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CONSTRUCTING ESSENTIAL LAMINATIONS IN 2-BRIDGE KNOT SURGERED 3-MANIFOLDS

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A nontrivial knot that can be drawn with only two relative maxima in the vertical direction is called a 2-bridge knot, and one that can be drawn on a torus is called a torus knot. Loosely speaking, a lamination in a manifold M is a foliation of M, except that it can have a nonempty open complement in M, and very loosely speaking, the lamination is essential if each leaf of it L is incompressible, i.e. inclusion of L into M induces an injective homomorphism from $\pi_1(L)$ into $\pi_1(M)$. Our main result is:

Theorem 2. ***Every* 3*-manifold obtained by surgery on a non-torus* 2*-bridge knot admits essential laminations.***

Some immediate corollaries are that these manifolds are covered by $\mathbf{R}^3$, and have infinite fundamental group. So Property P is true for non-torus 2-bridge knots, i.e. surgery on these knots never yields a homotopy 3-sphere, or a counterexample to the Poincare conjecture. We use general techniques which are not specific to 2-bridge knots to find and explicitly construct these laminations.

1. Introduction.

In trying to understand 3-manifolds with the hope of eventually classifying them, as with 2-manifolds, one approach that has turned out to be fruitful is to study objects of codimension one in them, more specifically, incompressible surfaces, Reebless foliations, and essential laminations.

For a 3-manifold M containing an incompressible surface, with some extra hypotheses, Waldhausen proved: The universal cover of M is $\mathbf{R}^3$, the topology of M is determined by its fundamental group, and homotopic homeomorphisms of M are isotopic. Similar theorems for manifolds containing Reebless foliations were proven by Haefliger, Novikov, Palmeira, Rosenberg, and others.

The essential lamination was developed in the 1980's as a generalization of the incompressible surface and the Reebless foliation, which themselves qualify as essential laminations. In fact they are just extreme cases of essential laminations: At one end we have compact properly embedded surfaces, and at the other end we have foliations, which have empty complement but can have noncompact leaves. A "typical" lamination is in general somewhere in between; it has non-empty and usually dense complement as in the case of surfaces, but can have non-compact leaves as in foliations. This seems to suggest that essential laminations might be more abundant than incompressible surfaces and Reebless foliations. And in fact, they are more abundant than incompressible surfaces: In contrast to Theorem 2 above and Theorem 1 below, Hatcher and Thurston [**HT**] show that for each 2-bridge knot only finitely many surgeries yield manifolds which contain incompressible surfaces. So far, however, we do not know of any manifolds which admit essential laminations but not foliations.

The study of essential laminations in 3-manifolds can be divided into two sub-disciplines:

1. *Which* 3*-manifolds admit essential laminations*?
2. *What can we say about such manifolds*?

Towards answering the second question, analogues of some of Waldhausen's theorems have been proven ([**GO**]) for closed manifolds admitting essential laminations: They are irreducible, have infinite fundamental group, and are covered by $\mathbf{R}^3$. Other questions such as whether homotopic homeomorphisms are isotopic are being worked on (with partial results by Gabai and Kazez [**GK**]).

Here we answer the first question for manifolds obtained by surgery on 2-bridge knots; in addition to Theorem 2 above, we also have:

Theorem 1. *Every manifold obtained by surgery with coefficient* $< q-2$ *on the* 2*-bridge torus knot* $T_{2,q}$ *admits essential laminations.*

Using results of Brittenham [**B**], Claus [**C**], and [**N2**], it immediately follows that Theorem 1 is sharp, i.e. for surgery coefficients $\geq q-2$ essential laminations do not exist. Thus we have a complete characterization of which 2-bridge knot surgered manifolds admit essential laminations.

Hatcher [**H**] constructed branched surfaces for $T_{2,q}$ which carried laminations for surgery coefficients $\in (-\infty, 0]$. The proof of Theorem 1 consists of showing that these branched surfaces in fact carry laminations for surgery coefficients $\in (-\infty, q-2)$. This construction is then used to prove part of Theorem 2; and the remaining part is proved using the method described further below.

The manifolds of Theorem 1 are easily shown to be Seifert fibered spaces with three singular fibers over S^2. Therefore, from the results of Eisenbud, Hirsch, Jankins, and Neumann [**EHN**], [**JN**], it follows that these manifolds admit foliations transverse to the Seifert fibers, which implies that they contain no Reeb components, and so they are essential. Theorem 2 was also proved (slightly later) by Delman [**D**] using different methods. And Roberts [**Rb**] constructs essential laminations for manifolds obtained by surgery on most alternating knots.

The main method used in this paper to find new essential laminations and branched surfaces can be summarized as follows: we start with two different surgery descriptions of a manifold, where for one of the surgery descriptions we have an essential lamination, but not for the other. Then we "pull back" this lamination through some Kirby Calculus to the second surgery description, and try to generalize it to other knots. This is done in Section 3.3. It is quite general and could potentially be as fruitful for many other knots and links.

This paper is a rewriting of my Ph.D. thesis [**N1**]. I would like to thank my advisor, Dave Gabai, for all his time, encouragement, enthusiasm, and advise.

2. Preliminaries.

For necessary definitions and terminology on Dehn Surgery, essential laminations, branched surfaces, and 2-bridge knots, we refer the reader to [**Rf**], [**GO**], [**H**], and [**HT**]. Here we only briefly explain by example how we describe a branched surface by "level curve diagrams".

Given a branched surface in $\mathbf{R}^3 \subset S^3$ we first isotope it so that the height function on it is Morse, i.e. the branched surface is transverse to horizontal planes in $\mathbf{R}^3$ (which represent level 2-spheres in S^3) except at isolated points, and then draw the intersection of the branched surface with these horizontal planes in successive stages. These are the "level curve diagrams".

Example. Figure 1(a) shows a surface (a branched surface with no branch points) whose boundary (thick lines) is the trefoil. The surface is described on the right, 1(b), by its intersection with horizontal planes, where the four dots in each diagram represent the knot's intersection with the planes. The dotted lines represent the saddles.

In a branched surface with nonempty branch locus, the intersections with all but finitely many horizontal planes are train tracks; Figure 2 shows an example. A lamination carried by this branched surface might be described as: we start with two vertical batches of sheets; the left batch splits into two parts or batches, one part continuing vertically down, the other joining the right batch in a saddle.

Because every closed 3-manifold M can be obtained by surgery on a knot or link $K \subset S^3$, in trying to construct essential laminations in M, we first try to do so in $S^3 - \mathring{N}(K)$, and then hope that the lamination extends to an essential lamination in M after the Dehn filling. More precisely, if

M is obtained by p/q surgery on a knot $K \subset S^3$, then we either want 1) essential laminations in $S^3 - \mathring{N}(K)$ which are disjoint from the boundary torus and remain essential after the Dehn filling, or 2) essential laminations in $S^3 - \mathring{N}(K)$ which intersect the boundary torus transversely in simple closed curves of slope p/q, and which remain essential after Dehn filling.

Of course the reason we want simple closed curves of slope p/q on the boundary torus is so that we can cap them off with the meridional discs in the solid torus of the Dehn filling. Making these boundary curves closed is crucial and sometimes difficult.

3. Main Results and Constructions.

3.1. Outline.

Theorem 1. *Surgery on a* 2*-bridge torus knot* $T_{2,q}$*, with coefficient* $\in (-\infty, q-2)$ *yields a manifold which admits essential laminations.*

Theorem 2. *Nontrivial surgery on a non-torus* 2*-bridge knot yields a manifold which admits essential laminations.*

Corollary. *Property P is true for* 2*-bridge knots.*

Proof. For non-torus knots, Theorem 2 above and [**GO**] imply that we get manifolds with infinite π_1. And for torus knots, we get Seifert fibered spaces, which are not homotopy spheres either; see [**M**]. (Takahasi [**Tk**] gives an approximately 100 page prove of this corollary using algebraic methods.) □

In this section we describe the constructions for the laminations of Theorems 1 and 2. *That these laminations are essential will be proved in the next section.*

In [**H**], Hatcher constructs branched surfaces which for "most" 2-bridge knots carry laminations of all boundary slopes. This construction is explained in Section 3.2 below. The knots for which not all boundary slopes were obtained consisted of three classes (m, n, and p are positive integers):

(a) Torus knots $T_{2,q}$; slopes $(-\infty, 0]$ were obtained.

(b) Knots of the form $[2m+1, -2n]$ (odd right-handed twists, even left-handed); slopes $(-\infty, -4n] \cup [0, \infty)$ were obtained.

(c) Knots of the form $[2m+1, -(2n+1), 2p+1]$; slopes $(-\infty, -2a] \cup [0, \infty)$ were obtained, where $a = \min(2m+1, 2p+1)$.

We show in Section 3.2 that the branched surface of (a) also carries the extra laminations claimed to exist in Theorem 1.

For Theorem 2, we find new branched surfaces in Section 3.3 which for knots of (b) and (c) either carry laminations of all boundary slopes or are disjoint from the knot and remain essential after Dehn filling.

3.2. Construction for Theorem 1.

3.2.1. Special Case: Trefoil.

We will first do this for the case of the trefoil $T_{2,3}$, which contains the main idea; then we will do the general case $T_{2,q}$, which is similar. Figure 3 shows [**H**]'s construction of a branched surface in the complement of the trefoil, carrying laminations with boundary slopes $\in (-\infty, 0]$. The numbers represent the "thickness" of the lamination and the saddles at the indicated points; for $0 \leq x \leq 1$, $\bar{x}$ denotes $1-x$. So we start with two batches of sheets of thickness 1 (Diagram 1), then go through a saddle with thickness 1 (full) on the right and $\epsilon < 1$ (partial) on the left. This saddle is linear, so that in Diagram 2, the two horizontal lines both represent the linear map from $[0, \epsilon]$ to $[0, 1]$. In fact all saddles in Figure 3 are linear. (In later constructions we will use nonlinear saddles also.) The saddle in Diagram 2 also is 1 (full) on the right and ϵ on the left. If $\bar{\epsilon}$ is an integer multiple of ϵ, then by repeating this saddle $\bar{\epsilon}/\epsilon$ times we will reach Diagram 5. If $\bar{\epsilon} = k\epsilon + r$, $0 < r < \epsilon$, however, then the last occurrence of this saddle will instead be r on the left and r/ϵ on the right, preserving the ϵ to 1 ratio.

The significance of this ϵ to 1 ratio and linearity of saddles is that we get the identity map from $[0, 1]$ to $[0, 1]$ for the two vertical lines in Diagram 7, and hence also in Diagrams 8 and 9. It is important to get this identity map in the last diagram in order for the boundary curves of the lamination on the torus boundary to be closed (i.e. no holonomy), so that we can cap them off with discs after Dehn surgery.

Now, to compute the boundary slope of this lamination as a function of ϵ, first note that for $\epsilon = 1$ we jump from Diagram 1 straight to Diagram 5 (but with only the horizontal lines, as $\bar{\epsilon} = 0$), and so the lamination just becomes $S \times [0, 1]$ where S is a Seifert surface for $T_{2,3}$, which has boundary slope 0 (Figure 1). Now for $\epsilon < 1$, we see that in going from Diagram 2 to 3, there is a batch of thickness 1 on the top right corner rotating counterclockwise and a batch of thickness ϵ on the top left corner rotating clockwise, so we get a contribution of $-1 + \epsilon$ to the total slope. Adding up all the repeats of this move ($\bar{\epsilon}/\epsilon$ times), we get a total addition of $+\bar{\epsilon} - \bar{\epsilon}/\epsilon$ to the slope of the $\epsilon = 1$ lamination (which by above is 0), so the slope is $\bar{\epsilon} - \bar{\epsilon}/\epsilon$. So as ϵ ranges over (0,1], the slope ranges over $(-\infty, 0]$. This was the construction of [**H**].

Now we claim that this branched surface in fact carries laminations with boundary slopes in $(-\infty, +1)$. To get the extra (0,1) range, the idea is that in Diagram 2 (still Figure 3) where we get a contribution of $+\epsilon - 1$ to the slope, we change the weights on the saddle to $\bar{\epsilon}$ (full) on the left, and $\delta > 0$ (small) on the right (Figure 4).

This gives a contribution of $+\bar{\epsilon} - \delta$ to the slope. This move cannot be repeated though, as we have already reached Diagram 5 of Figure 3. Thus by making ϵ and δ small, the total slope, $\bar{\epsilon} - \delta$, approaches 1 as desired.

But there is a problem: If we just make the saddles all linear, the boundary curves of the lamination will not be closed (Figure 5). In fact, because of the holonomy, the slope is really 0.

To avoid this holonomy, we proceed as follows. In Figure 4, do the saddle in Diagram 1 such that the horizontal lines in Diagram 2 look *qualitatively* as in Figure 6 (according to the size of the segments). We keep the saddle in Diagram 2 of Figure 4 linear, and make δ small enough as in Figure 6, on the right. We continue as in Diagrams 3 and 4 of Figure 4 (though the numbers/thicknesses are now different), so that Diagram 5 of Figure 5 will now look as in Figure 7, where α comes from the α to 1 ratio of Figure 6. Now we claim that the saddle in Diagram 5 can be chosen such that in Diagram 6, the maps in the two vertical lines will be mirror images, *so that in the last diagram both lines can be made identity.*

Proof of claim: The saddle of Diagram 5 of Figure 7 will be done in pieces (or stages). First we slice off $\bar{\epsilon}$ sheets from the right side of the bottom line and $\delta\alpha$ sheets from the right side of the top line. This yields Diagram 5.1 of Figure 8. Then we slice off x_1 from the right side of the bottom line and y_1 from the right side of the top line, getting Diagram 5.2. Note that the two vertical lines are starting to look the same, and in the *limit* will be mirror images. α, δ, and ϵ are picked to be small enough as follows: In the two linear regions of Figure 6 which have the α to 1 ratio, we not only make the left side small, but also make the right side large enough (i.e. close to 1) so that it is larger than Σx_i (which here happens to be $\bar{\epsilon}/\bar{\alpha}$); this is to assure that the bottoms of the saddles in Figure 8 do not "penetrate" into the second linear region.

Remark 3.1. Because the vertical line of Diagram 5 is linear, it turns out that here we could have described this saddle simply as *one* linear saddle, instead of in infinitely many pieces. However, in later constructions (Figures 11 and 26) we will *have* to use this "infinite saddle" construction, which works even if the vertical line is not linear.

Thus we get to Figure 9, where $x = \Sigma x_i = \bar{\epsilon}/\bar{\alpha}$, and $y = \Sigma y_i = \delta\alpha/\bar{\alpha}$. The top part of $h : \bar{y} \longrightarrow \bar{y}$ is not linear, but only piecewise linear (2 pieces in fact) because of the saddle in Diagram 2 of Figure 6. The important point however is that both g and h have no interior fixed points and have opposite directions: g moves points "down", h "up". Therefore there exists a homeomorphism $f_1 : \bar{x} \longrightarrow \bar{y}$ such that $f_1^{-1} h f_1 = g$, so $f_1 = h^{-1} f_1 g$ (this is not obvious, but not hard to show either; see for example [**EHN**, Lemma 2.2]), so we get Figure 10, which proves the claim, and completes the construction.

Remark 3.2. These laminations have a piecewise linear structure (though with maybe infinitely many pieces) since all the maps in all the diagrams are piecewise linear. (Even f_1 is piecewise linear, since g and h are.) Similarly, the laminations constructed in the next section for the general case $T_{2,q}$ have a piecewise linear structure.

3.2.2. General Case: $T_{2,q}$.

For the general case of $T_{2,q}$, avoiding holonomy needs a little more care, though basically it is the same idea. We do it for $T_{2,5}$; everything is the same for $T_{2,q}$, $q \geq 5$.

First, as with the trefoil, we draw $T_{2,5}$ as the 2-bridge knot $[-2, -2-2, -2]$, i.e. with four pairs of left-handed half-twists, as in Figure 11. (For $T_{2,q}$, we have $(q-1)/2$ pairs of half-twists.) Then we construct the branched surface described on the right in Figure 11. To prove Theorem 1, we will show that this branched surface carries laminations with boundary slopes in $[0, 3)$ $(q - 2 = 5 - 2 = 3)$.

As with the trefoil, we use the saddles of Diagrams 2, 5, and 8 to get slopes of up to 3, and try to kill the holonomy by appropriately picking the saddles of Diagrams 10 and 12.

In order to have no holonomy, i.e. identity maps for the horizontal lines in the last diagram, we want the maps of Diagram 11 to be mirror images. So we would like Diagram 10 to qualitatively look like Figure 6, so that we can carry out the same procedure as in Figures 8, 9, and 10.

To do this we use induction, the induction step being 'going from diagram 2 to 5' (and 5 to 8, and so on, when $q > 5$). More precisely, we start with diagram n of Figure 12 and want to reach the same position *qualitatively* (but rotated 90 degrees) in diagram $n + 3$, $n = 2$ or 5 (or 8, or $\cdots$, when $q > 5$). Because we are aiming for slopes close to the maximum of 3, all batches are either very thin or very thick, denoted by S for small, ~ 0, or L for large, ~ 1; the S's are not necessarily equal, nor are the L's.

This is achieved by doing the saddle in diagram n in such a way that the vertical line of diagram $n + 2$ looks as in Figure 12. This is the main difference with the case of the trefoil, where this saddle was just linear (Figure 6). □

Remark 3.3. This construction for Theorem 1 reduces the *missing* slopes of (a) in Section 3.1 from $(0, \infty)$ to $[q-2, \infty)$. The same construction works for (b) and (c), reducing the range of the missing slopes roughly by half. (More precisely, for (b) we can obtain slopes in $(-\infty, -4n+1) \cup (-2n+1, \infty)$, and for (c), $(-\infty, -a) \cup (-2n-1, \infty)$.) These laminations are proven to be essential along with the laminations of Theorem 1; see Remark 4.5.

3.3. Construction for Theorem 2. This will be done in two parts by dividing knots of (b) ∪ (c) of Section 3.1 into:

(I) Knots of the form[1] $[2m, 2n]$.

(II) The rest (i.e. (b ∪ c) - (I)).

Lemma 3.4. *For every* 2*-bridge knot* K *of type* (*I*) (*i.e. of the form* $[2m, 2n]$), $S^3 - \mathring{N}(K)$ *admits essential laminations which are disjoint from the boundary, and which remain essential for all nontrivial Dehn fillings.*

For knots of type (I) we have Figure 13. The knot $[2, 2n]$ with surgery coefficient $-1/m$ in the last diagram of Figure 13 is still of type (I). However, $[2, 2n] = [2n-1, -2]$, which is in class (b), so by Remark 3.3, laminations of boundary slope $-1/m$, $m \geq 1$, are realizable on it. Furthermore, the unknot with coefficient $r/(r+s)$ (last diagram) is disjoint from, and isotopic to, a leaf of the lamination (Figure 14). Therefore, by [**G1**] (Operation 2.4.2), for all surgeries but one[2] on the unknot, these laminations remain essential. So all surgeries but one (trivial surgery) on the original $[2m, 2n]$ knot yield manifolds which admit essential laminations. □

Lemma 3.5. *For every knot* K *of type* (II) (*i.e. of the form* $[2m+1, -2n]$, $m \geq 1$, $n \geq 2$, *or* $[2m+1, -2n+1, 2p+1]$, $m, n, p \geq 1$, *but not* $[2m, 2n]$), $S^3 - \mathring{N}(K)$ *admits branched surfaces transverse to the boundary which carry laminations of all finite boundary slopes.*

Proof.

Step 1. The 7_3 knot.

We start with the simplest knot of type (II), namely [3,-4], which is knot 7_3 in Rolfsen's tables [**Rf**], and will show that the branched surface of Figure 16 in its complement carries laminations of all boundary slopes.

Remark 3.6. This branched surface was discovered as follows. We first noticed that boundary slope -7 (among others) could not be obtained on

[1] At first glance, these may seem to be disjoint from (b) and (c), but they are not: In (b), $[2m+1, -2] = [2m, 2]$; in (c), $[2m+1, -1, 2p+1] = [2m+2, 2p+2]$. To check this, simply note that the brackets represent the same number (mod $\mathbf{Z}$) as continued fractions. See [**HT**] for more details.

[2] The unique surgery for which the lamination does not remain essential is represented by the "tangency curve" of the unknot with the leaf it is isotopic to, i.e. surgery 1. But $r/(r+s) = 1$ iff $r/s = 1/0 = \infty$, i.e. trivial surgery on the original $[2m, 2n]$ knot.

this knot using [**H**]'s constructions, and not even with the improvement of Remark 3.3. But it turns out that surgery -7 on the [3,-4] knot gives the same manifold as surgery -7/2 on the [3,-2] knot (5_2 in [**Rf**]), as in Figure 15. By Remark 3.3, -7/2 on [3,-2] does admit essential lamination. If we *pull back* this lamination, along with the branched surface carrying it, through the surgery and isotopy "moves" (Kirby Calculus) to -7 on [3,-4] (which is a bit of work, but we don't need to do it here, since we will prove its essentiality directly, in the next section), we get the branched surface of Figure 16.

Showing that this branched surface carries laminations is a bit involved. We construct the lamination so that in the "spiral regions around the knot" (Figure 17a) it has a triple spiral structure (Figure 17b): There are exactly two closed curves (the leaves containing these closed curves though are non-compact elsewhere), and the remaining curves spiral and limit on these two curves. The inner and outer spirals have the same sense; they spiral *in* as they turn clockwise. The middle spiral has the opposite sense; it spirals *out* as it turns clockwise. The "infiniteness" of the inner spiral will enable us to realize all boundary slopes.

The reason for this *triple* spiral structure is to make things work in going from Diagram 6 to 7 in Figure 16, which is the crucial point of the construction (there are only two other saddles, and they pose no problems: In Diagram 3, the saddle is arbitrary, as long as it's full on the right, and partial on the left; in Diagram 9, there is a unique saddle which will give identity for the vertical line in Diagram 10). The point is that in Diagram 7 a new spiral region is created (which is more visible in 8), and the lamination there must have the same structure as the other spiral (top left in 7) since the two spirals will eventually have to join (Diagram 11). It turns out that a *triple* spiral structure is the "minimum amount of structure" that will work, as explained below. (We can also construct laminations which are *finite* in the spiral regions. However, only finitely many boundary slopes can be realized this way.)

Figure 18 shows an enlarged picture of Diagram 7 with the details of the saddle. This is where we start building the lamination. We have to find maps F, G, and H which will give the new spiral the triple spiral structure. Because of the "old spiral" (i.e. the spiral in Diagram 6 to which the saddle is joining), F has to be an increasing homeomorphism, while G and H can be any homeomorphisms.

The points labeled $a \cdots f$, and x' occur in pairs, signifying that in Diagram 6, before the saddle, they are joined pairwise; the two curves through b and c are (were) the two closed curves of the old spiral.

Let $x = (b+c)/2$, $G(x') = H(x') = x$, $z = (x+c)/2$, and $y = (x+b)/2$, say. Now choose F "increasing enough" such that $F(z) = y$. (There will be other requirements imposed on F later.) The only requirement on G and H, which is easily satisfied, is that y' and z', where $y' = G^{-1}(y)$, $z' = H^{-1}(z)$, lie on the same curve (leaf) which goes all the way around on the right, so that the curve $z' \overset{H}{\longmapsto} z \overset{F}{\longmapsto} y \overset{G^{-1}}{\longmapsto} y' \longmapsto z'$ closes up. Call this curve γ. Now, x' (the lower one) $\overset{G}{\longmapsto} x \overset{F}{\longmapsto}$ between y and $b \overset{G^{-1}}{\longmapsto}$ between y' and $e \longmapsto$ between x' and $z' \overset{H}{\longmapsto} \cdots$. So the curve through the lower x' is an infinite spiral, limiting in the clockwise direction either to γ, or possibly some other closed curve γ', in which case we identify γ' with γ after removing all curves between them. Similarly for the other side of γ.

Now we modify F so that γ thickens up to an annulus A with two closed boundary curves γ_1 and γ_2, and with all interior curves of A spirals limiting on γ_1 and γ_2 (Figure 19). On A, F is of course still increasing, but "relative to γ_1 and γ_2", F looks decreasing inside A, which is what gives us the middle spiral with the opposite sense.

This *almost* proves the claim that the branched surface of Figure 16 carries laminations of all boundary slopes, since in truth, it fully carries only laminations of one slope (which happens to be -7 as in Figure 15). We can get *any* integer slope by twisting one of the inner spirals in Diagram 2 of Figure 16, before doing the saddle, the appropriate number of times. And then we can get all (rational) slopes between any two integers by branching the inner spiral and twisting a copy of it, as shown in Figure 20. This new branched surface clearly still fully carries laminations.

By modifying this branched surface appropriately, we get laminations for all knots of type (II):

Step 2. (II) $\cap$ (b).

First we start with those in class (b), namely $[2m+1, -2n]$, with $m \geq 1$ and $n \geq 2$ because we are excluding type (I). The $2m+1$ twists are realized in going from Diagram 2 to 3 (Figure 16), where instead of the $+3\pi$ rotation we have $(2m+1)\pi$. This obviously makes no difference in whether or not the branched surface carries laminations. The $-2n$ twists, on the other hand, are realized in Diagram 6 before doing the saddle by rotating its bottom portion counterclockwise the required number of times, i.e. $-(n-2)2\pi$ (for $n = 2$ we don't need any more twists than the $-2\pi - 2\pi$ already present in Diagrams 4-5 and 8-9) (Figure 21).

That the branched surface still carries laminations is no longer obvious, but easy to check, as follows. In Figure 18, the only thing that changes is that the position of the upper x', which has to be consistent with that of the lower x', moves up a bit, as in Figure 22. However, the argument given to show that the old branched surface carries laminations still goes through word for word.

Step 3. (II) $\cap$ (c).

For (c), the knot $[2m+1, -2n+1, 2p+1]$ can also be represented as $[2m+1, -2n, -2, -2, \cdots, -2]$ with $2p$ (-2)'s (see footnote 1), with $m, p \geq 1$ and $n \geq 2$ since $n = 1$ gives type (I). The $2m+1$ and the $-2n$ are taken care of the same way as for (b) above. The $2p$ (-2)'s are realized by adding $2p$ saddles and pairs of counterclockwise full-twists after Diagram 10 of Figure 16, before reaching Diagram 11, as in Figure 23. Obviously this does not affect whether or not the branched surface carries laminations. $\square$

4. Proofs of Essentiality.

In this section we finish the proof of Theorems 1 and 2 by checking that the branched surfaces (and therefore laminations) constructed in the previous section are essential.

4.1. Some Generalities. What we did in the previous section was to start with a given manifold M obtained by p/q surgery on a knot $K \subset S^3$, construct a branched surface $B \subset S^3 - \mathring{N}(K)$, and show B carries a lamination of boundary slope p/q, which is capped off with discs in the Dehn filling solid torus. Now to show this lamination is essential, we complete B to a branched surface $B' \subset M$ carrying the capped off lamination, and show B' is essential.

Construction 4.1. With notation as above, let $T = \partial(S^3 - \mathring{N}(K))$, and let V be the Dehn filling solid torus. Let $T \times [0, 1]$ be a one sided collar of T in $V \subset M$, with $T \times 0 = T = \partial V$. Now $\tau = \partial B$ is a train track in T which carries a lamination of slope p/q, so it can be split into a finite number of simple closed curves of slope p/q in T, or equivalently slope ∞ (i.e. meridians) in ∂V. So now the natural way to construct B' is to first extend B to the collar $T \times [0, 1]$ according to the splitting of τ, so that $\{B' \cap (T \times t) \mid t \in [0, 1]\}$ gives us a "movie" for the splitting of τ, and then cap it off with discs in $V - T \times [0, 1]$.

Actually though, we construct B' a little differently: Assume $\tau \subset T$ is "nontrivial enough" that $T - \tau$ consists of digons. Then, before splitting τ across the collar $T \times [0, 1]$, first we take a copy of one side of each digon and move it parallel to itself across the digon to the opposite side as t goes from 0 to 1/2 in the collar (Figure 24). Thus $B' \cap (T \times 1/2) = \tau$ again, as in level $t = 0$. Then for $1/2 \leq t \leq 1$ we extend B' as before, i.e. by splitting τ and then capping off with discs. $\square$

The reason for 'moving copies of sides of digons across from one side to the other' is to make the analysis of the complementary components of $B' \subset M$ (and therefore checking essentiality of $B' \subset M$) simpler, as explained in the following three lemmas (which are proven, although not in so many words, in Proposition 1 of [**H**]).

Note. We emphasize that in the following lemmas, we implicitly assume that B satisfies the hypothesis necessary for Construction 4.1, namely that ∂B is a train track in $\partial(S^3 - \mathring{N}(K))$ whose complementary regions are all digons.

Lemma 4.2. *With notation as above, every complementary component of B' which lies entirely in V is an open 3–ball.*

Proof. Let X be such a complementary component. Then $X \cap (T \times [0, 1/2])$ is a disjoint union of balls. As t goes from $1/2$ to 1, only a splitting of the kind shown in Figure 25 can possibly affect the topology of $X \cap (T \times [0, t])$: it either joins two disjoint balls into a new ball (corresponding to the joining of two digons), which does not change the topology, or it joins two "parts" of the same ball (corresponding to the joining of the two ends of the same digon) to form a solid torus, which intersects $T \times 1$ in an annulus. $B' \cap (T \times 1)$ consists of a finite union of parallel simple closed curves (of slope p/q), which are capped off with meridional discs in V. Thus to each solid torus $X \cap (T \times [0, 1])$ a 2-handle $D^2 \times I$ is being attached along an annulus in $T \times 1$, resulting in a 3-ball. □

Lemma 4.3. *With notation as above, if $B \subset \left(S^3 - \mathring{N}(K)\right)$ has essential horizontal boundary, then so does $B' \subset M$.*

Proof. Let X be a component of S^3- interior$(N(K) \cup N(B))$. Then $\partial X \subset \partial_h N(B) \cup \partial_V(B) \cup \partial N(K)$, and $\partial X - \partial_h N(B)$ is a disjoint union of annuli, which we call the *suture*; each annulus consists of a finite union of discs $D_i \simeq I \times I$ with each D_i contained alternately in $\partial_v N(B)$ and $\partial N(K)$.

Now, the result of 'moving copies of sides of digons across from one side to the other' (in the $T \times [0, 1/2]$ part of Construction 4.1) is that to each complementary component X we are gluing one $D^2 \times I$ for each $D_i \subset \partial N(K)$ in the suture of X. For each such $D^2 \times I$, half of $\partial D^2 \times I$ is in $\partial_v N(B')$, the other half glued to D_i. Therefore, X remains topologically the same, and further, its suture also remains the same except that it is now entirely $\subset \partial_v N(B')$.

This shows $\partial_h N(B')$ has no compressing discs or monogons in the enlarged X, since any such compressing disc or monogon could be isotoped to lie in the old X, violating the hypothesis that $\partial_h N(B)$ is essential. And $\partial_h N(B')$ has no properly embedded spheres or discs since neither does $\partial_h N(B)$.

So to finish the proof, we need to show $\partial_h N(B')$ has no compressing discs or monogons or properly embedded spheres or discs in the remaining complementary components, namely the ones which lie entirely in the Dehn filling solid torus V glued to $\partial(S^3 - \overset{\circ}{N}(K))$. But this follows immediately because by Lemma 4.2 all these complementary components are just $D^2 \times I$ products, each with the annulus $\partial D^2 \times I$ as suture. □

Lemma 4.4. *With notation as above, suppose* (*the closure of*) *every* $D^2 \times I$ *complementary component of* $B \subset \left(S^3 - \overset{\circ}{N}(K)\right)$ *intersects* T*, and* ∂B *is a train track transverse to the meridians of* T*. Then* B' *contains no Reeb branched surfaces.*

Proof. Suppose towards contradiction that there is a Reeb branched surface $C \subset B'$. Then by definition C is contained in a solid torus R with $\partial R \subset C$. Since by definition C carries a sub-lamination of a Reeb foliation of R, $R - C$ consists of $(D^2 \times I)$'s, so by hypothesis ∂R must intersect T (or $T \times \{1/2\}$, if $R \subset V$) in one or more simple closed curves α_i transverse to meridians of T. Let β be a simple closed curve on T intersecting α_1 transversely once. Then β must intersect ∂R in at least one more point (entering and exiting R), which shows there are at least two simple closed curves $\alpha_1 \cup \alpha_2 \subset T \cap \partial R$ transverse to meridians of T. By renumbering α_i if necessary, we can assume α_1 and α_2 are adjacent, i.e. bound an annulus A on T which is properly embedded in R. By hypothesis, A is a union of digons on T, transverse to meridians. So as a meridian m crosses A from α_1 to α_2, in each digon it goes from one side of a $D^2 \times I$ complementary component of $R - C$ to the other.

Let W be a vector field on R transverse to the Reeb foliation, and pointing *into* R on ∂R. Since W is also transverse to C, as we travel on m from α_1 to α_2, we always point in the same direction as W on the sides of the digons in A. But this gives us a contradiction, since when we reach α_2, we are exiting R, and must therefore be pointing opposite to W. □

Now we are ready to finish the proofs of Theorems 1 and 2. That the branched surfaces of Theorem 1 are essential is basically proven in Proposition 1 of [**H**]. There are two adjustments however, in Case 3 of Step 4, and in Step 5, below.

4.2. Proof of Theorem 1. Let $K \subset S^3$ be the $T_{2,q}$ torus knot, with exterior $X = S^3 - \overset{\circ}{N}(K)$, and M obtained by surgery on K, with surgery coefficient as in Theorem 1.

Step 1. First we enlarge the branched surfaces constructed in the previous section by adding, for each saddle, a "complementary saddle" in the same level 2-sphere; Figure 26 shows the case $K =$ trefoil.

This enlarged branched surface B contains the original branched surface B_0 as a subset. Therefore any lamination carried by B_0 is also carried by B; so it is enough to show B is essential. The reason for enlarging B_0 to B is that B has "smaller" complementary components, which are easier to work with. The disadvantage, however, is that it may be harder to show that B *fully* carries a lamination.

Step 2. B has essential horizontal boundary in X.

The two saddles of Diagram 1 (Figure 26) "add up" to the whole level 2-sphere at that level; therefore B has a complementary component "above" Diagram 1, which is just a ball $D^2 \times I$ with $\partial D^2 \times I$ as the suture; so this component's horizontal boundary contains no compressing discs or monogons or properly embedded spheres or discs.

The two saddles of Diagram 3 also add up to a level 2-sphere. The complementary component of B trapped between the level 2-spheres of Diagrams 1 and 3 is also seen (with a little more visualization) to be topologically a ball $D^2 \times I$ with $\partial D^2 \times I$ as the suture, as shown in Figure 27 (where the suture is drawn as just a curve rather than an annulus).

The complementary component between Diagrams 3 and 5 is exactly the same as the one between 1 and 3, except that it is upside down. (This can also be seen without any visualization by simply noting that to go "upward" from Diagram 6 to 5 one makes the same "saddle moves" as going "downward" from 1 to 3—the complementary components actually extend a little below and above the saddles that trap them; the one between 3 and 5, in particular, extends down to 6.)

The complementary component between Diagrams 5 and 7 is a solid torus $A \times I$ with the suture consisting of two parallel annuli $\partial A \times I$ of slope -1 on the boundary of the solid torus, as in Figure 28. Therefore the horizontal boundary $A \times \partial I$ is essential in $A \times I$.

Remark 4.5. In going from Diagram 9 to 10, if instead of the -2π rotation we had $2n\pi$ rotation, n a nonzero integer, we would still get an $A \times I$ solid torus complementary component, except that the suture would now consist of two parallel annuli $\partial A \times I$ of slope $1/n$. For a $(2n+1)\pi$ rotation, $n \geq 1$ or ≤ -2, we get an $A \times I$ solid torus complementary component, with suture consisting of one annulus $A \times 0$ of slope $2/(2n+1)$. Both these cases arise for the non-torus 2-bridge knots, which we are concerned with in Remark 3.3. In either case, the horizontal boundary ($A \times \partial I$ for the $2n\pi$ case, or $\partial(A \times I) - (A \times 0)$ for $(2n+1)\pi$) is essential in the solid torus.

Finally, the complementary component below Diagram 10 is the same as the one above Diagram 1, i.e. a ball.

For the other torus knots (and in fact all 2-bridge knots with the branched surfaces of [**H**]) we get the same complementary components since we are using the same "saddle moves", which yield branched surfaces made up of the same "building blocks". (There are nice pictures of these building blocks in Figure 3.1 of [**FH**]—for 2-bridge knots, only Σ_A, Σ_C, and Σ_D are used.)

Step 3. B' has essential horizontal boundary, and no Reeb branched surfaces.

First note that we *can* apply Construction 4.1 to B to get $B' \subset M$, since ∂B is a train track on ∂X transverse to meridians, with digons as complementary components. Now by Step 2, the hypotheses of Lemmas 4.2, 4.3, and 4.4 are satisfied, so B' has essential horizontal boundary in M, and has no Reeb branched surfaces.

Step 4. B' has no discs of contact.

Suppose towards contradiction that D is a disc of contact for B'.

Case 1. ∂D lies in the suture $\partial D^2 \times I$ of a $D^2 \times I$ complementary component of B'.

Since, by definition of disc of contact, ∂D is *embedded* in the annulus $\partial D^2 \times I$, it must be isotopic to its core $\partial D^2 \times 1/2$. Therefore D together with $D^2 \times 1/2$ give an embedded sphere in $N(B')$. This sphere cannot lie entirely in the Dehn filling solid torus, since if it did, we could split B' along D to get a $S^2 \times I$ complementary component, which contradicts Lemma 4.2 (the proof of the lemma does not change after this splitting of B'). Therefore B carries a genus zero surface in X, but this can happen only if in Figure 3, $\epsilon = 1$, i.e. B is one of the incompressible branched surfaces of [**HT**]. But all the incompressible surfaces in $S^3 - \mathring{N}(\text{2-bridge knot})$ are of genus ≥ 1 (except for Mobius bands in the case of torus knots), which gives us a contradiction.

Case 2. ∂D lies in the suture $A \times 0$ of an $A \times I$ complementary component (as is Remark 4.5).

Then D, together with the horizontal boundary annulus $\partial A \times I \cup A \times 1$, plus one more copy of D give a sphere carried by B, which again leads to a contradiction.

Case 3. ∂D lies in the suture $\partial A \times I$ of an $A \times I$ complementary component.

Let α be the core of one of the annuli in the suture, i.e. one of the components of $\partial A \times 1/2$. As noted in Remark 4.5, the annuli $\partial A \times I$ have slope $1/n$, $n \neq 0$, on the solid torus $A \times I$, and by Figure 28 this solid torus is unknotted in $S^3 = X \cup N(K)$, so α cannot bound a disc in $S^3 - (A \times I)$, and hence neither in $X - (A \times I)$. Nor can it bound a disc in M (after the Dehn filling), because of the following. If D is a disc of contact such that $\partial D = \alpha$, then after making it transverse to ∂X, $D \cap \partial X$ is a disjoint union of simple closed curves $c_1, \cdots, c_n$ on ∂X. By Figure 28, any curve on ∂X, and in particular any c_i, has linking number zero with the core of the solid torus $A \times I$ (since K is a 2-bridge knot, and so it goes up and down through the "hole" of the solid torus exactly four times). By renumbering if necessary, assume that $c_1, \cdots, c_m$, $1 \leq m \leq n$, are in the same component of $D \cap X$ as ∂D. Then ∂D is homotopic in $X - (A \times I)$ to a connected sum of $c_1, \cdots, c_m$, so ∂D also has linking number zero with the core of the solid torus, which contradicts the fact that α has linking number one with the core of the solid torus. (This part of the proof is different from that given in [**H**]; there, it is argued that because of a symmetry the branched surface has, the other suture on the solid torus must also bound a disc of contact, thus again yielding a sphere carried by the branched surface, which gives a contradiction. Here, however, the branched surfaces constructed for the non-trefoil torus knots, as in Figure 11, no longer possess this symmetry—the symmetry exists only as long as at most one of the saddles of Diagrams 2, 5, or 8 are present—and so we need a different argument.)

Step 5. B' fully carries a lamination.

This requires showing that B fully carries a lamination with *closed* boundary curves of slope p/q, since according to Construction 4.1, B' was obtained by first splitting ∂B into closed curves of slope p/q, and then capping off with discs in the Dehn filling solid torus. We showed in the previous section that B_0, the branched surface before the enlarging of Step 1, fully carries a lamination with closed boundary curves, and the whole argument was basically qualitative: the main point was to get the convergence in Figure 8, and fixed-point free self-homeomorphisms g and h of the interval in Figure 9; see Remark 3.1.

But this is achieved for B also, simply by assigning small enough weights to all portions of $B - B_0$, since under small enough perturbations g and h will still satisfy the (open) property of being fixed-point free. Also, in Figure 9 the conjugating map f_1 will now be distributed over the saddles of Diagram 5 of Figure 26. □

4.3. Proof of Theorem 2. The laminations of Lemma 3.4 were proven to be essential in the proof of Theorem 1 (Remark 4.5). So here we prove that the branched surfaces of Lemma 3.5 are essential. Let $K \subset S^3$ be a knot of type (II).

Step 1. For $K = [2m+1, -2]$ (i.e. in (II) $\cap$ (b)), we have the branched surface of Figure 16 modified according to Step 2 of proof of Lemma 3.5. With some visualization one sees that $N(K) \cup N(B)$ is just a solid torus with suture as shown in Figure 29.

For $K = [2m+1, -2n+1, 2p+1]$ (i.e. in (II) $\cap$ (c)), we further modify B according to Step 3 of proof of Lemma 3.5, as in Figure 23. Then $N(K) \cup N(B)$ is obtained by adding $2p$ 1-handles to Figure 29. Figure 30 shows the case $p = 1$.

Step 2. B has essential horizontal boundary in X.

For the case of Figure 29, B has exactly one complementary component. It is an unknotted solid torus $A \times I$ with one annulus as suture (the curve in Figure 29 is connected). A meridional disc of $A \times I$ must bound a longitude on the solid torus of Figure 29, which intersects the suture in $2m+1$ points, $m \geq 1$; so there are no compressing discs or monogons.

For the case of Figure 30 also, B has exactly one complementary component. It is a handlebody of genus $2p + 1$. We use the disc decomposition technology of [**G2**] (explained below in Section 4.4) to simplify Figure 30 to Figure 31 by filling in its $2p$ extra holes; by Lemma 4.6 one has essential horizontal boundary if and only if so does the other one. But Figure 31 is the same as Figure 29, so it has no compressing discs or monogons.

Step 3. B' has essential horizontal boundary, and no Reeb branched surfaces.

∂B is a train track on ∂X transverse to meridians, with digons as complementary components. And by Step 2 above, the hypotheses of Lemmas 4.2, 4.3, and 4.4 are satisfied, so B' has essential horizontal boundary in M, and has no Reeb branched surfaces.

Step 4. B' has no discs of contact.

To show essentiality for the particular laminations (carried by B') we have in mind, it is in fact enough to show B' has no disc of contact *which is isotopic to a leaf of the lamination* [**G3**]. This is clear since the laminations constructed in Lemma 3.5 have infinite spirals. However, we will also show that B' has no discs of contact (isotopic to a leaf, or not).

Assume D is a disc of contact for B'. As in Case 1 of Step 4 in the proof of Theorem 1, $D \cap X$ is not empty. Therefore the spiral regions of B must be finite spirals, i.e. they can be opened up by unspiraling, and B will still carry $D \cap X$. So again as in Case 1 of Step 4, B degenerates into one of the branched surfaces of [**HT**]; call it B_1. Then D is a disc of contact for B_1', which we know is impossible by proof of Theorem 1 (or Proposition 1 of [**H**]).

Step 5. B' fully carries a lamination.

This was already shown when B was constructed. □

4.4. Essentiality is Preserved by Disc Decomposition . For a general definition of disc decomposition see [**G2**]. Here we consider only the special case in Figure 32: We have a handlebody $H \subset S^3$ with a suture on it, and we attach to it a 2-handle $D^2 \times I$ along the annulus $A = \partial D^2 \times I$ such that the suture, thought of as a curve rather than an annulus, intersects A in two arcs $x \times I$ and $y \times I$, $x, y \in \partial D^2$. The suture in the new handlebody is obtained by replacing $x \times I$ and $y \times I$ by two (properly embedded) arcs in $D^2 \times 0$ and $D^2 \times 1$, connecting $x \times 0$ to $y \times 0$ and $x \times 1$ to $y \times 1$.

Suppose we want to show that $G = S^3 - \mathring{H}$ has no compressing discs or monogons (i.e a disc properly embedded in G, intersecting the suture on G at most once). Attaching a 2-handle to H as above is equivalent to removing a 1-handle from G (and changing the suture accordingly), which simplifies G. By the following lemma, it is enough to check that this simplified G has no compressing discs or monogons.

Lemma 4.6. *Let H_1 be obtained by disc decomposition from a sutured handlebody $H \subset S^3$ (as above). Then $G_1 = S^3 - \mathring{H}_1$ has compressing discs or monogons if and only if so does $G = S^3 - \mathring{H}$.*

Proof. The 'only if' direction is clear. For the converse, let D be a compressing disc or monogon in G. Let $D^2 \times I \subset G$ be the 1-handle being removed from G, and let $A = \partial D^2 \times I$. If D is disjoint from $D^2 \times I$, we are done. Otherwise, isotope D near its boundary so that ∂D is transverse to ∂A, has minimal intersection with it, and is disjoint from the suture inside A. Using a standard innermost disc argument and the irreducibility of G, we further isotope D, rel ∂, so that $D \cap (D^2 \times \partial I)$ has no circle components, so it is a union of disjoint arcs in D. Then $D \cap G_1$ is a finite union of properly embedded discs in G_1, at least two of which, say D_1 and D_2, each intersect $D^2 \times I$ in only one arc. We can assume each of these arcs intersects the (new) suture in G_1 in at most one point. But D has at most one point of intersection with the suture, so at least one of D_1 or D_2 is a compressing disc or monogon for G_1. (We needed to make $\partial D \cap \partial A$ minimal to ensure that D_1 and D_2 will have essential boundaries.) □

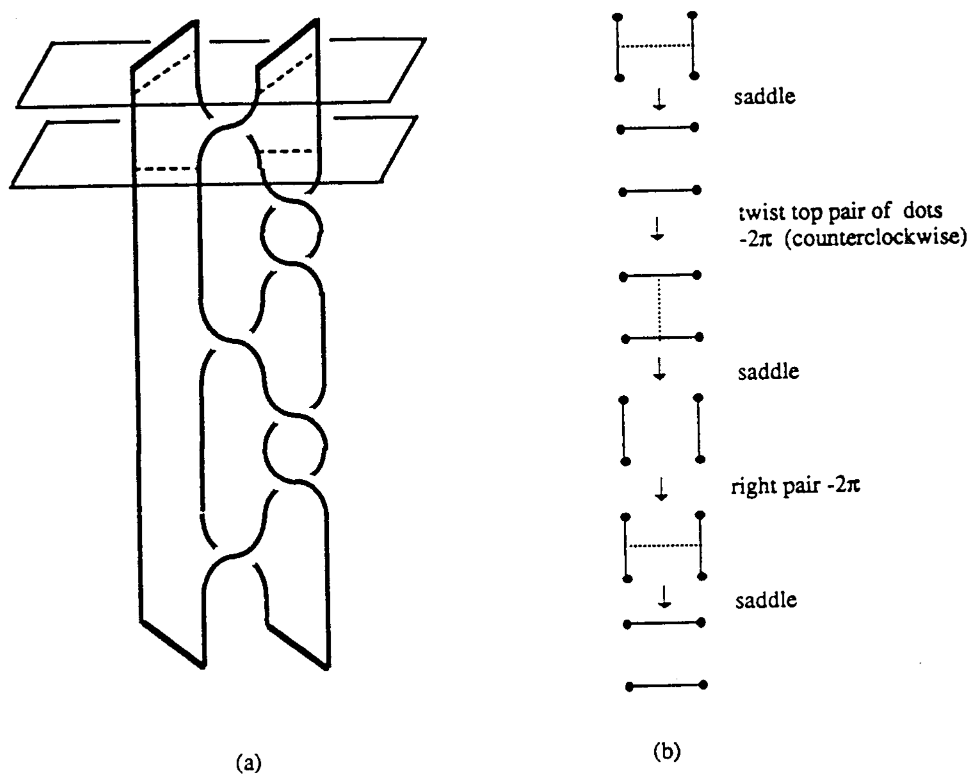

Figure 1. Seifert surface for trefoil.

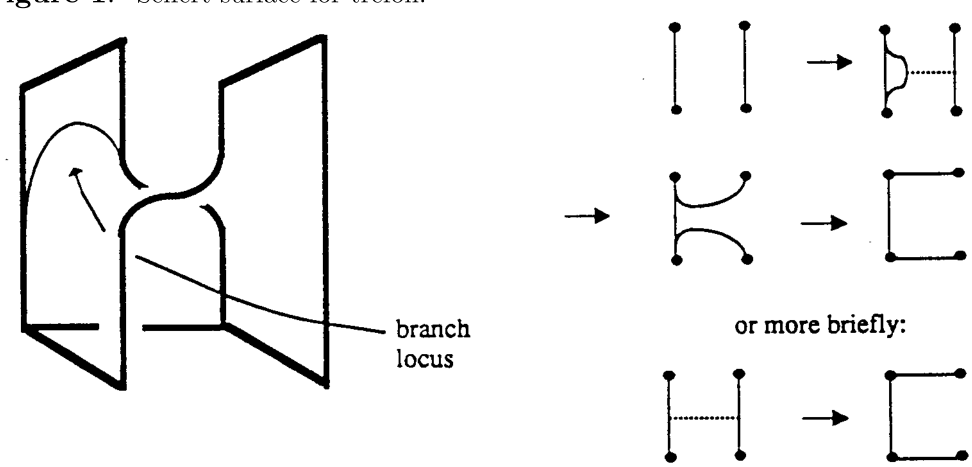

Figure 2. Intersection of a branched surface with horizontal planes.

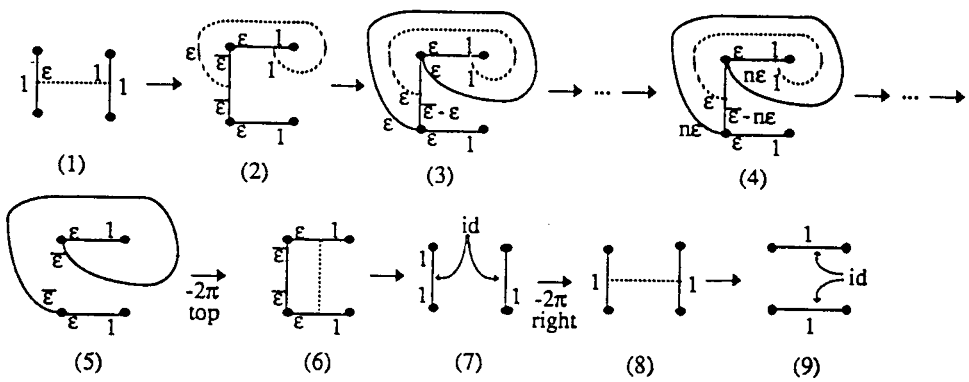

Figure 3. [H]'s construction of laminations of slopes $(-\infty, 0]$ on $T_{2,3}$.

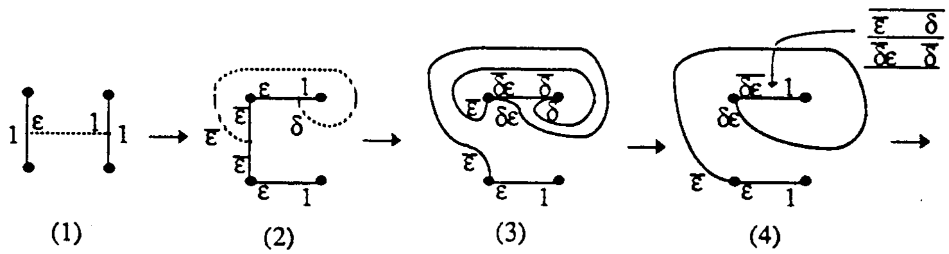

Figure 4. Modification of Figure 3 giving slopes (0,1) on $T_{2,3}$.

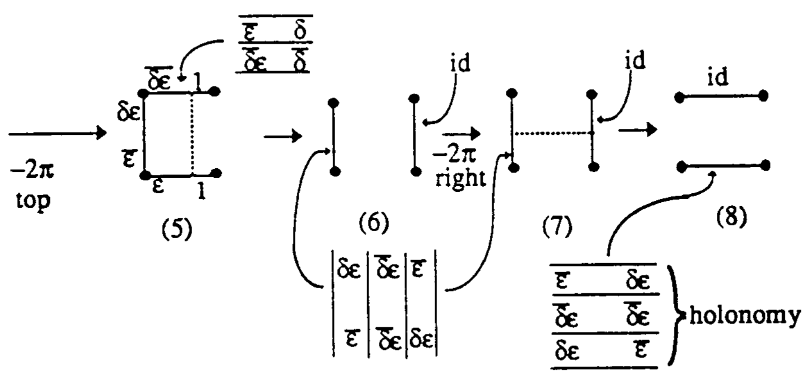

Figure 5. One gets holonomy if all saddles are just linear.

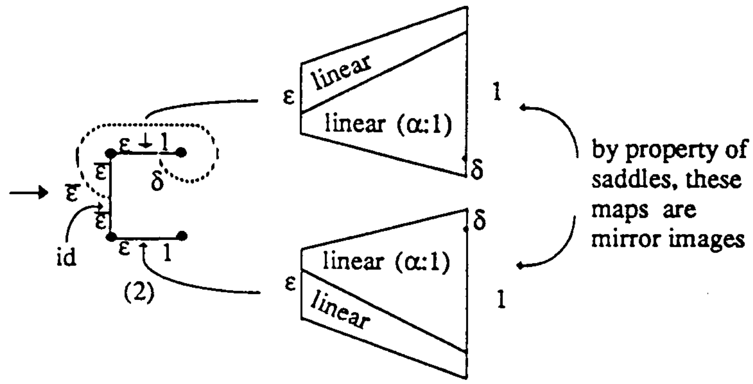

Figure 6. Avoiding holonomy.

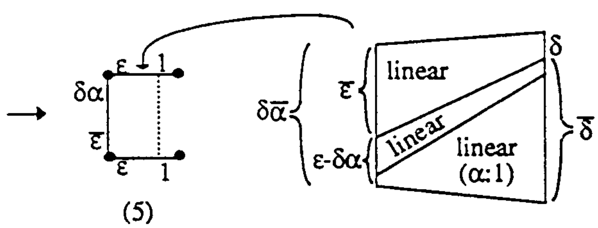

Figure 7. Modification of Diagram 5 of Figure 5 because of Figure 6.

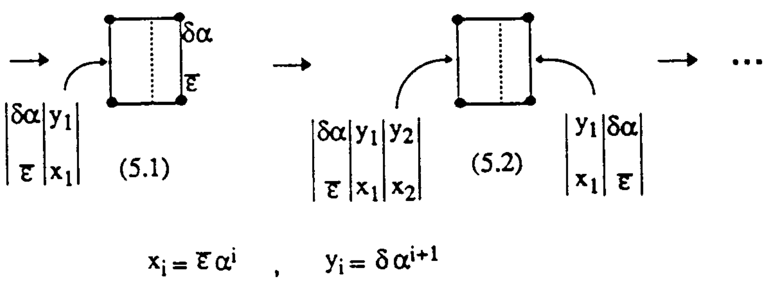

Figure 8. This process converges to Figure 9.

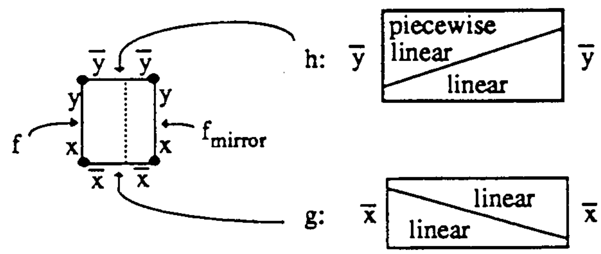

Figure 9. g and h have no interior fixed points.

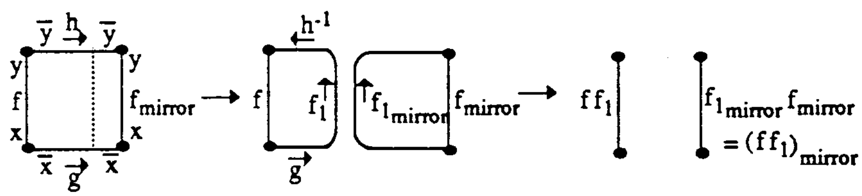

Figure 10. f_1 conjugates g to h.

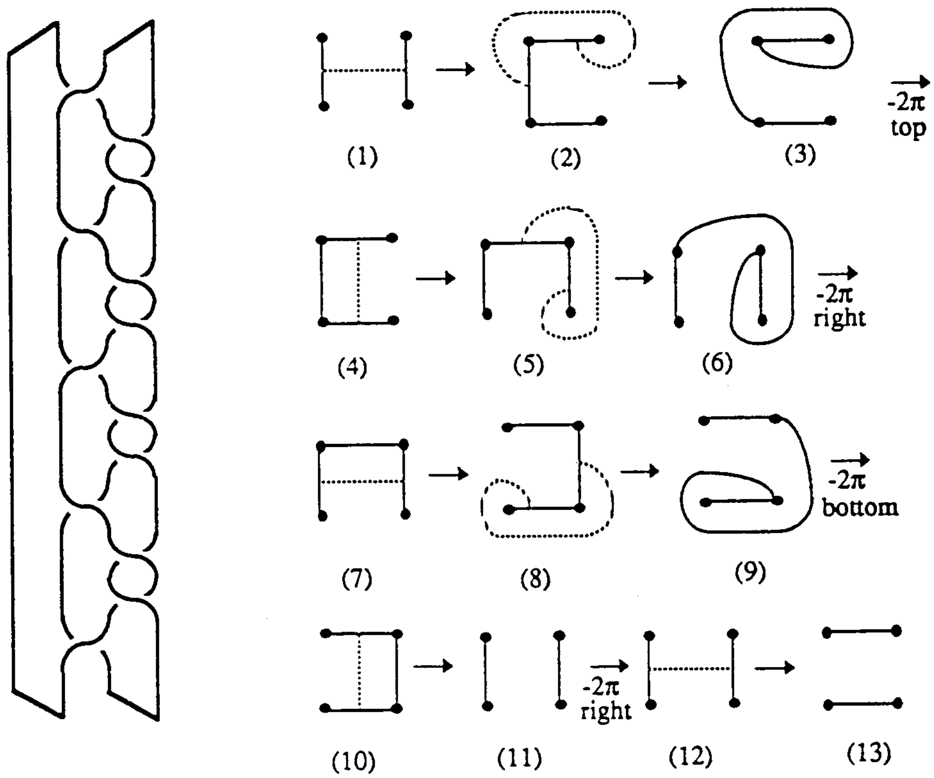

Figure 11. Branched Surface for $T_{2,5}$.

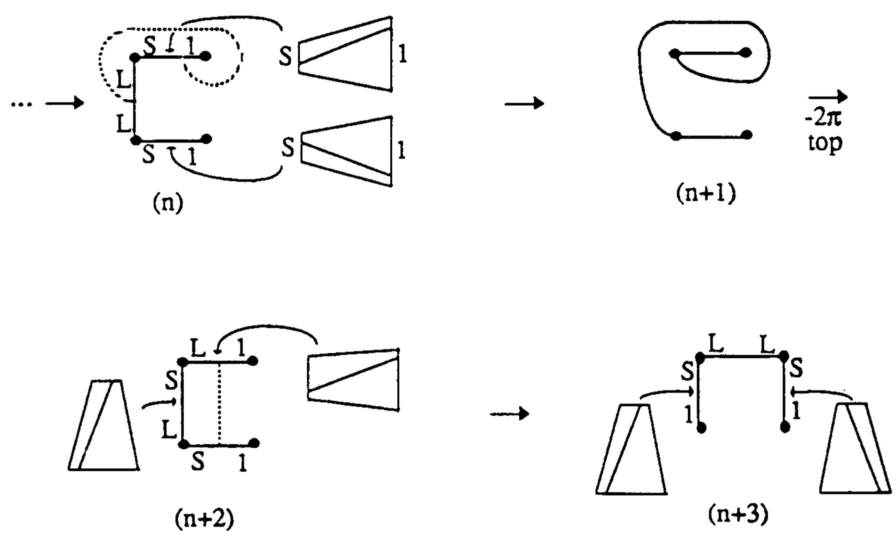

Figure 12. Induction step for avoiding holonomy for $T_{2,5}$.

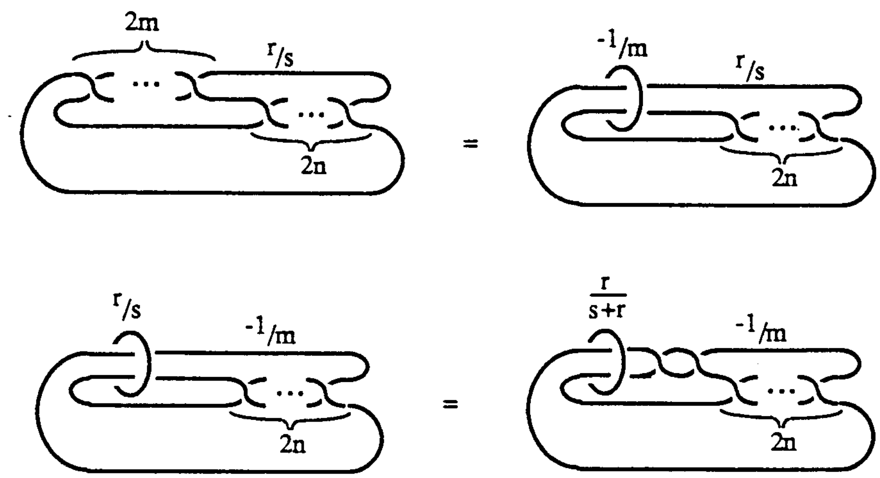

Figure 13. For knots of type (I).

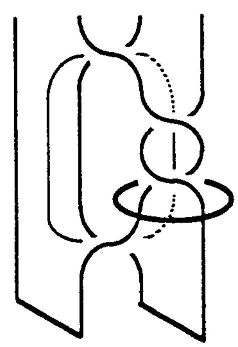

Figure 14. The unknot is isotopic to the curve on the surface.

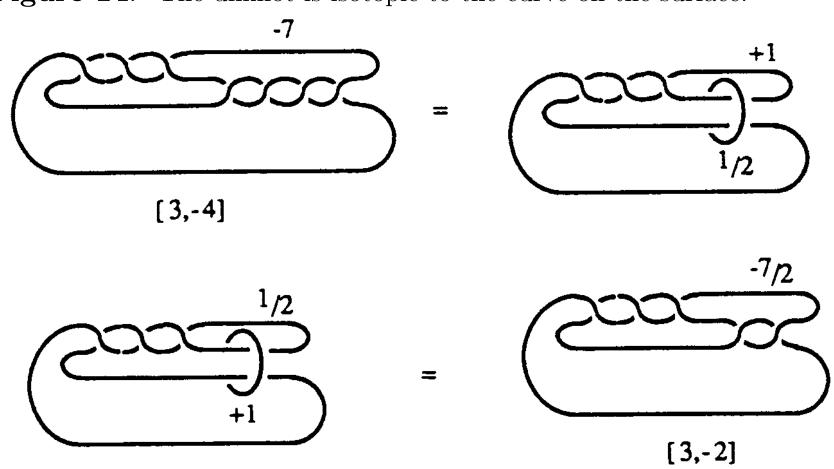

Figure 15. -7 on $7_3 = -7/2$ on 5_2.

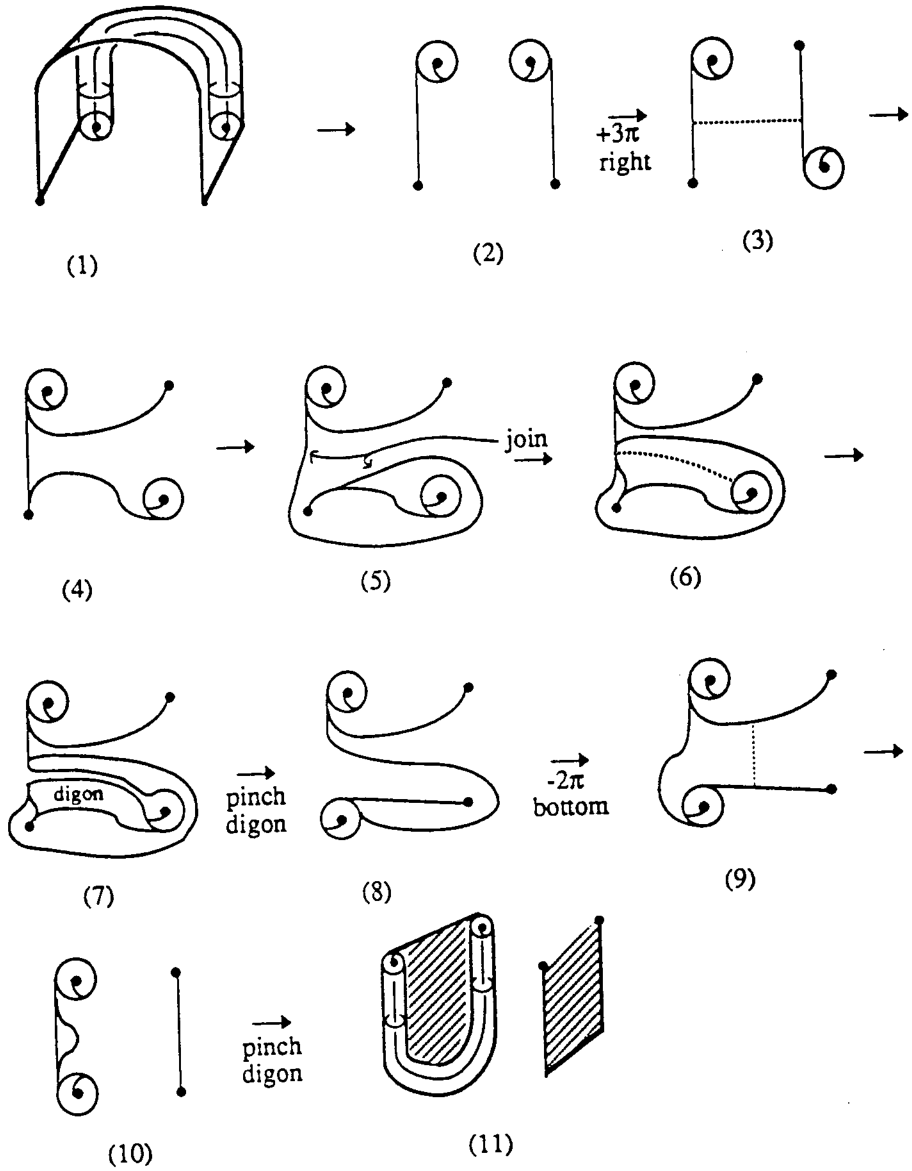

Figure 16. Essential branched surface for the [3,-4] knot (7_3 in [**Rf**]) carrying laminations of all (finite) boundary slopes.

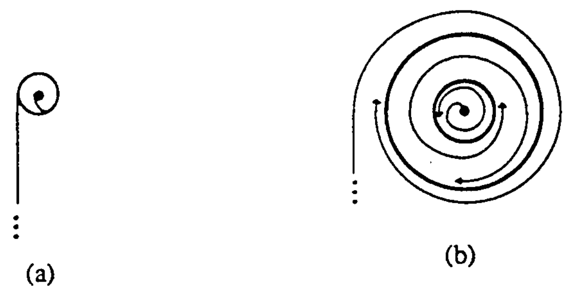

Figure 17. Triple Spiral Structure.

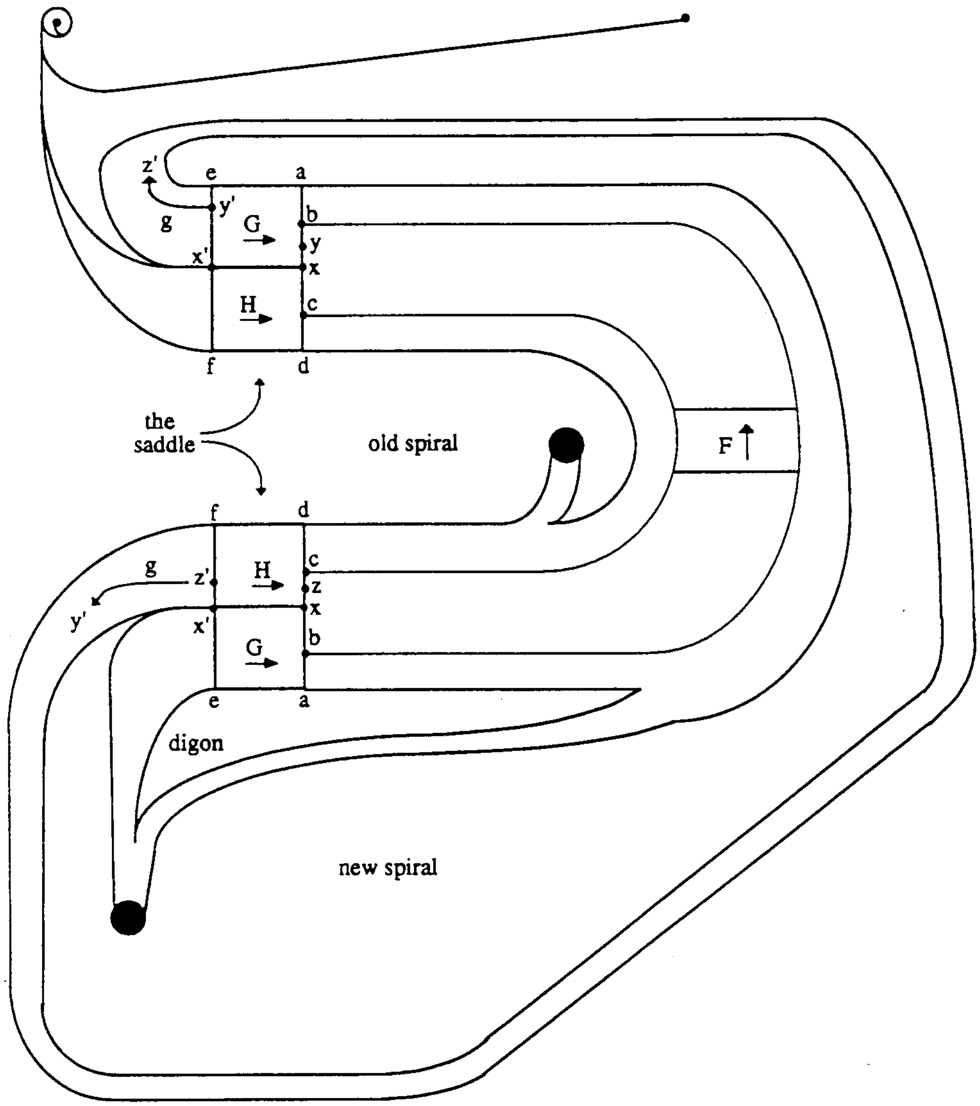

Figure 18. Blow-up of Diagram 7 of Figure 16.

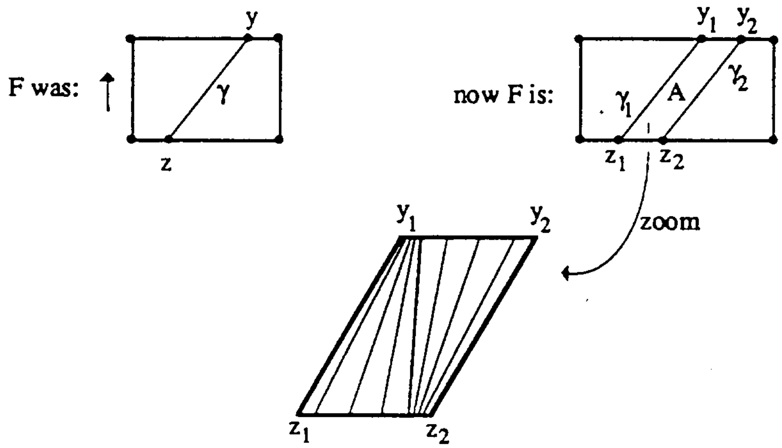

Figure 19. To get the middle spiral with opposite sense.

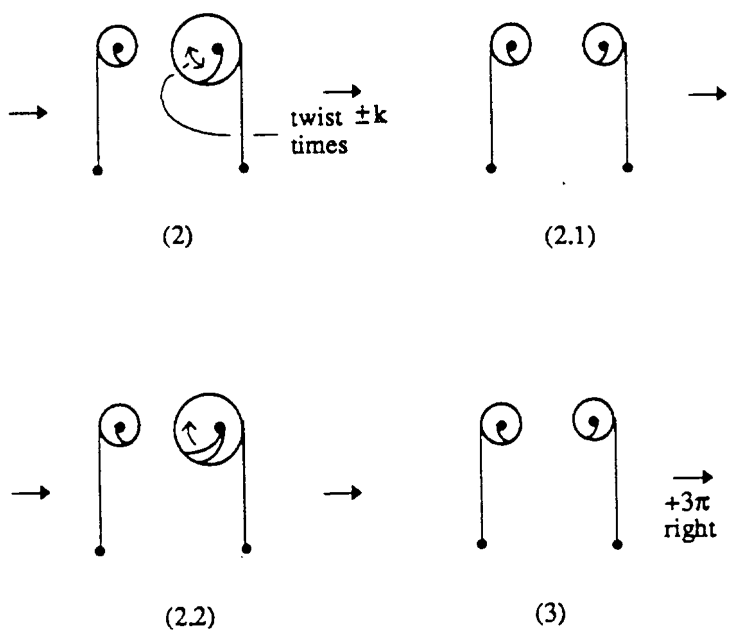

Figure 20. To get all slopes in Figure 16.

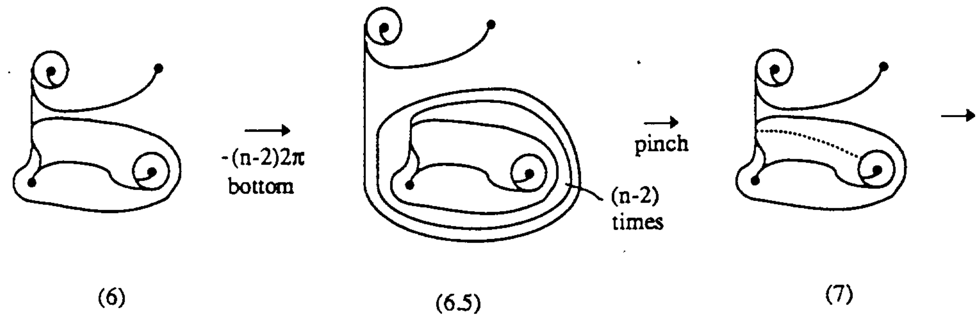

Figure 21. Modifying Figure 16 for knots in class (b).

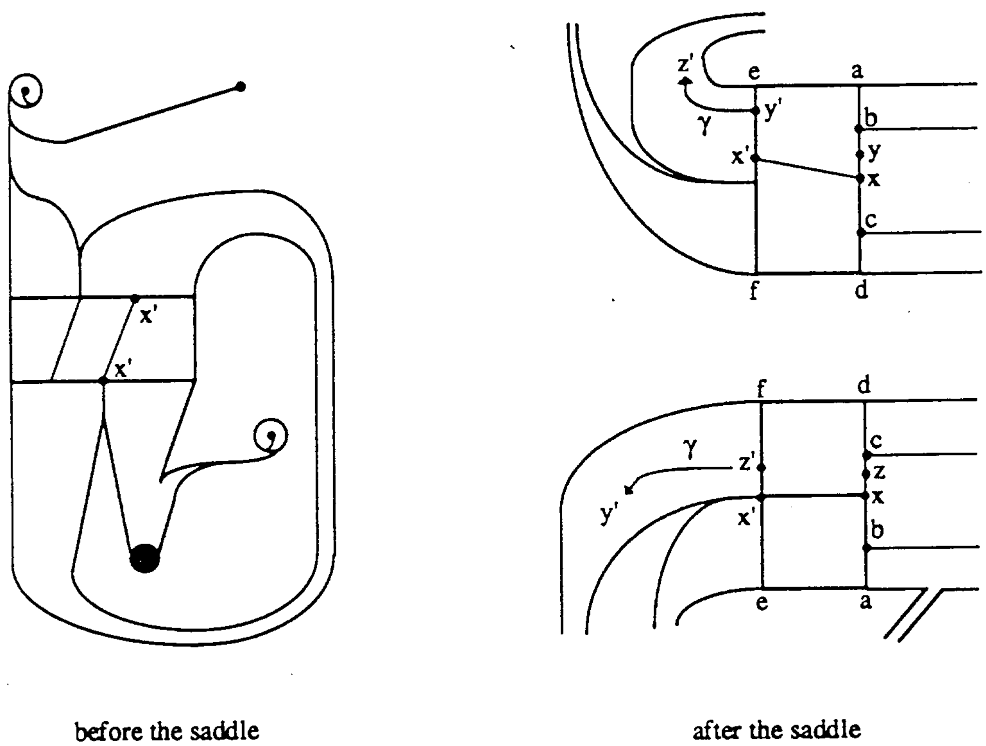

Figure 22. Modifying Figure 18 because of Figure 21.

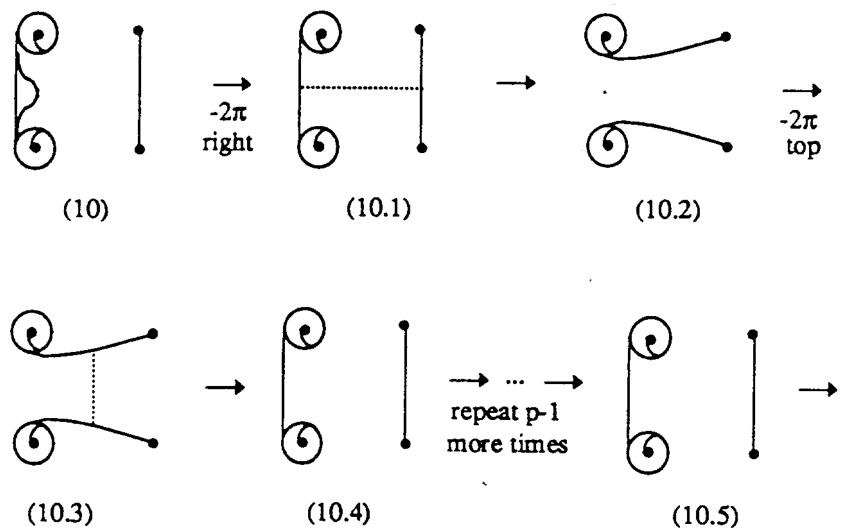

Figure 23. For knots in class (c).

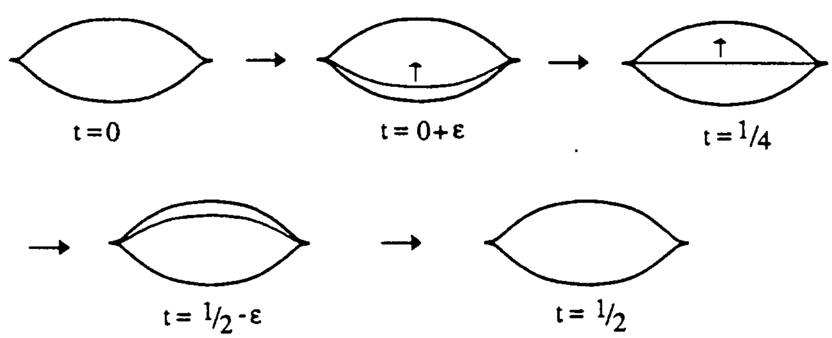

Figure 24. 'Moving copies of sides of digons across from one side to the other'.

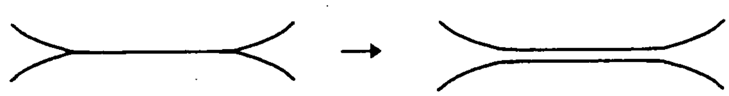

Figure 25. Only this type of splitting in Construction 4.1 can affect the topology of the complementary components.

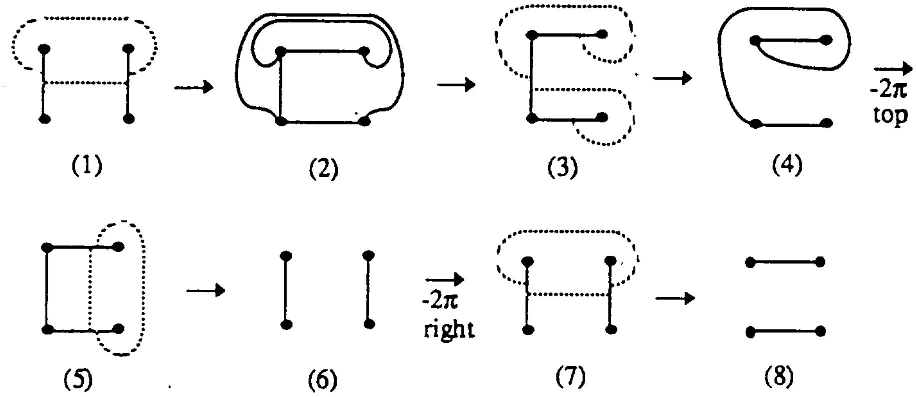

Figure 26. Branched surface for trefoil, with complementary saddles added in.

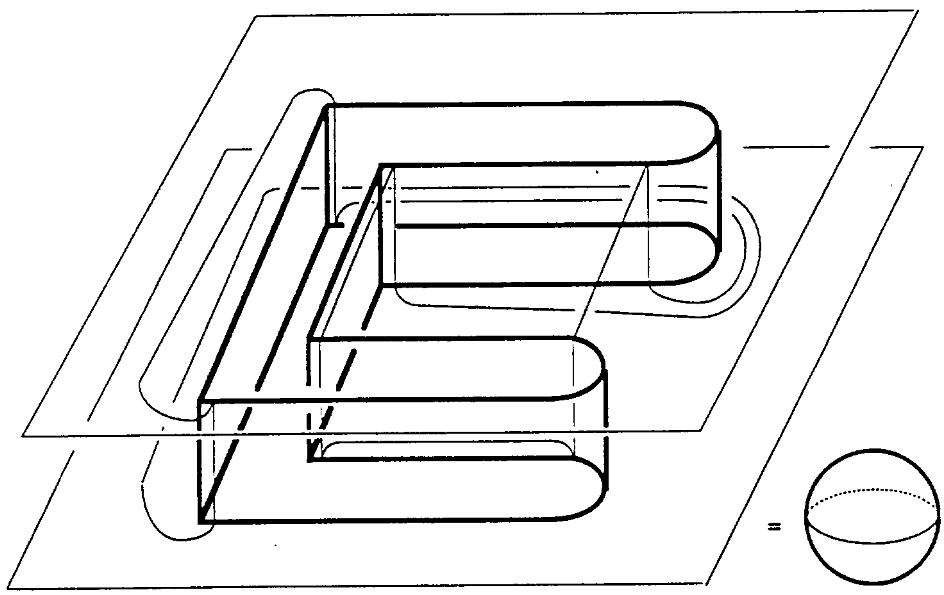

Figure 27. Complementary component of B between Diagrams 1 and 3 of Figure 26; the 6 thin horizontal arcs are $\subset \partial_V N(B)$, the 6 thin vertical arcs $\subset \partial N(K)$.

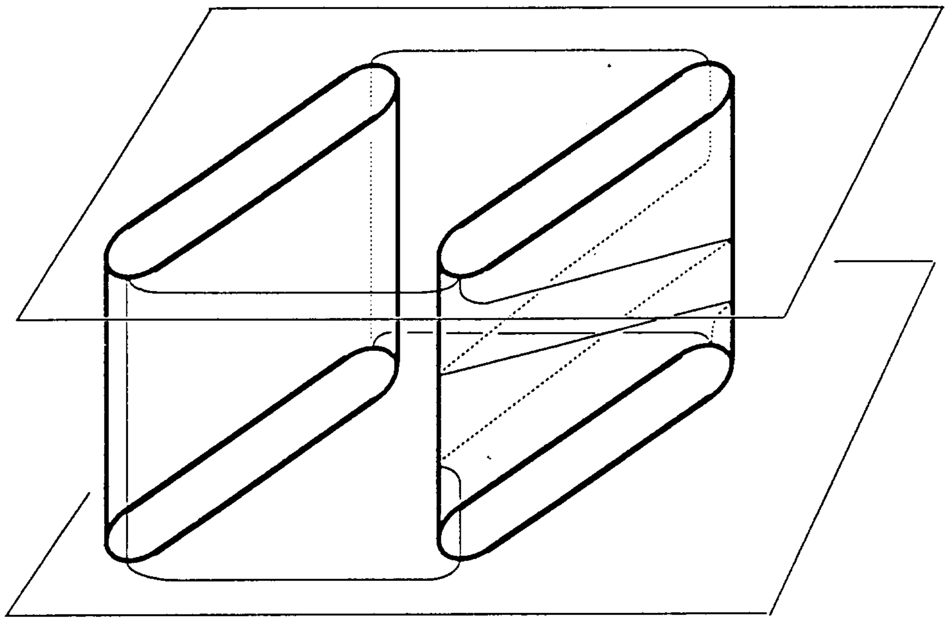

Figure 28. Complementary component of B between Diagrams 5 and 7 of Figure 26.

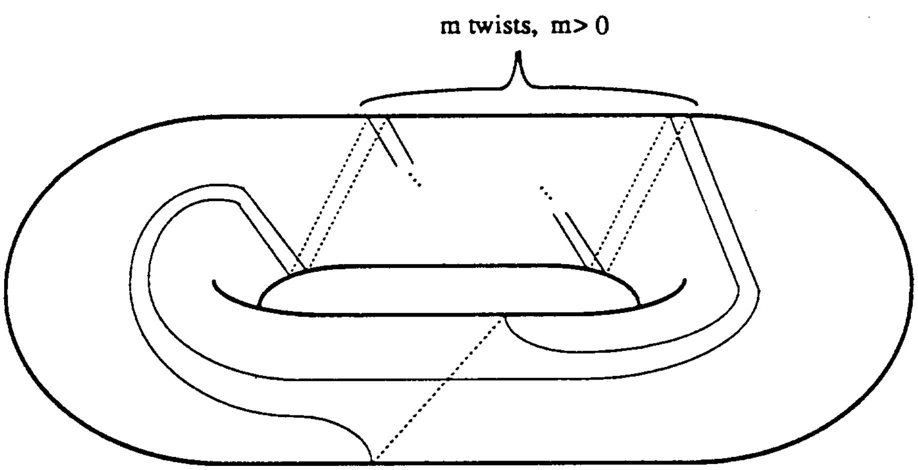

Figure 29. Neighborhood of branched surface of Figure 16, with Figures 20 and 21 added in.

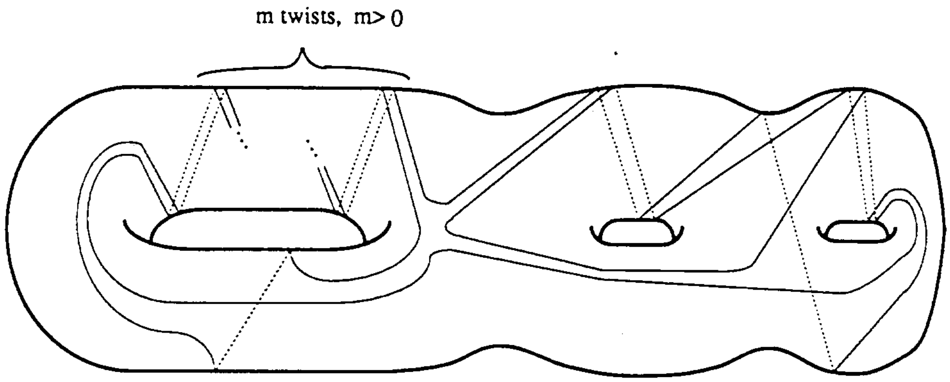

Figure 30. Neighborhood of branched surface for a knot in (II) ∩ (c).

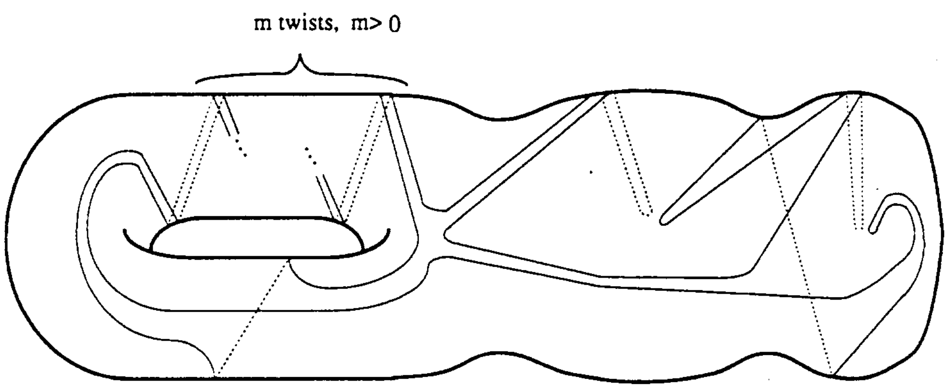

Figure 31. Result of disc decomposition on Figure 30.

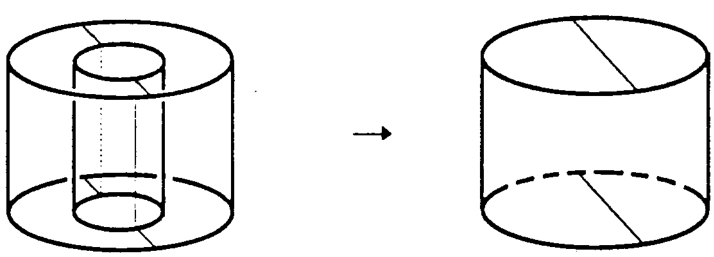

Figure 32. Disc Decomposition.

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