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ESSENTIAL NORMS OF COMPOSITION OPERATORS AND ALEKSANDROV MEASURES

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The essential norm of a composition operator on H^2 is calculated in terms of the Aleksandrov measures of the inducing holomorphic map. The argument provides a purely function-theoretic proof of the equivalence of Sarason's compactness condition for composition operators on L^1 and Shapiro's compactness condition for composition operators on Hardy spaces. An application is given relating the essential norm to angular derivatives.

§1.

If ϕ is a holomorphic map of the unit disk $\mathbb{D}$ into itself, it is a consequence of Littlewood's subordination principle [5] that composition with ϕ induces a bounded operator C_ϕ on each Hardy space H^p. A recurring theme in the study of composition operators has been the search for function theoretic conditions on ϕ which guarantee the compactness of C_ϕ on H^p. It was shown by Shapiro and Taylor [11] that if C_ϕ is compact on H^p for some $0 < p < \infty$, then C_ϕ is compact on H^p for all $0 < p < \infty$, and so it is enough to study compactness on H^2. In this context Shapiro [9] gave an expression for the essential norm of C_ϕ on H^2 in terms of the Nevanlinna counting function of ϕ, thus providing a complete function theoretic characterization of compact composition operators on H^2.

In a different direction Sarason [7] showed how to define the composition operator C_ϕ on the space M of complex Borel measures on the unit circle $\mathbb{T}$. Indeed, if u is the Poisson integral of a complex Borel measure, it is not difficult to see that $u \circ \phi$ is also, and then that the action of C_ϕ is bounded on M. He also showed that C_ϕ acts boundedly on L^1, and that compactness on M is equivalent to compactness on L^1. In the process he gave a function theoretic condition on ϕ equivalent to compactness on L^1. Since $H^1 \subset L^1$, it is evident from the above discussion that Sarason's condition implies Shapiro's. The reverse implication was established by Shapiro and Sundberg [10], and subsequently another more direct proof was found by Sundberg. However, Sarason [8] states that a direct function theoretic proof is still lacking. It is the main purpose of this note to provide such a proof.

Shapiro's expression for the essential norm of C_ϕ on H^2 is

$$\|C_\phi\|_e = \limsup_{|a|\to 1} \sqrt{\frac{N_\phi(a)}{\log \frac{1}{|a|}}},$$

where $N_\phi(z)$ is Nevanlinna's counting function for ϕ, given by

$$N_\phi(z) = \sum_{\phi(\zeta)=z} \log \frac{1}{|\zeta|}.$$

In particular C_ϕ is compact on H^2 if and only if $\limsup_{|a|\to 1} \frac{N_\phi(a)}{\log \frac{1}{|a|}} = 0$. In the course of proving this Shapiro established the inequality

$$\|C_\phi\|_e^2 \geq \limsup_{|a|\to 1} \|C_\phi f_a\|_2^2 \geq \limsup_{|a|\to 1} \frac{N_\phi(a)}{\log \frac{1}{|a|}},$$

where $f_a(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z}$ is the normalized kernel function for $a \in \mathbb{D}$. This, together with the rest of his proof, shows that

$$\|C_\phi\|_e = \limsup_{|a|\to 1} \|C_\phi f_a\|_2.$$

It is important to note that although Shapiro's proof of this equation is not purely function theoretic, his methods can be used to provide such a proof.

In the next section Sarason's condition will be derived from the alternate condition on the kernel functions. In the process a third expression for the essential norm of C_ϕ will be derived in terms of the singular parts of the Aleksandrov measures of ϕ. An application related to angular derivatives will be given in Section 3.

§2.

Sarason's compactness condition can be given in two equivalent formulations. In the first instance if $f \in L^1$ has harmonic extension u to the unit disk, then $C_\phi f$ is the boundary function of the harmonic function $u \circ \phi$. The Poisson formula gives

$$u(\phi(z)) = \int_{\mathbb{T}} \frac{1-|\phi(z)|^2}{|\zeta-\phi(z)|^2} f(\zeta)\, dm(\zeta), \qquad z \in \mathbb{D},$$

where m denotes the normalized Lebesgue measure on the unit circle $\mathbb{T}$. Sarason proceeds by analyzing the kernel

$$\frac{1-|\phi(\xi)|^2}{|\zeta-\phi(\xi)|^2}, \qquad \zeta, \xi \in \mathbb{T}.$$

He shows that C_ϕ is compact on L^1 if and only if

$$\int_{\mathbb{T}} \frac{1-|\phi(\xi)|^2}{|\zeta-\phi(\xi)|^2}\,dm(\xi) = 1 \tag{2.1}$$

for all $\zeta \in \mathbb{T}$, at least when $\phi(0) = 0$. The main ingredient in his proof is a theorem of Dunford and Pettis which asserts that a sequence of functions (f_n) in L^1 converges in norm to $f \in L^1$ if $f_n \to f$ and $\|f_n\|_1 \to \|f\|_1$. It is not difficult to show that in general C_ϕ is compact on L^1 if and only if

$$\int_{\mathbb{T}} \frac{1-|\phi(\xi)|^2}{|\zeta-\phi(\xi)|^2}\,dm(\xi) = \Re\left(\frac{\zeta+\phi(0)}{\zeta-\phi(0)}\right) \tag{2.2}$$

for all $\zeta \in \mathbb{T}$.

The other formulation, which is easily seen to be equivalent, results from consideration of measures studied by Aleksandrov [2]. For each $\alpha \in \mathbb{T}$, since $\|\phi\|_\infty \le 1$, $u_\alpha(z) = \Re\left(\frac{\alpha+\phi(z)}{\alpha-\phi(z)}\right)$ is a positive harmonic function, and so, by Herglotz's theorem, is the Poisson integral of a positive measure τ_α. This measure has total variation $\|\tau_\alpha\| = \Re\left(\frac{\alpha+\phi(0)}{\alpha-\phi(0)}\right) = u_\alpha(0)$ and Lebesgue decomposition $d\tau_\alpha = h_\alpha\,dm + d\sigma_\alpha$, where $h_\alpha \in L^1$ and $\sigma_\alpha \perp m$. Since

$$h_\alpha(\xi) = \lim_{r\to 1} u_\alpha(r\xi) = \frac{1-|\phi(\xi)|^2}{|\alpha-\phi(\xi)|^2},$$

for almost every $\xi \in \mathbb{T}$, it follows from (2.2) that C_ϕ is compact on L^1 if and only if $\sigma_\alpha = 0$ for all $\alpha \in \mathbb{T}$, or, what is the same thing, if and only if the Aleksandrov measures τ_α are all absolutely continuous. Sarason calls this the absolute continuity condition [8].

It follows from the Lebesgue decomposition of τ_α that

$$\begin{aligned}\|\sigma_\alpha\| &= \|\tau_\alpha\| - \int_{\mathbb{T}} h_\alpha(\xi)\,dm(\xi)\\ &= \Re\left(\frac{\alpha+\phi(0)}{\alpha-\phi(0)}\right) - \int_{\mathbb{T}} \frac{1-|\phi(\xi)|^2}{|\alpha-\phi(\xi)|^2}\,dm(\xi).\end{aligned}$$

On the other hand

$$\begin{aligned}\|C_\phi f_{r\alpha}\|_2^2 &= \int_{\mathbb{T}} \frac{1-r^2}{|\alpha-r\phi(\xi)|^2}\,dm(\xi)\\ &= \int_{\mathbb{T}} \frac{1-r^2|\phi(\xi)|^2}{|\alpha-r\phi(\xi)|^2}\,dm(\xi) - r^2\int_{\mathbb{T}} \frac{1-|\phi(\xi)|^2}{|\alpha-r\phi(\xi)|^2}\,dm(\xi)\\ &= \Re\left(\frac{\alpha+r\phi(0)}{\alpha-r\phi(0)}\right) - r^2\int_{\mathbb{T}} \frac{1-|\phi(\xi)|^2}{|\alpha-r\phi(\xi)|^2}\,dm(\xi). \qquad (2.3)\end{aligned}$$

Clearly, since $|\phi(0)| < 1$, $\lim_{r\to 1} \Re\left(\frac{\alpha+r\phi(0)}{\alpha-r\phi(0)}\right) = \Re\left(\frac{\alpha+\phi(0)}{\alpha-\phi(0)}\right)$ uniformly in α. Now if $0 < r < s \le 1$ and $|w| \le 1$, it is geometrically obvious that

$$\left|\frac{1}{s} - w\right| < \left|\frac{1}{r} - w\right|,$$

and so

$$\frac{r^2}{|1-rw|^2} < \frac{s^2}{|1-sw|^2}. \tag{2.4}$$

It follows that $r^2 \frac{1-|\phi(\xi)|^2}{|\alpha-r\phi(\xi)|^2}$ increases monotonically to $\frac{1-|\phi(\xi)|^2}{|\alpha-\phi(\xi)|^2}$ for almost every $\xi \in \mathbb{T}$, and so

$$\lim_{r\to 1} r^2 \int_{\mathbb{T}} \frac{1-|\phi(\xi)|^2}{|\alpha - r\phi(\xi)|^2}\, dm(\xi) = \int_{\mathbb{T}} \frac{1-|\phi(\xi)|^2}{|\alpha-\phi(\xi)|^2}\, dm(\xi). \tag{2.5}$$

Hence

$$\lim_{r\to 1} \|C_\phi f_{r\alpha}\|_2^2 = \|\sigma_\alpha\|. \tag{2.6}$$

In particular

$$\|\sigma_\alpha\| \le \limsup_{|a|\to 1} \|C_\phi f_a\|_2^2 = \|C_\phi\|_e^2 \tag{2.7}$$

for all $\alpha \in \mathbb{T}$, and so Shapiro's compactness condition implies Sarason's.

In order to prove the reverse inequality set

$$A = \limsup_{|a|\to 1} \|C_\phi f_a\|_2^2$$

and fix $\epsilon > 0$. For each r, $0 < r < 1$, let

$$E_r = \left\{ \alpha \in \mathbb{T} \mid \Re\left(\frac{\alpha+\phi(0)}{\alpha-\phi(0)}\right) - r^2 \int_{\mathbb{T}} \frac{1-|\phi(\xi)|^2}{|\alpha-r\phi(\xi)|^2}\, dm(\xi) \ge A - 2\epsilon \right\}.$$

By continuity each E_r is a closed set. Since $r^2 \int_{\mathbb{T}} \frac{1-|\phi(\xi)|^2}{|\alpha-r\phi(\xi)|^2}\, dm(\xi)$ is an increasing function of r for each α, it follows that $E_r \supset E_s$ whenever $r < s < 1$. Choose r_0 so that

$$\left|\Re\left(\frac{\alpha+\phi(0)}{\alpha-\phi(0)}\right) - \Re\left(\frac{\alpha+r\phi(0)}{\alpha-r\phi(0)}\right)\right| < \epsilon$$

for all α if $r_0 \le r < 1$. Now if $r_0 \le r < 1$, there exists r_1, $r \le r_1 < 1$, and $\alpha \in \mathbb{T}$ such that $\|C_\phi f_{r_1\alpha}\|_2^2 > A - \epsilon$, and so $\alpha \in E_{r_1} \subset E_r$. In particular each E_r is nonempty. By compactness there exists $\alpha_0 \in \cap_{0<r<1} E_r$. Hence, passing to the limit, $\|\sigma_{\alpha_0}\| \ge A - 2\epsilon$. Combining this with (2.7) yields

$$\|C_\phi\|_e = \sup_{\alpha\in\mathbb{T}} \sqrt{\|\sigma_\alpha\|}. \tag{2.8}$$

§3.

Equation (2.8) leads quickly to a lower bound for $\|C_\phi\|_e$ in terms of angular derivatives. The angular derivative $\phi'(\zeta)$ for $\zeta \in \mathbb{T}$ with $|\phi(\zeta)| = 1$ is the limit $\lim_{z\to\zeta} \frac{\phi(\zeta)-\phi(z)}{\zeta - z}$, provided the limit exists nontangentially. Let $S_\alpha = \{\, \zeta \in \mathbb{T} \mid \phi(\zeta) = \alpha \,\}$ for each $\alpha \in \mathbb{T}$. The Julia-Carathéodory theorem asserts that for each $\zeta \in S_\alpha$, $\phi'(\zeta)$ exists or is infinite. In any case the proof of the Julia-Carathéodory theorem on p. 11 of [1] shows that $\tau_\alpha(\{\zeta\}) = \frac{1}{|\phi'(\zeta)|}$ for each $\zeta \in S_\alpha$ (with the usual convention that $\frac{1}{|\phi'(\zeta)|} = 0$ if $\phi'(\zeta) = \infty$). In particular the quantity $\delta(\alpha) = \sum_{\zeta \in S_\alpha} \frac{1}{|\phi'(\zeta)|}$ is the variation of the purely atomic part of τ_α and hence is finite. Now (2.8) yields

$$\|C_\phi\|_e^2 \geq \sup_{\alpha\in\mathbb{T}} \delta(\alpha), \tag{3.1}$$

an estimate first obtained by Cowen [3, 4], who also provided the upper bound $2\sup_{\alpha\in\mathbb{T}} \delta(\alpha)$ if ϕ' is continuous on $\overline{\mathbb{D}}$. Actually, if τ_α has continuous singular part for no $\alpha \in \mathbb{T}$, then in fact

$$\|C_\phi\|_e^2 = \sup_{\alpha\in\mathbb{T}} \delta(\alpha). \tag{3.2}$$

Since σ_α is supported on $\overline{S_\alpha}$, this happens in particular whenever S_α is a finite set for each α. A theorem of Novinger and Oberlin [6] shows that this is the case if ϕ satisfies a Lipschitz condition of order 1 (see also [13]). Hence in this case (3.2) holds, improving Cowen's upper bound. It should be remarked that Shapiro has used his calculation of $\|C_\phi\|_e$ and the Julia-Carathéodory theorem to give a proof of the result of Novinger and Oberlin.

Finally it would be of interest to see a direct proof of (2.7). Since it is relatively easy to prove that $\|C_\phi\|_e^2 \geq \|\sigma_\alpha\|$ for each α, this is a question of providing a proof of the inequality $\sup_{\alpha\in\mathbb{T}} \|\sigma_\alpha\| \geq \|C_\phi\|_e$ which does not use Nevanlinna's counting function.

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