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K HOMOLOGY AND REGULAR SINGULAR DIRAC-SCHRÖDINGER OPERATORS ON EVEN-DIMENSIONAL MANIFOLDS

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We identify a class of Dirac-Schrödinger operators on incomplete manifolds and show that the index theory of these operators, including its expression in K homology, is parallel to that of Dirac-Schrödinger operators on complete manifolds.

Introduction.

The study of Fredholm indices of Dirac-Schrödinger operators (also known as perturbed Dirac operators and operators of Callias type) on complete manifolds arose in connection with questions in mathematical physics, [**10**, **18**, **21**]. It is now known that these indices carry information about the geometry of complete manifolds, [**1**, **9**, **22**]. The index theory of Dirac-Schrödinger operators on complete manifolds fits nicely in a K-theoretic framework, [**9**, **15**]. This observation and the observation that the resulting index formulas involve compactly supported data suggest that the index theory of Dirac-Schrödinger operators should be treated in a unified manner across a large class of noncompact manifolds, including complete and incomplete examples.

The present paper identifies a class of Dirac-Schrödinger operators on incomplete manifolds and shows that their index-theoretic and K-theoretic properties match those established for Dirac-Schrödinger operators on complete manifolds. The incomplete manifolds are those with asymptotically cone-like singularities, and the Dirac-Schrödinger operators are those we call regular singular (see Definition 2.1) that satisfy a property we call realizing their limiting indices (see Definition 1.1). In short these are sums of a Dirac operator and an order-zero "perturbation" that is invertible and large enough off a compact subset and whose pointwise norm grows in inverse proportion to the distance from the singularities.

We show that such a regular singular Dirac-Schrödinger operator defines a class in the K homology of the metric completion of the incomplete manifold on which it lives. We show that this class equals the Kasparov product of a K homology class (on the incomplete manifold) associated with the Dirac operator and a K cohomology class associated with the perturbation. Early in the argument one can veer off to establish an index formula for the Dirac-Schrödinger operator. Going on to K homology permits one to recover the full homology Chern character of the operator. The direct calculations of the Kasparov product in the complete case do not carry over to this incomplete setting because of the growth of the perturbation. However, this incomplete setting gives rise naturally to a compact manifold on which one can express the K-theoretic information carried by the Dirac-Schrödinger operator.

Among the tools used in this paper are the analysis of regular singular operators [**7**] and techniques for studying analytic K homology cycles on singular spaces and their open dense subsets [**3**, **17**]. The foundation of the paper is a theorem that allows one to conclude that certain Dirac-Schrödinger operators on very different manifolds have equal indices. This theorem is stated in a fairly general form in Section 1, but the proof follows exactly the reasoning used in [**16**] in the study of a special case.

0. Overview and example.

In this section we introduce the reasoning used in this paper by discussing it in the context of an explicitly worked example.

In this paper we study index theory on incomplete Riemannian manifolds associated with singularities. The asymptotically cone-like singularities (defined early in Section 2) that we consider can arise in two ways. If one attaches a finite-length cone to each component of a compact manifold with boundary, one gets a compact space which is singular (unless the original boundary components are spheres). The incomplete manifold is this space with the cone tips removed. Alternatively on a compact Riemannian manifold without boundary, one can encounter differential operators whose coefficients are singular at isolated points. (In our case the operators will be first-order elliptic with singularities only in the order zero terms.) Then the original manifold plays the role of compact space. The complement in the original manifold of the singular set is the incomplete manifold we work with. We refer to both the cone tips and the points where coefficients become singular as singular sets.

As indicated above, the operators we work with are first order elliptic differential operators with order zero potential terms exhibiting special behavior "near the edge" of the incomplete manifold, i.e., on the complement of a compact subset of the incomplete manifold. Roughly speaking, we require the potential to be invertible in a neighborhood of the singular set and to have in this neighborhood a pointwise norm that grows in inverse proportion to the distance from the singular set. A detailed description of these operators appears early in Section 2.

Example. We illustrate the subject of the paper with an example in which the singularities arise from the order zero term of the operator. Let X be a compact Riemann surface without boundary. Let $\mathcal{E}_1$ be a Hermitian holomorphic complex line bundle on X. Assume $\mathcal{E}_1$ has a meromorphic section for which the only pole, a simple pole, occurs at $z_0 \in X$. Choose such a section. Interpret this section as a bundle map $a : \mathcal{E}_0 \to \mathcal{E}_1$, where $\mathcal{E}_0$ is a trivial Hermitian complex line bundle on X. Assume X has a Riemannian structure so that in some neighborhood V of z_0, X is isometric and holomorphically equivalent to an open disk in the complex plane.

Let $\mathcal{S}_0$ denote the trivial line bundle on X and let $\mathcal{S}_1$ denote the bundle whose sections are differential forms of type $(0,1)$. The $\overline{\partial}$-operator maps sections of $\mathcal{S}_0$ to sections of $\mathcal{S}_1$. For $i \in \{0,1\}$, $\overline{\partial}$ extends to define operators $\overline{\partial}_i$ from sections of $\mathcal{S}_0 \otimes \mathcal{E}_i$ to sections of $\mathcal{S}_1 \otimes \mathcal{E}_i$. The incomplete manifold on which all of the above is defined is $M = X \setminus \{z_0\}$.

Restricting to M, we have the operator

$$\overline{\partial}_a = \begin{pmatrix} \overline{\partial}_0 & -a^* \\ a & \overline{\partial}_1^* \end{pmatrix}$$

mapping sections of $\mathcal{S}_0 \otimes \mathcal{E}_0 \oplus \mathcal{S}_1 \otimes \mathcal{E}_1$ to sections of $\mathcal{S}_1 \otimes \mathcal{E}_0 \oplus \mathcal{S}_0 \otimes \mathcal{E}_1$. This operator on M, $\overline{\partial}_a$, is the regular singular Dirac-Schrödinger operator of our example. (The difference in appearance between this operator and the operator in Definition 2.1 arises from our decision to write the summands in the range space in reverse order. This is merely an explicit implementation of one step of the unitary equivalence by which the operator is expressed in standard regular singular form.)

Analysis. The calculations we need to do with $\overline{\partial}_a$ depend on its behavior in $V \setminus \{z_0\}$. To see this behavior we use local coordinates in V that identify z_0 with 0. We choose local coordinates that respect the Riemannian and holomorphic structure of V. In V we use trivializations of the Hermitian holomorphic line bundles that respect all of their structure. For the sake of an example, we impose the further condition on a that in these coordinates a takes the form of multiplication by $1/z$. Then over $V \setminus \{z_0\}$ we can write $\overline{\partial}_a$ as

$$\begin{pmatrix} \partial/\partial\overline{z} & -1/\overline{z} \\ 1/z & -\partial/\partial z \end{pmatrix}.$$

In standard polar coordinates this operator is

$$\begin{pmatrix} e^{i\theta}/2 & 0 \\ 0 & -e^{-i\theta}/2 \end{pmatrix} \left\{ \begin{pmatrix} \partial/\partial r & 0 \\ 0 & \partial/\partial r \end{pmatrix} + r^{-1} \begin{pmatrix} i\partial/\partial\theta & -2 \\ -2 & -i\partial/\partial\theta \end{pmatrix} \right\}.$$

Ignoring the invertible first factor, we focus on

$$\begin{pmatrix} \partial/\partial r & 0 \\ 0 & \partial/\partial r \end{pmatrix} + r^{-1} \begin{pmatrix} i\partial/\partial\theta & -2 \\ -2 & -i\partial/\partial\theta \end{pmatrix}.$$

Standard polar coordinates involve a measure $rdrd\theta$ while the regular singular theory of [6, 7] is expressed in terms of $drd\theta$ in our context. The unitary operator from the L^2 space in one measure to the L^2 space in the other measure is multiplication by $r^{-1/2}$. Conjugating with this we can express the last operator in the $drd\theta$ setting as

$$\begin{pmatrix} \partial/\partial r & 0 \\ 0 & \partial/\partial r \end{pmatrix} + r^{-1} \begin{pmatrix} i\partial/\partial\theta - 1/2 & -2 \\ -2 & -i\partial/\partial\theta - 1/2 \end{pmatrix}.$$

This reveals that $\overline{\partial}_a$ can be studied using the techniques of regular singular theory as in [6, 7, 20].

In our analysis of $\overline{\partial}_a$, we focus on the above expression because the fundamental analytic questions that interest us revolve around the nature of the domain of the elliptic operator $\overline{\partial}_a$ on the incomplete manifold M. Using a smooth partition of unity consisting of a function with compact support in M and a function with support in V, we can break such questions of domain into well-understood questions about elliptic operators on compact manifolds and calculations on V.

An important step in our reasoning establishes that the index of $\overline{\partial}_a$ is equal to the index of a related Dirac-Schrödinger operator on a compact manifold. Our technique, as described in theorem 1.4 and its proof, establishes directly that the indices are equal when the order zero terms are sufficiently large. For this technique to have implications for index $(\overline{\partial}_a)$, we need to establish that the index is constant under a scaling of the order zero term. This is not guaranteed for an unbounded order zero term on an incomplete manifold, but we show that the following conditions (in addition to more easily verified conditions) imply this invariance for scale factors $s \geq 1$. The conditions are that for $s \geq 1$:

(i) The operator $\overline{\partial}_{sa}$, defined on smooth compactly supported sections, has a unique closed extension, which is Fredholm;

(ii) the domain of this closed extension is independent of s;

(iii) with this domain the Fredholm index of $\overline{\partial}_{sa}$ is independent of s.

To study these conditions we adopt the following notation.

$$t_{sa} = \begin{pmatrix} i\partial/\partial\theta - 1/2 & -2s \\ -2s & -i\partial/\partial\theta - 1/2 \end{pmatrix}$$

$$\partial_r + r^{-1}t_{sa} = \begin{pmatrix} \partial/\partial r & 0 \\ 0 & \partial/\partial r \end{pmatrix} + r^{-1}\begin{pmatrix} i\partial/\partial\theta - 1/2 & -2s \\ -2s & -i\partial/\partial\theta - 1/2 \end{pmatrix}.$$

To prove condition (i), we observe that the spectrum of the self-adjoint operator t_{sa} has empty intersection with $(-1/2, 1/2)$ and then we quote a result of [**6**, **7**]. The assertion about the spectrum follows from inspection of t_{sa}^2, which has $2s$ in the off-diagonal corners and has each diagonal corner equal to the sum of a nonnegative operator plus $4s^2$.

It follows from (i) and our use of a partition of unity that (ii) can be established via estimates of

$$\langle(\partial_r + r^{-1}t_{sa})\eta, (\partial_r + r^{-1}t_{sa})\eta\rangle \tag{0.1}$$

for η smooth and compactly supported on $\overline{V} \setminus \{z_0\}$. The expression in (0.1) equals

$$\begin{aligned} &\langle(\partial_r + r^{-1}t_{sa})^*(\partial_r + r^{-1}t_{sa})\eta, \eta\rangle \\ &= \langle(-\partial_r^2 + r^{-2}(t_{sa} + t_{sa}^2))\eta, \eta\rangle \\ &= \langle -\partial_r^2(\eta), \eta\rangle + \langle r^{-2}(t_{sa} + t_{sa}^2)\eta, \eta\rangle \\ &= \|\partial_r(\eta)\|^2 + \|r^{-1}(t_{sa} + t_{sa}^2)^{1/2}\eta\|^2. \end{aligned} \tag{0.2}$$

Again inspection of t_{sa}^2 shows that its spectrum has empty intersection with $(-2, 2)$ and hence that $t_{sa} + t_{sa}^2$ is positive and, in fact, bounded away from zero by a bound independent of s. In addition for each s, $t_{sa} + t_{sa}^2$ is a second order elliptic operator on a compact manifold (in this example, the circle). It follows that for $s_1, s_2 \geq 1$ each of $t_{s_1 a} + t_{s_1 a}^2$ and $t_{s_2 a} + t_{s_2 a}^2$ is relatively bounded with respect to the other. The condition (ii) follows.

Finally condition (ii), the description of domain $(\overline{\partial}_{sa})$ provided by the last line of (0.2), and the observation that $t_{sa} + t_{sa}^2$ is uniformly bounded away from zero show that $\begin{pmatrix} 0 & -a^* \\ a & 0 \end{pmatrix}$ is a bounded operator from domain $(\overline{\partial}_a)$ to the set of L^2 sections. It follows that for $s \geq 1$, $\overline{\partial}_{sa}$ is a norm continuous family of bounded operators from domain $(\overline{\partial}_a)$ to the set of L^2 sections. This establishes (iii).

Topology. In Section 3 we observe that letting

$$d_a = \begin{pmatrix} 0 & \overline{\partial}_a^* \\ \overline{\partial}_a & 0 \end{pmatrix},$$

we can use $d_a(1+d_a^2)^{-1/2}$ to define a class in the K homology of X, in particular in $KK(C(X), \mathbf{C})$. Our understanding of this K homology class is based on representing the same class by a cycle created from an elliptic operator on a compact manifold Y without boundary. There is some freedom in the choice of Y. Y need only satisfy the conditions given at the beginning of Section 3. In particular we require that there be a continuous map $f : Y \to X$ that identifies $f^{-1}(M)$ with M via a diffeomorphism. (The map f provides a means of viewing KK cycles defined on Y, M, or X in a single KK group. For instance f_* defines a map from the K homology of Y to the K homology of X.) We further require that Y carry an elliptic operator, vector bundles, and a vector-bundle map analogous to those on M. In fact the analogy we require is a strong one: We require (conditions 3.2 and 3.3) that for any compact subset of M it be possible to deform smoothly the structure of Y and the structures of the objects carried on Y so that over the interior of the compact subset, the identification arising from f is an isomorphism of these structures. Because of the singularity on X, it is not possible to impose structures on (the objects on) Y that can be identified over all of $f^{-1}(M)$ with those on M. However, to establish that the classes in $KK(C_0(M), \mathbf{C})$ defined by cycles on M equal the classes defined by the analogous cycles on Y, conditions 3.2 and 3.3 are sufficient. There are two steps in proving this sufficiency. One is the homotopy invariance of KK classes. (The homotopy arises from the deformations on Y.) The other (Lemma 3.13) is a KK theory exact sequence arising from an exhaustion of M by appropriate subsets.

A doubling construction described in Remark 2.16 guarantees that we can always find a Y satisfying the conditions stated in Section 3. However, in our example the manifold X, mapping to itself by the identity map, can play the role of Y. $\mathcal{S}_0$ and $\mathcal{S}_1$ take their previous meaning on Y, i.e. on X, as does $\mathcal{E}_0$. The vector bundle $\mathcal{E}_1^Y$ is defined as follows. Remove a small closed disk, centered at z_0 and contained in V, from X. Call the remaining open set U. $\mathcal{E}_1^Y$ arises by clutching $\mathcal{E}_1|_U$ with $\mathcal{E}_0|_V$ by the attaching map $a^{-1}|_{U\cap V}$. The vector bundle map $a^Y : \mathcal{E}_0 \to \mathcal{E}_1^Y$ is defined by $a^Y|U = a|U$ and $a^Y|_V = id|_{\mathcal{E}_0}$. As before, $\overline{\partial}$ extends to the resulting tensor product bundles; again we denote these extensions by $\overline{\partial}_0$ and $\overline{\partial}_1$. Then the new elliptic operator on $Y = X$ is

$$\overline{\partial}_a^Y = \begin{pmatrix} \overline{\partial}_0 & -(a^Y)^* \\ a^Y & \overline{\partial}_1^* \end{pmatrix}$$

mapping sections of $\mathcal{S}_0 \otimes \mathcal{E}_0 \oplus \mathcal{S}_1 \otimes \mathcal{E}_1^Y$ to sections of $\mathcal{S}_1 \otimes \mathcal{E}_0 \oplus \mathcal{S}_0 \otimes \mathcal{E}_1^Y$. With

$$d_a^Y = \begin{pmatrix} 0 & (\overline{\partial}_a^Y)^* \\ \overline{\partial}_a^Y & 0 \end{pmatrix},$$

$d_a^Y(1+(d_a^Y)^2)^{-1/2}$ is the operator from which we create a cycle defining a class in $KK(C(Y),\mathbf{C})$.

Remark. It is interesting to note that we impose few conditions on $Y\setminus f^{-1}(M)$ other than the requirement that the vector-bundle map be invertible over this set. This is because one can show by standard reasoning that a^Y represents a K theory class on Y which is in the image of the relative group $K^0(Y,Y\setminus f^{-1}(M))$ and that $d_a^Y(1+(d_a^Y)^2)^{-1/2}$ represents the cap product of this class with the K homology class represented by the $\overline{\partial}$-operator (in our example) on Y. (In Section 3 we recall the proofs of these assertions in the language of KK theory and Kasparov products.) The analogous statement for the cycles defined on M and the K homology of X is one of the goals of this paper, but the growth of the vector-bundle map near the singular set prevents direct calculation of the Kasparov products on X. Instead we show that the KK cycles defined on M represent the same classes over M or X (as appropriate) as the analogous cycles defined on Y. The product calculation with the cycles on Y then determines the product on X. In this way (or more directly through the equality of $KK(C(X),\mathbf{C})$ classes established by Proposition 3.15) one can use well-established techniques on Y to calculate the full homology Chern character in $H_*(X)$ of the class in $KK(C(X),\mathbf{C})$ represented by $d_a(1+d_a^2)^{-1/2}$.

Lemma 3.14 and Proposition 3.15 contain the heart of the reasoning showing that cycles defined on M and the analogous cycles defined on Y represent the same classes in $KK(C(X),\mathbf{C})$. An exact sequence shows that a class in $KK(C(X),\mathbf{C})$ is determined by its image in $KK(C_0(M),\mathbf{C})$ and by the index of its operator. That the classes defined by $d_a^Y(1+(d_a^Y)^2)^{-1/2}$ and $d_a(1+d_a^2)^{-1/2}$ have the same image in $KK(C_0(M),\mathbf{C})$ follows from conditions 3.2 and 3.3 and from Lemma 3.13. (In this reasoning the nonlocal operators $(1+d_a^2)^{-1/2}$ and $(1+(d_a^Y)^2)^{-1/2}$ provide a challenge that is met by the finite propagation speed techniques of [**3**].) Our analysis of $\overline{\partial}_a$ and standard properties of the operator $\overline{\partial}_a^Y$ on the compact manifold Y show that theorem 1.4 can be applied to establish that $\operatorname{index}(\overline{\partial}_a) = \operatorname{index}(\overline{\partial}_a^Y)$.

1. Index comparison.

In this section we define terms and discuss a theorem with which one can compare indices of Fredholm Dirac-Schrödinger operators. We later use this theorem to show that a Fredholm Dirac-Schrödinger operator on an incomplete manifold has index equal to that of a related Dirac-Schrödinger operator on a compact manifold without boundary.

Let W be a Riemannian manifold, not necessarily complete. Let F_0 and

F_1 be a pair of Hermitian vector bundles over W. Let

$$S + C : L^2(F_0) \to L^2(F_1)$$

be an operator in which S is a first-order elliptic differential operator and C arises from a vector bundle map. Assume that $S+C$ is closed. Throughout the paper we assume that bundles and their maps are smooth, that differential operators have smooth coefficients, and that the domains of these operators contain all smooth compactly supported sections.

Definition 1.1. We say that $S + C$ realizes its limiting index if $\{S + tC : t \in [1, \infty)\}$ is a family of Fredholm operators with constant domain and constant index.

Definition 1.2. Let S' be the restriction of S to smooth compactly supported sections. Denote the formal adjoint of S' by S'', which is also defined on smooth compactly supported sections. We say that $S+C$ satisfies the core condition if we can choose a positive t_0, a subspace $\mathcal{H}$ of $L^2(F_0) \oplus L^2(F_1)$, and a closed extension Σ of $\begin{pmatrix} 0 & S'' \\ S' & 0 \end{pmatrix}$ such that :

(a) $\mathcal{H}$ serves as a core for all operators $\begin{pmatrix} 0 & (S+tC)^* \\ S+tC & 0 \end{pmatrix}$ with $t \geq t_0$;

(b) $\begin{pmatrix} 0 & (S+tC)^* \\ S+tC & 0 \end{pmatrix}^2$ is defined on $\mathcal{H}$ for all $t \geq t_0$; and

(c) $\Sigma^*\Sigma$ is defined on $\mathcal{H}$.

Then for $t \geq t_0$ we define $R(S + tC)$ on $\mathcal{H}$ by

$$\begin{pmatrix} 0 & (S+tC)^* \\ S+tC & 0 \end{pmatrix}^2 = \Sigma^*\Sigma + R(S + tC).$$

Implicit in any use of $R(S + tC)$ is a set of choices as discussed above. In Sections 2 and 3 of this paper, operators $S+C$ will satisfy the core condition by virtue of having the set of smooth compactly supported sections serve as a core for $S + tC$ and for $(S + tC)^*$ for all $t \geq 1$. This is the core we will use, and there is then no ambiguity about the definition of $R(S + tC)$. In such a situation, it is suggestive and accurate to write the defining equation of $R(S + tC)$ as

$$\begin{pmatrix} 0 & (S+tC)^* \\ S+tC & 0 \end{pmatrix}^2 = \begin{pmatrix} S^*S & 0 \\ 0 & SS^* \end{pmatrix} + R(S + tC).$$

Definition 1.3. We say that $S+C$ satisfies the eventual positivity condition if for sufficiently large t, $R(S+tC)$ is a vector-bundle map and there is a compact set $K \subset W$ and a positive constant k such that for sufficiently large t, $R(S+tC) \geq k$ on the complement of K. (Suppose the set of smooth compactly supported sections is a core for $S+tC$ and $(S+tC)^*$ for all $t \geq 1$. Then because C^*C is a vector-bundle map, $R(S+C)$ is a vector-bundle map if and only if for arbitrary $t \geq 1$ $R(S+tC)$ is a vector-bundle map.)

The following theorem provides the idea unifying the index theory of Dirac-Schrödinger operators on a broad class of manifolds. The proof is based directly on the proof of a special case given in [**16**]. Some applications of this proof to a wide class of Dirac-Schrödinger operators were discussed in [**13**] under less general assumptions than are considered here.

Theorem 1.4. *Let W_1 and W_2 be Riemannian manifolds on which there are operators* (*of the kind discussed in the second paragraph of this section*) $S_1+C_1 : L^2(F_{0,1}) \to L^2(F_{1,1})$ *on* W_1 *and* $S_2+C_2 : L^2(F_{0,2}) \to L^2(F_{1,2})$ *on* W_2. *Assume that each of* C_1 *and* C_2 *is invertible off a compact set. Assume that each of* S_1+C_1 *and* S_2+C_2 *realizes its limiting index and that each satisfies the core condition and the eventual positivity condition. Assume that there are neighborhoods* V_1 *of the singular set of* C_1 *and* V_2 *of the singular set of* C_2 *that are isometric. Assume that the isometry is covered by isomorphisms of Hermitian vector bundles* $F_{i,1}|_{V_1} \to F_{i,2}|_{V_2}$ *and that the associated maps on sections intertwine the restrictions of* S_1 *and* S_2 *and intertwine the restrictions of* C_1 *and* C_2. *Then*

$$\operatorname{index}(S_1+C_1) = \operatorname{index}(S_2+C_2).$$

Proof. Let f be the map on sections defined by the isomorphisms of Hermitian vector bundles. Let ϕ be a nonnegative function that is identically one on the set where C_1 is not invertible and that has compact support in V_1. Let Φ denote multiplication by ϕ.

Because S_1+C_1 and S_2+C_2 realize their limiting indices, in order to show $\operatorname{index}(S_1+C_1) = \operatorname{index}(S_2+C_2)$ it suffices to show that for large enough t

$$\operatorname{index}\left(\begin{pmatrix} S_1+tC_1 & 0 \\ 0 & (S_2+tC_2)^* \end{pmatrix}\right) = 0. \tag{1.5}$$

Let

$$G_t = \begin{pmatrix} S_1+tC_1 & -(tf\circ\Phi)^* \\ tf\circ\Phi & (S_2+tC_2)^* \end{pmatrix}.$$

Because G_t is a relatively compact perturbation of the operator in (1.5), it suffices to show that for large enough t, G_t is invertible. We proceed to show that for large enough t, $G_t^* G_t$ has a positive lower bound. Analogous reasoning establishes the same property for $G_t G_t^*$.

Because ϕ has compact support in the region where f intertwines the operators, the off-diagonal entries of $G_t^* G_t$ are bounded vector-bundle maps multiplied by t. Note that these off-diagonal entries are zero outside the support of ϕ and its counterpart in W_2.

In the following analysis of $G_t^* G_t$, a subscript i, j denotes a row, column position in a two-by-two matrix. Inner products are those associated with L^2 spaces of sections. u is in the core used in the definition of $R(S_1 + tC_1)$, and v is in the core used in the definition of $R(S_2 + tC_2)$.

(1.6)
$$\left\langle G_t^* G_t \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle \geq \left\langle \left(R\left(S_1 + tC_1\right)_{1,1} + t^2 \left(\Phi f^* f \Phi\right) \right) u, u \right\rangle$$
$$+ \left\langle \left(R\left(S_2 + tC_2\right)_{2,2} + t^2 \left(f \Phi \Phi f^*\right) \right) v, v \right\rangle$$
$$+ \left\langle \left(G_t^* G_t\right)_{1,2} v, u \right\rangle + \left\langle \left(G_t^* G_t\right)_{2,1} u, v \right\rangle .$$

The maps appearing inside the inner products on the right side of the above inequality are all vector-bundle maps. Choose compact subsets K_1 of W_1 and K_2 of W_2 such that: for each i K_i plays the role of K in the definition of eventual positivity of $S_i + C_i$; K_1 contains support (ϕ); and K_2 contains the isometric image of support (ϕ).

The right side of (1.6) equals

(1.7)
$$\langle R(S_1 + tC_1)_{1,1} u|_{W_1 - K_1}, u|_{W_1 - K_1} \rangle + \langle R(S_2 + tC_2)_{2,2} v|_{W_2 - K_2}, v|_{W_2 - K_2} \rangle$$
$$+ \langle t^2 (C_1^* C_1 + \Phi f^* f \Phi) u|_{K_1}, u|_{K_1} \rangle + \langle t^2 (C_2 C_2^* + f \Phi \Phi f^*) v|_{K_2}, v|_{K_2} \rangle$$
$$+ \left\langle t\Psi \begin{pmatrix} u|_{K_1} \\ v|_{K_2} \end{pmatrix}, \begin{pmatrix} u|_{K_1} \\ v|_{K_2} \end{pmatrix} \right\rangle .$$

Here Ψ is defined to be whatever is necessary to make equality hold. The important properties to note are that Ψ is independent of t and that because the vector-bundle map Ψ is taking values only on $K_1 \times K_2$, Ψ is bounded. It follows that there exists a constant c_0 such that

$$\left| \left\langle t\Psi \begin{pmatrix} u|_{K_1} \\ v|_{K_2} \end{pmatrix}, \begin{pmatrix} u|_{K_1} \\ v|_{K_2} \end{pmatrix} \right\rangle \right| \leq tc_0 (\|u|_{K_1}\|^2 + \|u|_{K_1}\| \cdot \|v|_{K_2}\| + \|v|_{K_2}\|^2).$$

Observe that $C_1^* C_1 + \Phi f^* f \Phi$ has a positive lower bound on K_1 and that $C_2 C_2^* + f \Phi \Phi f^*$ has a positive lower bound on K_2. The eventual positivity condition and comparison of t^2 with t show that for large enough t, $G_t^* G_t$ has a positive lower bound. $\square$

2. Regular singular Dirac-Schrödinger operators.

In this section we define and study the properties of regular singular Dirac-Schrödinger operators on even-dimensional manifolds. We then carry out the construction which assigns to a regular singular Dirac-Schrödinger operator a related elliptic operator on a compact manifold without boundary. This construction is the foundation for the K-theoretic reasoning of the third section. In studying regular singular operators, we rely on [**7**], which was motivated by [**11**] and [**12**].

Let M be an oriented even-dimensional Riemannian manifold with asymptotically cone-like singularities. M is separated by a hypersurface N into two pieces that share their common boundary N: A compact manifold with boundary; and a piece $C_p(N)$ isometric to $(0,p] \times N$ with metric $dr \otimes dr + r^2 g_N(r)$. Here p is some positive number, $r \in (0,p]$, and $g_N(r)$ is a family of Riemannian metrics on N that is smooth on $[0,p]$. For $0 < x \leq p$ we let $C_x(N)$ denote the piece of M associated with $(0,x] \times N$, and we let M_x denote the complement in M of $C_x(N)$.

Let $\mathcal{D}$ be a Dirac operator acting on sections of a complex Dirac bundle S over M. (The terminology is from [**19**].) (At this point one can assume that the sections discussed are smooth and compactly supported. We will specify domains more carefully when that is necessary.) The Dirac bundle is graded by the positive and negative eigenspaces of the action of M's volume element into $S = S_+ \oplus S_-$. We assume that $\mathcal{D}$ is a first order regular singular elliptic operator (regular singular operator for short) in the sense of [**7**]. We make the further assumption that the unitary map by which $\mathcal{D}$ is realized as a regular singular operator respects the distance represented by the variable r. As shown in [**7**] and [**20**] the signature and spin Dirac operators satisfy these assumptions. It seems likely that many interesting Dolbeault operators (viewed as spin^c Dirac operators) will also. An example was discussed in Section 0.

Let E_0 and E_1 be Hermitian vector bundles with metric connections over M. For each i assume that there is a Hermitian bundle with metric connection $\widetilde{E_i} \to N$ and a chosen fixed isomorphism of all structures $E_i|_{C_p(N)} \cong (0,p] \times \widetilde{E_i}$. This isomorphism is henceforth implicit in our discussion of E_i. Let $\mathcal{A} : E_0 \to E_1$ be a vector bundle map that is invertible off some compact subset of M and that over $C_p(N)$ has the form $r^{-1}\widetilde{\mathcal{A}}$. Here $\widetilde{\mathcal{A}} : \widetilde{E_0} \to \widetilde{E_1}$ is a fixed invertible vector bundle map over N.

For each i let D_i denote the operator from sections of $S_+ \otimes E_i$ to sections of $S_- \otimes E_i$ that is defined using the definition of $\mathcal{D}$ and tensor product connections. Let A be the operator from sections of $S_\pm \otimes E_0$ to sections of $S_\pm \otimes E_1$ defined using the tensor product of the identity operator with $\mathcal{A}$.

Definition 2.1. Under the above assumptions we call the operator

$$D_A = \begin{pmatrix} A & D_1^* \\ D_0 & -A^* \end{pmatrix}$$

from sections of $S_+ \otimes E_0 \oplus S_- \otimes E_1$ to sections of $S_+ \otimes E_1 \oplus S_- \otimes E_0$ a regular singular Dirac-Schrödinger operator.

Remark 2.2. Let $\widetilde{S}$ be the positive eigenspace for the action of N's volume element on $S|_N$. Let $\widetilde{D_i}$ be the operator on the bundle $\widetilde{S} \otimes \widetilde{E_i}$ arising from the realization of $\mathcal{D}$ as a regular singular operator. Let $\widetilde{A}$ be the restriction of of A to N. As observed in [**13**] in a more restricted setting, one can reason as in [**20**] to show that over $C_p(N)$ D_A is unitarily equivalent to

$$\partial/\partial r + r^{-1} \begin{pmatrix} \widetilde{D_0} & \widetilde{A}^* \\ \widetilde{A} & -\widetilde{D_1} \end{pmatrix} + r^\beta \widetilde{T}(r)$$

as an operator on the Hilbert space of L^2 functions on $(0, p]$ with values in $L^2(\widetilde{S} \otimes \widetilde{E_0} \oplus \widetilde{S} \otimes \widetilde{E_1})$. It follows that D_A is a regular singular operator. Here $\beta > -1/2$ and $\widetilde{T}(r)$ is a smooth family of first order differential operators satisfying conditions described in [**7**]. $\widetilde{T}$ does not appear in the setting of [**20**] or of our Section 0, but [**7**] shows that it does occur for some examples of operators $\mathcal{D}$. For the moment one can assume that the domain of the operator consists of smooth compactly supported functions with values in smooth sections.

Notation 2.3.

$$T_A = \begin{pmatrix} \widetilde{D_0} & \widetilde{A}^* \\ \widetilde{A} & -\widetilde{D_1} \end{pmatrix}.$$

Let ψ be a positive compactly supported function on $C_p(N)$ that depends on r alone, that equals 1 in a neighborhood of $r = p$, and that takes all of its values in $[0, 1]$. Let $D_{A,\psi}$ be the operator that agrees with D_A off $C_p(N)$ and that over $C_p(N)$ is unitarily equivalent to

$$\partial/\partial r + r^{-1} T_A + \psi(r) r^\beta \widetilde{T}(r)$$

via the unitary equivalence used to give the analogous expression for D_A.

Remark 2.4. $D_{A,\psi}$ is also a regular singular operator. By focusing on a subset of $C_p(N)$ that has empty intersection with support(ψ), we see that $D_{A,\psi}$ has the form $\partial/\partial r + r^{-1} T_A$ on the asymptotically cone-like complement of a compact manifold with boundary.

By this remark, the analysis of [**13**], which we now recall, applies to $D_{A,\psi}$. We are interested in the index theory of D_A, but it is easier to do explicit analysis with the simpler operator $D_{A,\psi}$. Nothing is lost in this approach because we show, by an estimate in Lemmas 2.10 and 2.14, that the indices of D_A and $D_{A,\psi}$ are equal. Our exposition is based on the following guideline. We merely quote general results about regular singular operators. Detailed proofs appear in [**6**, **7**]. However, we provide proofs of statements whose validity or interest depends on the presence of the perturbation A.

Assumption 2.5. Henceforth in discussing regular singular Dirac-Schrödinger operators we assume that A has been chosen so that the spectrum of T_A has empty intersection with $(-1/2, 1/2)$.

Lemma 2.6. *For any A satisfying our earlier assumptions $\{s : sA$ satisfies Assumption 2.5$\}$ contains an interval of the form $[k_1, \infty)$.*

Proof. T_{sA}^2 is the sum of a nonnegative term, a bounded term that is multiplied by s, and a bounded term that is positive, bounded away from zero, and multiplied by s^2. To show that the term multiplied by s is bounded, calculate with the symbols of the operators involved. □

Lemma 2.7 [**7**]**.** *Under Assumption 2.5 $D_{A,\psi}$, defined on smooth compactly supported sections, has a unique closed extension as an operator on L^2 sections. This extension is Fredholm.*

Notation 2.8. Henceforth we use $D_{A,\psi}$ to denote this closed extension.

Lemma 2.9. *$\{s : D_{sA,\psi}$ realizes its limiting index$\}$ contains an interval of the form $[k_2, \infty)$.*

Proof. Lemmas 2.6 and 2.7 show that there is such an interval of values of s for which $D_{sA,\psi}$ is Fredholm. Reasoning like that used in the proof of Lemma 2.6 shows that for large enough s the spectrum of T_{sA} has empty intersection with $[-1, 1]$ and so $T_{sA} + T_{sA}^2$ has a positive lower bound. Thus for large enough s_1 and s_2, each of $T_{s_1A} + T_{s_1A}^2$ and $T_{s_2A} + T_{s_2A}^2$ is bounded relative to the other. It follows, by calculations analogous to those done explicitly in Section 0, that for large enough s domain$(D_{sA,\psi})$ is independent of s. The existence of the positive lower bound for $T_{sA} + T_{sA}^2$ also implies that A is a bounded operator from the common domain of the $D_{sA,\psi}$ to the space of L^2 sections of the appropriate bundle. Thus for large enough s $D_{sA,\psi}$ is a norm-continuous family of bounded Fredholm operators from the common domain of the $D_{sA,\psi}$ to L^2. □

Lemma 2.10. *With domain equal to* domain$(D_{A,\psi})$*, D_A is a closed operator.*

Proof. It suffices to show that the graph norms of $D_{A,\psi}$ and D_A are equivalent. Recall that $\beta > -1/2$. By assumption in [**7**], $\widetilde{T}(r)$ is bounded uniformly in r relative to $\begin{pmatrix} \widetilde{D_0} & 0 \\ 0 & -\widetilde{D_1} \end{pmatrix}$. Thus we can find p' such that on $C_{p'}(N)$ $r^\beta \widetilde{T}(r)$ is $r^{-1}T_A$-bounded with relative bound less than 1. It follows that the restrictions to $C_{p'}(N)$ of $D_{A,\psi}$ and D_A have equivalent graph norms. The graph norms of these elliptic operators are equivalent on any compact submanifold with boundary. A partition of unity argument finishes the proof. □

Lemma 2.11 [6]. *The restriction of D_A to the set of smooth compactly supported sections has a unique closed extension.*

Remark 2.12. By Lemma 2.10 the domain of this closed extension equals the domain of $D_{A,\psi}$. Henceforth we use D_A to denote this closed extension.

Lemma 2.13 [7]. *D_A is Fredholm.*

Lemma 2.14. $\operatorname{Index}(D_{A,\psi}) = \operatorname{index}(D_A)$.

Proof. By the estimates in the proof of Lemma 2.10, a homotopy of ψ to the function that is constantly 1 provides a norm-continuous family of operators from domain (D_A) to the space of L^2 sections. $D_{A,\psi}$ is at one end of the family, D_A at the other end. □

Lemma 2.15. *$D_{A,\psi}$ satisfies the eventual positivity condition.*

Proof. Because the eventual positivity condition permits us to ignore an arbitrary compact set, it suffices to analyze $R(D_{tA,\psi})$ on the subset of $C_p(N)$ where $\psi \equiv 0$. Let D_ψ denote the operator agreeing with $\begin{pmatrix} 0 & D_1^* \\ D_0 & 0 \end{pmatrix}$ off $C_p(N)$ and unitarily equivalent to

$$\partial/\partial r + r^{-1} \begin{pmatrix} \widetilde{D_0} & 0 \\ 0 & -\widetilde{D_1} \end{pmatrix} + \psi(r) r^\beta \widetilde{T}(r)$$

over $C_p(N)$. (Here the unitary equivalence is the same one used with D_A.) On smooth compactly supported sections over the subset of $C_p(N)$ where $\psi \equiv 0$, the upper left corner of $R(D_{tA,\psi})$ is unitarily equivalent to $D^*_{tA,\psi} D_{tA,\psi} - D^*_\psi D_\psi$, which equals

$$r^{-2} \begin{pmatrix} t^2 \widetilde{A}^* \widetilde{A} & t(\widetilde{D_0}\widetilde{A}^* - \widetilde{A}^* \widetilde{D_1}) \\ t(\widetilde{A}\widetilde{D_0} - \widetilde{D_1}\widetilde{A}) & t^2 \widetilde{A}\widetilde{A}^* \end{pmatrix}.$$

A calculation with principal symbols shows that the off-diagonal blocks are vector-bundle maps. Much as in the proof of Theorem 1.4, the terms with positive lower bound that are multiplied by t^2 dominate the terms that are linear in t, and so for sufficiently large t the operator has a positive lower bound.

The lower right corner of $R(D_{tA,\psi})$ admits a similar analysis. □

Remark 2.16. Following [16] we can form on a compact manifold $\widehat{M}$ a Dirac-Schrödinger operator $\widehat{D_{\widehat{A}}}$ that is closely related to D_A. Deform the metric on M (preserving distances in the variable r) so that M has a cylindrical end. Deform $\mathcal{D}$ so that it remains a Dirac operator. (See Chapter 2 of [5].) Let $\widehat{M}$ be the double of the manifold with boundary of which the new M is the interior. Let $\widehat{E_0}$ be the vector bundle on $\widehat{M}$ formed by clutching two copies of E_0 by the identity map. (The formal requirement that clutching be done over the intersection of open sets can be met by extending slightly each of the manifolds with boundary making up $\widehat{M}$.) Let $\widehat{E_1}$ be the vector bundle formed by clutching E_1 on the first copy of the new M with E_0 on the second copy via a map that is homotopic to $(\mathcal{A}/|\mathcal{A}|)^{-1}$. There is a natural extension (actually a family of such, all homotopic) of $\mathcal{A}|_{M_p}$ on the first copy of M to a vector bundle map $\widehat{\mathcal{A}}$ that maps $\widehat{E_0}$ to $\widehat{E_1}$, that is invertible off the first copy of M_p, and that is the identity map over the second copy of M. Following the conventions used in the description of D_A, we can define an operator

$$\widehat{D_{\widehat{A}}} = \begin{pmatrix} \widehat{A} & \widehat{D_1}^* \\ \widehat{D_0} & -\widehat{A}^* \end{pmatrix}$$

from sections of $\widehat{S_+} \otimes \widehat{E_0} \oplus \widehat{S_-} \otimes \widehat{E_1}$ to sections of $\widehat{S_+} \otimes \widehat{E_1} \oplus \widehat{S_-} \otimes \widehat{E_0}$. Here the Dirac operators $\widehat{D_i}$ and the Dirac bundle $\widehat{S}$ can be constructed with the help of the doubling construction in [5]. Note that for any $x \in (0, p]$ this construction can be done in a way that preserves all structures over M_x in the first copy of M. Furthermore the constructions can be done so that for any x_1 and x_2 all structures used in making the result preserve structures over M_{x_1} are homotopic to the corresponding structures used in making the result preserve structures over M_{x_2}.

Lemma 2.17. *$\widehat{D_{\widehat{A}}}$ realizes its limiting index and satisfies the eventual positivity condition.*

Proof. Because $\widehat{M}$ is compact without boundary, the order zero term does not affect the index of a first order elliptic operator. A calculation with principal symbols shows that $R(\widehat{D_{t\widehat{A}}})$ is a vector-bundle map. The other part of the eventual positivity condition is vacuous because $\widehat{M}$ is compact. □

Theorem 2.18. *If $D_{A,\psi}$ realizes its limiting index, then* $\operatorname{index}(D_{A,\psi}) = \operatorname{index}(\widehat{D_{\widehat{A}}})$.

Proof. By Lemmas 2.15 and 2.17, this is a consequence of Theorem 1.4. □

Remark 2.19. The Atiyah-Singer index theorem provides a formula for the index of $\widehat{D_{\widehat{A}}}$.

3. K homology.

In this section we show that the regular singular Dirac-Schrödinger operator D_A described in Section 2 defines a class in the K homology of the metric completion of M. We show that if D_A realizes its limiting index, this class is the Kasparov product of classes defined by $\mathcal{D}$ and by $\mathcal{A}$. Without the limiting index assumption, the product result need only be corrected by a K homology class supported on a point.

In this section we assume for simplicity that M is connected. The reasoning in this section can be applied one component at a time to a manifold that is not connected. Let X be the metric completion of M. X arises by adjoining to M a point, corresponding to $r = 0$, for each connected component of N.

Let Y denote an oriented even-dimensional compact Riemannian manifold without boundary. Let $\mathcal{D}^Y$ be a Dirac operator on Y that acts on sections of a complex Dirac bundle S^Y. The positive and negative eigenspaces of the action of Y's volume element grade $S^Y = S^Y_+ \oplus S^Y_-$. Let E^Y_0 and E^Y_1 be Hermitian vector bundles on Y. Let $\mathcal{A}^Y : E^Y_0 \to E^Y_1$ be a vector bundle map. In this section the notation introduced in this paragraph refers to any such structures that satisfy the following conditions.

Condition 3.1. There is a continuous $f : Y \to X$ whose restriction to $f^{-1}(M)$ is an orientation-preserving diffeomorphism from $f^{-1}(M)$ to M.

Condition 3.2. For any $x \in (0, p]$, when we give Y a (new) metric making f an isometry on $f^{-1}(M_x)$, the restriction of f to $f^{-1}(M_x)$ can be covered by a vector bundle map that intertwines all structures for S and $\mathcal{D}$ with the (new) structures for S^Y and $\mathcal{D}^Y$. (To view S^Y as a Dirac bundle over Y with new metric, scale the actions of tangent vectors on S^Y by the changes in their lengths and follow the reasoning in Chapter 2 of [**5**].)

Condition 3.3. For any $x \in (0, p]$ we can give E^Y_0 and E^Y_1 metrics and metric connections and we can find a vector bundle map homotopic to $\mathcal{A}^Y$ and invertible over $Y \setminus f^{-1}(M_p)$ such that the following is true. There is a vector bundle map covering the restriction of f to $f^{-1}(M_x)$ that intertwines all of this structure with that of E_0, E_1, and $\mathcal{A}$. (In passing from the notation $\mathcal{A}^Y$ to the notation A^Y, we will follow the same convention as in passing from $\mathcal{A}$ to A. This convention was described in the sentence preceding definition 2.1.)

Remark 3.4. When we have made the choices described in Conditions 3.2 and 3.3, we will say that all structures on Y and M agree over M_x.

Remark 3.5. A Y satisfying these conditions arises from the doubling construction of Remark 2.16. There are interesting examples of such Y that do not arise as doubles. Examples can occur where M is the complement of of a finite subset of a closed manifold and Y is that manifold. Such an example was discussed in Section 0.

Notation 3.6. We use the notation $D^Y_{A^Y}$ in a manner analogous to the notation D_A.

Notation 3.7. For an operator γ we use the notation $\underline{\gamma}$ to denote the operator $\begin{pmatrix} 0 & \gamma^* \\ \gamma & 0 \end{pmatrix}$.

Proposition 3.8. *The operators we have studied define the following KK cycles.*

(a) $\left(\mathcal{D}\,(1+\mathcal{D}^2)^{-1/2}, L^2(S_+)\oplus L^2(S_-)\right)$ *defines a class we denote* $[\mathcal{D}]$ *in* $KK(C_0(M),\mathbf{C})$. *To construct this cycle we need to choose a closed extension of* $\mathcal{D}$, *but all choices define the same* KK *class,* [**3**].

(b) $\left(\mathcal{D}^Y\,(1+(\mathcal{D}^Y)^2)^{-1/2}, L^2(S^Y_+)\oplus L^2(S^Y_-)\right)$ *defines a class we denote* $[\mathcal{D}^Y]$ *in* $KK(C(Y),\mathbf{C})$.

(c) $\left(\underline{A}\,(1+(\underline{A})^2)^{-1/2}, C_0(E_0)\oplus C_0(E_1)\right)$ *defines a class we denote* $[A]$ *in* $KK(C(X),C_0(M))$.

(d) $\left(\underline{A^Y}\,(1+(\underline{A^Y})^2)^{-1/2}, C(E^Y_0)\oplus C(E^Y_1)\right)$ *defines a class we denote* $[A^Y]$ *in* $KK(C(Y),C(Y))$.

(e) $$\left(\underline{D^Y_{A^Y}}\left(1+(\underline{D^Y_{A^Y}})^2\right)^{-1/2}, L^2(S^Y_+\otimes E^Y_0\oplus S^Y_-\otimes E^Y_1)\oplus L^2(S^Y_+\otimes E^Y_1\oplus S^Y_-\otimes E^Y_0)\right)$$
defines a class we denote $[D^Y_{A^Y}]$ *in* $KK(C(Y),\mathbf{C})$.

In all cases algebras of functions act by pointwise multiplication, and inner products arise from the pointwise inner products on the Hermitian bundles.

Proof. See [**4**] for definitions. The assertions involving the first order elliptic differential operators (assertions (a), (b) and (e)) are proven in [**3**]. (The assertions (b) and (e) concerning the compact manifold Y were known previous to [3]. They can be proven by using the pseudodifferential calculus and Rellich's lemma. The focus of the part of [3] to which we refer is the extension of these results to the noncompact case.) To prove assertions (c) and (d) about the vector-bundle maps, one needs to know that if F is a finite-dimensional Hermitian vector bundle over a manifold Z, the compact operators on $C_0(F)$, viewed as a Hilbert $C_0(Z)$-module, are the elements of $C_0(End(F))$. □

Lemma 3.9. *The classes associated with $\mathcal{D}^Y$, A^Y, and $D^Y_{A^Y}$ in the above proposition are not changed by the deformations described in Conditions 3.2 and 3.3.*

Proof. The effect of the deformations is limited to homotopies of the KK cycles. □

Lemma 3.10 [7], [8]**.** *Negative powers of $1 + (\underline{D_A})^2$ are compact on L^2 sections.*

Theorem 3.11. *$\left(\underline{D_A}\left(1 + (\underline{D_A})^2\right)^{-1/2}, L^2(S_+ \otimes E_0 \oplus S_- \otimes E_1) \oplus L^2(S_+ \otimes E_1 \oplus S_- \otimes E_0)\right)$ is a KK cycle defining a class we denote $[D_A]$ in $KK(C(X), \mathbf{C})$. Here functions in $C(X)$ act by pointwise multiplication.*

Proof. We are working with a self-adjoint operator. Lemma 3.10 shows that its square differs from the identity by a compact operator. To show that it has compact commutator with each element of $C(X)$, it suffices (because the algebra of compact operators is norm-closed) to calculate explicitly with functions that are smooth on M and locally constant in some neighborhood of $X \setminus M$. The calculations proceed by the commutator identity and "integral trick" of [2]. Details are analogous to those in the proof of Lemma 1.1 of [17]. □

We will use a subscript $*$ on a map of C^* algebras to denote the associated map on KK groups that is contravariant with respect to the algebra map on first entries. We will use a superscript $*$ for the map that is covariant on the second entries. Our map $f : Y \to X$ defines via composition a map $F : C(X) \to C(Y)$. The inclusions $M_x \subset M \subset X$ define, via extension by zero, maps we will call $R_0 : C_0(M) \to C(X)$ and $R_x : C_0(M_x) \to C_0(M)$. We proceed to establish relationships between the KK classes represented by cycles defined on M and the classes represented by analogous cycles on Y. The relationships are expressed in terms of the effects of F_* and F^*.

Lemma 3.12. *$F_*([A^Y]) = (F \circ R_0)^*([A]) \in KK(C(X), C(Y))$.*

Proof. By Condition 3.3 both are the image under $F^*_{p/2}$ of the same class in $KK(C(X), C_0(M_{p/2}))$. To see this use a homotopy that takes invertibility to unitarity over a neighborhood of $Y \setminus f^{-1}(M_{p/2})$ and use excision. □

Lemma 3.13. *Suppose elements α and β of $KK(C_0(M), \mathbf{C})$ are such that for each x $(R_x)_*(\alpha) = (R_x)_*(\beta)$. Then $\alpha = \beta$.*

Proof. The proof follows the proof of Proposition 2.3 of [**17**] in relying on the exact sequence

$$0 \to \lim{}^1 KK_{-1}(C_0(M_{p/n}), \mathbf{C}) \to KK(C_0(M), \mathbf{C}) \\ \to \varprojlim KK(C_0(M_{p/n}), \mathbf{C}) \to 0$$

of [**23**]. Here the limits are associated with $n \to \infty$. Because for all x_1 and x_2 $C_0(M_{x_1})$ and $C_0(M_{x_2})$ are homotopically equivalent, the $\lim^1$ term in the sequence vanishes. □

Lemma 3.14. $(F \circ R_0)_*([\mathcal{D}^Y]) = [\mathcal{D}] \in KK(C_0(M), \mathbf{C})$. $(F \circ R_0)_*([D^Y_{A^Y}]) = [D_A] \in KK(C_0(M), \mathbf{C})$.

Proof. The reasoning is the same for both statements. In our notation we focus on the first statement. We now establish that the cycles we are considering satisfy the hypotheses of Lemma 3.13. Choose arbitrary $x \in (0, p]$. Using Conditions 3.1-3.3, choose a representative of $[D^Y]$ arising from structures on Y that agree with those on M over $M_{x/2}$. To show that the two KK classes have the same image under $(R_x)_*$ we use the finite propagation speed argument of [**3**] to handle the nonlocal operators arising from the $-1/2$ powers of the differential operators. We briefly summarize the argument below. (For more details see [**3**] or the exposition in [**17**] based on the preprint of [**3**].) The finite propagation speed argument allows us to write the operator in each of the KK cycles we are considering as the sum of an operator that increases supports by less than $x/2$ and an operator that is continuous from L^2 to the domain of an arbitrarily high power of $1 + \mathcal{D}^2$, respectively $1 + (\mathcal{D}^Y)^2$. Moreover the terms with limited support increase can be chosen so that their compositions with the action of any $h \in C_0(M_x)$ are equal. (This follows from the uniqueness of solutions of the relevant system of differential equations.) It follows that for any $h \in C_0(M_x)$ the composition of h's action with the difference of the operators arising in the KK cycles is compact. Thus the images under $(R_x)_*$ of the classes of our cycles represent the same class, and Lemma 3.13 applies. □

Proposition 3.15. *If D_A realizes its limiting index, then $F_*([D^Y_{A^Y}]) = [D_A] \in KK(C(X), \mathbf{C})$.*

Proof. There is an exact sequence $\to KK(C(X \backslash M), \mathbf{C}) \to KK(C(X), \mathbf{C}) \to KK(C_0(M), \mathbf{C}) \to$. Because M is connected, the image of $KK(C(X\backslash M), \mathbf{C})$ in $KK(C(X), \mathbf{C})$ maps isomorphically to the image of $KK(C(X), \mathbf{C})$ in $KK(\mathbf{C}, \mathbf{C}) = \mathbf{Z}$ under the index homomorphism. Lemma 3.14 shows that $F_*([D^Y_{A^Y}])$ and $[D_A]$ have the same image in $KK(C_0(M), \mathbf{C})$. The indices of D_A and $D^Y_{A^Y}$ are equal by Lemma 2.14 and the reasoning proving Lemma 2.17 and Theorem 2.18. (Although Lemma 2.17 and Theorem 2.18 were stated for the operator $\widehat{D_{\widehat{A}}}$ on the manifold $\widehat{M}$, this operator and manifold can be replaced by $D^Y_{A^Y}$ and Y, respectively. This assertion follows from the observation that the proofs of Lemma 2.17 and Theorem 2.18 depend only on $\widehat{M}$ being compact without boundary and on the Dirac-Schrödinger operator $\widehat{D_{\widehat{A}}}$ matching $D_{A,\psi}$ in a neighborhood of the set where A is not invertible.) □

Remark 3.16. The above proof shows that if D_A does not realize its limiting index, one need only correct Proposition 3.15 by a term supported on a point that reflects the difference in indices.

Lemma 3.17. *$[D^Y_{A^Y}]$ equals the Kasparov product $[A^Y] \otimes_{C(Y)} [\mathcal{D}^Y]$ in $KK(C(Y), \mathbf{C})$.*

Proof. By homotopy we can ignore the vector bundle maps. This result is a consequence of the connection approach to products, the pseudodifferential calculus, and Rellich's lemma. (In fact this standard result is the motivation for the terminology "connection approach to products.") □

Theorem 3.18. *If D_A realizes its limiting index, then $[D_A]$ equals the Kasparov product $[A] \otimes_{C_0(M)} [\mathcal{D}]$ in $KK(C(X), \mathbf{C})$.*

Proof. $[D_A] = F_*([D^Y_{A^Y}]) = F_*([A^Y] \otimes_{C(Y)} [\mathcal{D}^Y]) = F_*([A^Y]) \otimes_{C(Y)} [\mathcal{D}^Y] = (F \circ R_0)^*([A]) \otimes_{C(Y)} [\mathcal{D}^Y] = [A] \otimes_{C_0(M)} (F \circ R_0)_*([\mathcal{D}^Y]) = [A] \otimes_{C(X)} [\mathcal{D}]$. The equalities arise from Proposition 3.15, Lemma 3.17, associativity of the Kasparov product, Lemma 3.12, associativity of the Kasparov product, and Lemma 3.14. The second application of associativity of the Kasparov product is due to the observation (see [4]) that $(F \circ R_0)^*$ can be represented by a Kasparov product on the right with a KK class whose Kasparov product on the left represents $(F \circ R_0)_*$. □

References

[1] N. Anghel, *On the index of Callias-type operators*, Geom. and Funct. Anal., **3** (1993), 431-438.

[2] S. Baaj and P. Julg, *Théorie bivariante de Kasparov et opérateurs non bornés dans les C^*-modules Hilbertiens*, C. R. Acad. Sci., Paris, **296** (1983), 875-878.

[3] P. Baum, R. Douglas and M. Taylor, *Cycles and relative cycles in analytic K-homology*, J. Diff. Geom., **30** (1989), 761-804.

[4] B. Blackadar, *K-Theory for Operator Algebras*, Springer-Verlag, New York, 1986.

[5] B. Booss-Bavnbek and K. Wojciechowski, *Elliptic Boundary Problems for Dirac Operators*, Birkhäuser, Boston, 1993.

[6] J. Brüning, *L^2-index theorems on certain complete manifolds*, J. Diff. Geom., **32** (1990), 491-532.

[7] J. Brüning and R. Seeley, *An index theorem for first order regular singular operators*, Am. J. Math., **110** (1988), 659-714.

[8] ———, *The resolvent expansion for second order regular singular operators*, J. Funct. Anal., **73** (1987), 369-429.

[9] U. Bunke, *A K-theoretic relative index theorem and Callias-type Dirac operators*, Math. Ann., **303** (1995), 241-279.

[10] C. Callias, *Axial anomalies and index theorems on open spaces*, Commun. Math. Phys., **62** (1978), 213-234.

[11] J. Cheeger, *Spectral geometry of singular spaces*, J. Diff. Geom., **18** (1983), 575-657.

[12] A. Chou, *The Dirac operator on spaces with conical singularities and positive scalar curvatures*, Trans. of the Amer. Math. Soc., **289** (1985), 1-40.

[13] J. Fox and P. Haskell, *Comparison of perturbed Dirac operators*, Proceedings of the Amer. Math. Soc., **124** (1996), 1601-1608.

[14] ———, *Index theory for perturbed Dirac operators on manifolds with conical singularities*, Proceedings of the Amer. Math. Soc., **123** (1995), 2265-2273.

[15] C. Gajdzinski, *L^2-index for perturbed Dirac operator on odd dimensional open complete manifold*, preprint, 1993.

[16] E. Guentner and N. Higson, *A note on Toeplitz operators*, preprint, 1993.

[17] P. Haskell, *Index theory of geometric Fredholm operators on varieties with isolated singularities*, K-Theory, **1** (1987), 457-466.

[18] S. Klimek and A. Lesniewski, *Local rings of singularities and $N = 2$ supersymmetric quantum mechanics*, Commun. Math. Phys., **136** (1991), 327-344.

[19] H. B. Lawson and M.-L. Michelsohn, *Spin Geometry*, Princeton Univ. Press, Princeton, 1989.

[20] M. Lesch, *Deficiency indices for symmetric Dirac operators on manifolds with conic singularities*, Topology, **32** (1993), 611-623.

[21] J. Råde, *Singular Yang-Mills fields* II. *Global theory*, preprint, 1993.

[22] J. Roe, *Coarse Cohomology and Index Theory on Complete Riemannian Manifolds*, Memoirs of the Amer. Math. Soc., **497**, Amer. Math. Soc., Providence, 1993.

[23] J. Rosenberg and C. Schochet, *The Künneth and universal coefficient theorem for Kasparov's generalized K-functor*, Duke Math. J., **55** (1987), 431-474.

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