* $\mathcal{A}[\mathbf{X}]$ *extending the obvious* $\mathcal{A}$*-module structure and such that for* $s \in S_1$ *and* $L, L_1 \in \mathbf{X}$

(10.4j)
$$T_r \circ L = \begin{cases} \frac{{}^rL - LL_r}{L_r - 1} + q^{\lambda(r)} \frac{LL_r - {}^rLL_r^{-1}}{L_r - 1} & \textit{if } r \textit{ is type I}, \\ \frac{{}^rL - LL_r^2}{L_r^2 - 1} + q^{\lambda(r)} \frac{LL_r^2 - {}^rLL_r^{-2}}{L_r^2 - 1} & \\ \quad - L_r^{-1} \left( q^{\frac{\lambda(r) + \lambda^*(r)}{2}} - q^{\frac{\lambda(r) - \lambda^*(r)}{2}} \right) \frac{{}^rL - L_r^2 L}{L_r^2 - 1} & \textit{if } r \textit{ is type II}. \end{cases}$$

(10.4k)
$$\theta_{L_1} \circ L = L_1 L.$$

*Proof.* An elementary calculation shows that

$$T_r \circ L = (\phi^+)^{-1} T_r^* \bullet (\phi^+ L), \tag{10.4l}$$

and

$$\theta_{L_1} \circ L = (\phi^+)^{-1} (\theta_{L_1}^* \bullet (\phi^+ L)). \tag{10.4m}$$

Here $*$ is as in 6.22 and $\phi^+$ is as in 10.2. We note that to perform the calculation one must use the fact that $\mathcal{A}[\mathbf{X}]$ is a an integral domain. Now the lemma follows from 10.3 and 6.22. $\square$

**10.5. A Computation.** In this section we compute $T_r\xi$ explicitly for $r \in S_1$ and $\xi \in {}^\sigma\mathbf{K}_0^M(\mathcal{B} \times \mathcal{B})$ of the form $L_1 \boxtimes L_2$ (see 9.6).

*Case* 1. $r$ is of type I. Let $\xi = L_1 \boxtimes L_2$. Then by 6.14d (and 5.12)

$$({}_r\phi)^*({}_r\phi)_*(\xi) = \left(\frac{(L_r)^{-d} - 1}{(L_r)^{-1} - 1} L_1\right) \boxtimes L_2, \tag{10.5n}$$

where $d$ is the twisted Euler characteristic (see 5.21) of $L_1$ restricted to any fiber of the map $\pi_r : \mathcal{B} \to \mathcal{P}_r$ (6.3); and ${}_r\phi : \mathcal{B} \times \mathcal{B} \to \mathcal{P}_r \times \mathcal{B}$ is $\pi_r \times 1$. Using 6.16m and 7.11e we calculate

(10.5o)

$$\begin{aligned} T_r\xi &= (\mathbf{q}^{\lambda(r)} - {}^r\tau)(\xi) \\ &= \mathbf{q}^{\lambda(r)} L_1 \boxtimes L_2 - (1 - \mathbf{q}^{\lambda(r)} L_r^{-1} \boxtimes 1)({}_r\phi)^*({}_r\phi)_*(L_1 \boxtimes L_2) \\ &= \mathbf{q}^{\lambda(r)} L_1 \boxtimes L_2 + \left(\frac{L_r^{-d} - 1}{1 - L_r^{-1}} L_1 - \mathbf{q}^{\lambda(r)} L_r^{-1} \frac{L_r^{-d} - 1}{1 - L_r^{-1}} L_1\right) \boxtimes L_2 \\ &= \left(\frac{L_r^{-d} - 1}{1 - L_r^{-1}} L_1 + \mathbf{q}^{\lambda(r)} \left(L_1 + L_r^{-1} \frac{1 - L_r^{-d}}{1 - L_r^{-1}} L_1\right)\right) \boxtimes L_2. \end{aligned}$$

Now multiply the last line by $\frac{L_r}{L_r}$ and use the fact that ${}^rL = LL_r^{-d+1}$ (see 6.13c) and continue

(10.5p)

$$\begin{aligned} &= \left(\frac{{}^rL_1 - L_1 L_r}{L_r - 1} + \mathbf{q}^{\lambda(r)} \left(\frac{L_1(L_r - 1)}{L_r - 1} - \frac{L_1(1 - L_r^{-d})}{L_r - 1}\right)\right) \boxtimes L_2 \\ &= \left(\frac{{}^rL_1 - L_1 L_r}{L_r - 1} + \mathbf{q}^{\lambda(r)} \frac{L_1 L_r - {}^rL_1 L_r^{-1}}{L_r - 1}\right) \boxtimes L_2, \end{aligned}$$

which proves Case 1.

*Case* 2. $r$ is of type II.

As in Case 1, we consider $\xi \in {}^{\sigma}\mathbf{K}_0^M(\mathcal{B}\times\mathcal{B})$ of the form $\xi = L_1 \boxtimes L_2$. By 6.14d we have

$$(10.5q)\qquad ({}_r\phi)^*({}_r\phi)_*\xi = \left(\frac{(L_r)^{-2d}-1}{(L_r)^{-2}-1}\right) L_1 \boxtimes L_2,$$

where $d$ is the twisted Euler characteristic of $L_1$ restricted to any fiber of $\pi_r : \mathcal{B} \to \mathcal{P}_r$. Then using 6.14d we have

(10.5r)
$$\begin{aligned}
T_r\xi &= (\mathbf{q}^{\lambda(r)} -^r \tau)(L_1 \boxtimes L_2)\\
&= \mathbf{q}^{\lambda(r)} L_1 \boxtimes L_2 - \Big[1 - \left(\mathbf{q}^{\frac{\lambda(r)+\lambda^*(r)}{2}} - \mathbf{q}^{\frac{\lambda(r)-\lambda^*(r)}{2}}\right) L_r^{-1}\\
&\qquad - \mathbf{q}^{\lambda(r)} L_r^{-2}\Big] \frac{L_r^{-2d}-1}{L_r^{-2}-1} L_1 \boxtimes L_2\\
&= \left[\frac{L_r^{-2d}-1}{1-L_r^{-2}} L_1 + \mathbf{q}^{\lambda(r)}\left(\frac{1-L_r^{-2}}{1-L_r^{-2}} L_1 - L_r^{-2}\frac{L_r^{-2d}-1}{1-L_r^{-2}} L_1\right)\right.\\
&\qquad \left. - \left(\mathbf{q}^{\frac{\lambda(r)+\lambda^*(r)}{2}} - \mathbf{q}^{\frac{\lambda(r)-\lambda^*(r)}{2}}\right) L_1 L_r^{-1} \frac{L_r^{-2d}-1}{1-L_r^{-2}}\right] \boxtimes L_2.
\end{aligned}$$

Now multiply the last line by $\frac{L_r^2}{L_r^2}$ and use the fact that ${}^rL = LL_r^{-2d+2}$ (see 6.13c) and continue

(10.5s)
$$\begin{aligned}
&= \left[\frac{L_r^{-2d+2}L_1 - L_r^2 L_1}{L_r^2-1} + \mathbf{q}^{\lambda(r)}\left(\frac{L_1L_r^2 - L_1 - L_r^{-2d}L_1 + L_1}{L_r^2-1}\right)\right.\\
&\qquad \left. -L_r^{-1}\left(\mathbf{q}^{\frac{\lambda(r)+\lambda^*(r)}{2}} - \mathbf{q}^{\frac{\lambda(r)-\lambda^*(r)}{2}}\right) \frac{L_r^{-2d+2}L_1 - L_r^2L_1}{L_r^2-1}\right] \boxtimes L_2\\
&= \left[\frac{{}^rL_1 - L_r^2L_1}{L_r^2-1} + \mathbf{q}^{\lambda(r)}\left(\frac{L_r^2L_1 -^r L_1L_r^{-2}}{L_r^2-1}\right)\right.\\
&\qquad \left. -L_r^{-1}\left(\mathbf{q}^{\frac{\lambda(r)+\lambda^*(r)}{2}} - \mathbf{q}^{\frac{\lambda(r)-\lambda^*(r)}{2}}\right) \frac{{}^rL_1 - L_r^2L_1}{L_r^2-1}\right] \boxtimes L_2,
\end{aligned}$$

which proves Case 2.

**10.6. Proof of Proposition 10.2.** We identify ${}^{\sigma}\mathbf{K}_0^M(\mathcal{B})$ with $\mathcal{A}[\mathbf{X}]$ as in 6.10 and so obtain ${}^{\sigma}\mathbf{K}_0^M(\mathcal{B}\times\mathcal{B}) = \mathcal{A}[\mathbf{X}] \underset{\mathcal{A}[\mathbf{X}]^{W^\Gamma}}{\otimes} \mathcal{A}[\mathbf{X}]$ by 9.6. With these identifications and the calculations of 10.5 we see that the endomorphism $\xi \to T_r\xi$ (resp. $\xi \to \xi T_r$) in 10.2 is just

$$L_1 \boxtimes L_2 \to (T_r \circ L_1) \boxtimes L_2 \text{ (resp. } L_1 \boxtimes L_2 \to L_1 \boxtimes (T_r \circ L_2)).$$

Similarly the endomorphism $\xi \to \theta_L\xi$ (resp. $\xi \to \xi\theta_L$) in 10.2 is just $L_1 \boxtimes L_2 \to (\theta_L \circ L_1) \boxtimes L_2$ (resp. $L_1 \boxtimes (\theta_L \circ L_2)$).

Therefore 10.2(a) follows from 10.4 (for the left action) and from 10.4 and 6.21 (for the right action). The fact that in the resulting $\mathbf{H}$-module structure the action of the centre $\mathcal{A}[\mathbf{X}]^{W^\Gamma} = {}^\sigma\mathbf{R}_M$ coincides with the natural ${}^\sigma\mathbf{R}_M$-module structure on ${}^\sigma\mathbf{K}_0^M(\mathcal{B} \times \mathcal{B})$ follows immediately from the definitions.

We also see that the $\mathbf{H}$-module structures in 10.2(a) satisfy

$$h(L_1 \boxtimes L_2) = (h \circ L_1) \boxtimes L_2, \tag{10.6t}$$

$$(L_1 \boxtimes L_2)h = L_1 \boxtimes (\tilde{h} \circ L_2), \tag{10.6u}$$

for $h \in \mathbf{H}, L_1, L_2 \in \mathbf{X}, h \to \tilde{h}$ as in 6.21.

**10.7. A Lemma.** Following [**KL2**] for each $L \in \mathbf{X}$ we set $\mathrm{Alt}(L) = \sum_{w\in W^\Gamma} \epsilon_w {}^wL$, where $\epsilon_w$=sign of $w$.

Let $L_\rho \in \mathbf{X}$ be the element such that $L_\rho^2$ is the product of all positive roots in $\Sigma_1$. We consider the $\mathcal{A}[\mathbf{X}]^{W^\Gamma}$-bilinear symmetric pairing $(\ ,\ ) : \mathcal{A}[\mathbf{X}] \times \mathcal{A}[\mathbf{X}] \to \mathcal{A}[\mathbf{X}]^{W^\Gamma}$ defined by

$$(L_1, L_2) = \mathrm{Alt}(L_1L_2L_\rho) \cdot \mathrm{Alt}(L_\rho)^{-1},$$

which coincides under the identification $\mathcal{A}[\mathbf{X}] = {}^\sigma\mathbf{K}_0^M(\mathcal{B})$ with the pairing given in the proof of 9.6. Then as in [**KL2**, Sec. 3.12] we easily check that

$$(T_r^* \bullet L_1, L_2) = (L_1, T_r \circ L_2), \text{ for all } r \in S_1,\ L_1, L_2 \in \mathbf{X}. \tag{10.7v}$$

**Lemma 10.5.** *For all $h \in \mathbf{H}, \eta, \eta' \in \mathcal{A}[\mathbf{X}]$, we have*

$$((\theta_{\phi^+}h) \circ \eta, \eta') = (\eta, (\theta_{\phi^+}\tilde{h}) \circ \eta').$$

*Proof.* Follows exactly as in [**KL2**]. It is enough to check on the generators $T_r$ and $\theta_L$ of $\mathbf{H}$. The case where $h = \theta_L$ is trivial. The case where $h = T_r$ follows from 10.4l. □

**10.8. Proof of Theorem 10.1.** We now prove 10.1(b). By 10.5, for any $h \in \mathbf{H}$, the $\mathcal{A}[\mathbf{X}]^{W^\Gamma}$ endomorphism $\phi : \eta \mapsto (\theta_{\phi^+}h) \circ \eta$ of $\mathcal{A}[\mathbf{X}]$ has as transpose with respect to $(\ ,\ )$ the endomorphism ${}^t\phi : \eta \mapsto \widetilde{h\theta_{\phi^+}} \circ \eta$ of $\mathcal{A}[\mathbf{X}]$. By 9.3, we have $(\phi \otimes 1)A = (1 \otimes {}^t\phi)A$ where $A$ is regarded as an element of $\mathcal{A}[\mathbf{X}] \underset{\mathcal{A}[\mathbf{X}]^{W^\Gamma}}{\otimes} \mathcal{A}[\mathbf{X}]$. In view of 10.6t, the last equality is: $\theta_{\phi^+}hA = Ah\theta_{\phi^+}$. This proves 10.2(b). Thus 10.2 and therefore also 10.1(a) and (b) are proved.

**10.9. Preparation for Proof of Theorem 10.1(c).** The rest of this chapter is concerned with the proof of 10.1(c). To simplify notations, *in the rest of this section we shall write* $\mathbf{K}()$ *instead of* ${}^{\sigma}\mathbf{K}^{M}()$ *and* $\mathbf{K}_1()$ *instead of* ${}^{\sigma}\mathbf{K}_1^{M}()$. This section, with the exception of the ${}^{\sigma}\mathbf{K}$-functor, is [**KL2**], pp. 178-182, and has been reproduced here with the permission of Kazhdan and Lusztig.

We define $Z_{\leq w} = \underset{y\leq w}{\cup} Z_y$. We define similarly $Z_{<w}, Z_{\geq w}, Z_{>w}$.

A subset $I$ of $W$ is said to be closed (resp. open) if $w_1 \in I, w_2 \leq w_1 \Rightarrow w_2 \in I$ (resp. if $w_1 \in I, w_2 \geq w_1 \Rightarrow w_2 \in I$). A subset $I$ of $W$ is said to be locally closed if it is of the form $I' \cap I''$ where $I' \subset W$ is closed and $I'' \subset W$ is open. For a locally closed subset $I$ of $W$ we define $Z_I = \underset{y\in I}{\cup} Z_y$. Then $Z_I$ is a locally closed subvariety of $Z$; moreover, $Z_I$ is a closed (resp. open) subvariety if $I$ is a closed (resp. open) subset of $W$. For example $Z_{\leq w}, Z_{<w}$ above are closed and $Z_{\geq w}, Z_{>w}$ are open. Note that $Z_I$ an $M$-stable subvariety of $Z$ if and only if $I$ is a $\sigma$-invariant subset of $W$.

**Lemma 10.6.** *If $I$ is a locally closed $\sigma$-invariant subset of $W$, then* $\mathbf{K}_1(Z_I) = 0$ *and* $\mathbf{K}(Z_I)$ *is a projective* ${}^{\sigma}\mathbf{R}_M$*-module of rank* $|I \cap W^{\Gamma}| \cdot |W^{\Gamma}|$.

*Proof.* Using the partition of $Z_I$ into the pieces $Z_y$ $(y \in I \cap W^{\Gamma})$ and $Z_{\mathcal{O}(y)}$ $(y \notin I \cap W^{\Gamma};\ \mathcal{O}(y) = \sigma\text{-orbit of } y)$ we may proceed as in the proof of Proposition 9.4. $\square$

**10.10. Facts I.** We now state some results which are immediate consequences of 10.6 and the exact sequences 5.16.

Let $w \in W^{\Gamma}$. Then the imbeddings

$$i_{\leq w} : Z_{\leq w} \hookrightarrow Z \text{ and } j_w : Z_w \hookrightarrow Z_{\geq w}$$

induce injective maps

$$\text{(10.10w)} \quad (i_{\leq w})_* : \mathbf{K}(Z_{\leq w}) \to \mathbf{K}(Z), \quad (j_w)_* : \mathbf{K}(Z_w) \to \mathbf{K}(Z_{\geq w}).$$

**10.11. Facts II.** The embeddings $i_{\geq w} : Z_{\geq w} \to Z$ and $j'_w : Z_w \hookrightarrow Z_{\leq w}$ induce surjective maps

$$\text{(10.11x)} \quad (i_{\geq w})^* : \mathbf{K}(Z) \to \mathbf{K}(Z_{\geq w}), \quad (j'_w)^* : \mathbf{K}(Z_{\leq w}) \to \mathbf{K}(Z_w).$$

$$(i_{\geq w})^*(i_{\leq w})_* = (j_w)_* j'^*_w : \mathbf{K}(Z_{\leq w}) \to \mathbf{K}(Z_{\geq w}), \quad \text{by base change, (5.14).}$$

**10.12. Facts III.** If $I \subset W$ is closed and $\sigma$-invariant, then the image of the injective map $(i_I)_* : \mathbf{K}(Z_I) \to \mathbf{K}(Z)$ induced by $i_I : Z_I \to Z$ coincides with the sum of the images of $(i_{\leq w})_*$ in 10.10w for all $w \in I \cap W^\Gamma$.

**10.13. Conclusion of Proof of Theorem 10.1(c).** We fix $r \in S_1$ (see 3.20) and $w \in W^\Gamma$ such that $rw < w$. Note that $l(w) = l(rw) + l(r)$. Here $l$ represents the length function on $w$.

It is clear that $Z_{\leq w}$ and $Z_{\geq w}$ are left-$r$-saturated subvarieties of $Z$ (see 7.8) hence we have natural operations

$$ {}^r\tau : \mathbf{K}(Z_{\leq w}) \to \mathbf{K}(Z_{\leq w}), \quad {}^r\tau : \mathbf{K}(Z_{\geq w}) \to \mathbf{K}(Z_{\geq w}), $$

and these are compatible with ${}^r\tau : \mathbf{K}(Z) \to \mathbf{K}(Z)$ via $(i_{\leq w})_*(i_{\geq w})^*$.

**Lemma 10.7.** *Let $r \in S_1, w \in W^\Gamma$ be such that $rw < w$. Then there is a unique ${}^\sigma\mathbf{R}_M$-isomorphism $\rho_1 : \mathbf{K}(Z_{rw}) \xrightarrow{\sim} \mathbf{K}(Z_w)$ such that the diagram*

$$\begin{tikzcd}
& \mathbf{K}(Z_{rw}) \arrow[rr, "\rho_1", "\approx"'] \arrow[dl, "(j_{rw})_*"] & & \mathbf{K}(Z_w) \arrow[dr, "(j_w)_*"] & \\
\mathbf{K}(Z_{\geq rw}) \arrow[rr, "{}^r\tau"] & & \mathbf{K}(Z_{\geq rw}) \arrow[rr, "d^*"] & & \mathbf{K}(Z_{\geq w}) \\
& \mathbf{K}(Z) \arrow[ul, "i^*_{\geq rw}"] \arrow[rr, "{}^r\tau"] & & \mathbf{K}(Z) \arrow[ul] \arrow[ur, "i^*_{\geq w}"'] &
\end{tikzcd}$$

*is commutative.* (*Here, $d : Z_{\geq w} \hookrightarrow Z_{\geq rw}$ is the inclusion.*)

*Proof.* The uniqueness of $\rho_1$ is clear since $(j_w)_*$ is injective 10.10w. To prove existence we introduce some notations. Let

$$\begin{aligned}
\tilde{Z}_{\geq rw} &= \text{set of all } (u, B', B_1, B'') \text{ with } (u, B_1, B'') \in Z_{\geq rw} \text{ and } B' \xrightarrow{\leq r} B_1, \\
{}_u\tilde{Z}_{\geq rw} &= \{(u, B', B_1, B'') \in \tilde{Z}_{\geq rw} | u \in B'\}, \\
\tilde{Z}_{rw} &= \text{set of all } (u, B', B_1, B'') \text{ with } (u, B_1, B'') \in Z_{rw} \text{ and } B' \xrightarrow{\leq r} B_1, \\
{}_u\tilde{Z}_{rw} &= \{(u, B', B_1, B'') \in \tilde{Z}_{rw} | u \in B'\}, \\
Z'_{rw} &= \{(u, B', B_1, B'') \in \tilde{Z}_{rw} | B' \xrightarrow{r} B_1\}, \\
{}_uZ'_{rw} &= \{(u, B', B_1, B'') \in Z'_{rw} | u \in B'\}.
\end{aligned}$$

Let $\tilde{\mathcal{E}}$ be the complex of $M$-equivariant vector bundles on $\tilde{Z}_{\geq rw}$ obtained by taking the pullback of $\hat{\mathcal{E}}^r$ (see 7.3) under the map $\tilde{Z}_{\geq rw} \to \Lambda^r$,

$$ (u, B', B_1, B'') \mapsto (u, B'). $$

Then $\tilde{\mathcal{E}}$ is acyclic on $\tilde{Z}_{\geq rw} \smallsetminus {}_u\tilde{Z}_{\geq rw}$. The restriction of $\tilde{\mathcal{E}}$ to the subvariety $\tilde{Z}_{rw}$ (resp. $Z'_{rw}$) is denoted $\tilde{\mathcal{E}}_1$ (resp. $\tilde{\mathcal{E}}_2$). By 5.6 we have

$$\tilde{\mathcal{E}}\otimes : \mathbf{K}(\tilde{Z}_{\geq rw}) \to \mathbf{K}({}_u\tilde{Z}_{\geq rw}),$$

$$\tilde{\mathcal{E}}_1\otimes : \mathbf{K}(\tilde{Z}_{rw}) \to \mathbf{K}({}_u\tilde{Z}_{rw}),$$

$$\tilde{\mathcal{E}}_2\otimes : \mathbf{K}(Z'_{rw}) \to \mathbf{K}({}_uZ'_{rw}).$$

These fit into the following commutative diagram

$$\begin{tikzcd}
\mathbf{K}(Z_{\geq rw}) \arrow[r, "f^*"] & \mathbf{K}\left(\tilde{Z}_{\geq rw}\right) \arrow[r, "\tilde{\mathcal{E}}\otimes"] & \mathbf{K}\left({}_u\tilde{Z}_{\geq rw}\right) \arrow[r, "g_*"] & \mathbf{K}(Z_{\geq rw}) \arrow[r, "d^*"] & \mathbf{K}(Z_{\geq w}) \\
\mathbf{K}(Z_{rw}) \arrow[u] \arrow[r, "f_1^*"] \arrow[dr, "f_2^*"'] & \mathbf{K}\left(\tilde{Z}_{rw}\right) \arrow[u] \arrow[r, "\tilde{\mathcal{E}}_1\otimes"] \arrow[d] & \mathbf{K}\left({}_u\tilde{Z}_{rw}\right) \arrow[u] \arrow[r, "d_1^*"] \arrow[ur, "(g_1)_*"] \arrow[d] & \mathbf{K}(Z_w) \arrow[ur] & \\
 & \mathbf{K}(Z'_{rw}) \arrow[r, "\tilde{\mathcal{E}}_2\otimes"] & \mathbf{K}({}_uZ'_{rw}) \arrow[ur, "(g_2)_*"'] & &
\end{tikzcd}$$

Here $f_1, f_2$ are the restrictions of the flag variety bundle

$$f : \tilde{Z}_{\geq rw} \to Z_{\geq rw}, \quad ((u, B', B_1, B'') \mapsto (u, B_1, B''))$$

to $\tilde{Z}_{rw}, Z'_{rw}$; $g : {}_u\tilde{Z}_{\geq rw} \to Z_{\geq rw}$ is defined by $g(u, B', B_1, B'') = (u, B', B'')$ and $g_1, g_2$ are its restrictions to ${}_u\tilde{Z}_{rw}, {}_uZ'_{rw}$. The map $d_1 : Z_w \hookrightarrow {}_u\tilde{Z}_{rw}$ is defined by $d_1(u, B', B'') = (u, B', B_1, B'')$ where $B_1$ is such that $B' \xrightarrow{r} B_1 \xrightarrow{rw} B''$. The remaining un-named upward arrows are direct image maps induced by obvious closed imbeddings; the un-named downward arrows are inverse image maps induced by obvious open imbeddings.

The compositions of the arrows on the highest line in the diagram is the same (by the equivalence of the definitions 7.9a, 7.9) as the composition of the arrows $d^{*r}\tau$ in the first diagram. We define $\rho_1$ to be the composition of the arrows in the second to highest horizontal line in the above diagram. It is clear that with this choice of $\rho_1$, as we have just defined it, the first diagram is commutative. From the above diagram we see that it is enough to show that $f_2^*, \tilde{\mathcal{E}}_2\otimes, (g_2)_*$ are isomorphisms. For $(g_2)_*$ this is clear since $g_2$ is an isomorphism of varieties. The map $f_2$ is a unipotent bundle see 5.5; hence $f_2^*$ is an isomorphism by (5.5). Let $Z''_{rw}$ be the set of all $(u, B', B_1, B'', T)$ where $(u, B', B_1, B'') \in Z'_{rw}$ and $T$ is a maximal torus in $B_1 \cap B''$. Let ${}_uZ''_{rw}$ be the subvariety of $Z''_{rw}$ defined by the condition $u \in B'$. Let $\pi : Z''_{rw} \to Z'_{rw}$

be the unipotent bundle $(u, B', B_1, B'', T) \to (u, B', B_1, B'')$ and let $\pi_1$ be its restriction ${}_u Z''_{rw} \to {}_u Z'_{rw}$ (again a unipotent bundle). Let $\tilde{\mathcal{E}}_3$ be the complex on $Z''_{rw}$ obtained by pulling back $\tilde{\mathcal{E}}_2$ under $\pi$. It gives rise by (5.6) to a natural map

$$\tilde{\mathcal{E}}_3 \otimes : \mathbf{K}(Z''_{rw}) \to \mathbf{K}({}_u Z''_{rw}),$$

and we have a commutative diagram

$$\begin{array}{ccc}
\mathbf{K}(Z'_{rw}) & \xrightarrow{\tilde{\mathcal{E}}_2 \otimes} & \mathbf{K}({}_u Z'_{rw}) \\
{\scriptstyle \pi^*}\big\downarrow \approx & & {\scriptstyle \pi^*}\big\downarrow \approx \\
\mathbf{K}(Z''_{rw}) & \xrightarrow{\tilde{\mathcal{E}}_3 \otimes} & \mathbf{K}({}_u Z''_{rw})
\end{array} .$$

To show that $\tilde{\mathcal{E}}_2 \otimes$ is an isomorphism it is then enough to show that $\tilde{\mathcal{E}}_3 \otimes$ is an isomorphism. Now $Z''_{rw}$ is in a natural way a vector bundle over ${}_u Z''_{rw}$ and the natural inclusion ${}_u Z''_{rw} \hookrightarrow Z''_{rw}$ is the zero section of this vector bundle; we are in the situation considered in 5.13 hence $\tilde{\mathcal{E}}_3 \otimes$ is an isomorphism. This completes the proof of the lemma. □

**Lemma 10.8.** *Let $r \in S_1$, $w \in W^\Gamma$ be such that $rw < w$. Then*

(a) *The image of $(i_{\le w})_* : \mathbf{K}(Z_{\le w}) \hookrightarrow \mathbf{K}(Z)$ is stable under ${}^r\tau$.*

(b) *For any $x \in \mathbf{K}(Z_{\le w})$ there exists $x' \in \mathbf{K}(Z_{\le rw})$ such that $\overline{x} - {}^r\tau\overline{x}'$ is in the image of $(i_{<w})_* : \mathbf{K}(Z_{<w}) \to \mathbf{K}(Z)$. (Here $i_{<w} : Z_{<w} \hookrightarrow Z$ is the inclusion and $\overline{x}, \overline{x}'$ denote the images of $x, x'$ under $(i_{\le w})_*, (i_{\le rw})_*$ respectively.)*

*Proof.* (a) follows from 10.13. We now prove (b). Let $x \in \mathbf{K}(Z_{\le w})$. By 10.11x we have $(i_{\ge w})^*\overline{x} = (j_w)_* x_1$ for some $x_1 \in \mathbf{K}(Z_w)$. Let $x_2 = \rho_1^{-1}(x_1) \in \mathbf{K}(Z_{rw})$ (see 10.7). Using again 10.11,10.11x for $rw$, we see that $(j_{rw})_* x_2 = i^*_{\ge w}\overline{x}''$ for some $x'' \in \mathbf{K}(Z_{\le rw})$. From the diagram in the statement of lemma 10.7 we have

$$\begin{aligned}
i^*_{\ge w}{}^r\tau\overline{x}'' &= d^{*r}\tau i^*_{\ge rw}\overline{x}'' = d^{*r}\tau(j_{rw})_* x_2 \\
&= (j_w)_*\rho_1 x_2 = (j_w)_* x_1 = (i_{\ge w})^*\overline{x}.
\end{aligned}$$

Thus $i^*_{\ge w}({}^r\tau\overline{x}'' - \overline{x}) = 0$. Since $\overline{x}, \overline{x}''$ are in the image of $(i_{\le w})_*$, we see from (a) that ${}^r\tau\overline{x} - \overline{x} = (i_{\le w})_* y$ for some $y \in \mathbf{K}(Z_{\le w})$. We have $i^*_{\ge w}(i_{\le w})_* y = 0$ hence by 10.11x, $(j_w)_*(j'_w)^* y = 0$ and by 10.10w we have $(j'_w)^* y = 0$. From the exactness of $\mathbf{K}(Z_{<w}) \to \mathbf{K}(Z_{\le w}) \xrightarrow{(j'_w)^*} \mathbf{K}(Z_w)$, 5.16, we see that $y$ is the image of some $z \in \mathbf{K}(Z_{<w})$ under $\mathbf{K}(Z_{<w}) \to \mathbf{K}(Z_{\le w})$. We have $(i_{\le w})_* y = (i_{<w})_* z$ and the lemma is proved. □

We can now prove the following result which is actually stronger than 10.1a(c).

**Lemma 10.9.** *For any $w \in W^\Gamma$, let $\mathbf{H}_{\leq w}$ be the $\mathcal{A}$-submodule of $\mathbf{H}$ spanned by all $T_{w'}\Theta_L$ $(w' \leq w, w' \in W^\Gamma, L \in \mathbf{X})$ or, equivalently, by all $\Theta_L T_{w'}$ $(w' \leq w, w' \in W^\Gamma, L \in X)$. Let $K_{\leq w}$ be the image of $(i_{\leq w})_* : \mathbf{K}(Z_{\leq w}) \to \mathbf{K}(Z)$. Then $h \mapsto h\mathbf{1}$ defines an isomorphism of $\mathbf{H}_{\leq w}$ onto $K_{\leq w}$.*

*Proof.* In the case where $w = e$, the result follows from 6.11. Assume now that $w \neq e$ and that the result is known for all $w'$ such that $w' < w$ with $w' \in W^\Gamma$. Let $r \in S_1$ be such that $rw < w$. We first show that $\mathbf{H}_{\leq w}\mathbf{1} \subset K_{\leq w}$. Since $K_{\leq w}$ is clearly stable by multiplication by $\Theta_L (L \in \mathbf{X})$ it is enough to show that $T_y\mathbf{1} \in K_{\leq w}, \forall y \leq w$. If $y < w$, then by the induction hypothesis, $T_y\mathbf{1} \in K_{\leq y} \subset K_{\leq w}$. Assume now that $y = w$. By the induction hypothesis, $T_{rw}\mathbf{1} \in K_{\leq w}$. By 10.8(a), we have $T_r K_{\leq w} \subset K_{\leq w}$ hence $T_w\mathbf{1} = T_r T_{rw}\mathbf{1} \subset K_{\leq w}$. Hence our map $\mathbf{H}_{\leq w} \to K_{\leq w}$ is well defined.

Let $x \in K_{\leq w}$. By 10.8(b), there exists $x' \in K_{\leq w}$ such that $x - (\mathbf{q} - T_r)x'$ is in the image of $(i_{<w})_*$, hence, by 10.12 it is of the form $\sum_{i=1}^m x_i''$ where $x_i'' \in K_{\leq w_i}$, $w_i < w$, $(i = 1, 2, \ldots, m)$. By the induction hypothesis we have $x' \in \mathbf{H}_{\leq rw}\mathbf{1}$. Note that $T_r\mathbf{H}_{\leq rw} \subset \mathbf{H}_{\leq w}$. Hence $x \in \mathbf{H}_{\leq w}\mathbf{1}$. Thus, our map $\mathbf{H}_{\leq w} \to K_{\leq w}$ is surjective.

It is an ${}^\sigma\mathbf{R}_M$-linear map between projective ${}^\sigma\mathbf{R}_M$-modules of the same rank $|I \cap W^\Gamma||W^\Gamma|$, where $I = \{w' | w' \leq w\}$. (For $\mathbf{H}_{\leq w}$ this is clear from 6.18,6.20, and proof of 9.6(1), for $K_{\leq w} \cong \mathbf{K}(Z_{\leq w})$ this follows from 10.6.) This implies that $\mathbf{H}_{\leq w} \to K_{\leq w}$ must be an isomorphism. The lemma is proved. □

Since 10.1(c) is the special case of the lemma when $w$ is the longest element of $W$, we see that 10.1(c) is proved.

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